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Abstract

We derive several comparative-static results for Cournot games when firms have nonconstant marginal-cost curves which shift exogenously. The results permit us to rank certain vectors of equilibrium marginal costs with the same component sum according to their associated social surplus or industry profit. We arrange the components of each vector in ascending order and then construct from the resulting ordered vector its associated Lorenz curve. We show that if two Lorenz curves do not cross, the one reflecting greater inequality is associated with higher social surplus and industry profit. A duality result permits a corresponding ranking of equilibrium output vectors. The same partial ordering is used in the literature on income inequality to rank certain distributions of income and in the literature on decision-making under uncertainty to compare the riskiness of certain probability distributions with the same mean.

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I Introduction

In comparisons of two distributions of income with the same mean, one distribution is said to exhibit greater income inequality if its associated Lorenz curve lies everywhere weakly below the Lorenz curve of the other (Lorenz, 1905; Dalton, 1920). In comparisons of two random variables with the same mean, one random variable is said to be more variable than the other if the former can be obtained from the latter by a finite sequence of mean-preserving spreads (Rothschild-Stiglitz, 1970, 1971). Although developed independently in the economics literature on income inequality and the literature on uncertainty, the Lorenz criterion as a measure of income inequality and mean-preserving spreads as a measure of riskiness are closely related. Their relationship was first clarified by Atkinson (1970) with subsequent contributions by Dasgupta et al. (1973) and Rothschild-Stiglitz (1973): the Lorenz curves of two ascending vectors do not cross if and only if the vector whose Lorenz curve reflects greater inequality can be obtained from the other vector by a finite sequence of mean-preserving spreads and permutations. Fields-Fei (1978) showed that this same result can be obtained without any permutations by using a subset of mean-preserving spreads which preserve the order of the vector. These economics literatures were foreshadowed by mathematics literature on the theory of majorization first formulated by Muirhead (1903) at the turn of the century and then generalized by Hardy et. al. (1934,1952) three decades later.[1]

In this paper, we show how these same tools of mean-preserving spreads and Lorenz curves, so useful in other contexts, can be utilized to address two unresolved issues in industrial organization. Previous literature on Cournot oligopoly (Bergstrom-Varian, 1985a,b; Salant-Shaffer, 1998, 1999) has established that the qualitative effects of certain exogenous shifts in marginal-cost curves on industry profit and social surplus can be deduced from the sign of the change in the variance of the equilibrium marginal-cost vector. As shown in the next section, however, the change in variance ceases to be a reliable guide once the assumption is relaxed that marginal-cost curves are linear with a weakly positive, common slope. The first contribution of our paper is to show for a more general class of marginal-cost curves that analogous positive and normative conclusions can *sometimes*

[1]For a survey of the mathematics literature with a particularly accessible introduction, see Marshall-Olkin (1979).

be drawn by examining instead the Lorenz curves associated with the equilibrium marginal-cost vectors or, by duality, with the equilibrium output vectors. Although Lorenz curves provide no guidance about changes in industry profit and social surplus when such curves cross, they provide reliable guidance when they do not.

The second contribution of our paper concerns the economic implications of changes in indices of industrial concentration. Previous literature on industrial concentration (Encaoua-Jacquemin, 1980) treated as axiomatic that an index of concentration should increase if the distribution of the market shares of firms in an industry undergoes a mean preserving spread toward the tails of the distribution. The so-called Herfindahl-Hirschman and the entropy indices both satisfy this condition. An increase in such indices is typically taken by the antitrust authorities as a signal that economic welfare has declined.[2] Tirole (1989, p. 223) has pointed out, however, that such indices have "no systematic relationship with economic variables of interest..." and thus may be unreliable as a welfare indicator. Our analysis reinforces Tirole's observation. Within the context of our model, *any* shifts in marginal-cost curves which result in a mean-preserving spread in realized marginal costs or outputs must strictly increase *every* Encaoua-Jacquemin index of industrial concentration while at the same time *strictly increasing* social surplus. This finding casts doubt on the use of any index in this class to signal a problem in industrial performance.

The paper proceeds as follows. In section II, we show which of the conventional results in the special case of linear marginal-cost curves with a weakly positive, common slope generalize and which do not. In section III, we discuss the concepts of order-creating progressive permutations and order-preserving regressive transfers and derive their effects on industry costs and hence industry profit and social surplus. We also consider the effect of sequences of these transformations and relate them to the mathematical concept of majorization, which has been widely used in the literatures on income inequality and uncertainty. In section IV, we show how our results can be applied to vectors of equilibrium outputs. Section V concludes the paper.

[2] The Herfindahl-Hirschman index has received greater prominence than other measures of industrial concentration in the deliberation of the federal enforcement agencies and the courts. See *Antitrust Law Developments* (American Bar Association Antitrust Section, 1992 pp. 302-03), the U.S. Department of Justice, *Merger Guidelines (1992)* [reprinted in 4 Trade Reg. Rep. (Commerce Clearing House, 1992 ¶13,103)] and *FTC v. Bass Bros. Enters.,* 1984-1 Trade Cases (Commerce Clearing House, 1994 p. 68,609, ¶66,041)(N.D. Ohio 1984).

II What Comparative-Statics Results Extend in the Absence of Constant Marginal Costs?[3]

Suppose n (≥ 2) firms play a Cournot game. Assume firm $i \in \{1, \ldots n\}$ has no fixed costs and denote its marginal cost as $c_i(q_i)$. Assume $c_i(q_i) \geq 0$ for all $q_i \geq 0$. Denote the inverse demand as $P(Q)$, where $Q = \sum_{i=1}^{n} q_i$ is industry output. Assume $P(Q) \geq 0$ for all $Q \geq 0$. Then firm i's profit is $P(Q)q_i - \int_0^{q_i} c_i(u)du$. Assume $c_i(q_i)$ and $P(Q)$ are twice continuously differentiable, with $c_i'(q_i) \geq 0$ for all $q_i \geq 0$ and $P'(Q) < 0$ for all Q such that $P(Q) > 0$. Assume firm i's marginal revenue is everywhere decreasing in each rival's output: $P'(Q) + P''(Q)q_i < 0$ for all $q_i \in [0, Q]$. These assumptions ensure the existence of a unique pure-strategy Cournot equilibrium (Gaudet-Salant, 1991).

Assume each firm produces a strictly positive output ($q_i > 0$) in equilibrium. We refer to this as an "interior equilibrium." Denote equilibrium quantities by asterisks (Q^*, q_i^*). These quantities are determined by the n first-order conditions: $P(Q^*) + P'(Q^*)q_i^* - c_i(q_i^*) = 0$. We refer to firm i's marginal cost evaluated at its equilibrium quantity as its "realized marginal cost." It follows that a firm with a strictly higher realized marginal cost will have a strictly lower equilibrium output and two firms with the same realized marginal cost will have the same equilibrium output. Summing the n first-order conditions to obtain $nP(Q^*) + Q^*P'(Q^*) = \sum_{i=1}^{n} c_i(q_i^*)$ yields a second implication:

Proposition 1: Industry output in any interior Cournot equilibrium depends only on the *sum* of the realized marginal costs and not on the distribution of those costs.

Proposition 1 generalizes the result in Bergstrom-Varian (1985 a,b) to non-constant marginal costs. It follows that if exogenous shifts in the marginal-cost curves cause the realized marginal costs of the n firms to change in a way which preserves their sum, then industry output will be unchanged (assuming an interior equilibrium). Moreover, output will contract at each firm experiencing an increase in its realized marginal cost and expand at each firm experiencing a decrease in its realized marginal cost; there will be no change in the output of a firm with unchanged realized marginal

[3] Everything we say about the case of constant marginal costs holds as well in the more general case of linear marginal-cost curves with a weakly positive, common slope.

cost. When the shifts in the marginal-cost curves are caused by changes in production technologies, these characteristics of Cournot-equilibrium have an important comparative-static implication for industry profit (industry revenue less industry production costs) and social surplus (gross consumer surplus less industry production costs). We state this implication in the following proposition.

Proposition 2: Suppose exogenous real shifts in the marginal-cost curves cause the realized marginal costs of the n firms to change in a way which (1) preserves their sum and (2) results in a new interior Cournot equilibrium. Then industry profit and social surplus both increase (respectively, decrease) by the amount of the reduction (respectively, increase) in industry costs.

Proof: Industry profit and social surplus are respectively defined as $QP(Q) - \sum_{i=1}^{n} \int_0^{q_i} c_i(u)du$ and $\int_0^Q P(u)du - \sum_{i=1}^{n} \int_0^{q_i} c_i(u)du$, where $QP(Q)$ is industry revenue, $\int_0^Q P(u)du$ is gross consumer surplus, and $\sum_{i=1}^{n} \int_0^{q_i} c_i(u)du$ is industry cost. The result follows immediately since, under the two hypotheses, the first term in each expression is constant (Proposition 1). $\square$

Proposition 2 generalizes the result in Salant-Shaffer (1999, p. 588) to non-constant marginal costs. When the cost shifts are real, it permits a complete ordering in terms of industry profit and social surplus of all industry-wide vectors of realized marginal costs with the same number of components and component sum (provided they induce an interior Cournot equilibrium); the ordering of industry profit and social surplus is inversely related to the ordering of industry costs.

In our previous paper, we showed (Corollary 4 of Salant-Shaffer, 1999) that a Cournot equilibrium in which n firms have identical constant marginal costs has larger industry cost and hence smaller industry profit and social surplus than any interior equilibrium with the same marginal-cost sum where the variance of the marginal costs is strictly positive. A related result can be obtained in the absence of constant marginal costs by iteratively applying the following proposition:[4]

Proposition 3: Assume that in the initial n-firm Cournot equilibrium there exists some subset of firms with identical realized marginal costs. Assume, moreover, that within that subset there exist two firms (i and j) such that if firm i increases its output by x and firm j decreases its output by x for some $x > 0$, the sum of the induced changes in their two marginal costs would be weakly positive. Assume, finally, that the marginal-cost curves of firm i and j are vertically shifted (with

[4]Proposition 8 below generalizes Corollary 4 of Salant-Shaffer (1999).

no shift in the marginal-cost curves of the other firms) in such a way that in the new equilibrium firm i expands by $x > 0$ and firm j contracts by x. Then such shifts must strictly lower industry cost and must strictly raise industry profit and social surplus.

Proof: See Appendix.

Proposition 3 has broad application even if marginal-cost curves are not everywhere differentiable. Its conditions always hold if marginal-cost curves are weakly increasing and weakly convex. Indeed, these conditions can sometimes be satisfied even when marginal-cost curves are weakly concave.[5]

We can apply Proposition 3 to an initial vector of realized marginal costs with identical components. We can then continue to apply it *iteratively* until no two firms remain with the same realized marginal cost. This final vector of marginal costs will have the same component sum as the initial vector. But since industry cost strictly declines at every iteration, the final equilibrium has a strictly larger profit and social surplus than the initial vector. This does not establish that *every* vector with a strictly positive variance and the same component sum will have a strictly larger profit and social surplus than the vector of identical realized marginal costs, but it leaves open that possibility.

In the well-studied case of constant marginal costs, comparisons can be made even when neither marginal cost vector has identical components. In the case of constant marginal costs, whenever *any* two marginal-cost vectors have the same component sum, the one with the larger variance is associated with the smaller industry cost and hence the larger industry profit and social surplus. To establish whether this result generalizes, we express industry cost as a function of the n realized marginal costs, which we henceforth denote $\theta_i = c_i(q_i^*)$ for $i = 1, \ldots, n$. To do so, simply sum each firm's total cost, expressed as the area under its marginal-cost curve:

$$C = \sum_{i=1}^{n} \int_0^{\frac{P-\theta_i}{-P'}} c_i(u)du, \tag{1}$$

where the upper limit of the definite integral equals firm i's equilibrium output ($q_i^* = \frac{P-\theta_i}{-P'}$ for

[5]If the marginal-cost curve of every firm is a weakly convex, increasing function, then one can pick any two firms with the same initial realized marginal cost and any $x > 0$. By designating as i the firm which would experience the larger marginal-cost increase if it expanded by x, one can always satisfy the conditions of the proposition. The conditions can sometimes be satisfied when marginal-cost curves are increasing but concave if x is suitably chosen.

$i = 1, \ldots, n$). When each firm has a constant marginal cost of production, (1) simplifies dramatically to $C = \frac{1}{-P'}(P \sum \theta_i - \sum \theta_i^2)$, yielding the following implication: when the marginal costs of the n firms change in a way that preserves their sum and results in a new interior Cournot equilibrium, industry costs are strictly lower if and only if the sum of the squared marginal costs of the n firms or, equivalently, the variance of their marginal costs is larger (Proposition 1 of Salant-Shaffer, 1999).

Given the straightforward extension of the results so far when the assumption of constant marginal costs is relaxed, one might anticipate that this "variance" proposition extends in a similar way. Unfortunately it does not. For assume marginal-cost curves have the following form: $c_i(q_i) = a_i + dq_i^k$, where $a_i \geq 0$, $d \geq 0$, and $k \geq 1$.[6] Then, for $d = 0$ or $d > 0$ and $k = 1$, C is strictly decreasing in $\sum_{i=1}^n \theta_i^2$ and does not depend on terms of θ_i to higher powers. One can then rank the industry cost associated with alternative realized marginal-cost vectors with the same sum using the variance of such vectors as the literature suggests (Salant-Shaffer, 1999).[7] If $k = 2$, however, industry cost is still strictly decreasing in $\sum_{i=1}^n \theta_i^2$ but is also *strictly increasing* in $\sum_{i=1}^n \theta_i^3$. Then just because two realized marginal-cost vectors have the same sum of squared components and the same component sum does not imply that industry costs are identical. Consider the realized marginal-cost vectors (2,2,5) and (1,4,4). Even though these vectors have a common component sum ($\sum_{i=1}^n \theta_i = 9$) and a common sum of squared components ($\sum_{i=1}^n \theta_i^2 = 33,$) nonetheless (2,2,5) has the larger sum of *cubed* components ($\sum_{i=1}^n \theta_i^3 = 141 > 129$). It follows that if $k = 2$, the vector (2,2,5) must induce a strictly larger industry cost in the interior Cournot equilibrium.[8] Conversely,

[6] Substituting into (1) and simplifying we obtain:

$$C = \frac{1}{-P'} \left(P \sum_{i=1}^n \theta_i - \sum_{i=1}^n \theta_i^2 - \frac{d}{(-P')^k} \frac{k}{k+1} \sum_{i=1}^n (P - \theta_i)^{k+1} \right).$$

To derive this conclusion, let C_i denote firm i's total cost in the Cournot equilibrium. Substituting for $c_i(q_i)$ and evaluating the definite integral in (1), we obtain: $C_i = a_i(\frac{P-\theta_i}{-P'}) + \frac{d}{k+1}(\frac{P-\theta_i}{-P'})^{k+1}$. Substituting for a_i, this becomes: $C_i = \left[\theta_i - d\frac{(P-\theta_i)^k}{(-P')^k}\right](\frac{P-\theta_i}{-P'}) + \frac{d}{k+1}(\frac{P-\theta_i}{-P'})^{k+1}$, which simplifies to $C_i = \frac{1}{-P'}\left(P\theta_i - \theta_i^2 - \frac{d}{(-P')^k}\frac{k}{k+1}(P-\theta_i)^{k+1}\right)$. Summing over the n firms to get $C = \sum_{i=1}^n C_i$ yields the displayed equation.

[7] Salant-Shaffer (1999, footnote 23) point out that when marginal-cost curves are upward-sloping lines of the same slope, $c_i(q_i) = a_i + dq_i$, the industry cost can be written as a function of $\sum_{i=1}^n a_i$ and $\sum_{i=1}^n a_i^2$. The assertion here is that realized marginal costs can be used instead of the intercept terms.

[8] To elaborate on this example, assume the inverse demand is $P(Q) = 15 - Q$, firm i's marginal cost is $c_i(q_i) = a_i + .04q_i^2$, $i = 1, 2, 3$, and $(a_1, a_2, a_3) = (1.36, 1.36, 4.96)$. Then in equilibrium the output vector is $(4, 4, 1)$, the vector of realized marginal costs is $(2, 2, 5)$, and industry cost is 17.56. Suppose the marginal-cost functions shift exogenously so that (a_1, a_2, a_3)=(0,3.84,3.84). Then in the new equilibrium, the output vector changes to $(5, 2, 2)$, the realized marginal-cost vector becomes $(1, 4, 4)$ and industry costs fall to 17.24. Industry costs are thus higher when the realized

if one vector of realized marginal costs has a strictly higher variance but the same component sum, one cannot conclude that it is associated with the lower industry cost. Consider the realized marginal-cost vectors $(2-\epsilon, 2, 5+\epsilon)$ and $(1,4,4)$, obtained by perturbing the previous example. Then the sum of the components in the new initial vector is unchanged but the sum of its squared components is increasing in ϵ for $\epsilon \geq 0$. Since the industry cost associated with $(2-\epsilon, 2, 5+\epsilon)$ is continuous in ϵ, industry cost must remain strictly larger not merely for $\epsilon = 0$ but for sufficiently small $\epsilon > 0$. These two examples suffice to establish the following proposition.

Proposition 4: Suppose exogenous real shifts in the marginal-cost curves cause the realized marginal costs of the n firms to change in a way which (1) preserves their sum and (2) results in a new interior Cournot equilibrium. Then a decrease in industry costs can occur without an increase in the sum of squared realized marginal costs; conversely, if the sum of the squared realized marginal costs does increase, industry costs need not be lower.

Although in the familiar case of parallel linear marginal-cost curves, a change which preserves the sum of the components of the vector of realized marginal costs strictly lowers industry costs *if and only if* the sum of the squared components increases, *neither* the "if" *nor* the "only if" part of this statement extends when the slope of the marginal-cost curves is not constant.

One can, of course, use the functional in (1) to determine the *exact* magnitude of industry costs. But the inference requires knowing the entire marginal-cost function of each firm. An alternative approach would be to compute the total differential dC from (1):

$$dC = \frac{1}{P'} \sum_{i=1}^{n} \theta_i d\theta_i. \tag{2}$$

This simple equation is useful since it relates local changes in realized marginal costs to local changes in industry cost. One of its implications is that if two realized marginal costs are unequal, raising *infinitesimally* the higher of two realized marginal costs while lowering the other by an offsetting amount must reduce industry costs (since $P' < 0$). Using an alternative approach, we will verify (in Proposition 6) that this qualitative result also holds for non-local changes in realized marginal cost.

marginal costs are (2,2,5) even though the sum of the squared realized marginal costs is the same for the two vectors.

However, the usefulness of this total differential has its limits. Serious error can result if it is treated as valid for non-local changes. In fact, $\Delta C \neq \frac{1}{P'}\sum_{i=1}^{n}\theta_i\Delta\theta_i$. To demonstrate that the two sides are unequal, regard $(2,2,5)$ as the initial vector of realized marginal costs and $(1,4,4)$ as the final vector of realized marginal costs. Then, as we have shown above with $k=2$, the change in industry cost is negative, $\Delta C < 0$; but the right-hand side of this formula is strictly positive.[9]

To summarize, determining whether industry profit and social surplus have increased when real exogenous shifts in marginal-cost curves leave industry output unchanged (and result in an interior Cournot equilibrium) requires knowing the sign of the change in industry costs. When the slope of each marginal-cost curve is a common constant, the sign of the change in industry costs may be inferred from the sign of the change in the sum of the squared realized marginal costs (or variance of the realized marginal costs). When the slope of the marginal-cost curves is not constant, however, this statistic can be misleading. One alternative is to use instead the statistic $\frac{1}{P'}\sum\theta_i\Delta\theta_i$. But unless changes are infinitesimal, this statistic will also be misleading. Another alternative is to use the functional in (1). However, that formula requires knowing the entire marginal-cost function of each firm, information that may not be readily available to analysts and policy makers.

Below we propose a *new* way to determine the sign of the change in industry costs. A limitation of the proposed method is that it sometimes yields no answer. When it does yield an answer, however, that answer is correct. The method we propose involves comparing Lorenz curves constructed from ordered vectors of realized marginal costs. If the Lorenz curves do not cross (or equivalently, if the vector of partial sums of each ordered vector can be Pareto ranked), the curve reflecting greater inequality is associated with higher social surplus and industry profit. Since marginal-cost realizations may be difficult to observe, we show in Section V a duality result: the same ordering also applies to Lorenz curves constructed from ordered vectors of equilibrium firm outputs.

[9] Indeed both the sign and the magnitude of the right-hand side remain the same ($\frac{3}{-P'} > 0$) even if the role of the initial and final vectors are *interchanged*. But this is obvious nonsense! Clearly each industry cost cannot be strictly larger than the other.

III Transformations of the Initial Vector of Realized Marginal Costs

Preliminaries

In any comparison of two vectors of realized marginal costs with the same number of components and component sum, what can we infer about industry costs in the two Cournot equilibria? As we have seen, nothing can be said using the sum of squared components or differentials if the change induced in realized marginal costs is non-local. However, if the realized marginal costs change because the marginal-cost curves of firms are shifted vertically, it is sometimes possible merely by inspecting the components of these vectors to determine which yields the lower industry costs, and hence, the higher industry profit and social surplus.

Let $\mathbf{c^I} = (c_1(q_1), \ldots, c_n(q_n))$ denote the initial vector of marginal-cost curves (evaluated at an arbitrary $(q_1, \ldots, q_n)$) and $\mathbf{c^F} = (c_1(q_1) + \Delta_1, \ldots, c_n(q_n) + \Delta_n)$ the final vector of marginal-cost curves, where we consider final vectors which are obtained from initial vectors by exogenous vertical shifts in the components of $\mathbf{c^I}$. We denote these vertical shifts by the vector $(\Delta_1, \ldots, \Delta_n)$.

Our previous assumptions imply that $c_i(q_i) \geq 0$ for all $q_i \geq 0$, $i = 1, \ldots, n$. We restrict attention to exogenous vertical shifts that belong to the set $\{(\Delta_1, \ldots, \Delta_n) | c_i(0) + \Delta_i \geq 0 \text{ for } i = 1, ...n\}$. Since marginal-cost curves are assumed to be increasing, $c_i(q_i) + \Delta_i \geq 0$ for all $q_i \geq 0$, $i = 1, ...n$.

Let $\mathbf{q^I} = (q_1^I, \ldots, q_n^I)$ and $\mathbf{q^F} = (q_1^F, \ldots, q_n^F)$ denote the initial and final vectors of Cournot equilibrium quantities, and let $\theta^{\mathbf{I}} = (\theta_1^I, \ldots, \theta_n^I)$ and $\theta^{\mathbf{F}} = (\theta_1^F, \ldots, \theta_n^F)$ denote the initial and final vectors of realized marginal costs, where $\theta_i^I = c_i(q_i^I)$ and $\theta_i^F = c_i(q_i^F) + \Delta_i$, for $i = 1, \ldots, n$. Our assumptions imply that $\theta^{\mathbf{I}} \in \Re_+^n$ and $\theta^{\mathbf{F}} \in \Re_+^n$, where $\Re_+^n \equiv \{(x_1, \ldots, x_n) | x_i \geq 0 \text{ for all } i\}$.

Lemma 1: Assume the initial and final equilibrium are interior. Then if the n initial marginal-cost curves are vertically shifted in any way that results in a new interior equilibrium with the same realized marginal-cost sum, the vector of shifts $(\Delta_1, \ldots, \Delta_n)$ can be uniquely inferred from the final vector of realized marginal costs $(\theta_1^F, \ldots, \theta_n^F)$. Moreover, if $\theta_i^F \gtreqless \theta_i^I$ then $\Delta_i \gtreqless 0$.

Proof: Since the realized marginal-cost sum is unchanged, the industry output remains Q^*. Given θ_i^F and the fact that the equilibrium is interior, we can infer firm i's output from the first-order condition: $q_i^F = \frac{P(Q^*)-\theta_i^F}{-P'(Q^*)}$. But this final output would induce a marginal cost different from the given θ_i^F unless firm i's original marginal-cost curve shifted vertically by exactly $\Delta_i = \theta_i^F - c_i(q_i^F)$. Hence, the underlying exogenous shifts $(\Delta_1, \ldots, \Delta_n)$ are unique.[10] If the realized marginal cost at firm i strictly increases ($\theta_i^F > \theta_i^I$) then that firm must strictly reduce production in order for its perceived marginal revenue $(P + q_i P')$ to continue to equal the realized marginal cost. But since its marginal cost is weakly increasing in its output, the realized marginal cost could increase when output falls only if the marginal-cost curve of firm i shifts vertically upward ($\Delta_i > 0$). Analogous arguments establish that if $\theta_i^F \gtreqless \theta_i^I$, then $\Delta_i \gtreqless 0$. $\square$

In what follows, we assume throughout that the marginal-cost curves of firms 1 to n belong to the set of marginal-cost curves ξ, whose components are non-negative, twice continuously differentiable, nondecreasing and weakly convex such that the slope of the marginal-cost curve of a firm with a lower index is weakly larger than the slope of the marginal-cost curve of a firm with a higher index when evaluated at the same point. More formally,

$$\xi \equiv \left\{(c_1(q_1), \ldots, c_n(q_n)) | \, c_i(s) \geq 0, c_i''(s) \geq 0, c_i'(s) \geq c_j'(s) \geq 0, \text{ for all } s \geq 0, i < j, \; i, j \in (1, \ldots, n)\right\}.$$

The well-studied case of constant marginal costs is a member of this set, as is the class of functions $c_i(q_i) = a_i + dq_i^k$, $k \geq 1$ and $i = 1, ..., n$, from which was drawn the example illustrating that industry costs do not necessarily decrease when the variance of realized marginal costs increases. Both of these examples fall within that subset of ξ where each firm's marginal-cost curve is a vertical translation of the same weakly increasing, weakly convex function.[11]

Order-Creating Progressive Permutations

One type of transformation that preserves the sum of the realized marginal costs occurs when two marginal-cost curves shift vertically in opposite directions in such a way that in the new equilibrium a permutation of the two realized marginal costs occurs, with no change in the realized marginal

[10] Reconsider the example in footnote 5, where $P(Q) = 15 - Q$ and $c_i(q_i) = a_i + .04q_i^2$, $i = 1, 2, 3$. Recall that initially (a_1, a_2, a_3)=(1.36,1.36,4.96) and the resulting vector of realized marginal costs was (2,2,5). When the final vector of realized marginal costs was $(1, 4, 4)$, (a_1, a_2, a_3)=(0,3.84,3.84). Hence, the unique vertical shifts which generates this final vector of realized marginal costs is $(\Delta_1, \Delta_2, \Delta_3)$=(0,3.84,3.84)-(1.36,1.36,4.96)=(-1.36, 2.48, -1.12).

[11] On first reading, some readers may prefer to focus on this subset of ξ before considering the more general case where slopes of marginal-cost curves of different firms may differ at some common output.

costs of the other firms. If in this transformation the marginal-cost curve of the firm with the smaller index shifts down so that its realized marginal cost decreases, then we call the transformation an "order-creating progressive permutation."

Definition: A transfer of realized marginal cost from firm i to firm j, $i < j$, is an *order-creating progressive permutation* if (1) firm i's realized marginal cost decreases ($\theta_i^F < \theta_i^I$) and firm j's realized marginal cost increases ($\theta_j^F > \theta_j^I$) such that $\theta_i^F = \theta_j^I$ and $\theta_i^I = \theta_j^F$, and (2) there is no change in the remaining firms' realized marginal costs ($\theta_k^F = \theta_k^I$, $k \in \{1, ...n\}$, $k \neq i, j$).

The transformation of (13,5,22,3,10,7) to (3,5,22,13,10,7), for example, is an order-creating progressive permutation since only two components have been permuted and the firm with the smaller index experiences a decrease in its realized marginal cost.[12] The transformation is termed "progressive" because—like a progressive income tax—it lowers something higher and raises something lower (realized marginal cost instead of income). Finally, the transformation is termed "order-creating" because, as shown in Lemma 2 below, its repeated application can rearrange the initial vector of realized marginal costs into ascending order—in our example to (3,5,7,10,13,22).

To demonstrate this, we adopt the following notation. For any vector $\mathbf{g} = (g_1, \ldots, g_n)$, let $\mathbf{g}_\uparrow$ be obtained from $\mathbf{g}$ by reordering the components of $\mathbf{g}$ from lowest to highest. Denote the components of $\mathbf{g}_\uparrow$ as $\left(g_{(1)}, \ldots, g_{(n)}\right)$, where $g_{(1)} \leq \cdots \leq g_{(n)}$. We now present our lemma:

Lemma 2: Any vector $\mathbf{g}$ can be transformed to $\mathbf{g}_\uparrow$ by a finite sequence of order-creating progressive permutations.

Proof: Find the smallest component of $\mathbf{g}$. If it does not have the index 1, assign to it the index 1 and assign its index to the component previously indexed as 1. Now find the second smallest component of this transformation of $\mathbf{g}$. If it does not have the index 2, assign to it the index 2 and assign its index to the component previously indexed as 2 . . . After at most $n-1$ such order-creating

[12] On the other hand, the transformation of the first vector to (22,5,13,3,10,7) is not an order-creating progressive permutation because the realized marginal cost of the firm with the smaller index shifts up.

progressive permutations, $\mathbf{g}_{\uparrow}$ will result. $\square$

Consider now the economic consequences of such a transformation. Since an order-creating progressive permutation leaves Q^* and the equilibrium price unaffected, and since the largest realized marginal cost is the same in $\theta^{\mathbf{I}}$ and $\theta^{\mathbf{F}}$, it follows that if the final Cournot equilibrium is interior then the initial Cournot equilibrium must also be interior.

Consider next the effects of such a transformation on the output of each firm. Since each firm always adjusts its output in equilibrium so its perceived marginal revenue $(P + qP')$ equals its realized marginal cost and since under this transformation $\theta_i^F = \theta_j^I < \theta_i^I = \theta_j^F$, it follows that $q_i^F = q_j^I > q_i^I = q_j^F$. Hence the output of firm i, which is initially smaller than that of firm j, *expands* in response to the downward shift in its marginal-cost curve until it reaches the initial output level of firm j; similarly, the output of firm j *contracts* in response to the upward shift in its marginal-cost curve until it reaches the initial output level of firm i.

We conclude our discussion of the economic effects of order-creating progressive permutations by showing their effect on social surplus, industry profit, and industry cost.

Proposition 5: In any order-creating progressive permutation, if the final equilibrium is interior then industry costs weakly decrease and hence industry profit and social surplus weakly increase.

Proof: Since the realized marginal costs of $n-2$ of the firms do not change, the equilibrium outputs of these firms do not change—nor do their equilibrium costs of production. When calculating the effect of an order-creating progressive permutation on industry costs, therefore, we can focus solely on its effect on firms i and j. Thus, we can write the actual change in the n firms' costs as:

$$\begin{aligned} &\int_0^{q_i^F} (c_i(s) + \Delta_i)\, ds + \int_0^{q_j^F} (c_j(s) + \Delta_j)\, ds - \int_0^{q_i^I} c_i(s)\, ds - \int_0^{q_j^I} c_j(s)\, ds \\ &\qquad = \int_0^{q_i^I} (\Delta_i + \Delta_j)\, ds + \int_{q_i^I}^{q_j^I} (c_i(s) + \Delta_i - c_j(s))\, ds, \end{aligned} \tag{3}$$

where in the second line we have replaced final outputs using the fact that $q_i^F = q_j^I > q_i^I = q_j^F$.

The right-hand side of (3) decomposes the change in industry costs into the sum of two definite integrals. Since the upper limit of each exceeds the lower limit, one can establish that industry

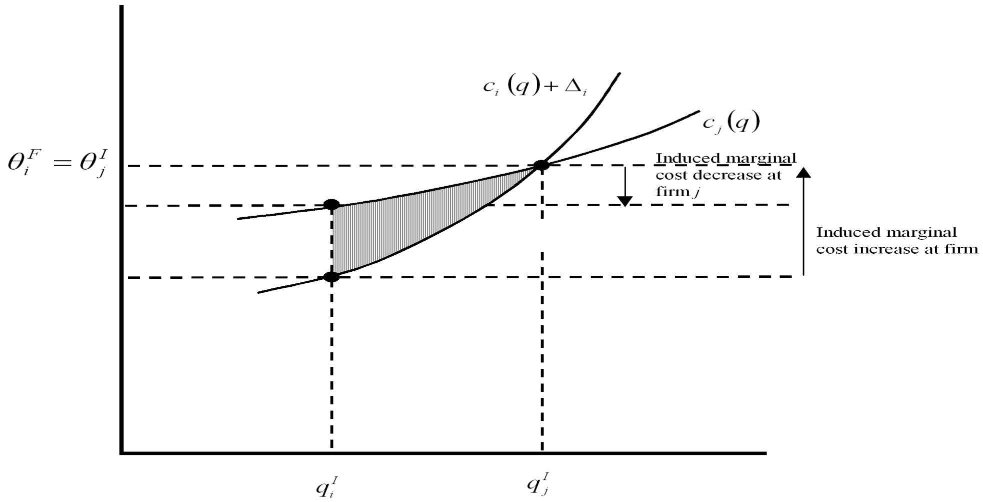

Figure 1. **Illustration of the cost effects of an order-creating progressive permutation**

costs weakly decrease if one can show that each integrand is weakly negative over its range of integration. Figure 1 will facilitate this demonstration.

Consider the first integrand in the second line. As discussed above, firm i expands its output from q_i^I to q_j^I while firm j $(> i)$ contracts its output over this same range—from q_j^I to q_i^I. Since each marginal-cost curve is weakly convex and the marginal-cost curve of firm i is everywhere weakly steeper, its output expansion must increase its induced marginal cost by at least as much as the output contraction of firm j reduces its induced marginal cost. Consequently, the sum of the *induced* marginal cost changes must be weakly positive as is reflected in the vertical distances indicated in Figure 1. In order for the sum of the realized marginal costs nonetheless to remain unchanged, the sum of the *vertical shifts* in the two marginal-cost curves $(\Delta_i + \Delta_j)$ must be weakly negative. Hence, the first definite integral in (3) must be weakly negative.

The second definite integral reflects the change in industry costs that occurs from having firm i and not firm j produce the additional units from q_i^I to q_j^I. Its integrand is also weakly negative since the final marginal-cost curve of firm i is everywhere weakly steeper and therefore must lie below the initial marginal-cost curve of firm j to the left of $q_j^I = q_i^F$ (where they must coincide).

See the shaded area in Figure 1. $\square$

In the special case where *every* marginal-cost curve is a vertical translation of the *same* function, each integrand on the right-hand side of (3) is zero. The first integrand is zero since the net downward shift in the two marginal-cost curves must offset the net induced increase in the two marginal costs; and since the latter must be zero under the circumstance posited, the former must be zero as well. As for the second integrand, it will be zero since firm i's marginal-cost curve after its shift down will, in the circumstance posited, coincide with firm j's initial marginal-cost curve. As a consequence, order-creating progressive permutations of marginal-cost curves in ξ have no effect on industry cost, profit, or social surplus. This implies that in the special cases of constant marginal costs or the weakly increasing, weakly convex cost curves used in the discussion of Proposition 4, such vertical shifts in the marginal-cost curves have no aggregate economic effects.

Order-Preserving Regressive Transfers

Another type of transformation that preserves the sum of the realized marginal costs occurs when two marginal-cost curves shift vertically in opposite directions in such a way that in the new equilibrium the larger of two realized marginal costs increases by as much as the smaller of them decreases, with no change in the realized marginal costs of the other firms. We define this transformation only on realized marginal-cost vectors which weakly ascend. If the transformed vector also weakly ascends, then we call the transformation an "order-preserving regressive transfer."

Definition: Suppose the initial vector of realized marginal cost is in ascending order ($\theta^{\mathbf{I}} = \theta^{\mathbf{I}}_{\uparrow}$). Then a transfer of realized marginal cost from firm i to firm j, $i < j$, is an *order-preserving regressive transfer* if (1) firm i's realized marginal cost decreases ($\theta_i^F < \theta_i^I$), firm j's realized marginal cost increases ($\theta_j^F > \theta_j^I$), and $\theta_i^F + \theta_j^F = \theta_i^I + \theta_j^I$, (2) there is no change in the remaining firms' realized marginal costs ($\theta_k^F = \theta_k^I$, $k \in \{1, ...n\}$, $k \neq i, j$), and (3) the vector of realized marginal cost remains in ascending order ($\theta^{\mathbf{F}} = \theta^{\mathbf{F}}_{\uparrow}$).

The transformation of (3,5,7,10,13,22) to (2,5,7,10,14,22), for example, is an order-preserving regressive transfer since two components change in equal and opposite directions and the smaller of the two components is reduced.[13] The transformation is termed "regressive" because—like a regressive income tax—it raises something higher and lowers something smaller. Finally the transformation is termed "order-preserving" because it transforms one weakly ascending vector into another.

Given two ascending vectors of realized marginal costs, Fields-Fei (1978) have shown that it possible to begin with one vector and, after a finite sequence of order-preserving regressive transfers, to generate the other vector if and only if an easily verified condition holds.

Before presenting their condition, we need two definitions:

Definition: For $\mathbf{g} \in \Re^n$, the vector of partial sums of $\mathbf{g}$ is the $n-$tuple with k^{th} component $\sum_{i=1}^{k} g_{(i)}$ for $i = 1, \ldots, k$, where $g_{(i)} \leq g_{(j)}$ if $i < j$.

For example, to compute the vector of partial sums of $\mathbf{g}$=(3,2,4), we first transform it to $\mathbf{g}_{\uparrow}$=(2,3,4) so that $g_{(1)} = 2,\ g_{(2)} = 3,$ and $g_{(3)} = 4$. We then compute the partial sums $\sum_{i=1}^{1} g_{(i)} = 2, \sum_{i=1}^{2} g_{(i)} = 5,$ and $\sum_{i=1}^{3} g_{(i)} = 9$. Hence, the vector of partial sums of $\mathbf{g}$ is (2,5,9).

Definition: For $\mathbf{y}, \mathbf{x} \in \Re^n$, $\mathbf{y}$ is *majorized* by $\mathbf{x}$ (or, equivalently, $\mathbf{x}$ majorizes $\mathbf{y}$), written $\mathbf{y} \prec \mathbf{x}$, if and only if either $\mathbf{x}_{\uparrow} = \mathbf{y}_{\uparrow}$ or (1) the vector of partial sums of $\mathbf{y}$ Pareto dominates the vector of partial sums of $\mathbf{x}$ and (2) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.[14]

For example, if $\mathbf{y}$=(3,3,3) and $\mathbf{x}$=(3,2,4) then $\mathbf{x}$ majorizes $\mathbf{y}$ since (3,6,9)—the vector of partial sums of $\mathbf{y}$—Pareto dominates (2,5,9)—the vector of partial sums of $\mathbf{x}$ and since $\mathbf{y}$ and $\mathbf{x}$ have the same component sum.[15]

[13] On the other hand, the transformation of the first vector to $(3, 5, 4, 10, 16, 22)$ is not an order-preserving regressive transfer. While a smaller realized marginal cost in the first vector is reduced by as much as a larger realized marginal cost is increased, the components of the transformed vector are no longer in ascending order.

[14] Condition (2) in this definition may seem unduly restrictive. When only the first condition holds, $\mathbf{x}$ is said to weakly majorize $\mathbf{y}$. The Appendix shows that our results do not generalize to weak majorization. Section V suggests, however, how to use our majorization results when the sum of the realized marginal costs changes.

[15] The theory of majorization has previously been applied in the economics literature to decision-making under uncertainty because of its relationship to mean-preserving spreads, and to income inequality because the partial sums in its definition have a special relationship to the Lorenz curve. For example, if $\mathbf{y} \prec \mathbf{x}$, then the distribution of incomes in $\mathbf{y}$ is said to *Lorenz dominate* the distribution of incomes in $\mathbf{x}$, leading to lower income inequality. The precise relationship between majorization and Lorenz curves is as follows. Suppose $\mathbf{y}, \mathbf{x} \in \Re^n$. Then $\mathbf{y}$ is majorized

The precise relationship between majorization and order-preserving regressive transfers is given in the following lemma which we attribute[16] to Fields-Fei (1978, p. 309).[17]

Lemma 3: For $\mathbf{x}, \mathbf{y} \in \Re^n$, if $\mathbf{x}_\uparrow \neq \mathbf{y}_\uparrow$ then $\mathbf{y}$ is majorized by $\mathbf{x}$ if and only if $\mathbf{x}_\uparrow$ can be obtained from $\mathbf{y}_\uparrow$ by a finite sequence of order-preserving regressive transfers.

Given two ascending vectors, one of which majorizes the other, the "only if" part of this lemma establishes the existence of a finite sequence of order-preserving regressive transfers which will transform the majorized vector into the majorizing vector.

Consider now the economic consequences of any order-preserving regressive transfer. Since each transformation leaves Q^* and the equilibrium price unaffected, and since the largest realized marginal cost in $\theta^{\mathbf{F}}$ weakly exceeds the largest realized marginal cost in $\theta^{\mathbf{I}}$, it follows that if the final Cournot equilibrium is interior then the initial Cournot equilibrium must also be interior.

Consider next the effects of such a transformation on the output of each firm. Since each firm always adjusts its output so its perceived marginal revenue $(P + qP')$ equals its realized marginal cost and since, under this transformation, $\theta_i^F < \theta_i^I < \theta_j^I < \theta_j^F$, it follows that $q_i^F > q_i^I > q_j^I > q_j^F$. Hence, the output of firm i, which is initially larger than that of firm j, *expands* in response to the downward shift in its marginal-cost curve while the output of firm j, which was already smaller initially, *contracts* in response to the upward shift in its marginal-cost curve.

We conclude our discussion of the economic effects of order-preserving regressive transfers by showing their effect on social surplus, industry profit, and industry cost.

Proposition 6: In any order-preserving regressive transfer, if the final equilibrium is interior, then

by $\mathbf{x}$ if and only if the Lorenz curve of $\mathbf{y}$ dominates the Lorenz curve of $\mathbf{x}$. See Rothschild-Stiglitz (1973).

[16] For the intellectual history of this result, see Foster (1985, p.49-50). Foster traces this proposition to Hardy et. al. (1952) and discusses its tangled evolution over the ensuing 33 years.

[17] Fields-Fei (1978) showed that the Lorenz curve of $\mathbf{y}$ Lorenz dominates that of $\mathbf{x}$ if and only if $\mathbf{x}_\uparrow$ can be obtained from $\mathbf{y}_\uparrow$ by a finite sequence of order-preserving regressive transfers. Lemma 3 then follows from the equivalence of Lorenz domination and majorization (see footnote 11). Alternatively, we could have used the result in Rothschild-Stiglitz (1973) and written Lemma 3 as "For $\mathbf{x}, \mathbf{y} \in \Re^n$, if $\mathbf{x}_\uparrow \neq \mathbf{y}_\uparrow$ then $\mathbf{y}$ is majorized by $\mathbf{x}$ if and only if $\mathbf{x}_\uparrow$ can be obtained from $\mathbf{y}_\uparrow$ by a finite sequence of mean-preserving spreads and order-creating progressive permutations." Our results would be unchanged since applying a mean-preserving spread to an ascending vector followed immediately by an order-creating progressive permutation will also always result in an ascending vector with a strictly lower associated industry cost (Proposition 5 and 6).

industry costs strictly decrease and hence industry profit and social surplus strictly increase.

Proof: When calculating the effect of a regressive transfer on industry costs, we can focus solely on its effect on firms i and j. Thus, we can write the actual change in the n firms' costs as:

$$\int_0^{q_i^F} (c_i(s) + \Delta_i)\, ds \;+ \int_0^{q_j^F} (c_j(s) + \Delta_j)\, ds \;-\; \int_0^{q_i^I} c_i(s)\, ds \;- \int_0^{q_j^I} c_j(s)\, ds$$

$$= \int_{q_j^I}^{q_i^I} \Delta_i\, ds \;+\; \int_0^{q_j^I} (\Delta_i + \Delta_j)\; ds + \int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds \;-\; \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds. \tag{4}$$

The right-hand side of (4) decomposes the change in industry costs into four definite integrals. The first pair of integrals reflects the change in industry costs that would occur if the marginal-cost curves were vertically shifted with no change in outputs. The upper limit of each integral exceeds the lower limit. Hence, to show that each integral is negative we need merely establish that each integrand is negative over its range of integration. The first integrand is strictly negative since the marginal-cost curve of firm i shifts vertically downward. The second integrand is the net *vertical shift* in the marginal costs curves of firms i and j. Since this net vertical shift must be exactly offset by the net induced change in marginal costs, we can show that the net shift is weakly negative ($\Delta_i + \Delta_j \leq 0$) by establishing that the net *induced* change in marginal costs is weakly positive. That it is weakly positive follows since marginal costs curves are weakly convex and the firm with the weakly steeper marginal-cost curve expands its initially larger output by as much as the other firm contracts its initially smaller output.

The final two terms reflect the additional change in industry costs that would occur when the outputs of the two firms re-equilibrate. Combined, these two terms are strictly negative. To see this, note first that since $q_i^F - q_i^I = q_j^I - q_j^F > 0$, each term is integrated over an interval of the same width. Denote the common width as $\delta > 0$. Since marginal-cost curves are weakly increasing,

$$\int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds \;-\; \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds \;\leq\; \delta\left[\left(c_i(q_i^F) + \Delta_1\right) \;-\; \left(c_j(q_j^F) + \Delta_j\right)\right].$$

We complete the proof by verifying that the factor on the right in square brackets is strictly negative. Since the two firms begin with equal outputs and firm i expands while firm j contracts, firm i produces a strictly larger final output and must have a strictly smaller final realized marginal cost: $[(c_i(q_i^F) + \Delta_i) - (c_j(q_j^F) + \Delta_j)] < 0$. Hence, $\int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds \; - \; \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds < 0$. Since the term in square brackets is $\theta_i^F - \theta_j^F$, which is strictly negative, we conclude that the change in industry costs resulting from a regressive transfer must be strictly negative.[18] This establishes that a regressive transfer must strictly decrease industry costs and hence must strictly increase industry profit and social surplus (Proposition 2). $\square$

Figure 2 illustrates why a regressive transfer must strictly lower industry production costs. The horizontal rectangular areas shaded *light* gray depict the changes in production costs at each firm if outputs remained fixed at their initial levels. The vertical areas depict the changes in production costs at each firm which occur after outputs adjust to their final equilibrium levels.

The area of the lower horizontal rectangle measures the cost reduction that would occur following the downward shift of Δ_i in the marginal-cost curve of the lower marginal-cost firm (firm i). The area of the upper horizontal rectangle measures the cost increase that would occur following the upward shift of Δ_j in the marginal-cost curve of the higher marginal-cost firm (firm j). The area of the increase in cost is smaller since the magnitude of the height Δ_j is smaller than the magnitude of the height Δ_i and the width of the upper rectangle is smaller than the width of the lower rectangle.

When the quantities re-equilibrate, firm i will expand while the firm j will contract. The vertical areas under the two final marginal-cost curves in Figure 2 measure the induced change in each firm's production costs. The cost reduction induced at the contracting firm clearly exceeds the cost increase induced at the expanding firm. To verify this, note that even if we overstated the cost increase and understated the cost reduction, the induced industry cost change would be negative. To overstate the cost increase, replace the "trapezoidal" actual area under the lower final marginal-cost curve by the right-hand vertical rectangle; the white "triangular" area measures the size of the overstatement of the true cost increase. To understate the cost reduction, replace the actual 'trapezoidal" area under the higher final marginal-cost curve by the left-hand vertical rectangle; the white "triangular" area measures the size of the understatement of the true cost

[18] Recall that, when discussing the total differential in equation (2), we reached a similar conclusion when the shifts in realized marginal costs were local.

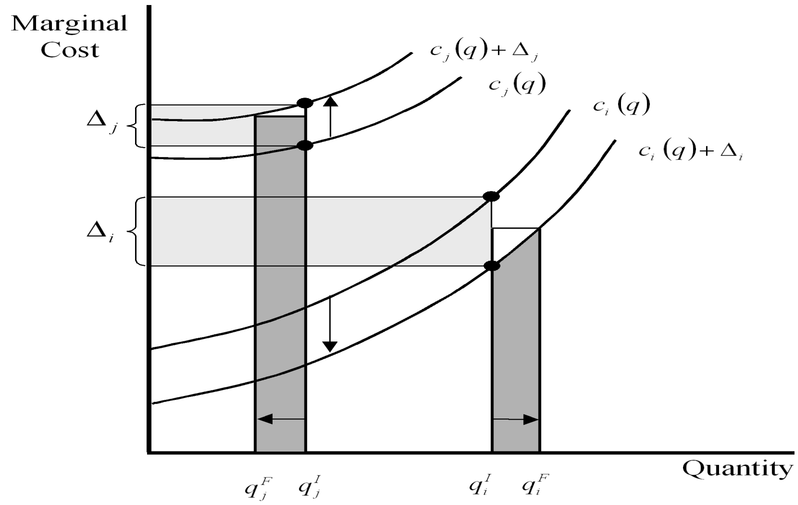

Figure 2. Illustration of the cost effects of an order-preserving regressive transfer

decrease. Both rectangles have the same width but the left-hand rectangle must have a larger height (since in the final equilibrium, the firm with the lower realized marginal cost must have the larger output). Hence, the cost change reflected in these rectangular areas must be negative. The true induced cost change must be even more negative than these two rectangular areas (by the sum of the two white "triangular" areas). Since industry costs would fall even if the outputs of the two firms did not change in response to the shifts in the marginal-cost curves shift and must fall further when outputs re-equilibrate, a regressive transfer of realized marginal costs necessarily lowers industry costs.

Harnessing the Two Types of Transformations

We now present the key result in the paper.

Proposition 7: Given two vectors of realized marginal costs generated in Cournot equilibrium by vertically shifting the marginal-cost curves of two or more firms, if (1) one vector (denoted $\mathbf{x}$) majorizes the other (denoted $\mathbf{y}$), (2) $\mathbf{x}$ is generated in an interior equilibrium, (3) $\mathbf{x}_{\uparrow} \neq \mathbf{y}_{\uparrow}$, and (4) when $\mathbf{x}$ is re-indexed so its components ascend, the underlying marginal-cost curves are contained in ξ, then $\mathbf{x}$ is associated with a strictly lower industry cost and hence a strictly higher industry profit and social surplus.

Proof: Re-index $\mathbf{y}$ so that its components correspond to those in the (re-indexed) majorizing vector. The set of marginal-cost curves underlying the reindexed $\mathbf{y}$ will then also be elements of ξ. We can vertically shift those curves via a finite sequence of order-creating progressive permutations until an ascending permutation of the re-indexed majorized vector is generated (Lemma 2). We can then further vertically shift these curves via a finite sequence of order-preserving regressive transfers until $\mathbf{x}_{\uparrow}$ is generated (Lemma 3). Since permutations do not alter the largest component of a vector while regressive transfers weakly increase that component, it follows that the largest component of $\mathbf{x}_{\uparrow}$ weakly exceeds the largest component of every prior vector in this sequence. Since by assumption $\mathbf{x}$ is generated in an interior equilibrium, every previous equilibrium in the sequence must also be interior. Since no transformation in this sequence raises industry cost and since some transformations in the sequence strictly reduce it (Propositions 5 and 6), the majorizing vector must have associated with it a lower industry cost and hence a strictly higher industry profit and social surplus. $\square$

To illustrate, consider the following three vectors of realized marginal costs generated in interior Cournot equilibrium with stationary demand by vertically shifting the marginal-cost curves of the six firms: $(27, 3, 4, 17, 7, 2), (3, 5, 22, 13, 10, 7), (10, 10, 10, 10, 10, 10)$. It is straightforward to verify that the first vector majorizes the second and the second majorizes the third (and hence, by transitivity, that the first majorizes the third).[19] Denote the three vectors, respectively, as $\mathbf{x}, \mathbf{y},$ and $\mathbf{z}$. Re-index the majorizing vector $\mathbf{x}$ so that its components are in ascending order. Then, if the re-indexed underlying marginal-cost curves are contained in ξ, it follows from Proposition 7 that the Cournot equilibrium associated with $\mathbf{x}$ has a strictly higher social surplus and industry profit than the Cournot equilibrium associated with $\mathbf{y}$ which in turn has a strictly higher social surplus and industry profit than the Cournot equilibrium associated with $\mathbf{z}$.

In the previous example, the vector with equal components was majorized by each of the other

[19]The vector of partial sums of $\mathbf{x}_{\uparrow}$ is (2,5,9,16,33,60) and of $\mathbf{y}_{\uparrow}$ is (3,8,15,25,38,60) and of $\mathbf{z}_{\uparrow}$ is (10,20,30,40,50,60). Since the first is Pareto dominated by the second and the second by the third, $\mathbf{z} \prec \mathbf{y} \prec \mathbf{x}$.

two vectors. This is not a coincidence, and it suggests the following result:

Lemma 4: For $\mathbf{x}, \mathbf{y} \in \Re^n$, if $\mathbf{y}$ is a vector of equal components and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then $\mathbf{y}$ is majorized by $\mathbf{x}$.

Proof: Suppose not. Then there exists some $j < n$ such that $\sum_{i=1}^j y_{(i)} < \sum_{i=1}^j x_{(i)}$. For, otherwise, the definition of majorization would be satisfied. Suppose this jth partial sum is, in fact, the first such partial sum. Then it must be the case that $y_{(j)} < x_{(j)}$. Since the components of $\mathbf{y}_\uparrow$ are the same, whereas those of $\mathbf{x}_\uparrow$ are weakly increasing, it follows that $y_{(k)} \leq x_{(k)}$ for all $j < k \leq n$. But this implies $\sum_{i=1}^n y_{(i)} < \sum_{i=1}^n x_{(i)}$, which contradicts the assumption that $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$. $\square$

The economic significance of Lemma 4 is stated in the following proposition.

Proposition 8: For all $(c_1(q_1), \ldots, c_n(q_n)) \in \xi$ that result in an interior equilibrium with the same realized marginal-cost sum, the equal-component vector of realized marginal costs *maximizes* industry costs and hence *minimizes* industry profit and social surplus.

Proof: The result follows immediately from Proposition 7 and Lemma 4. $\square$

Proposition 8 generalizes Corollary 4 in Salant-Shaffer (1999) to non-constant marginal-cost curves within ξ.[20]

Given the link between the profitability and social surplus of Cournot equilibria and majorization, it would be useful to have a graphical test to determine whether one vector of realized marginal costs majorizes another. Lorenz curves of realized marginal-cost vectors serve this purpose.

We illustrate how the Lorenz curve is constructed for the realized marginal-cost vector $\mathbf{x}$. Plot the following points: $\left(k/n, S_k^x/S_n^x\right)$, for $k = 0, \ldots, n$ where $\mathbf{x}_{(0)} \equiv 0$, and $S_k^x = \sum_{i=0}^k x_{(i)}$. Then connect adjacent points with straight line segments. That is, vertical components of the Lorenz curve consist of $n+1$ points: zero and the n partial sums of $\mathbf{x}$, normalized by the component sum.

The Lorenz curves for $\mathbf{y}$ and $\mathbf{z}$ are constructed in a similar fashion (with the same normalization

[20] A related result can be obtained when marginal-cost curves are not in ξ by beginning with a vector of identical realized marginal costs and applying Proposition 3 iteratively until no two firms have the same realized marginal cost.

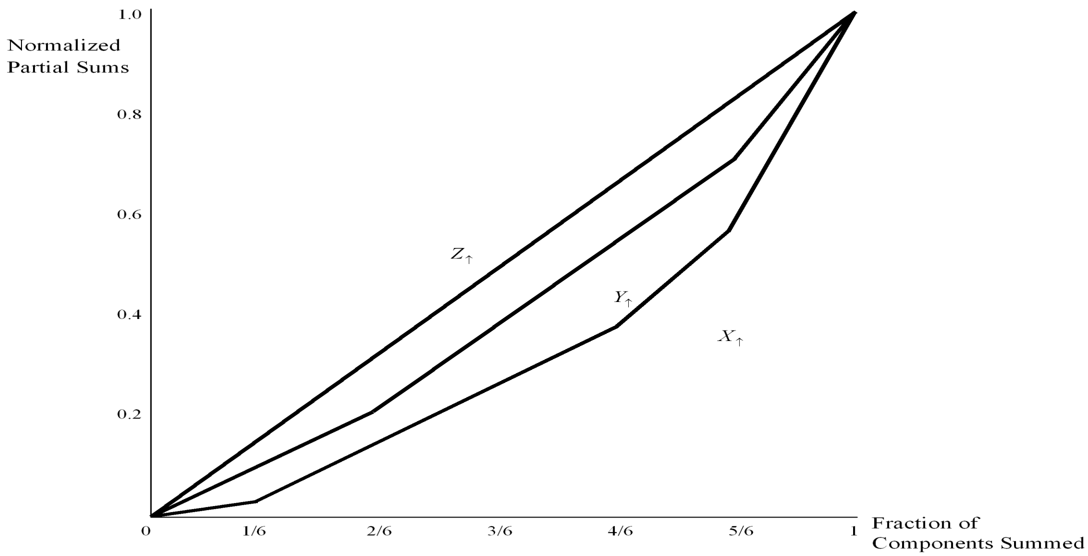

Figure 3: Lorenz Curves

factor since $S_n^y = S_n^z = S_n^x$). Each curve is a graphical representation of the normalized partial sums corresponding to the given ordered vector, in this case, $\mathbf{x}_\uparrow$, $\mathbf{y}_\uparrow$, and $\mathbf{z}_\uparrow$, respectively. Thus, for instance, to say $\mathbf{y}$ is majorized by $\mathbf{x}$ implies that the graph associated with $\mathbf{y}_\uparrow$ lies everywhere weakly above the Lorenz curve associated with $\mathbf{x}_\uparrow$, as illustrated in Figure 3 above.

IV Duality Between Realized Marginal Costs and Outputs

In any regressive transfer of realized marginal costs, the firm with the smaller initial output will contract and the firm with the larger initial output will expand, while none of the other firms will change its output. Since industry output remains constant, a regressive transfer of output also occurs. Similarly, in any regressive transfer of output (firm i's output expands and firm h's output contracts by offsetting amounts with no change in the outputs of the remaining firms), a regressive transfer of realized marginal cost also occurs. This suggests there may be a duality relationship between vectors of realized marginal costs and the associated vectors of outputs.

We now state our main result of this section:

Proposition 9: Suppose the initial and final equilibrium are interior. Then $\theta^{\mathbf{I}}$ is majorized by $\theta^{\mathbf{F}}$

if and only if $\mathbf{q^I}$ is majorized by $\mathbf{q^F}$.

Proof: Since $\mathbf{q^I}, \mathbf{q^F} \in \Re^n_{++}$, the first order conditions hold with equality. That is, $\theta_i^J = P + P'q_i^J$ for $J = I, F,$ and $i = 1, \ldots, n$ where $P > 0$ and $P' < 0$ are constants. The result then follows since if the vector $\mathbf{x}$ majorizes the vector $\mathbf{y}$ and every component of each vector is subjected to the same linear affine transformation so as to produce two transformed vectors, then the transformation of the vector $\mathbf{x}$ majorizes the transformation of the vector $\mathbf{y}$ (Marshall-Olkin, 1979, p.9).[21] □

Proposition 9 establishes that when the initial and final equilibrium are interior, an initial vector of realized marginal cost is majorized by a final vector of realized marginal cost if and only if the associated initial vector of output is majorized by the associated final vector of output. Thus, for example, the realized marginal cost vector $(3, 2, 4)$ majorizes the vector $(3, 3, 3)$. Suppose that $P = 10$ and $P' = -1$. Then the associated output vectors will be $(7, 8, 6)$ and $(7, 7, 7)$. As can be verified, the output vector associated with the majorizing realized marginal-cost vector itself majorizes the other output vector. This duality result expands the scope of our earlier analysis since inferences about industry cost, profit, and social surplus can be drawn from observations of firm outputs instead of their realized marginal costs. Proposition 7 can be recast as follows:

Proposition 10: Given two output vectors generated in Cournot equilibrium by vertically shifting the marginal-cost curves of two or more firms, if (1) one vector (denoted $\mathbf{q}$) majorizes the other (denoted $\mathbf{r}$), (2) $\mathbf{q}$ is generated in an interior equilibrium, (3) $\mathbf{q}_\uparrow \neq \mathbf{r}_\uparrow$, and (4) when $\mathbf{q}$ is re-indexed so its components descend, the underlying marginal-cost curves are contained in ξ, then $\mathbf{q}$ is associated with a strictly lower industry cost and hence a strictly higher industry profit and social surplus.

Since a realized marginal-cost vector of equal components implies a corresponding output vector with equal components, Proposition 8 implies that the vector of equal outputs minimizes industry

[21] Since $P' < 0$, if $\theta^{\mathbf{J}}$ is arranged in ascending order then the corresponding output vector will be arranged in descending order. This implies that the vector of partial sums of $\theta^{\mathbf{J}}$ Pareto dominate the vector of partial sums of $\theta^{\mathbf{K}}$ if and only if $\sum_i^m q_{[i]}^J \leq \sum_i^m q_{[i]}^K$, for all $m = 1, ..., n$, where $q_{[i]}^J$ (respectively, $q_{[i]}^K$) denotes the ith component of the output vector which is obtained from $\mathbf{q^J}$ (respectively, $\mathbf{q^K}$) by arranging it in descending order. Thus, as Marshall and Olkin note, this equivalence implies that the two conditions that are necessary and sufficient for the vector $\mathbf{x}$ to be majorized by the vector $\mathbf{y}$ can also be written as (1) $\sum_i^m x_{[i]} \leq \sum_i^m y_{[i]}$, $m = 1, ..., n$, and (2) $\sum_i^n x_{[i]} = \sum_i^n y_{[i]}$.

profit and social surplus. We state this implication in the following proposition:

Proposition 11: For all $(c_1(q_1), \ldots, c_n(q_n)) \in \xi$ that result in an interior equilibrium with the same output sum, the equal-component equilibrium vector of outputs *maximizes* industry costs and hence *minimizes* industry profit and social surplus.

Since realized marginal costs must decrease (respectively, increase) at any firm whose output increases (respectively, decreases), a rearrangement of output among firms with realized marginal costs that are initially equal is tantamount to transferring output from a higher cost firm to a lower cost firm. This lowers industry costs and hence raises industry profit and social surplus. Thus, for example, if firms in a triopoly facing stationary demand experience vertical shifts in their respective marginal-cost curves so that their equilibrium outputs change from $(5, 5, 5)$ to $(1, 1, 13)$, then industry profit and social surplus must increase.

Schur-Convexity and Concentration Indices

Industry profit and social surplus are examples of functions that increase (over the restricted domain of vectors associated with interior equilibria) when one vector in the domain is replaced by another which majorizes it. Following standard usage, we refer to any function that preserves the ordering of majorization as "Schur-convex."[22] Dalton (1920) showed that the sum of squared components of a vector is also Schur-convex: if one Lorenz curve lies weakly below another, the sum of the squared components of the underlying vector must be larger.

Our duality result has an important implication for concentration indices such as the Herfindahl-Hirschman index which are used to signal changes in industrial performance. The Herfindahl-Hirschman index is the sum of the squares of each firm's output expressed as a proportion of industry output. As shown by Encaoua-Jacquemin (1980), the Herfindahl-Hirschman index is a

[22] As Marshall-Olkin (1979, p. 14) point out, "Schur-increasing" would be clearer terminology were it not for the fact that "Schur-convex" is in widespread use. Schur (1923) showed that a function $\phi : \Re^n_+ \to \Re$, with continuous first partial derivatives, is convex in the sense that $\Big(\bar{\mathbf{c}} \prec \hat{\mathbf{c}} \text{ implies } \phi(\bar{\mathbf{c}}) \leq \phi(\hat{\mathbf{c}})\Big)$ if and only if the function takes on the same value for all permutations of $\bar{\mathbf{c}} \in \Re^n_+$ and satisfies $(\bar{c}_1 - \bar{c}_2)\Big(\frac{\partial\phi(\bar{\mathbf{c}})}{\partial\bar{c}_1} - \frac{\partial\phi(\bar{\mathbf{c}})}{\partial\bar{c}_2}\Big) \geq 0$. For example, if firm 1's realized marginal cost is strictly larger than firm 2's realized marginal cost, so that $\bar{c}_1 - \bar{c}_2 > 0$, then this condition is satisfied if a local increase in $\bar{c}_1$ and corresponding decrease in $\bar{c}_2$ does not decrease the value of the function. If it strictly increases the value of the function—as it does for industry profit and social surplus–the function is said to be strictly Schur-convex. See Berge (1963) at pages 219-227 for a more complete discussion of Schur-convexity.

member of a *class* of "allowable" concentration indices which are invariant to permutations of market shares between firms but which increase whenever the distribution of market shares in the industry experiences a mean preserving spread. All such indices are, therefore, Schur-convex. It follows that:

Proposition 12: Under the four hypotheses of Proposition 10, the majorizing vector of outputs will cause *every* allowable concentration index to increase even though industrial performance *improves* as measured not only by industry profit but also by social surplus.

The conventional practice of interpreting an increase in the Herfindahl-Hirschman index, the "entropy index" (Tirole, p. 222) or any other allowable concentration index as a signal of a decline in industrial performance, therefore, lacks theoretical support.

Proposition 12 considerably generalizes Corollary 2 in Salant-Shaffer (1999), which is restricted to changes in the Herfindahl-Hirschman index under constant marginal costs.

V Conclusion

In this paper, we have described a new way to determine qualitatively how industry cost, industry profit and social surplus change when vertical shifts displace the marginal-cost curves of Cournot competitors. Previous results (Salant-Shaffer, 1999) required that (1) both equilibria be interior, (2) both sums of realized marginal costs be identical, and (3) the marginal-cost curve of each firm be linear with a common slope. Under these three assumptions, the qualitative changes of interest can *always* be inferred from the change in the variance of the realized marginal costs. The requirement that each marginal-cost curve be linear with a common slope (typically zero), however, seems to us very restrictive and we have relaxed it (while maintaining the other two requirements). Under our weaker assumptions we have shown how qualitative conclusions about industry cost, profit, and surplus can nonetheless *sometimes* be reached.

Admittedly, the new criterion that we discuss cannot be used to compare *any* two vectors of realized marginal costs with the same component sum since their associated Lorenz curves may cross. Majorization is a partial ordering. In this respect, it may appear inferior to the "variance criterion." After all, given two vectors with the same component sum one can always determine which has the larger variance even if neither vector Lorenz dominates the other. One must remember, however, that changes in the variance of the realized marginal-cost vector do not always have economic significance.

An illustration will be helpful. Recall from section II the pair of realized marginal-cost vectors $(2-\epsilon, 2, 5+\epsilon)$ and $(1, 4, 4)$. They have the same component sum and, for $\epsilon > 0$, the former vector has the larger sum of squared components and variance. Nonetheless, for $0 < \epsilon < 1$, their Lorenz curves cross and provide no guidance about which vector is associated with the higher industry profit and social surplus.[23] Hence, one can rank the vectors by the sum of their squared components (or, equivalently, by their variance) but one cannot rank them by majorization. If these two realized marginal-cost vectors were generated by vertical shifts of marginal-cost curves which are weakly increasing parallel lines, then being able to rank the vectors by the variance criterion is *advantageous* because whenever the variance increases industry costs decline (Salant-Shaffer, 1999)—whether or not the associated Lorenz curves cross. However, if the two realized marginal-cost vectors were instead generated by vertical shifts of marginal-cost curves of the form $c_i(q_i) = a_i + dq_i^2$, then being able to rank them by the variance criterion is *disadvantageous* since, as Proposition 4 reflects, changes in that criterion are unrelated to changes in economic variables of interest.

In principle, there may exist some *other* complete ordering of realized marginal-cost vectors with the same component sum which does correctly indicate in all cases which vector is associated with the larger industry cost. The existence or non-existence of such a ranking is an open question to be addressed in future work.

In conclusion, something should be said about the requirement that for comparisons to be valid using the Lorenz criterion (or the variance), the sum of realized marginal costs (or realized output) must not change. Comparisons may *still* be possible using these approaches when the sum of marginal costs *does* change; but the comparisons then must rely not merely on these criteria but on additional information. Suppose, for example, that vertical shifts in the marginal-cost curves

[23] The vectors of partial sums are, respectively, $(2-\epsilon, 4-\epsilon, 9)$ and $(1, 5, 9)$, so for $0 < \epsilon < 1$, neither vector Pareto dominates the other.

of firms result in a decline in the sum of realized marginal costs. Denote the initial marginal-cost vector as $\mathbf{x}$ and the final marginal-cost vector as $\mathbf{y}$. It may be possible to devise a *hypothetical* marginal-cost vector $\hat{\mathbf{y}}$ which has the same component sum as the final vector $\mathbf{y}$ but which can be compared *by some other method* to the initial vector $\mathbf{x}$ to determine whether industry profit or social surplus has increased. In many such cases, we can compare $\mathbf{x}$ to $\mathbf{y}$ using the Lorenz criterion even though the sum of realized marginal costs changes: we would compare $\mathbf{x}$ to $\hat{\mathbf{y}}$ using the other method, $\hat{\mathbf{y}}$ to $\mathbf{y}$ using the Lorenz criterion and then, by transitivity, would indirectly be able to compare $\mathbf{x}$ to $\mathbf{y}$ despite the change in their marginal-cost sum. Such comparisons are analogous to those made all the time in demand theory (Allen-Hicks, 1934): the sign of the substitution that would occur at *unchanged* real income is useful in determining the sign of the change in demand induced by a price increase even though the price increase causes a *change* in real income.[24]

[24] By analogy to our case, one supplements the information that the Hicksian substitution effect is negative with additional information about the income effect and then combines the two using transitivity. In particular, the substitution effect indicates how demand would change if the price increased and yet utility did not change (the analog to our discussion of the profit and surplus consequences if the marginal-cost sum did not change); one then brings in supplemental informatin that the good is "normal." Combining the two pieces of information using transitivity, we conclude that the final demand will be strictly smaller than the initial demand.

Appendix

Proof of Proposition 3

By hypothesis, firm i's marginal-cost curve shifts down and firm j's marginal-cost curve shifts up (with no shifts to the marginal-cost curves of other firms) so that firm i's output expands by $x > 0$ and firm j's output contracts by x. Denote the vertical shift in firm i's marginal-cost curve as $\Delta_i < 0$ and the vertical shift in firm j's marginal-cost curve as $\Delta_j > 0$. Then since industry output does not change and the marginal-cost curves of the other $n-2$ firms do not shift, the change in industry costs can be written as:

$$\int_0^{q_i^F} (c_i(s) + \Delta_i)\, ds + \int_0^{q_j^F} (c_j(s) + \Delta_j)\, ds - \int_0^{q_i^I} c_i(s)\, ds - \int_0^{q_j^I} c_j(s)\, ds$$

$$= \int_{q_j^I}^{q_i^I} \Delta_i\, ds + \int_0^{q_j^I} (\Delta_i + \Delta_j)\, ds + \int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds - \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds. \tag{A.1}$$

We now show that the right-hand side of this expression is strictly negative.

The first definite integral is zero since its upper and lower limits are identical. In each of the other three definite integrals, the upper limit strictly exceeds the lower limit.

The second integral is weakly negative since its integrand is weakly negative. To verify this, note that the overall change in the realized marginal cost at either firm can be decomposed into (1) the change that would occur if its marginal-cost curve shifted vertically with no change in output and (2) the additional change that occurs when its output re-equilibrates. By hypothesis, firm i's expansion by x and firm j's contraction by x would weakly increase the marginal-cost sum if neither marginal-cost curve shifted. But since, by a second hypothesis of Proposition 3, the sum of the realized marginal costs at the two firms does not change, the weakly positive induced change in the sum of the two marginal costs must be exactly offset by a weakly negative sum of the exogenous *shifts* in the two marginal-cost curves: $\Delta_i + \Delta_j \leq 0$.

The final two terms combined are strictly negative. To see this, note first that since $q_i^F - q_i^I = q_j^I - q_j^F > 0$, each term is integrated over an interval of the same width. Denote the common width as $\delta > 0$. Since marginal-cost curves are weakly increasing,

$$\int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds - \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds \leq \delta \left[\left(c_i(q_i^F) + \Delta_1 \right) - \left(c_j(q_j^F) + \Delta_j \right) \right].$$

We complete the proof by verifying that the factor on the right in square brackets is strictly negative. Since the two firms begin with equal outputs and firm i expands while firm j contracts, firm i produces a strictly larger final output and must have a strictly smaller final realized marginal cost: $[(c_i(q_i^F) + \Delta_i) - (c_j(q_j^F) + \Delta_j)] < 0$. Hence, $\int_{q_i^I}^{q_i^F} (c_i(s) + \Delta_i)\, ds \; - \; \int_{q_j^F}^{q_j^I} (c_j(s) + \Delta_j)\, ds < 0$.

We conclude, therefore, that the industry cost change (the right-hand side of (A.1)) is strictly negative. Under the hypotheses of Proposition 3, the initial equilibrium with equal marginal costs at every firm can be improved upon in terms of both profit and social surplus. $\square$

Weak Majorization

We have considered a strong form of majorization in which the sum of the components in each vector must be the same. Under *weak majorization*, the sum of the components in each vector is allowed to differ. The reader may thus wonder whether the relationship we have identified between the strong form of majorization and industry profit and social surplus would also hold for weak majorization. To address this, we define weak majorization as (Marshall-Olkin, 1979):

Definition: For $\mathbf{x}, \mathbf{y} \in \Re^n$, $\mathbf{x}$ is *weakly majorized* by $\mathbf{y}$ (or equivalently, $\mathbf{y}$ weakly majorizes $\mathbf{x}$), written $\mathbf{x} \prec^w \mathbf{y}$, if and only if the vector of partial sums of $\mathbf{x}$ Pareto dominates the vector of partial sums of $\mathbf{y}$.

The reader might conjecture that if $\mathbf{x}$ *weakly* majorizes $\mathbf{y}$, then the final equilibrium would still involve strictly higher social surplus and industry profit. This conjecture is false. Suppose two ascending vectors of realized marginal costs differ only in their last component. If the final vector has the smaller n^{th} component then it weakly majorizes the initial vector. Nonetheless, we now show that the final equilibrium may have a strictly *smaller* social surplus and industry profit.[25]

To see this, consider the following example where $c_i(q) = a_i + b(q)$, for $a_i \geq 0, b'(\cdot) \geq 0,$ and $b''(\cdot) \geq 0$. Differentiate the expression for social surplus with respect to the vertical intercept of the n^{th}

[25] This not only generalizes to the non-constant marginal-cost case the main result in Yuan-Khan (2000) but also shows that a proposition corresponding to their social-surplus result holds for industry profit as well.

marginal-cost curve. Since $W = \int_0^Q P(u)du - \sum_{i=1}^n \int_0^{q_i} [a_i + b(u)]du$,

$$\frac{dW}{da_n} = \sum_{i=1}^{n-1} \left(P(Q) - c_i(q_i)\right) \frac{dq_i}{da_n} + \left(P(Q) - c_n(q_n)\right) \frac{dq_n}{da_n} - q_n, \tag{A.2}$$

where we have used the fact that $\frac{dQ}{da_n} = \sum_{i=1}^{n-1} \frac{dq_i}{da_n} + \frac{dq_n}{da_n}$. Consider an interior Cournot equilibrium among $n-1$ firms with different realized marginal costs. Now add one more firm with a realized marginal cost equal to the price in this equilibrium. That firm will produce zero and the equilibrium price will remain the same. Now vertically shift the marginal-cost curve of the additional firm downward marginally so that its realized marginal cost is *lower*. Its output will expand and the output of each of the other $n-1$ firms will contract. The resulting marginal change in social surplus is given by (A1). By hypothesis, the last two terms in (A1) are zero: $q_n = 0$ and $P(Q) - c_n(q_n) = 0$. Since the first term is strictly positive, $\frac{dW}{da_n} > 0$. Thus, when a_n is reduced, social surplus is reduced even though the final vector of realized marginal costs weakly majorizes the initial vector. Intuitively, the marginal expansion at firm n has no first-order effect on net social surplus since the marginal cost at firm n equals the market price; however, the induced marginal contraction at each of the other firms strictly lowers net surplus since each of those firms generates strictly positive net surplus.

To verify that *industry profit* will be smaller as well, rewrite it as $\Pi = W - \int_0^Q P(u)du + QP(Q)$ and differentiate to obtain: $\frac{d\Pi}{da_n} = \frac{dW}{da_n} + QP'(Q)\frac{dQ}{da_n}$. Since the sum of the realized marginal costs declines, industry output must increase when a_n is marginally reduced. Hence, $\frac{dQ}{da_n} < 0$. Since both terms on the right-hand side of the equation defining $\frac{d\Pi}{da_n}$ are strictly positive, $\frac{d\Pi}{da_n} > 0$. Indeed, since $\frac{dW}{da_n}$ and $\frac{d\Pi}{da_n}$ are continuous they will remain strictly positive for any a_n in a neighborhood below the old equilibrium price. Thus, extension of our comparative-static results to weak majorization is not straightforward. Nevertheless, as explained in Section V, our majorization results can be useful even in situations where the marginal-cost sum changes.

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