TOO MUCH INVESTMENT: A PROBLEM OF COORDINATION FAILURE

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Too Much Investment: A Problem of Coordination Failure*

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Abstract

This paper shows that coordination failure and contractual incompleteness can lead to socially excessive investment. Firms and workers choose investment levels, then enter a stochastic matching process. If investment levels are discrete, and match frictions are low, high-investing workers (firms) impose a negative pecuniary externality on any worker (firm) who cuts investment. Specifically, an agent cutting investment subsequently bargains with a partner with a binding outside option due to the fact that it can easily match with another high investor. The deviant thus bears the full loss in revenue from its action. However, given enough complementarity in investments, when one agent cuts investment it is efficient that its partner also does so. So, only part of the cost saving accrues to the deviant, with the implication that the net private gain to cutting investment is less than the social gain. A similar argument establishes that over-investment can occur when agents are heterogenous i.e. differ in their cost of investing, even if investments are continuous. Then, over-investment occurs because low-cost investors have a private incentive to invest to shift rent away from high-cost investors. Our model can also explain some recent trends in graduate/non-graduate wage differentials.

Keywords: hold-up, coordination failure, matching, over-investment.
JEL Classification Numbers: D23, D62, J31

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1. Introduction

A familiar policy concern is that coordination failure may strand an economy in a Pareto dominated low-investment equilibrium. For example, beneficial innovations may never get off the ground since there is no point in firms upgrading plant if workers lack the skills to use it and workers have no reason to train if the plant to make use of the skills is not built. The possibility of such equilibria arise when investment decisions are taken independently and are only individually profitable when enough agents invest. Variations on this theme of strategic complementarity and market failure are Scitovsky (1954), Murphy Shleifer and Vishny (1989), Redding (1996), Acemoglu (1996), and Masters (1998) amongst others.

A further source of underinvestment is hold-up. First analysed by Williamson (1975, 1985)[1], this arises when contractual incompleteness allows one party to bargain for a share of the quasi-rents created by the complementary investments of another. If *ex ante* contracts are insufficiently complete to prevent the threat of noncooperation being subsequently exercised, the division of the returns must be bargained *ex post* and the investor fails to capture all the social returns. Hold up can thus be viewed as a tax on investment.

Taken together, coordination failure and hold-up suggest that market economies may be vulnerable to under-investment. This paper shows to the contrary that the two problems may interact in such a way as to lead to *over-investment*. When enough agents invest, non investors are put at so much of a bargaining disadvantage that they too invest to redress the balance, a private but not a social gain.

In our model, firms and workers make complementary general investments before the two types of agents are allocated to each other via a matching process. This sequence reflects the inability of workers and firms to anticipate all their future partners and so contract with them.[2] Once matched, firms and workers bargain over the division of the surplus using a standard alternating-offers bargaining protocol. In the bargaining, crucially, agents have an outside option: at each stage, the responder can return to searching for a new partner. For simplicity, suppose that investment is binary: agents can invest or not. In this set-up, if enough agents invest, this undermines the bargaining power of

[1] Other notable references are Klein, Crawford and Alchian (1978), Grout (1984), Crawford (1988), Grossman and Hart (1986), North and Weingast (1989), Hart and Moore (1990), and Shleifer and Vishny (1997).

[2] In the formal modelling there is no exogenous match break up. Introducing this would add to the plausibility of assuming that long-lived general investments are not governed by contracts, but is otherwise innocuous.

non investors. Specifically, when search frictions are low, an investor matched with a non investor has a binding outside option and thus appropriates most of the surplus.

So, investment generates a negative pecuniary externality which works through the outside option; the investments of other workers (firms) allows any particular firm (worker) who invests to appropriate more of the surplus from a match with a non-investor than would be the case with bilateral monopoly. Because of this negative externality, equilibrium investment can be inefficiently high[3]: specifically, the private loss to one partner (firm or worker) from not investing can exceed the cost saving, even when it is socially efficient not to invest.[4] In perfectly competitive economies these issues do not arise for an agent's payoff does not depend on the characteristics of the particular partner with whom they produce. Yet in few markets can vacancies be filled instantaneously or job offers found without delay, as an extensive theoretical and empirical literature attests. When there are search frictions, even small ones, our conclusions follow.

For the case of homogeneous agents we identify two features that are generally required for equilibrium investment to be excessive. First, investments must be complementary (the cross partial of the production function is positive). Second, investments must be discrete rather than infinitely divisible. However, the size of the physical unit of investment (step-size) needed is bounded below *only relative to the level of matching friction*: as the matching process becomes frictionless, the minimum step-size required goes to zero.

The second part of the paper shows that the same kind of pecuniary externality can lead to overinvestment when agents are heterogenous, even if investments are continuous and output is concave in investments. Suppose that a fraction of firms and workers have a low cost of investment (low-cost agents) and the complementary fraction have a high cost of investment (high-cost agents). Moreover, high-cost agents do not invest in equilibrium,

[3] The possibility of hold up leading to overinvestment has been noted by Grossman and Hart (1986) and Hart (1995). They point out that, under Nash bargaining, if an increment in investment augments disagreement payoffs by more than the relationship surplus, its private return exceeds the social return. Yet it seems implausible that this condition would be met, and indeed Hart (1995) assumes it away. This does appear very reasonable. If investment has a higher return within another easily accomplished match, with the result that walking out presents a credible threat in bargaining with the current partner, then the alternative relationship is an obvious candidate to be the equilibrium match. It is possible that the marginal return to investment is high in another relationship though the total surplus is relatively low, but it is not easy to tell a convincing story.

[4] Che and Sakovics (2003) and Pitchford and Snyder (2004) show that in a two-agent model in which investment occurs post matching but can be postponed at little cost there is an efficient equilibrium in addition to outcomes in which investment is inefficiently low. In our model, investment is chosen prior to the relationship forming so their "punishment" mechanism does not apply.

because the cost of investment is too high. With sufficient search frictions, a low-cost agent will match with a high-cost agent (non-assortative matching). Assume that in this event, the investments are complementary in the sense that the investment of the low-cost agent is unproductive[5]. Nevertheless, an investment by the low-cost agent in such a match enhances his bargaining power by creating - or increasing the value of - a binding outside option through the opportunity to find a match with another low-cost agent who has also invested. This rent-transfer opportunity is privately profitable but not socially beneficial.

Our results are applicable to a wide range of settings; workers and employers, lenders and borrowers, and indeed buyers and sellers of all sorts may invest prior to meeting their future trading partners. A particularly interesting case is general education and training. Our results have two interesting implications here, one normative, and one positive.

The first point is that in our model, equilibrium wage differentials are not necessarily a guide to the social desirability of further investment in education. In the simplest version of our model, with binary investments, interpret investment by workers as the acquisition of a college degree. For some parameter values, there will be two equilibria, one where workers acquire degrees, and one where they do not. At the investment equilibrium, the graduate/non-graduate wage differential will exceed the cost of a degree and thus exceed the same differential at the non-investment equilibrium. But again for some parameter values, the investment equilibrium is inefficient. In this case, it is efficient to *cut* education levels even though the private return to education appears high.

Second, the model developed here can also help explain the puzzling phenomenon, documented by Acemoglu (1999), that in the US and elsewhere an upsurge in the number of graduates has been accompanied by an increase in their wages *and* a decrease in those of non-graduates. A possible explanation is that exogenous technical progress has increased the relative productivity of graduates (Katz and Murphy (1992) and Card and Lemieux (2001) develop this line of argument), but the fall in the absolute wage of non graduates is not so easily accounted for.[6]

Acemoglu(1999) provides an explanation based on search costs[7]. According to this firms must commit to investment prior to matching. When there are relatively few graduates in the population there is a pooling equilibrium. Firms hire both types of worker

[5] This will occur, for example, if technology is Cobb-Douglas: then zero investment by one party implies zero marginal product of investment by any other party.

[6] Substitutability between the two forms of labor could result in biased technical progress yielding the a decrease in both the wage and number of non graduates but the conditions are stringent.

[7] Machin and Manning (1997) offer a model with related features though it is based on preference rather than productivity differences between groups of workers.

having committed to an investment level intermediate between that appropriate for non graduates and the higher level that is optimal for a graduate. When there are more graduates in the working population, the expected search cost of finding a graduate declines. Now a separating equilibrium emerges; there are high-investment graduate-only jobs and low investment non-graduate jobs. The increased number of graduates causes the wage of non-graduates to fall because they have less capital to work with. Our analysis supplements this account by showing that *even if firms do not decrease investment in non graduate jobs*, the outside-option principle implies that having more graduates depresses the wages of non graduates.

Suppose initially that firms have no choice over investment levels so there is no issue of pooling or separation. Also, simplify our model by taking the number of graduates as exogenous. When the proportion of graduates is low and a firm bargains with a non-graduate, the firm's outside option does not bind as the probability of finding a graduate is low. When the proportion of graduates is sufficiently high, though, the firm's outside option starts to bind: from this point on, any increase in the number of graduates depresses the wage of the non-graduate. Contrary to Acemoglu, this is not due to any change in investment by firms.

Endogenising investment by firms however, augments the effect. When there are many graduates our analysis implies that firms may also invest more. At first sight this is good news for the non-graduates who have more capital to work with. Assuming complementarity, the extra investment raises the output of a graduate by more than a non graduate so the firm's outside option may increase by more than the revenue generated by a match with a non graduate. Hence the extra investment further depresses the wage of non graduates whilst increasing that of graduates.

The remainder of the paper is organized as follows. In Section 2, which follows, we illustrate the main results in a simple numerical example with binary investments which are perfect complements. The generality of the over-investment result is then investigated. Section 3 analyses a full dynamic matching and bargaining model where investors have outside options, and studies equilibrium investment levels in this setting, allowing investments to be discrete or infinitely divisible, and permitting any degree of complementarity. Stability issues are addressed and wage patterns in different equilibria are compared. Section 4 extends the model to allow for heterogenous agents and shows that this introduces a new source of overinvestment that arises even when investment is a continuous variable. Section 5 discusses related literature, and Section 6 concludes.

2. An Example

This simple example illustrates the working of the pecuniary externality discussed in the Introduction. At $t = 0$, there are equal numbers of new workers and firms who simultaneously make binary investment decisions. A worker (resp. firm) can choose training (resp. investment) at a cost of 1.5. A firm and worker produce a present value of 8 if both invest but output is 6 otherwise (investments are perfect complements).

Each subsequent period $t = 1, 2...$ a random process pairs unmatched firms and workers. Matching frictions are "small". Paired firms and workers bargain over the division of revenue from production. Once they reach agreement, they start producing. In this event, they permanently exit the matching process. The bargaining protocol between a matched firm and worker is alternating offers, with the proviso that in each round of bargaining the responder can also choose to re-enter the matching process. So, as the length of the period tends to zero the two parties split the surplus equally unless if one agent has an outside option in excess of the equal division (a binding outside option). In this event, that agent has his outside option payoff and the other agent gets the remainder i.e. is the residual claimant.

In this model, investment is inefficient: investment by both boosts revenue by $8-6 = 2$ but costs 3. Nevertheless, there is an equilibrium where all agents invest. In equilibrium, the payoff to each investor is $\frac{1}{2}8 - 1.5 = 2.5$. Consider a deviant choosing not to invest. If subsequently matched with an investor, the two bargain over the division of 6 units of revenue. As the investor can to break off the bargaining and almost instantly match with another investor, the investing agent's outside option is approximately 4 and thus binds. So, a deviant non-investor obtains approximately $6 - 4 = 1.5$. This confirms that it is an equilibrium for all to invest. Note that there is also an equilibrium where no-one invests. So, the investment equilibrium is inefficient - and also Pareto-dominated by the no-investment equilibrium.

Note that this example, while it clearly establishes the intuition for our general results, is limited in several ways. First, there are some special assumptions: binary investments which are perfect complements. Second, the key claim that the outside option of the investor is binding in the match with the non-investor, thus penalising non-investment, is - while plausible - not rigorously established. We now turn to a general model that relaxes the special assumptions, and also carefully defines the set of agents, the timing of events, strategies and the equilibrium concept.

3. The Model

3.1. Preliminaries

There are two types of agents: firms and workers. Time is discrete, with a period length of Δ, so $t = 0, \Delta, 2\Delta, ..$ and all agents have a discount factor $\delta = e^{-r\Delta}$. At period 0, a unit measure of each of workers and firms make investments $e,\ i$ respectively. Investments may be discrete or continuous. If investments are continuous, $e, i \in \Re_+$. If investments are discrete, $e, i \in \{0, \frac{1}{n}, \frac{2}{n}..\} = S_n$ where $\frac{1}{n}$ is the investment step size. Whether continuous or discrete, investments have a cost of $c(e), c(i)$, where $c(.)$ is strictly increasing, differentiable, and convex with $0 < c'(0) < \infty$.

In periods $t = \Delta, 2\Delta, ...$, the following events occur in each period t. First, a fraction Δa of the measure of as yet unmatched[8] firms and workers, μ_t, are randomly matched with each other. That is[9], every worker is matched with a firm (and vice versa) with probability Δa. If both firm and worker are matched, they decide simultaneously and independently whether to accept or reject the match. If they both accept, one of them is then randomly selected to be proposer in bargaining over revenue. A firm and worker can produce present value of revenue of $y(e, i)$: the properties of this revenue function are discussed below.

Then, all such proposers, *plus* all proposers in matches formed in previous periods that have not yet reached agreement over the division of revenue, can propose a division of revenue. The responder can accept, reject, or terminate the match. If the proposal is accepted, the matched firm and worker start producing in the following period. If the responder rejects, then he is proposer at $t + \Delta$. If the responder terminates the match, both parties return to the unmatched state at the beginning of the next period, $t + \Delta$.

Note that in contrast to the bilateral monopoly case (with just one firm and one worker), agents have two *outside options* in this model. First, an agent can reject a match, and continue searching. Second, a responder can exit back to the searching state. It is these outside options that drive our results.

[8] As any firm must exit matched with a worker, in any period, the measures of firms and workers in the unmatched state are the same. Also, $\mu_\Delta = 1$.

[9] For concreteness, think of a two-stage matching process where measure Δa agents on either side of the market are randomly selected from the pool of the unmatched, and then these Δa workers and firms are randomly matched with each other. The existence of such procedure even with a continuum on each side of the market is guaranteed by the arguments of Alos-Ferrer(2002).

3.2. The Revenue Function

Our assumptions on the revenue function are more easily stated assuming that investments are continuous: the same properties are assumed to hold, appropriately restated, when there is a fixed investment step-size. We assume that $y(e,i)$ is non-negative, symmetric, and strictly increasing and strictly concave in its arguments for all (e,i) in $\Re_+^2$. In addition to these baseline assumptions, we will additionally make one of two assumptions. First:

A1. *$y(e,i)$ is twice continuously differentiable in $e, i \in \Re_+$, and $y_{12} \geq 0$.*

Here, subscripts on y denote partial derivatives. Given differentiability, $y_{12} \geq 0$ indicates that investments are complements. If $y_{12} > 0$, we say that the investments are *strict complements*: otherwise, they are *weak complements*.

Clearly, A1 limits the degree of complementarity, as then by concavity, $y_{12} < \sqrt{y_{11} \cdot y_{22}}$. In particular, A1 rules out the case of perfect complements (as in the Example above). So it is also desirable to consider the case of perfect complements, which is most generally expressed as A2:

A2. *$y(e,i) = f(\min\{e,i\})$ where f is strictly increasing, differentiable, and concave on $\Re_+$.*

Finally, we assume the following limit conditions. If A1 holds, we assume that $\lim_{e \to 0} y_k(e,e) = \infty$, $\lim_{e \to \infty} y_k(e,e) = 0$ where $k = 1,2$ is the kth derivative of y, and if A2 holds, we assume that $\lim_{e \to 0} f'(e,e) = \infty$, $\lim_{e \to \infty} f'(e,e) = 0$.

3.3. Strategies and Equilibrium

Section 3.1 above describes a stochastic game played at periods $t = 0, \Delta...$ by a continuum of players. In this game, we restrict attention to equilibria where all agents invest e^* at $t = 0$. So, in any continuation game, we can assume that all but a measure zero of agents have invested e^*, as any potential deviant is insignificant i.e. measure zero. So, whatever the match acceptance and bargaining strategies of the agents, we note that at $t \geq 1$, almost all unmatched agents have invested e^*. So, if a firm f and a worker w are matched at the beginning of period t, the only payoff-relevant aspects of the history of play for this pair are (i) their two investment levels[10] e_w, e_f : (ii) the equilibrium investment e^* made by almost all agents.

We will say that the match acceptance strategy of an agent is *Markov* if it only depends on e_w, e_f, e^*. Also, we will say that if bargaining, a strategy for the proposer or responder

[10] By definition, one of $e_w, e_f = e^*$.

is *Markov* if at any date, the amount of revenue offered to the responder, or if the choice to accept, reject, or opt out only depends on e_w, e_f, e^*. Note that if (almost) all agents follow Markov strategies, the expected payoff to being unmatched in at the beginning of any period $t = \Delta, 2\Delta, ...$ will only depend on that agent's own investment $e' = e_w, e_f$ and e^*, and not on any other aspect of the history of the game: let this payoff be $v(e', e^*)$.

Within this class of strategies, we will focus on *perfect* match acceptance and bargaining strategies of the agents. Such a match acceptance strategy is one where an agent accepts a match at any date iff doing so gives a higher payoff than continued search. Bargaining strategies are perfect if they are subgame-perfect in the alternating-offers bargaining game between the two partners f and w, with outside options[11] $v(e_f, e^*)$, $v(e_w, e^*)$ generated when these same Markov-perfect strategies are played by almost all agents. It is well-known that such a game has a unique subgame-perfect equilibrium[12], where agreement is immediate (Muthoo(1999)).

Note that as $\Delta \to 0$, $v(e', e^*)$ is also the equilibrium continuation payoff in period Δ onwards, (discounted back to $t = 0$) to a deviant agent if he invests some level e' at $t = 0$, and all other agents invest at the equilibrium level e^*. This function is key, as it will determine equilibrium investments. To characterise $v(e', e^*)$ further, we have:

Lemma 1. *Conditional on (almost) all agents investing e^*, there is a unique continuation equilibrium in subgame-perfect Markov strategies and thus $v(e', e^*)$ is uniquely defined. Moreover, in the limit as $\Delta \to 0$,*

$$v(e', e^*) = \begin{cases} 0.5\phi y(e', e^*), & \text{if } e' \geq \underline{b}(e^*) \\ \phi(y(e', e^*) - 0.5\phi y(e^*, e^*)) & \text{if } \underline{b}(e^*) > e' \geq \underline{\underline{b}}(e^*) \\ 0 & \text{if } \underline{\underline{b}}(e^*) > e' \end{cases} \tag{3.1}$$

where $\phi = \frac{a}{a+r}$, and $\underline{b}(e^*)$, $\underline{\underline{b}}(e^*)$ solve

$$y(\underline{b}(e^*), e^*) = \phi y(e^*, e^*), \;\; y(\underline{\underline{b}}(e^*), e^*) = 0.5\phi y(e^*, e^*)$$

and thus satisfy $e^* > \underline{b}(e^*) > \underline{\underline{b}}(e^*) > 0$.

From (3.1), it is clear that the continuation payoff to deviation from equilibrium investment embodies the two outside options discussed above, at the match acceptance stage, and the bargaining stage. The formula for $v(e', e^*)$ says that if the deviation e' is

[11] Given that strategies are Markov, these outside options are stationary i.e. time independent.

[12] In the limit as $\Delta \to 0$, the "outside option principle" applies: the two agents split the revenue equally, unless one of them has an outside option greater than half the revenue, in which case that agent gets his outside option, and the other agent gets what remains i.e. is the residual claimant.

not too low ($e' \geq \underline{b}(e^*)$), the deviant will find a match with a non-deviant, and will get half the revenue from a subsequent match - the *equal division* case. If e' is intermediate ($\underline{b}(e^*) > e' \geq \underline{\underline{b}}(e^*)$), the deviant again finds a match but will face a binding outside option because the revenue $y(e', e^*)$ to be shared is small relative to the outside option of the non-deviant, which is $v(e^*, e^*) = 0.5\phi y(e^*, e^*)$ from the above formula - the *residual claimant* case. If e' is very small ($\underline{\underline{b}}(e^*) > e'$), the deviant cannot even find a match because total revenue to be divided is $y(e', e^*)$ is smaller than the outside option of the non-deviant, $0.5\phi y(e^*, e^*)$.– the *no-match* case. Note that all revenues are "discounted" by the parameter ϕ, which measures match friction: with match friction, there is an expected delay before a match and thus before revenue can be generated.

We are now in a position to define an equilibrium investment. An *equilibrium investment* is an $e^* \in \mathcal{F}$ such that

$$v(e^*, e^*) - c(e^*) \geq v(e', e^*) - c(e'), \text{ all } e' \in \mathcal{F},$$

where $v(e', e^*)$ is defined in (3.1), $\mathcal{F} = \Re$ if investments are continuous, and $\mathcal{F} = S_n$ if investments are discrete. The net payoff to deviation from equilibrium investment e^*, is illustrated in Figure 1 below.

Figure 1 in here

Note that our definition of equilibrium investment is "supported" by the assumption of a particular continuation equilibrium in the sub-game once investments have been chosen. We have three defenses of this. First, our main objective in this paper is to demonstrate the possibility of overinvestment equilibria, not to characterise all the equilibria in the model. Second, the restrictions on strategies seem reasonable, and are widely used in the matching and bargaining literature (e.g. Osborne and Rubinstein(1990), Coles and Muthoo(1998)). Thirdly, this particular equilibrium is "focal": it is stationary, and the relationship between the outside options (at the match and bargaining stages) and the incentive to invest is particularly clear.

4. Efficient and Equilibrium Investments

4.1. Efficient Investments

In this model, the ability of firms and workers to start production is constrained by search frictions, and this should be taken into account when defining efficient investments. As payoffs are linear in consumption (i.e. quasi-linear), the natural efficiency criterion is the sum of the payoffs to search net of investment costs at some common[13] level of investment e, given that all matches are accepted. Following the proof of Lemma 1, it can be shown that the payoff to search net of investment costs for either firm or worker in this scenario is $0.5\phi y(e,e) - c(e)$. So, this criterion is

$$W(e) = 2[0.5\phi y(e,e) - c(e)] = \phi y(e,e) - 2c(e)$$

The efficient level of investment $\hat{e}$ or $\hat{e}_n$ maximizes $W(e)$ subject to $e \in \Re_+$ or S_n, depending on whether investments are continuous or discrete. The limit conditions on the derivatives of the production and cost functions, ensure that this problem has an interior solution[14]. Then, given the strict concavity of y - and therefore W - we can be sure that this solution $\hat{e}$ is characterised by the relevant first-order condition in the continuous case, and in the discrete case, with step-size $\frac{1}{n}$, the solution $\hat{e}_n$ is the closest one - in terms of payoff W - to $\hat{e}$. So, we have:

Proposition 1. *Assume that investments are continuous. If A1 holds, the efficient level of investment $0 < \hat{e} < \infty$ is characterised by $y_1(\hat{e},\hat{e}) = c'(\hat{e})$. If A2 holds, the efficient level of investment $0 < \hat{e} < \infty$ is characterised by $f'(\hat{e}) = 2c'(\hat{e})$. If investments are discrete, then $\hat{e}_n \in (\hat{e} - \frac{1}{n}, \hat{e} + \frac{1}{n})$.*

4.2. Equilibria with Continuous Investments

Consider the artificial static game where a single firm and worker have "equal division" payoffs $\phi 0.5 y(e,i) - c(i)$, $\phi y 0.5(e,i) - c(e)$, and choose $i, e \in \Re_+$. In the continuous case, let $e = i = e_H$ denote any symmetric equilibrium of this game: we will call such an equilibrium a *hold-up equilibrium*. Then, we have the following result:

Proposition 2. *Assume that investments are continuous.*

(i) If A1 holds, exactly one hold-up equilibrium $e_H > 0$ exists[15]*, and at this equilibrium, investment is inefficiently low i.e. $e_H < \hat{e}$.*

(ii) If A2 holds, there is a continuum of hold-up equilibria $e_H \in [0, \hat{e}]$.

[13] It is easy to show that it cannot be efficeint for workers and firms to invest asyummetrically. Suppose that $\hat{e} \neq \hat{\imath}$: then, by symmetry of $W(e,i) = \phi y(e,i) - c(e) - c(i)$, both $(\hat{e},\hat{\imath})$ and $(\hat{\imath},\hat{e})$ maximise $W(e,i)$ on $\Re^2_+$. But then by strict concavity, $W(0.5\hat{e} + 0.5\hat{\imath},\ 0.5\hat{e} + 0.5\hat{\imath}) > 0.5W(\hat{e},\hat{\imath}) + 0.5W(\hat{e},\hat{\imath})$, a contradiction.

[14] For example, if A1 holds, we have $\lim_{e\to 0}[0.5\phi y_1(e,e) - c'(e)] = \infty$, $\lim_{e\to\infty}[\phi y_1(e,e) - c'(e)] \le -c'(0)$, and a similar condition applies if A2 holds.

[15] This Proposition does not fully answer the question of whether a hold-up equilibrium $e_H = 0$ also exists. This is clearly possible: if for example, $y = \frac{1}{\alpha}(ei)^{\alpha}$, $c(e) = e$, $0 < \alpha < 0.5$, then clearly a hold-up equilibrium $e_H = 0$ exists, and this example satisfies all the assumptions on y, plus A1.

(iii) If either A1 or A2 holds, e^* is a equilibrium investment iff $e^* = e_H$.

So, this result says that an investment level is an equilibrium level in the dynamic matching game iff it is also an equilibrium of the associated static game, with "equal division" payoffs: i.e. a hold-up level of investment. The result also characterises hold-up investments. Note first the striking difference between cases of imperfect complements (Assumption A1) and perfect complements (Assumption A2). In particular, only in the latter case, there is a hold-up investment that is efficient in the dynamic matching game.

The intuition for part (iii) of this result is as follows. First, (as argued above - see Figure 1), the net payoff to deviation from e_H is bounded above by the discounted payoff to half the revenue in a subsequent match minus the cost of investment i.e. $\phi 0.5y(e', e_H) - c(e_H)$. By definition of e_H, this expression is maximized at e_H, so there cannot be any incentive to deviate from e_H. Second, if a deviant (say) worker changes his initial investment slightly from some e_0, he does not subsequently cause an outside option to bind in any match with a non-deviant firm. So, he obtains half the revenue even after this deviation, and so if $e_0 \neq e_H$, some deviation (either up or down) from e_0 must be profitable.

4.3. Equilibria with Discrete Investments

We now turn to the case of discrete investments. Our main finding is that if the search friction is "small" relative to the investment step size, then we have multiple equilibria above the efficient level. To obtain easily stated results, we assume here that the cost of investment is linear i.e. $c(e) = ce$, $c > 0$: all results generalize easily to the more general cost function assumed so far.

Assume first that A1 holds. From strict concavity of $y(e, i)$, it is true that the revenue gain to a unilateral one-unit increase in investment, $y(e + \frac{1}{n}, e) - y(e, e)$ is continuous and strictly decreasing in e for any step-size $\frac{1}{n}$. So, fixing the step-size, there is a unique critical $\underline{e}^n$ defined by

$$0.5y(\underline{e}_n, \underline{e}_n - \frac{1}{n}) - y(\underline{e}_n - \frac{1}{n}, \underline{e}_n - \frac{1}{n}) > \frac{c}{n} \geq 0.5y(\underline{e}_n + \frac{1}{n}, \underline{e}_n) - 0.5y(\underline{e}_n, \underline{e}_n) \qquad (4.1)$$

So, $\underline{e}^n$ is *the smallest e such that an agent does not (weakly) wish to unilaterally increase investment by one unit when (i) revenue is shared equally, and (ii) the matching process is approximately frictionless i.e. when $\phi \simeq 1$.* It is easily verified that as $n \to \infty$, $\underline{e}^n \to e_H$, where $e_H > 0$ is the unique positive hold-up equilibrium in the continuous game. If A2 holds, $y(e + \frac{1}{n}, e) - y(e, e) = 0$, so no agent ever wants to unilaterally increase investment, so we define $\underline{e}^n = 0$.

Again from concavity of $y(e,i)$, it is true that the revenue loss to a unilateral one-unit cut in investment $y(e,e)-y(e-\frac{1}{n},e)$, is decreasing in e for any step-size $\frac{1}{n}$. So, fixing the step-size, there is a unique critical $\overline{e}_n$ defined by[16]

$$y(\overline{e}_n,\overline{e}_n)-y(\overline{e}_n-\frac{1}{n},\overline{e}_n)>\frac{c}{n}\geq y(\overline{e}_n+\frac{1}{n},\overline{e}_n+\frac{1}{n})-y(\overline{e}_n,\overline{e}_n+\frac{1}{n}) \tag{4.2}$$

So, $\overline{e}_n$ *is the largest* e *such that the residual claimant does not (strictly) wish to cut investment by one unit when the matching process is approximately frictionless i.e. when* $\phi\simeq 1$. Then, our main result is:

Proposition 3. *(i) For a fixed investment step-size* $\frac{1}{n}$, *there is a match friction parameter* $\phi_n<1$ *such that for all* $\phi>\phi_n$, *any feasible level of investment* e^* *is an equilibrium level of investment iff* $\underline{e}^n\leq e^*\leq\overline{e}_n$

(ii) If investments are weakly complementary, $\overline{e}_n\geq\hat{e}_n$ *i.e. any feasible level of investment between* $\underline{e}^n$ *and the efficient level can be an equilibrium level.*

(iii) If investments are strictly complementary, there exists a range of investment costs such that $\overline{e}_n\geq\hat{e}_n+\frac{1}{n}$. *i.e. it is always possible to choose costs so that there is an equilibrium with overinvestment.*

This is a major result of the paper and deserves some comment. First, part (ii) is reminiscent of folk theorems in repeated games, particularly as one way of taking ϕ to the limit of 1 is to let r go to zero. One interpretation is that it provides a partial solution to the hold-up problem. As long as there is *any* degree of discreteness (e.g. smallest physical unit) in investment, if the matching process is sufficiently frictionless, an equilibrium investment level can be found that is "close" (i.e. one physical unit) away from the efficient level.

Second, part (iii) is really the key result. It says that the possibility of overinvestment identified in the example above is quite general. An overinvestment equilibrium can exist whenever investment levels are discrete and investments are strictly complementary.

The intuition for this result is illustrated in Figure 2 below, which builds on Figure 1. Consider some investment level e_0 between $\underline{e}^n$ and $\overline{e}_n$. An agent deviating "upwards" i.e. to $e'>e_0$, does not subsequently face a binding outside option, and so only gets half the return on his additional investment. As $e_0\geq\underline{e}^n$, this deviation will ensure him a strictly lower payoff than at e_0, so he does not want to deviate upwards.

[16] Let e' be the largest e for which the equilibrium payoff with neglible frictions is non-negative i.e. $0.5y(e,e)\geq ce$ (again, from concavity, any $e\leq e'$ satisfies this). For convenience, let $\overline{e}_n\leq e'$. Our results are modified in an obvious way if the surplus is exhausted first: in this case, $\min\{\overline{e}_n,e'\}$ replaces $\overline{e}_n$.

Figure 2 in here

Now suppose that an agent deviates "downwards" i.e. to $e' < e_0$. For fixed n, we can always find a small enough level of friction such that *any* downward deviation will face the deviant with a binding outside option in any subsequent match, as shown in Figure 2 - what is required is that output at the smallest possible deviation, $y(e_0 - \frac{1}{n}, e_0)$ is less than the non-deviator's outside option, $\phi y(e_0, e_0)$. But then any downward deviation will make the deviating agent residual claimant in any subsequent match (or worse, give him no match at all). In this case, , as $e_0 \leq \overline{e}_n$, downward deviation does not pay, either.

Note now that the claim in Proposition 3 is that c can be chosen so that overinvestment *by one increment* is always possible in equilibrium. This raises the question: is overinvestment by more than one unit possible? The following example shows that it is. This example also shows how Proposition 3 can be applied to an example.

Example 1. Investments cost 10 per increment i.e. $c = 10$ and are perfect complements. The revenue function is strictly concave, with revenue from the first three increments as follows.

e, i	0	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{3}{n}$
y	32	48	60	68

Note here, efficient investment is zero ($\hat{e}_n = 0$), but equilibrium investments can be 0, $\frac{1}{n}$ or $\frac{2}{n}$ with small enough search frictions. This can be seen in two ways. One is to apply Proposition 3. First, as investments are perfect complements - in fact, A2 holds - $\underline{e}_n = 0$. Second, for $\phi \simeq 1$, $\overline{e}_n = \frac{2}{n}$, as if an agent is residual claimant, he loses approximately 12 units of revenue by cutting his investment to $\frac{1}{n}$ while saving only 10 in costs. So, applying Proposition 3, equilibrium investments can be 0, $\frac{1}{n}$ or $\frac{2}{n}$ with small enough search frictions.

Given the simplicity of the example, the same conclusion can be established using a more direct argument. Note that if equilibrium investment is $\frac{2}{n}$, equilibrium payoffs are approximately 60/2-20=10. A deviant who deviates to investment of $\frac{1}{n}$ will therefore face a binding outside option in any subsequent match can therefore anticipates a payoff of approximately 48-30-10=8. A deviant who deviates to 0 will get, by the same argument, approximately 32-30=2. If equilibrium investment is $\frac{1}{n}$, the equilibrium payoff is approximately 48/2-10=14. A deviant who deviates to investment of 0 will get 32-24=8. In no case will any agent with to invest more than the equilibrium level, as investments are perfect complements. //

In general, we cannot say how much above $\hat{e}_n$ is $\overline{e}_n$ i.e. how much over-investment is possible in equilibrium. However, in the case of perfect complements, it is possible to get a general result of this kind. This result generalizes the example just shown.

Proposition 4. *Assume that A2 holds. Then, defining* $f'(\overline{e}) = c$, $\overline{e}_n \in [\overline{e} - \frac{1}{n}, \overline{e})$.

An example applying this result is as follows. Let $f(e) = 2\sqrt{e}$, $c = 1$. Then from Proposition 4, $\overline{e} = 1$. Moreover, from Proposition 1, efficient investment in the continuous case, $\hat{e}$, solves $f'(\overline{e}) = 2c$, implying $\hat{e} = \frac{1}{4}$. So, for small enough investment step-size, $\hat{e}_n \simeq \frac{1}{4}$, $\overline{e}_n \simeq 1$ i.e. equilibrium investment can be far above the efficient level.

The final question is whether this characterization of $\overline{e}_n$ in Proposition 4 relies on perfect complementarity. Our last result indicates that it does. With less than perfect complementarity, in the limit, as investment increments become small, the maximum equilibrium investment converges to the efficient investment.

Proposition 5. *Assume that y is twice continuously differentiable in its arguments i.e. A1 holds. Then, as $n \to \infty$, $\overline{e}_n \to \hat{e}$.*

Proposition 5 indicates that even with A1, there is still a discontinuity in the limit as investment indivisibility goes to zero. That is, with continuous investments, only the hold-up level of investment e_H is an equilibrium. However, as investment step-size $\frac{1}{n} \to 0$, any level of investment between e_H and the efficient level $\hat{e}$ can be approximated by an equilibrium level of investment.

4.4. Stability of Overinvestment Equilibrium

A possible objection to the overinvestment equilibrium characterised in Proposition 3 is that it is unstable in the game-theoretic sense that it does not survive arbitrarily small perturbations in the behavior of investing firms or workers. Consider the example of Section 2 with binary investments which are perfect complements. Suppose that a small exogenous fraction ε of firms and workers do not invest (this may be due to mistakes or because the costs of doing so are prohibitive) in any equilibrium. Does this eliminate the overinvestment equilibrium? If not, we will say that the overinvestment equilibrium is stable: otherwise, unstable.[17]

As search frictions go to zero for ε fixed, the equilibrium will always be unstable in

[17] Acemoglu (1997) applies a similar analysis to show that in the frictionless case with Bertrand-style competition the (low investment) coordination failure equilibrium is unstable. When search frictions are present coordination failure equilibria are stable with respect to "trembles". Our findings are the same except that coordination failure involves overinvestment.

this sense. Take for example, the numbers of Section 2, where two investors can produce 8 and output otherwise is 6. Then, any agent who can invest would prefer not to, as by waiting for a match with a non-investor she can expect at least half of 6, compared to approximately $4 - 1.5$ if she invests and waits for a match with an investor.

However, holding ε fixed and taking the frictionless limit is a very strong stability requirement. It is more in the spirit of game-theoretic stability (e.g. trembling-hand perfection) to take all parameters, including the search friction parameter, ϕ, as fixed, and perturb the model so that a small exogenous fraction ε of firms and workers do not invest. In this case, it is generally true[18] that given other parameter values, there is a $\overline{\varepsilon} > 0$ such that for all $\varepsilon < \overline{\varepsilon}$, there is an equilibrium where all agents who can invest, do invest, *if* there is such an equilibrium when $\varepsilon = 0$. So, any overinvestment equilibrium is stable, given this weaker notion of stability.

Interestingly, overinvestment equilibrium can persist even if quite a large fraction of the population of agents can be constrained from investing: such an example is given in Appendix 2, where, for the numerical example presented in Section 2, it is shown that overinvestment can persist when up to 1/4 of the agents cannot invest. This shows that the mechanism leading to over-investment is relatively robust.

4.5. Equilibrium Wages

So far, we have focussed only on the efficiency properties of equilibrium. Here, we investigate a more positive consequence of the basic pecuniary externality that gives rise to the inefficiency. Suppose for simplicity that there are just two possible investment levels, 0 and 1. Then, in equilibrium, workers who invest (graduates) lower the wages of those who do not (non-graduates). As a result, the graduate/non-graduate wage differential[19] is higher in an equilibrium where all invest (an investment equilibrium) than in a non-investment equilibrium.

This is consistent with some of the stylized facts about the US job market summarised

[18] The proof of this is simple. Let y_1 be revenue of both invest, and y_0 otherwise. If ε is very small, the return to search in investment equilibrium is approaximately $\phi y_1/2$, and the return to a deviant is approaximately $\phi(y_0 - \phi y_1/2)$ if the deviant faces a binding outside option i.e. if $\phi > y_1/y_0$. So, if $\phi y_1/2 - \phi(y_0 - \phi y_1/2) = \phi\left[\frac{(1+\phi)}{2}y_1 - y_0\right] > c$, where c is the cost of investment, then there is an equilibrium with over-investment for ε small enough. If also $\phi(y_1 - y_0) < 2c$, the equilibrium is inefficient for ε small enough..

[19] Of course, in investment (non-investment) equilibrium, the wage of a non-graduate (graduate) is a hypothetical wage. However, by introducing a few agents who cannot invest, or must invest, this wage differential will be observed in both equilibria.

in Acemoglu (1999). As Katz and Murphy (1992) document, between 1979 and 1987 the real wage of college graduates increased some 30% whilst the wage of young US high school graduates fell 20% . Further evidence along these lines is provided by Machin and Manning (1997) who find that a large increase in the relative supply of more educated workers in the late eighties did not result in falling wage differentials of the more educated.

Our results on wages can be established as follows. Consider two identical economies A and B of the type analysed above. Suppose that parameters are such that both a non-investment equilibrium and a investment equilibrium are possible[20] and suppose that economy A (B) is in the investment (non-investment) equilibrium. Denote by w_1^A, w_0^B the wages (i.e. share of revenue) in the two equilibria in the two economies. By definition, $w_1^A = 0.5y(1,1)$, $w_0^B = 0.5y(0,0)$. Now let $\tilde{w}_0^A, \tilde{w}_1^B$ be the (hypothetical) wages to a deviant worker in each of these two economies i.e. a worker who decides not to invest in A, or invest in B. If c is the cost of investing (in a college degree), then by the fact that A (B) is in the investment (non-investment) equilibrium, in must pay a worker in economy A to invest, but not one in economy B i.e. $w_1^A - \tilde{w}_0^A \geq c \geq \tilde{w}_1^B - w_0^B$. So the graduate/non-graduate wage differential is *higher* in the economy A where all workers are graduates, than it is in economy B where none are[21].

Finally, note that when search frictions are small ($\phi \cong 1$), the non-graduate wage in economy B is higher than it is in A:

$$\begin{aligned} w_0^B &= 0.5y(0,0) > y(0,1) - 0.5y(1,1) \\ &\simeq y(0,1) - \phi 0.5y(1,1) = \tilde{w}_0^A \end{aligned}$$

where the first inequality follows directly from the definition of strict complementarity. So, we have the very counter-intuitive result that non-graduate wages fall *absolutely* as the economy moves to an equilibrium with (a) more graduates, and (b) more investment by firms.

Each of these changes contributes to the fall. With the supply of graduates plentiful, a firm bargaining with a non-graduate has a binding outside option and so the non-graduate is residual claimant. The easier it is for a firm to locate graduates, the higher its outside option so the lower the non-graduate wage. Moreover, under complementarity,

[20] Let y_1 be revenue if both invest, and y_0 otherwise. This requires $\phi\left[\frac{(1+\phi)}{2}y_1 - y_0\right] \geq c$ for the investment equilibrium, and $\frac{\phi}{2}\left[y_1 - y_0\right] \leq c$ for the non-investment equilibrium.

[21] One might object that $\tilde{w}_0^A$, $\tilde{w}_1^B$ are not observed in equilibrium. This objection can easily be dealt with by introducing measures ε of workers who always invest or who never invest. For ε small, we will have an investment equilibrium in A and a non-investment equilibrium in B, where all four wages are observed in equilibrium.

the more graduates there are the greater is the firm's incentive to invest. At first sight, this would seem good news for the non-graduates whose productivity generally rises with investment. However, as the productivity of the graduate match increases even more, the firm's outside option rises relative to the surplus available from a match with a non-graduate whose residual income consequently falls.

Finally, note two other applications of these results. First, our model indicates that a widening gap between the wages of graduates and non graduates need not indicate a high *social* return to education, as some governments apparently do (the UK government is an example). In the example developed here, in economy A, discouraging higher education is the correct policy response.

Another phenomenon that this model sheds light on is over-qualification where many people are employed in jobs for which their educational qualifications are unnecessary (see, for example Sicherman (1991), Goos and Manning (2003)). Suppose that in the model here, there are a few firms that have no investment opportunities (these firms can only offer non graduate jobs). This will not preclude an equilibrium in economy A in which all agents that can invest do so (see the stability analysis above). Graduates take non graduate jobs and are paid their binding outside option based on the split the surplus division in graduate jobs.

5. Heterogenous Firms and Workers

5.1. Preliminaries

Our model is that of Section 3, with the modification that agents on either side of the market may have different costs of investment, following Cole, Malaith and Postlethwaite (2002). A measure λ_i of both workers and firms have investment cost $c_i(e)$, $i = h, l$ with $\lambda_h + \lambda_l = 1$ and $c_h(e) > c_l(e)$, $c'_h(e) > c'_l(e)$, all e. So, $h-$types have a higher cost of investment than $l-$types. We will assume that A1 holds and moreover that investments are strict complements ($y_{12} > 0$), and that the other conditions on revenue and cost functions assumed above continue to hold. The order of events is as in the homogenous model. Again, we focus on symmetric equilibrium investments, in the sense that all agents of type $i = h, l$ invest at level e_i. We again assume that match acceptance and bargaining strategies are Markov-perfect in the sense defined above.

In this setting, there is one complication relative to the homogenous case. With the restriction to Markov-perfect strategies, and assuming all agents of type i invest the same, it is easy to see that there are two kinds of match acceptance strategies that could occur in equilibrium. *Assortative matching* occurs when agents of some type $i = h, l$ only agree to match with agents of the same type. *Non-assortative* matching occurs when agents of both types h, l agree to match with agents of either type[22].

With non-assortative matching (i.e. all agents accept all matches) the proportion of type i in the pool of unmatched at time t, λ_{it} is clearly time-invariant ($\lambda_{it} = \lambda_i$), as both types are exiting at the same rate - every agent exists with probability Δa over a time period. However, if matching is assortative, there is the potential problem that the search environment could be non-stationary, in that λ_{it}, could change over time. However, it turns out that $\lambda_{it} = \lambda_i$ also the case if matching is assortative, as explained in Appendix B.

The general conditions defining a symmetric investment equilibrium are exactly as in the homogenous case (section 3.2 above). An *equilibrium investment* is a pair $e^* = (e_h^*, e_l^*)$, $e^* \in \Re_+^2$ such that

$$v_i(e_i^*, e^*) - c_i(e_i^*) \geq v_i(e', e^*) - c_i(e'), \text{ all } e' \in \Re_+, \ i = h, l$$

where $v_i(e', e^*)$ is is the expected discounted payoff to search of an agent of type i who deviates to e', given that almost all (i.e. all but a measure zero) of agents of type i choose e_i^*, and that match acceptance and bargaining strategies are Markov-perfect in the continuation game.

In the homogenous case, we could obtain quite an explicit characterization of $v_i(e', e^*)$ i.e. (3.1) which could be used to prove general results. Here due to the number of different cases generated by heterogeneity, it is very tedious to do this. Rather, we prove general results only for the liming cases as match frictions go to zero ($a \to \infty$) or as match frictions become large ($a \to 0$). In these limiting cases, a general characterization of $v_i(e', e^*)$ is not required. As before, we begin with a characterization of efficient investments.

5.2. Efficient Investments

Again, as payoffs are linear in consumption (i.e. quasi-linear), the natural efficiency criterion is the sum of the payoffs to search net of investment costs at some levels of investment for each type e_l, e_h. But now, we have to distinguish between assortative and

[22] The last possibility (given Markov acceptance strategies and symmetric investments) is that agents of type h will only match with agents of type l, and vice versa. However, these cannot be equilibrium strategies, as (assuming for example that $l-$types invest more) an $l-$type can expect a higher payoff to a match with another $l-$type than with an $h-$type and so if he is willing to match with an $h-$type, he must be willing also to match with an $l-$type.

non-assortative matching i.e. efficient investments are conditional on the type of matching. If matching is assortative, outside options can never be binding at the bargaining stage[23], so by the arguments already used, an agent of type i has expected discounted payoff[24] of

$$v_i = 0.5\phi_i y(e_i, e_i),\ i = l, h \tag{5.1}$$

where $\phi_i = \frac{a\lambda_i}{r+\lambda_i}$ is a type-specific friction parameter. So, the efficiency criterion is

$$W(e_h, e_l) = \sum_{i=h,l} \lambda_i[0.5\phi_i y(e_i, e_i) - c_i(e_i)] \tag{5.2}$$

Efficient investments maximise (5.2). The existence of an interior solution to this problem is guaranteed by the conditions on revenue and cost functions. So, $\hat{e}_h, \hat{e}_l$ are characterised by

$$\phi_i y_1(\hat{e}_i, \hat{e}_i) = c_i'(\hat{e}_i),\ i = l, h \tag{5.3}$$

Now with non-assortative matching, in a match between a type h and l, the outside options of agents who have invested more (presumably the $l-$types) may or may not be binding in subsequent bargaining. However, because the efficiency criterion *sums* the payoff of the two agents in the match, this is irrelevant for the calculation of W. So, assuming no binding outside options, an agent of type i has expected discounted payoff[25] of

$$v_i = \phi[\lambda_i 0.5y(e_i, e_i) + (1-\lambda_i)0.5y(e_i, e_j)],\ i = l, h \tag{5.4}$$

where $\phi = a/(a + r)$, so

$$W(e_h, e_l) = \sum_{i=h,l} \phi[0.5\lambda_i y(e_i, e_i) + (1-\lambda_i)0.5y(e_i, e_j) - c_i(e_i)] \tag{5.5}$$

Efficient investments maximise this expression, and so are characterised by

$$\phi[\lambda_i y_1(\hat{e}_i, \hat{e}_i) + (1-\lambda_i)y_1(\hat{e}_i, \hat{e}_j)] = c_i'(\hat{e}_i),\ i = l, h \tag{5.6}$$

5.3. General Results

Our first general result is for the limiting case where match frictions go to zero. An argument similar to that used to prove part (i) of Proposition 2 above establishes that

[23] This is because no agent is ever in a match with another with an investment level lower than his own.

[24] This is because as $\Delta \to 0$, v_i solves the dynamic programming equation $rv_i = a\lambda_i(0.5y(e_i, e_i) - v_i)$.

[25] This is because as $\Delta \to 0$, v_i solves the dynamic programming equation $rv_i = a[\lambda_i 0.5y(e_i, e_i) + \lambda_j 0.5y(e_i, e_j) - v_i]$.

there exactly one pair (e_h^*, e_l^*) of investments[26] $e_i^* > 0$ which that solve

$$\phi_i \frac{y_1(e_i^*, e_i^*)}{2} = c_i^{'}(e_i^*),\ i = h, l \tag{5.7}$$

Then, we have:

Proposition 6. *Assume that*

$$\frac{y(e_h^*, e_h^*)}{2} - c_h(e_h^*) \geq \frac{y(e_l^*, e_l^*)}{2} - c_l(e_l^*) \tag{5.8}$$

Then, for a large enough, there is an equilibrium in Markov-perfect strategies with where all agents of type i invest e_i^*, where e_i^* solves (5.7). In this equilibrium there is assortative matching. There is no other equilibrium in Markov-perfect strategies with strictly positive investments. In equilibrium, investments are inefficiently low i.e. $e_i^* < \hat{e}_i$.

So, we have a general limit result: for low enough market friction, there is *always* underinvestment in equilibrium[27]. The intuition is straightforward, and is indeed the same as in the two-agent case without outside options: each agent underinvests, anticipating he will only obtain half the additional surplus i.e. the hold-up problem applies.

Finally, we should remark on equilibrium condition (5.8). This is a kind of self-selection constraint: it requires that at hold-up investment levels, it should not pay for a high-cost type to imitate a low-cost type. This condition is not implied by the concavity of individual payoffs in investment, as the high-cost type agent can effectively "free-ride" on the higher level of investment by low-cost agents on the other side of the market by imitating a low-cost agent on his own side of the market.

Now we turn to the other limit case where a is very small, and so frictions are large. For convenience, we will assume that there is a unique pair (e_h^{**}, e_l^{**}) of investments with $e_i^{**} > 0$ which that solve

$$\phi\left[\lambda_h y(e_i^{**}, e_h^{**}) + \lambda_l y(e_i^{**}, e_l^{**})\right] = c_i^{\prime}(e_i^{**}),\ i = h, l \tag{5.9}$$

Then, we have:

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[26] For example, $y(e, e') = \frac{1}{\alpha}(e.e')^{\alpha}$, $c_i(e) = c_i e$, with $\alpha < 0.5$, it is easy to see that $(e_h^*, e_l^*) = ((\frac{2c_h}{\phi_h})^{1/(2\alpha-1)}, (\frac{2c_l}{\phi_l})^{1/(2\alpha-1)})$.

[27] It is also worth noting that Proposition 6 sidesteps the question of equilibria with zero investments. In fact, under certain conditions on y, such equilibria always exist - for example, when $y = \frac{1}{\alpha}(e_i.e_j)^{\alpha}$, as then $i's$ marginal product is zero if j invests zero. In this case, there are all agents are identical ex post and so the question of whether matching is assortative does not arise. However, zero investment equilibria are of limited interest - the crucial question is whether equilibria with *positive* investment levels have suboptimally low - or high - investment.

Proposition 7. *For a small enough, there is an equilibrium in Markov-perfect strategies with where all agents of type i invest e_i^{**}, where e_i^{**} solves (5.9). In this equilibrium there is non-assortative matching. There is no other equilibrium in Markov-perfect strategies with strictly positive investments. In equilibrium, investments are inefficiently low i.e. $e_i^{**} < \hat{e}_i$.*

So, we again have a general limit result: for high enough market friction, there is *always* underinvestment in equilibrium. The intuition is straightforward, and is indeed the same as in the case with low market frictions: each agent underinvests, anticipating he will only obtain half the additional surplus.

So, Propositions 6 and 7 taken together strongly indicate that a necessary condition for overinvestment in equilibrium to be possible is that outside options must bind in equilibrium. The example of the next section shows in fact that when outside options are binding in equilibrium, we can always choose functional forms and parameter values so that there will be overinvestment.

5.4. An Example with Overinvestment

We will suppose that the cost of investment for $l-$types is prohibitive ($c(e) = \infty$) so that $e_l = 0$ is both efficient and part of any equilibrium outcome. We will construct an example where matching is non-assortative and where, when an $h-$type matches with an $l-$type, the outside option of the $l-$type binds. The outside option of the $l-$type is of course the present value expected payoff to continued search which increases with e_l. This is the key to the example: the $l-$type has an additional incentive to invest in order to increase the value of this outside option and thus shift rent away from the $h-$type partner. This rent-shifting incentive may lead to overinvestment relative to the efficient level.

The details are as follows. We suppose that if a match between an i and $j-$type occurs, revenue is $y = y_0 + \frac{1}{\alpha}(e_i e_j)^\alpha,\ \alpha < 0.5$: so, all the assumptions made above on the revenue function are satisfied. The cost of investment by the $l-$type is c_l.

Recall that v_h, v_l are the present value expected payoffs to continued search for the two types. Then, as the outside option of the $l-$type is assumed to bind in a non-assortative match, and the latter kind of match generates a present value of output of y_0 (as $e_h = 0$), we require

$$v_l > \frac{y_0}{2} \tag{5.10}$$

The payoffs v_h, v_l satisfy the following dynamic programming equations in the limit as

$\Delta \to 0:$

$$\begin{aligned} rv_l &= a\lambda_h(v_l - v_l) + a\lambda_l(\frac{y(e_l, e_l)}{2} - v_l) \quad (5.11) \\ rv_h &= a\lambda_h(\frac{y_0}{2} - v_h) + a\lambda_l(y_0 - v_l - v_h) \quad (5.12) \end{aligned}$$

The first equation follows because when matched with an $h-$type (which occurs with probability $\Delta a\lambda$) the $l-$type gets no surplus from the match. The second follows because when matched with an $l-$type (which occurs with probability $\Delta a\lambda_l$) the $h-$type is residual claimant.

Solving (5.11), (5.12), we get

$$\begin{aligned} v_l(e_l, e_l) &= \phi_l \frac{y(e_l, e_l)}{2} \quad (5.13) \\ v_h(e_l, e_l) &= \phi[\lambda_h \frac{y_0}{2} + \lambda_l(y_0 - \phi_l \frac{y(e_l, e_l)}{2})] \quad (5.14) \end{aligned}$$

Now, in equilibrium, e_l must be an optimal investment for a firm (or worker) of type l, given that the partner is investing e_l. This requires $v_l(e, e_l) - c(e)$ to be maximized at $e = e_l$. Given the assumptions on y, f for this it is sufficient that the first-order condition holds i.e.

$$\phi_l \frac{(e_l)^{2\alpha-1}}{2} - c_l = 0 \Longrightarrow e_l^* = \left[\frac{\phi_l}{2c_l}\right]^{\frac{1}{1-2\alpha}} \quad (5.15)$$

The description of the equilibrium is completed by giving the condition under which matching is non-assortative, which is

$$y_0 \geq v_h + v_l \quad (5.16)$$

So, the equilibrium is fully characterized by (5.10)-(5.16).

Now, consider the efficient investment, given that matching is non-assortative. At any date $t \geq 0$, conditional on identical but otherwise arbitrary investments e by all $l-$agents, aggregate surplus is the sum of individual surpluses minus investment costs

$$\begin{aligned} W(e, e) &= \lambda_h v_h(e, e) + \lambda_l v_l(e, e) - \lambda_l c(e) \\ &= \lambda_h \phi \left[\lambda_h \frac{y_0}{2} + \lambda_l y_0\right] + \lambda_l(1 - \lambda_h \phi)\phi_l \frac{y(e_l, e_l)}{2} - \lambda_l c(e) \end{aligned}$$

So, efficient investment maximises this expression[28]. Given that $\alpha < 0.5$, $y(e, e)$ and thus W is concave in e, so a necessary condition for the efficient $\hat{e}_l$, is the first-order condition

$$[1 - \lambda_h \phi]\phi_l(\hat{e}_l)^{2\alpha-1} - c_l = 0 \Longrightarrow \hat{e}_l = \left[\frac{[1 - \lambda_h \phi]\phi_l}{c_l}\right]^{\frac{1}{1-2\alpha}} \quad (5.17)$$

[28] Bear in mind that in the efficient case, v is differentiated twice with respect to e, as the social planner internalizes the positive external effect of firms' investments on workers and vice-versa.

So, comparing (5.15) and (5.17), we see that $\hat{e}_l < e_l^*$ if $1 - \lambda_h \phi < 0.5$, or

$$\lambda_h \phi > 0.5 \tag{5.18}$$

In this case, *equilibrium investment is inefficiently high.* This condition (5.18) has an intuitive explanation. The social return to an increment in e_l is composed of two parts. The first is the gain due to the additional payoffs to $l-$agents in assortative matches. This is double the gain to the private gain to investors in those matches, and explains the "0.5" in the formula. But the second effect is to reduce the payoffs of $h-$agents in non-assortative matches: this is captured by the term $\lambda_h \phi$, which as expected, is higher the more such agents there are, and the more efficient the matching process as then the loss occurs sooner.

It remains to show that we can choose parameter values such that (5.18) can be satisfied simultaneously with the equilibrium conditions (5.10)-(5.16). Note first using (5.13), (5.14), the condition for non-assortative matching (5.16) can be written

$$y_0 \geq \left(\frac{1 - \phi + \phi \lambda_h}{1 - \phi + 0.5 \phi \lambda_h} \right) v_l \tag{5.19}$$

So, from (5.16),(5.19), the condition for the binding outside option and non-assortative matching conditions to be satisfied together become

$$2v_l > y_0 \geq \left(\frac{1 - \phi + \phi \lambda_h}{1 - \phi + 0.5 \phi \lambda_h} \right) v_l \tag{5.20}$$

Now note by substitution that at equilibrium with $e_l^* = \left[\frac{\phi_l}{2c_l} \right]^{\frac{1}{1-2\alpha}}$ that

$$v_l = \frac{\phi_l}{2} \left[y_0 + \left(\frac{\phi_l}{2c_l} \right)^{\frac{2\alpha}{1-2\alpha}} \right] \tag{5.21}$$

There certainly exist parameter values for which (5.18), (5.20),(5.21) are simultaneously satisfied. For example, take $a = 0.8$, $r = 0.2$, $\lambda_h = 0.7$. Then $\phi = 0.8$, so $\phi \lambda_h = 0.56$ so (5.18) is satisfied. Moreover, take $c_l = \phi_l$, $\alpha = 0.25$, so $\left(\frac{\phi_l}{2c_l} \right)^{\frac{2\alpha}{1-2\alpha}} = 0.5$. Then, noting $\phi_l = \frac{6}{11}$, $v_l = \frac{3}{11}[y_0 + \frac{1}{2}]$. So, (5.20) becomes

$$\frac{6}{11} y_0 + \frac{3}{11} > y_0 > \frac{76}{48} \left(\frac{3}{11} y_0 + \frac{1.5}{11} \right)$$

Taking $y_0 = 0.5$, for example, these last two inequalities are satisfied.

6. Related Literature

Our work is related to a number of recent papers. First, in a sequence of papers, Acemoglu studies ex ante investments by workers and/or firms in a market where workers and firms are subsequently matched. In Acemoglu (1996), search frictions are responsible for an imperfectly competitive labour market. Both firms and workers can make complementary investments. Once paired, the parties have no effective opportunity to rematch and wages are decided by bilateral bargaining. The model is rich in externalities. As more firms invest, the greater the chance a worker is in match where their investment really pays off. So investment by any one firm stimulates training and thus the chance that non investing firms encounter trained workers. Such cumulative causation yields multiple equilibria and provides the basis for increasing social returns. A version of this model is utilized in Acemoglu (1997) to consider investment distortions. As the bargaining created by search frictions means investment returns are split with partners, there is a positive externality associated with the *ex ante* acquisition of education, leading to the unambiguous conclusion of under provision.[29] In contrast to our model, in these treatments rematching is not relevant, the outside-option principle does not apply and so overinvestment results are precluded.[30] Re-matches are possible in Acemoglu and Shimer (1999) but as investment is one sided (and continuous) so with random search there is still underinvestment.

The reason our model comes to a different conclusion is twofold. First, both sides of the market have investment opportunities. Second there are some (possibly small) investment indivisibilities. If graduate degrees raise productivity and firms can invest to take further advantage of these skills, the *laissez faire* equilibrium of the economy may involve too much education. When investment complementarities are not so great but there is some indivisibility in investment, the equilibrium may be arbitrarily close to the efficient level. If not everyone has the ability to benefit from graduate education then, even if education is a continuous variable, it may be excessive for other than signalling reasons.

Our work is also related to a number of other papers. First, Burdett and Coles (2001) have a matching model of the marriage market where both men and women can make investments in their "pzazz" (sex appeal) prior to matching. Utility is non transferable,

[29] Stevens (2001) uses a version of the model to analyse efficiency enhancing educational and training subsidies

[30] An unpublished version of Acemoglu (1996) has an analysis in the spirit of this paper though the framework and results are not identical.

so relationship surplus is not bargained over. More pzazz means it is possible to match with a higher pzazz partner and an overinvestment equilibrium may emerge. This result does though depend on the impossibility of bargaining. Investments are additive, so our results (Proposition 3) show that were bargaining possible, so that (say) a low pzazz man could "buy" a high pzazz woman by a cash transfer, overinvestment could not occur.

Felli and Roberts (2000) analyse a model with heterogeneous types in which there are no post-investment search frictions. With workers sufficiently close substitutes, the Bertrand-style game with firms posting wages involves agents receiving close to competitive returns. Efficient equilibria always exist but when workers have *ex ante* investment decisions there may be (bounded) coordination failures with attendant inefficiencies. What is shown is that the ranking of workers by equilibrium investment levels may differ from the ranking by optimum investment levels. A worker with lower intrinsic quality may nevertheless invest sufficiently more than an able worker that in the Bertrand equilibrium they are hired by a more productive firm. The distribution of investment is then suboptimal but it is unclear what happens to the total; indeed with heterogeneous types this may not be such an interesting question. At all events, our mechanism for inefficiency is different. The issue is not so much bargained versus Bertrand price determination, but complementarity and discreteness. This can be established by noting that the appendix of Acemoglu (1997) shows that for the Bertrand game with homogeneous types and continuous single sided investment overinvestment is precluded.[31] Translating our double-sided discrete investment game into a static Bertrand framework still yields overinvestment equilibria. The key point is that a non investing deviant may leave the wage of investors unchanged (similarly to the binding outside option in the dynamic matching and bargaining model) allowing equilibria with excessive investment.

Cole, Mailath and Postlewaite (2001a,b) consider a matching model in which buyers and sellers make investment decisions non cooperatively prior to entering a frictionless matching process. The rule sharing the surplus from trade is exogenously fixed, rather than emerging from a bargaining protocol, as in our paper : it is only constrained by the requirement that the matching be stable i.e. no worker and firm can leave their current matches, match together, and both be better off. There are multiple sharing rules that satisfy this requirement. As the sharing rule is not unique, there are several possible

[31] Notice that with continuous investment our model that the unique investment equilibrium is independent of the level of search frictions with the exception of a singularity at zero. The standard assumption that it is impossible to bargain with more than one partner may be responsible for this odd feature which does not however matter for our main result.

equilibrium investment profiles (at least one for each stable sharing rule).

It is possible to choose a sharing rule so that investments are efficient ("solving" the hold-up problem), but one can also choose rules that generate underinvestment and over-investment equilibria. When investments are continuous, their overinvestment examples require that less investment by one agent increases the payoff to the other agents. This is inherently implausible and cannot occur, for example, in our model. With discrete investments, they have an example[32] where overinvestment arises without this feature, but there is no general analysis of conditions under which this can occur.

7. Conclusions

As typically represented, incomplete contracting creates hold-up problems causing investors to anticipate expropriation of their returns. The result is underinvestment. This paper shows that with endogenous outside options, it is non investors that are held up and their best defence is to invest. When investment indivisibilities and search frictions are low there is always an equilibrium close to the first-best level. When complementarities are high, cumulative causation may result in a quasi-competitive overinvestment equilibrium that is locally but not globally efficient. At heart this is a coordination problem which economic policy should discourage.

There is a second mechanism leading to overinvestment. For example, education, by widening employment opportunities, may enhance bargaining power even in occupations in which it has little or no effect on productivity. This constitutes a private but not a social gain. The outcome may be that education spreads like a contagion. Nevertheless, from a social perspective the costs do not justify the returns and education should be limited. If firms do not lower investment in response the wage of non graduates is higher if there are fewer graduates and if firms invest less as a result this effect is all the greater.

For investment to be excessive complementarities must be involved. Such synergies are often thought to be the basis of a "new economy" virtuous circle of investment. On this view the policy problem is to ensure that coordination problems do not prevent the good equilibrium from emerging with government intervention is justified to kick start the economy into a better, high-investment equilibrium. Our paper is a warning that misery may result; the low investment outcome may be the good equilibrium.

[32] Figure 6 in Cole, Mailath and Postlewaite (2001a,b)

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A. Proofs of Propositions

Proof of Lemma 1. Consider a match between a firm and a worker, and assume w.l.o.g. that the deviant agent is a worker i.e. $e_f = e^*$, $e_w = e'$. Let $v^* \equiv v(e^*, e^*)$ and $v' = v(e', e^*)$ respectively. If the match is accepted, the firm and worker play an alternating-offers bargaining game with time-invariant outside options v^*, v'. There is a unique subgame-perfect equilibrium in this game, with immediate agreement and the payoff u' for the deviant worker in this game, in the limit as $\Delta \to 0$, is[33] :

$$u' = \begin{cases} 0.5y(e^*, e') & v, v' \leq 0.5y(e^*, e') \\ y(e^*, e') - v^* & v^* > 0.5y(e^*, e') \geq v' \end{cases} \tag{A.1}$$

The payoff to the firm matched with a deviant worker is thus $y(e', e^*) - u'$. So, the two parties will only accept the match if $u' \geq v'$, $y(e', e^*) - u' \geq v^*$ which reduces to

$$y(e', e^*) \geq v^* + v' \tag{A.2}$$

Next, in the limit as $\Delta \to 0$, a standard dynamic programming argument implies that the payoff to search for the deviant worker solves

$$rv' = a(u' - v') \tag{A.3}$$

i.e. the return to search, rv', is equal to the expected capital gain due to making a transition to the matched state, $a(u' - v^*)$, where u' is defined in (A.1) if the match is accepted i.e. (A.2) holds, and $u' = 0$ otherwise. So, solving (A.3) for v', and using this definition of u', we get:

$$v(e', e^*) = \begin{cases} \phi 0.5y(e', e^*) & \text{if } v^* \leq 0.5y(e', e^*) \\ \phi(y(e', e^*) - v^*) & \text{if } y(e', e^*) > v^* \geq 0.5y(e', e^*) \\ 0 & \text{if } v^* > y(e', e^*) \end{cases} \tag{A.4}$$

where $\phi = \frac{a}{a+r}$. The special case where $e' = e^*$ therefore solves to give $v^* = \phi 0.5y(e^*, e^*)$.

Uniqueness follows from the fact that conditional on e^*, e', there is a unique solution v', u' to (A.1), (A.2), (A.3).

To get formula (3.1) from (A.4), note that given definitions of $\underline{b}(e^*), \underline{\underline{b}}(e^*)$, and the definition of v^*, then the three conditions on the RHS of (A.4) are each equivalent to the corresponding conditions on the RHS of (3.1). □

[33] There is, logically, another possibility , namely that $v' > 0.5y(e^*, e') \geq v^*$. However, this cannot occur in equilibrium, as all the firms with which the deviant worker might potentially be matched have invested at the same level e^*.

Proof of Proposition 2. (i) Assume A1 holds. Note that in the static game, the condition defining a hold-up equilibrium e_H is

$$g(e_H) = \phi 0.5 y_1(e_H, e_H) - c'(e_H) = 0$$

which is the first-order condition for the optimal choice of investment at equilibrium (the second-order condition is satisfied by assumptions on y, c). So, it suffices to show that $g(e) = 0$ has at exactly one root on $[0, \infty)$. First, note that $g(.)$ is continuous by assumption. Second, note that

$$g'(e_H) = \phi 0.5[y_{11}(e_H, e_H) + y_{12}(e_H, e_H)] - c''(e_H)$$

which is strictly negative as y is assumed strictly concave. Finally, by the limit assumptions on y and c, $\lim_{e \to 0} g(e) > 0$, $\lim_{e \to \infty} g(e) < 0$. So, $g(e)$ has exactly one root as required.

(ii) Assume A2 holds. Let e_H be a candidate for symmetric equilibrium in the static game. Then if any agent increases his investment by a small amount Δ, his additional payoff is simply $-\Delta c'(e_H) < 0$. On the other hand, if he cuts his investment by a small amount Δ, his change in payoff is $-\Delta[0.5 f'(e_H) - c'(e_H)]$, which is non-positive as long as $f'(e_H) \geq 2c'(e_H)$, which of course requires $e_H \in [0, \hat{e}]$. This establishes that any $e_H \in [0, \hat{e}]$ is a hold-up equilibrium.

(iii) Consider now the dynamic matching game. Assume that A1 holds and that all agents except for a deviant set $e = e_H$. By (3.1), the deviant's payoff is bounded above by $\sigma(e', e_H) = \phi 0.5 y(e', e_H) - c(e_H)$. But, by part (i) above, $\sigma(e', e_H)$ is maximised on $\Re_+$ at $e = e_H$. So, deviation from e_H does not pay, establishing that e_H is an equilibrium.

Now suppose that all agents except for a deviant set $e_0 \neq e_H$. Then, for fixed $\phi < 1$, noting that $\underline{b}(e) < e$, for any $e_0 \in \Re_+$ there is a neighborhood $N(e_0) = (e_0 - \delta, e_0 + \delta)$ around e_0 such that for $e' \in N(e_0)$, a deviant's outside option is not binding, so that his payoff is $\sigma(e', e_0) = \phi 0.5 y(e', e_0) - c(e_0)$. Then,

$$\left. \frac{\partial \sigma(e, e_0)}{\partial e} \right|_{e = e_0} \neq 0 \tag{A.5}$$

So, from (A.5), and the preceding discussion, a deviation $e' \in N(e_0)$ exists such that $\sigma(e', e_0) > \sigma(e_0, e_0)$. So, $e_0 \neq e_H$ cannot be an equilibrium in the dynamic matching game. A similar argument applies if A2 holds. □

Proof of Proposition 3. Let e^* be a candidate equilibrium level of investment, with $\underline{e}_n \leq e^* \leq \overline{e}_n$.

(i) Consider a deviation to $e' > e^*,\ e' \in S_n$. Then, from (3.1), the deviant's outside option cannot bind at e'. Also, $e' = e^* + \frac{k}{n}$ by definition, so for some $k \geq 1$, his gain to deviation is

$$\begin{aligned}
\Delta &= \phi y(e^* + \frac{k}{n}, e^*) - \phi y(e^*, e^*) - c\frac{k}{n} \\
&\leq \phi y(e^* + \frac{1}{n}, e^*) - \phi y(e^*, e^*) - \frac{c}{n} \\
&< y(e^* + \frac{1}{n}, e^*) - y(e^*, e^*) - \frac{c}{n} \\
&\leq 0
\end{aligned}$$

where the first inequality follows from strict concavity of y, the second as $\phi < 1$, and the third from (4.1) and the fact that $\underline{e}_n \leq e^*$.

(ii) Now consider a deviation to $e' > e^*,\ e' \in S_n$. Assume $e^* - \frac{1}{n} < \underline{b}(e^*)$, i.e. $y(e^* - \frac{1}{n}, e^*) < \phi y(e^*, e^*)$ which is equivalent to assuming $\phi > \phi_n^0(e^*)$, for some $\phi_n^0(e^*) < 1$. Now consider a deviation $e' < e^*$. By assumption, $e' \leq e^* - \frac{1}{n} < \underline{b}(e^*)$. So, There are then two possibilities.

(a) First, $e' < \underline{\underline{b}}(e^*)$ in which case from (3.1) the net payoff to deviation is $-c(e') \leq 0$. As A1 holds, the payoff at equilibrium is bounded below by $y(0, e^*) \geq 0$. So, deviation does not pay.

(b) The other possibility is that $\underline{\underline{b}}(e^*) \leq e' < \underline{b}(e^*)$.Then, from (3.1), the deviant is residual claimant at e', so the gain to deviation to $e' = e^* - \frac{k}{n}$ is

$$\begin{aligned}
\Delta(e', e^*) &= v(e', e^*) - c(e') - [v(e^*, e^*) - c(e^*)] \qquad \text{(A.6)} \\
& \phi(y(e', e^*) - \phi 0.5 y(e^*, e^*)) - ce' - [\phi 0.5 y(e^*, e^*) - ce^*] \\
&= \left\{c\frac{k}{n} - \phi\left(y(e^*, e^*) - y(e^* - \frac{k}{n}, e^*)\right)\right\} + \phi y(e^*, e^*)(1 - 0.5(1 + \phi)) \\
&< k\left\{\frac{c}{n} - \phi\left(y(e^*, e^*) - y(e^* - \frac{1}{n}, e^*)\right)\right\} + \phi y(e^*, e^*)(1 - 0.5(1 + \phi))
\end{aligned}$$

where the second line follows from (3.1), the third by rearrangement, and the fourth by strict concavity of y in its first argument. So, taking the limit as $\phi \to 1$ in (**??**), we see that

$$\lim_{\phi \to 1} \Delta(e', e^*) < k\left\{\frac{c}{n} - \left(y(e^*, e^*) - y(e^* - \frac{1}{n}, e^*)\right)\right\} \qquad \text{(A.7)}$$

Now, directly from the definition of $\overline{e}_n$, it follows that

$$y(\overline{e}_n, \overline{e}_n) - y(\overline{e}_n - \frac{1}{n}, \overline{e}_n) > \frac{c}{n}$$

But then as $e^* \leq \overline{e}_n$,

$$y(e^*, e^*) - y(e^* - \frac{1}{n}, \overline{e}_n) > \frac{c}{n} \tag{A.8}$$

So, from inspection of (A.7), and $(A.8)$, we see that $\lim_{\phi \to 1} \Delta(e', e^*) < 0$. We conclude that there exists a $\phi_n^1(e^*) < 1$ such that $\Delta(e^* - \frac{1}{n}, e^*) \leq 0$ for all $\phi > \phi_n^1(e^*)$

Finally, take $\phi_n(e^*) = \max\{\phi_n^0(e^*), \phi_n^1(e^*)\}$and set $\phi_n = \max_{e^* \in S_n} \phi_n(e^*)$. We have thus shown that any $\underline{e}_n \leq e^* \leq \overline{e}_n$ is an equilibrium, as required.

(iii) To show that no $e^* \notin [\underline{e}_n, \ \overline{e}_n]$ can be an equilibrium, suppose first that $e^* > \overline{e}_n$, and consider a downward deviation to $e' = e^* - \frac{1}{n}$. If $\phi > \phi_n$, then $e^* - \frac{1}{n} < \underline{b}(e^*)$ i.e. this deviation induces a binding outside option. As $e^* > \overline{e}_n$,

$$\phi y(e^* - \frac{1}{n}, e^*) - c(e^* - \frac{1}{n}) > \phi y(e^*, e^*) - c(e^*)$$

so a downward deviation makes any agent strictly better off, contradicting the assumption that e^* is an equilibrium. A similar argument shows that if $e^* < \underline{e}_n$ for ϕ close enough to 1, an upward deviation can make the deviant strictly better off.

(iv) To prove $\overline{e}_n \geq \hat{e}_n$, suppose the contrary i.e.$\overline{e}_n = \hat{e}_n - \frac{1}{n}$. Note that as $\hat{e}_n$ is the most efficient feasible investment level, and y is strictly concave, $\hat{e}_n$ satisfies

$$y(\hat{e}_n + \frac{1}{n}, \hat{e}_n + \frac{1}{n}) - y(\hat{e}_n, \hat{e}_n) \leq \frac{2c}{n} < y(\hat{e}_n, \hat{e}_n) - y(\hat{e}_n - \frac{1}{n}, \hat{e}_n - \frac{1}{n}) \tag{A.9}$$

Then from(A.9), and from (4.2), using $\overline{e}_n = \hat{e}_n - \frac{1}{n}$, we get:

$$2y(\hat{e}_n, \hat{e}_n) - 2y(\hat{e}_n - \frac{1}{n}, \hat{e}_n) \leq \frac{2c}{n} < y(\hat{e}_n, \hat{e}_n) - y(\hat{e}_n - \frac{1}{n}, \hat{e}_n - \frac{1}{n}) \tag{A.10}$$

Or, rearranging (A.10),

$$y(\hat{e}_n, \hat{e}_n) - y(\hat{e}_n - \frac{1}{n}, \hat{e}_n) < y(\hat{e}_n - \frac{1}{n}, \hat{e}_n) - y(\hat{e}_n - \frac{1}{n}, \hat{e}_n - \frac{1}{n})$$

which violates (weak) complementarity.

(v) Finally, to prove $\overline{e}_n \geq \hat{e}_n + \frac{1}{n}$ if investments are strictly complementary, there exists a $\tilde{c}$ such that

$$2y(\hat{e}_n + \frac{1}{n}, \hat{e}_n + \frac{1}{n}) - 2y(\hat{e}_n, \hat{e}_n + \frac{1}{n}) > \frac{2\tilde{c}}{n} > y(\hat{e}_n + \frac{1}{n}, \hat{e}_n + \frac{1}{n}) - y(\hat{e}_n, \hat{e}_n)$$

So,

$$y(\hat{e}_n + \frac{1}{n}, \hat{e}_n + \frac{1}{n}) - y(\hat{e}_n, \hat{e}_n + \frac{1}{n}) > \frac{\tilde{c}}{n}$$

implying that for cost $\tilde{c}$, $\hat{e}_n + \frac{1}{n} \leq \overline{e}_n$. $\square$.

Proof of Proposition 4. From the properties of the production function, $y(e,e)-y(e-\frac{1}{n},e)=f(e)-f(e-\frac{1}{n})$ so $\overline{e}_n$ is the largest feasible e such that $f(e)-f(e-\frac{1}{n})>\frac{c}{n}$. So, if $\overline{e}$ is feasible, $\overline{e}_n=\overline{e}-\frac{1}{n}$. Otherwise, there is a feasible investment level in $(\overline{e}-\frac{1}{n},\overline{e})$ in which case $\overline{e}_n$ is equal to that investment level. □

Proof of Proposition 5. First, from (4.2), and denoting the limit of $\overline{e}_n$ by $\overline{e}$, we have

$$\frac{y(\overline{e}_n,\overline{e}_n)-y(\overline{e}_n-\frac{1}{n},\overline{e}_n)}{1/n}>c\Longrightarrow y_1(\overline{e},\overline{e})\geq c$$

where we obtain the derivative by taking the limit as $n\to\infty$. Next, from the fact that $\hat{e}_n$ is the efficient level of investment, we have:

$$\frac{y(\hat{e}_n+\frac{1}{n},\hat{e}_n+\frac{1}{n})-y(\hat{e}_n,\hat{e}_n)}{1/n}\leq 2c\Longrightarrow y_1(\hat{e},\hat{e})\leq c$$

where again we obtain the derivative by taking the limit as $n\to\infty$. But then, as $y_1(e,e)$ is strictly decreasing in e (by concavity), we see that $\overline{e}\leq\hat{e}$.Finally, as $\overline{e}_n\geq\hat{e}_n$ from Proposition 5, $\overline{e}\geq\hat{e}$ also. So, $\overline{e}=\hat{e}$. □

Proof of Proposition 6. (i) first we show that given that matching is assortative, $e_l^*>e_h^*$. With assortative matching, from (5.1), an agent of type $i=h,l$ gets expected present value payoff of search of $\phi_i 0.5y(e_i^*,e_i^*)$. So, a necessary condition for equilibrium is that $\phi_i 0.5y(e,e_i^*)-c_i(e)$ is maximized at $e=e_i^*$. By the convexity of c, and concavity of y, the first-order necessary condition for this is

$$\phi_i 0.5y_1(e_i^*,e_i^*)=c_i'(e_i^*),\ i=h,l$$

Moreover, from the concavity of y in both variables, and convexity of c_i, $e_l^*>e_h^*$ if $c_h'(e)/\phi_h>c_l'(e)/\phi_l$. For a large enough, $\phi_i\simeq 1$, and so this last condition will hold.

(ii) Now we show that given $e_l^*>e_h^*$, matching is assortative for ϕ close enough to 1. For suppose not. Then for non-assortative matches to occur, there are two possible cases. The first is where the $l-$type$'$s outside option is not binding in a match with an $h-$type. In this case, the $l-$type can expect $0.5y(e_l^*,e_h^*)$ from the match, and (from (5.4)) $\phi\left[\lambda 0.5y(e_l^*,e_h^*)+(1-\lambda)0.5y(e_l^*,e_l^*)\right]$ from continued search. As $e_l^*>e_h^*$ $\lambda y(e_l^*,e_h^*)+(1-\lambda)y(e_l^*,e_l^*)>y(e_l^*,e_h^*)$, for ϕ close enough to 1, the $l-$type's outside option will bind in a match with an $h-$type.. Then, the $h-$type can expect $y(e_l^*,e_h^*)-v_l$ from accepting a match with an $h-$type, and $0.5\phi y(e_h^*,e_h^*)$ from rejecting it. As $\phi\to 1$, $v_l\to 0.5y(e_l^*,e_l^*)$, so the gain to accepting for the $h-$type is approximately $y(e_l^*,e_h^*)-0.5y(e_l^*,e_l^*)-0.5y(e_h^*,e_h^*)$. But by $y_{12}>0$, and $e_l^*>e_h^*$, this is strictly negative. So, for ϕ close enough to 1, equilibrium matching cannot be non-assortative.

(iii) Finally, in equilibrium, it must be the case that

$$\phi_i \frac{y(e_i^*, e_i^*)}{2} - c_i(e_i^*) \geq \phi_j \frac{y(e_j^*, e_j^*)}{2} - c_i(e_j^*),\ j \neq i \tag{A.11}$$

holds. For if it did not, an $i-$type could profitably change his investment to e_j^* and thus match with $j-$types only. We can split the gain to not deviating as follows i.e.

$$\begin{aligned} \phi_i \frac{y(e_i^*, e_i^*)}{2} - c_i(e_i^*) - \phi_j \frac{y(e_j^*, e_j^*)}{2} - c_i(e_j^*) \ &= \ \left[\phi_i \frac{y(e_i^*, e_i^*)}{2} - c_i(e_i^*) - \left(\phi_i \frac{y(e_j^*, e_i^*)}{2} - c_i(e_j^*)\right)\right] + \\ &\quad \left[\phi_i \frac{y(e_j^*, e_i^*)}{2} - \phi_j \frac{y(e_j^*, e_j^*)}{2}\right] \end{aligned}$$

The term in the first square brackets on the RHS is always positive as from (5.7), e_i^* is a global maximizer of $\phi_i \frac{y(e, e_i^*)}{2} - c_i(e)$. Moreover, for a large enough, $\phi_i, \phi_j \simeq 1$, so the second square bracket is positive as long as $e_i^* > e_j^*$. So, (A.11) holds for $i = l$, but must be imposed for $i = h$.

(iv) Parts (i)-(iii) establish that an equilibrium with assortative matching always exists, subject to (5.8) holding, and that at this equilibrium, investments are given by the solutions to (5.7). It remains to show that no other equilibrium can exist for ϕ close enough to 1. Such an equilibrium must have non-assortative matching. Part (ii) of the proof has shown that non-assortative matching is not possible when $e_l^* > e_h^*$. So, we must have have $e_l^* \leq e_h^*$. But in equilibrium, using (5.4), it is straightforward to establish that $e_l^* > e_h^*$ if matching is non-assortative, a contradiction.

(v) To prove $e_i^* < \hat{e}_i$, compare (5.7) and (5.3), and use the concavity of y and convexity of c. $\square$

Proof of Proposition 7. (i) Assume that investments are at their hold-up levels with non-assortative matching. We show that if a is small enough, matching is non-assortative. Suppose the contrary. Then, from (5.1), $v_i = \phi_i 0.5 y(e_i, e_i)$, as outside options can never bind with non-assortative matching. So, a type i will always accept a match with a type j if $0.5y(e_i, e_j) > v_i$, which surely holds if a is small enough, as $\phi_i \to 0$ as $a \to 0$. Contradiction.

(ii) Assume non-assortative matching. Then, we show that for a small enough, the only equilibrium investment levels are those for which (5.9) hold. But these are the equilibrium investment levels given that no outside options bind in equilibrium. So, it suffices to show that for a small enough, outside options do not bind in equilibrium. Clearly, only the outside option of the agents who have the higher investment can bind in equilibrium. W.l.o.g., let equilibrium investment levels be $e_l > e_h$. Then, for a binding outside option, we require that $v_l > 0.5y(e_l, e_h)$; but as $v_i \to 0$ as $a \to 0$, this cannot hold for a small enough.

(iii) To show that no other equilibrium can exist, we argue as follows. An equilibrium with non-assortative matching and binding outside options has already been ruled out by (ii). An equilibrium with assortative matching cannot exist either. For suppose that there is such an equilibrium: then an argument as in (ii) establishes that for a low enough, type i agents will accept a match with a $j-$type as the option of continued search is too low.

(iv) To prove $e_i^{**} < \hat{e}_i$, compare (5.9) and (5.6), and use the concavity of y and convexity of c.□

B. Example of Overinvestment with a Non-Negligible Fraction of Non-Investors

Let $a = 1$ (this is w.l.o.g. as all formulae only depend on ratio r/a), and let ε be the fraction of agents in the population who for some exogenous reason do not invest in equilibrium. Also, revenues and costs are as in the example in Section 2. We will first characterise an equilibrium where (i) all the agents who can invest do so; (ii) there is non-assortative matching (NAM) i.e. investors will accept matches with non-investors and (iii) in a match with an investor and a non-investor, the outside option of the investor is binding. The question of interest is how high ε can be for such an equilibrium to exist.

Let v, w be the equilibrium payoffs to search for investors and non-investors respectively. A simple dynamic programming argument[34] implies that

$$v = \frac{1-\varepsilon}{1+r-\varepsilon}4, \ w = \frac{1}{1+r}[6 - 3\varepsilon - (1-\varepsilon)v] \tag{B.1}$$

These formulae are intuitive. In particular, when there are no exogenous non-investors, $v = \frac{4}{1+r}$ i.e. half the discounted revenue from a match between two investors, and when $\varepsilon \simeq 1$, $v \simeq 0$, as by hypothesis, an investor gets no surplus from a match with a non-investor.

[34] The dynamic programming equations defining v, w are

$$\begin{aligned} rv &= (1-\varepsilon)(4-v) + \varepsilon(v-v) \\ rw &= (1-\varepsilon)(6-v-w) + \varepsilon(3-w) \end{aligned}$$

Each states that the return to search (rv or rw) is equal to the expected capital gain from accepting a match. The gain to an investor from accepting a match with another investor is $4-v$ i.e. half the revenue generated minus the payoff to search. The gain to a non-investor from accepting a match with another non-investor is $3-w$ i.e. half the revenue generated minus the payoff to search. The gains to non-investor and investor respectively from a non-assortative match are $3-v-w$ and $v-v=0$ respectively, as the outside option of the investor is assumed binding. Solving these two equations gives (B.1).

Now NAM requires that $v+w \leq 6$, the condition for the outside option of the investor to be binding is $v > 3$, and finally the condition that investment occurs in equilibrium is $v - w \geq 2$. The NAM condition requires

$$\frac{6r+3\varepsilon}{r+\varepsilon} \geq v = \frac{1-\varepsilon}{1+r-\varepsilon}4 \tag{B.2}$$

which always holds as long as $r \geq \varepsilon$. If $r = \varepsilon$, $v > 3$ if $\varepsilon \leq \frac{1}{4}$. Finally, the investment condition is

$$v = \frac{1-\varepsilon}{1+r-\varepsilon}4 \geq \frac{7.5+1.5(r-\varepsilon)}{2+r+\varepsilon} \tag{B.3}$$

This is easiest to evaluate if $r = \varepsilon$, in which case it reduces to $\varepsilon \leq \frac{1}{4}$. So, if $r = \varepsilon$, we have an overinvestment equilibrium even when up to $\frac{1}{4}$ of the agents cannot invest.

C. Proof that $\lambda_t = \lambda$ with Assortative Matching

Let ν_{it} be the number of those unmatched at time t of type i. Then, ν_{it} follows the following dynamics

$$\begin{aligned}
v_{1t} &= v_{1t-1}(1 - a\frac{v_{1t-1}}{v_{1t-1}+v_{2t-1}}) \\
v_{2t} &= v_{2t-1}(1 - a\frac{v_{2t-1}}{v_{1t-1}+v_{2t-1}})
\end{aligned}$$

Explanation: in the first equation, at time t, the probability of exiting the process at $t-1$ is a times the probability of finding a type 1 to match with, which is $\frac{v_{1t-1}}{v_{1t-1}+v_{2t-1}}$. Rearranging, we see that the percentage change in the number of each type in the pool is the same i.e.

$$\frac{v_{1t}-v_{1t-1}}{v_{1t-1}} = \frac{v_{2t}-v_{2t-1}}{v_{1t-1}} = \frac{a}{v_{1t-1}+v_{2t-1}}$$

So, the ratio $\frac{v_{1t}}{v_{2t}}$ must remain unchanged. Thus, λ_t is constant. So, the search environment is stationary whether matching is assortative or not.

Figure 1 – Graph of Net Payoff to Deviant Investor

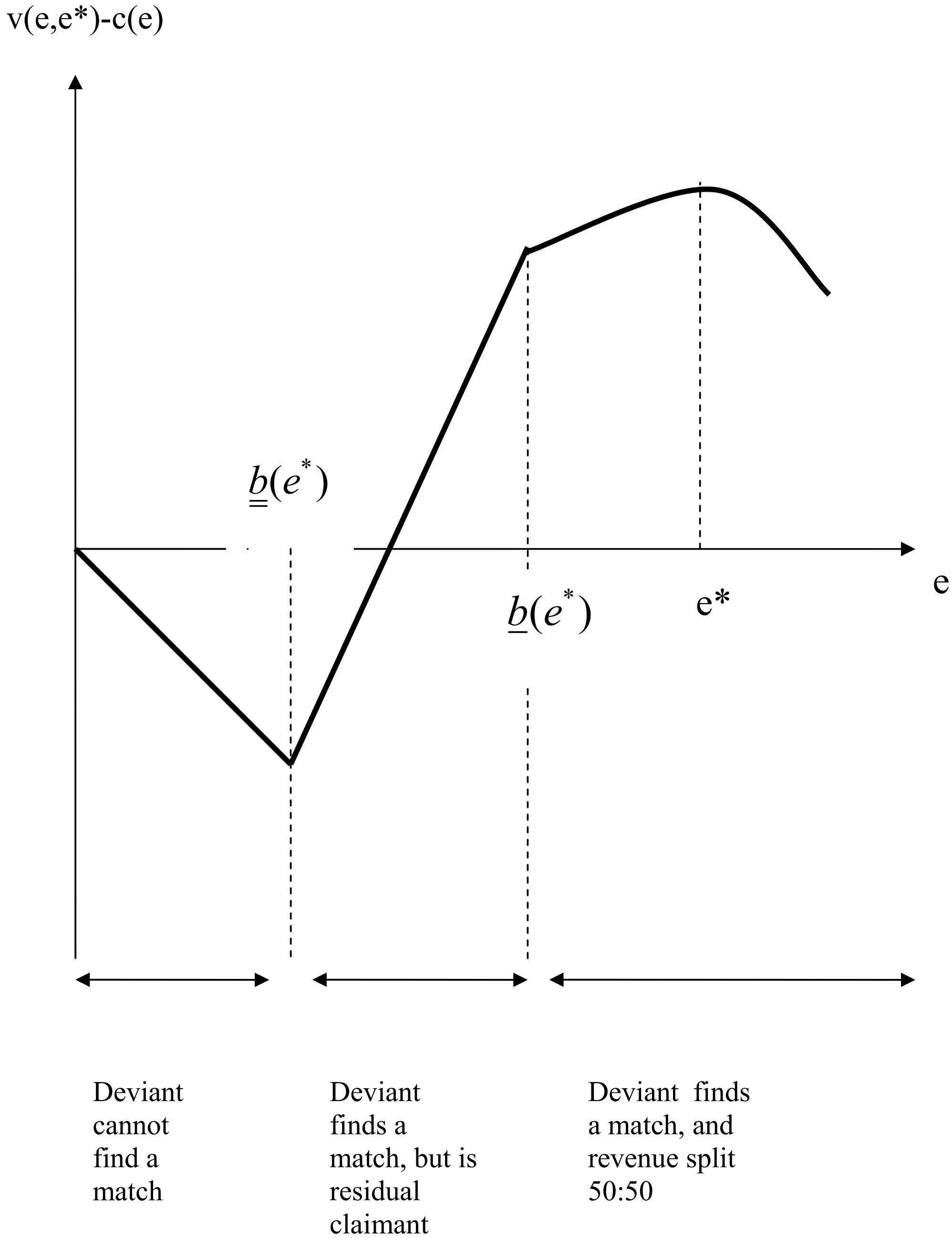

Figure 2 – Equilibrium with Discrete Investments