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Abstract: The paper analyses coupon bonds linked to variable interest rate in a contingent claim approach such that it can be decomposed in elementary options on interest rate and options to default. It is considered the case of continuous arithmetic average of interest rate in a simple capitalization to value the variable coupon paid by the bonds at maturity. The paper determines the expected interest rate on the bonds and the risk spread due to the default risk.

The model and its assumptions

Traditionally a coupon bond linked to variable interest rate is seen by the management as a bond plain vanilla. Indeed, it can be decomposed in elementary options on interest rate such that it is a structured product. We will show how a coupon linked to variable interest rate is a combination of Call option and Put option. The value of the coupon is computed as follows:

$$R_T = [1/(T - t)] \int_t^T r_t \, dt$$

$$\mathrm{B}_T = R_T \, (T - t) \text{ Face Value}$$

Where R_T denotes the continuous arithmetic average of interest rate r_t and B_T denotes the coupon paid by the bonds at maturity. Now we see how to decompose the pay-off of the bonds:

$$R_T = r_t + \max[R_T - r_t, 0] - \max[r_t - R_T, 0]$$

The third and the fourth term are respectively the pay-off of a Call option on interest rate and a Put option on interest rate. Now we denote the expectation of the pay-off as follows:

$$\mathrm{C}(T) = \mathrm{E}\{\max[R_T - r_t, 0]\}$$

$$\mathrm{P}(T) = \mathrm{E}\{\max[r_t - R_T, 0]\}$$

Along Zhang(1998) and Brigo,Mercurio(2006), we have the following:

$$\mathrm{C}(T) = \mathrm{F}(r_t) \, k \, q \, \mathrm{N}[\mathrm{d}_2] - r_t \, \mathrm{N}[\mathrm{d}_1]$$

$$\mathrm{P}(T) = r_t \, \mathrm{N}[-\mathrm{d}_1] - \mathrm{F}(r_t) \, k \, q \, \mathrm{N}[-\mathrm{d}_2]$$

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Where:

N[…] denotes the cumulative normal distribution

$$d_1 = \frac{\ln[kF(r_t)/\, r_t] - ½\, \sigma^2_F\, (T-t)/2}{\sigma_F \sqrt{(T-t)/3}}$$

$$d_2 = d_1 + \sigma_F \sqrt{(T-t)/3}$$

$$k = 1 + (1/24)(-½\, \sigma^2_F)^2\, (T-t)^2 + (1/576)\, (-½\, \sigma^2_F)^4\, (T-t)^4$$

$$q = exp[-\sigma^2_F\, (T-t)/12]$$

$F(r_t)$ denotes the future on the interest rate r_t

The dynamic of the future $F(r_t)$ is given by the following stochastic continuous process:

$$dF(r_t) / F(r_t) = \sigma_F\, dW_r$$

σ_F denotes the instantaneous volatility of the future $F(r_t)$

dW_r denotes a standard Wiener process capturing the interest rate risk

Now the price of the options at initial time is given by:

$$C(t) = C(T)\, D(t,T)$$

$$P(t) = P(T)\, D(t,T)$$

We use the price of a default free zero coupon bond $D(t,T)$ with the same maturity of the coupon bonds as discount factor. Hence, the expected interest rate R^e on the coupon is given by:

$$R^e = r_t + C(T) - P(T) + \textit{Risk Spread}$$

Thus, the expected value of the coupon B^e is:

$$B^e = R^e\, (T - t)\ \text{Face Value}$$

Whereas, the expected total value of the debts L^* is given by:

$$L^* = (\text{Face Value} + B^e)\ \text{Number Bonds}$$

Now, along Black-Scholes(1973) and Merton(1974), we can value the debts by using the option to default. We can see how to apply the default option by using the pay-off. Therefore, we have the following final pay-off for the coupon bonds, where we denote with L_t the initial value of the debt and with A_t the initial value of assets:

$$L_T = \min [\, L^*, A_T\,]$$

Where:

$$L_T - L^* = \min [\, A_T - L^* , 0 \,]$$

At this point, we can note that this is the same final pay-off of a short position on a Put option written on the company value A_t and with exercise price L^*. Thus, we have:

$$L_T - L^* = -\max [\, L^* - A_T , 0 \,]$$

This result is intuitive; it reflects the option of stakeholders to walk away if things go wrong. Straightforward, the initial value of the debt is:

$$L_t = L^* D(t,T) - P(A_t , L^*, T - t)$$

We denote with P(…) the value of European Put option on the firm's underlying asset A_t , maturing at time T and with exercise price L^*.
Now we assume that the dynamic of assets is given by the following stochastic continuous process:

$$dA_t / A_t = \mu_t \, dt + \sigma_A \, dW_A$$

μ_t denotes the drift of the process

dW_A denotes a standard Wiener process capturing the assets-risk

σ_A denotes the instantaneous volatility of assets

Thus, we have the following price for the Put option by using $D(t,T)$ as *numeraire*:

$$P(A_t , L^*, T - t) = D(t,T)\, L^*\, N[h_2] - A_t\, N[h_1]$$

Where:

N[…] denotes the cumulative normal distribution

$$h_2 = \frac{\ln(L^* D(t,T) / A_t) + ½\, \sigma^2_{(t,T)} (T - t)}{\sigma_{(t,T)} \sqrt{(T - t)}}$$

$$h_1 = h_2 - \sigma_{(t,T)} \sqrt{(T - t)}$$

$$\sigma^2_{(t,T)} = [1 / (T - t)] \int_t^T \sigma_A^2 + \sigma_D^2 - 2\, \rho(A,D)\, \sigma_A(t)\, \sigma_D \, dt$$

σ_D denotes the instantaneous volatility of the discount factor

$\rho(A,D)$ represents the correlation between the total value of assets and the discount factor

Conclusion

We have showed how coupon bonds can be decomposed in elementary options on interest rate and options to default. Obviously, if the interest rate paid by the coupon is not computed continuously the options on interest rate become European options. We have to note that for sake of simplicity we assumed that the coupon will be paid just at maturity of the bonds but it is possible to extend the model to multiple coupon paid during the life of the bonds. In fact, it can be seen as multiple debts model developed in Giandomenico(2006), the basic idea is that the coupons are seen as debts maturing before the time of maturity, straightforward, the default options become an option on option. We have to observe that the expected interest rate of the coupon is determined essentially by the future rate. Thus, the formula permits to calibrate the model to the market expectations. Indeed, some problems can arise in the use of the model; for instance, it is possible to have a future rate just for short maturity but in this case we can use the interest rate swap. Moreover, it is possible to don’t have a future rate on the interest-rate; in this case, along Heath-Jarrow-Morton(1992) it is possible to extract the forward rate implied by the yield curve straightforward from the interest rate tem structure. In fact, they take the observed yield curve as initial condition for the forward rate curve, they assume that the forward rate curve reflects the expectation of the market on the future interest-rates such that to avoid arbitrage opportunity it determines the yield curve. We definitely conclude by noting that the model can be extended to price various kinds of coupon bonds from default free bonds, where essentially we don’t have the options to default, to structured bonds.

References

Bjòrk, T. : Arbitrage Theory in Continuous Time
Oxford University Press, 1998

Black, F., Scholes, M. : The Pricing of Options and Corporate Liabilities
Journal of Political Economy, pag. 637-659, 1973

Brigo, D.,Mercurio, F. : Interest Rate Model
Springer Finance, 2006

Giandomenico, R. : Asset Liability Management in Insurance Company
Social Science Research Network, 2006

Heath, D., Jarrow, R., Morton. A. : Bond Pricing and the Term Structure of Interest Rates: a New Methodology for Contingent Claims Valuation
Econometrica, 60(1), pag. 77-105, 1992

Merton, R. : On the Pricing of Corporate Debt: The Risk Structure of Interest Rate
The Journal of Finance, Vol. 29, pag. 449-470, 1974

Zhang, P. : Exotic Options
World Scientific, 1998