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Abstract

I show that if an agent is risk neutral over a set of alternatives contained in a Euclidean space, then her utility function decreases linearly in the city block distance to her ideal point. Given a set of alternatives which is not contained in a Euclidean space, I find simple necessary and sufficient conditions on preferences such that, for any $p \geq 1$, there exists a mapping of the set of alternatives into a Euclidean space where the utility of the agent is a decreasing function of the l_p distance to her ideal point. City block and quadratic Euclidean utilities are the special cases $p = 1$ and $p = 2$. For these cases I extend the result to a society with multiple agents, finding additional conditions such that a common space exists in which the preferences of every agent are representable by city block utilities, or by quadratic Euclidean utilities.

JEL Classification: D81, D72.

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In decision theory, game theory or social choice, a multidimensional spatial model is used to represent preferences and choices over objects that have multiple attributes or dimensions and values within each attribute have a natural order so that objects can be ordered according to their values in any given attribute. Political competition over multiple

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policy issues is one application of spatial models. Each policy issue corresponds to a given dimension on a multidimensional vector space. A standard assumption on preferences is that each agent has an ideal policy bundle, represented by a most preferred point in a multidimensional policy space, and that preferences over policy bundles are representable by a utility function defined over the vector space that is decreasing in the Euclidean distance to the ideal policy bundle, either linearly, or in quadratic or exponential form. If preferences are Euclidean or more generally smooth, for a generic distribution of ideal points the core of simple majority is generically empty and there exists no stable policy outcome under simple majority voting rule, as shown by Plott [19]. On the other hand, if preferences are linearly decreasing in the l_1 or city block distance to an ideal policy bundle instead of in the Euclidean distance, then under general conditions, the majority rule core is not empty and there exists a stable policy outcome, as shown by Rae and Taylor [21], Wendell and Thorson [26], McKelvey and Wendell [15] and, more recently, Humphreys and Laver [8].

In this paper, I provide theoretical foundations for the assumption of city block preferences, and hence, indirectly, for the existence of core outcomes in multidimensional political competition under majority voting.

Empirical work by Grynaviski and Corrigan [7] finds that a model based on the city block or l_1 metric outperforms a model based on the Euclidean or l_2 metric in explaining the choices of US voters. Using data from Norway, Westholm [27] obtained similar results in favor of the city block metric over a model based on the square of the Euclidean distance. A city block metric calculates the distance between two points in a multidimensional space by calculating the absolute value of the difference of the two vectors on each dimension, and then aggregating across all dimensions by simple addition or by a weighted sum. Research on artificial intelligence and cognitive sciences such as the work of Shepard [23] and other psychology papers reviewed by Arabie [1] argues that given objects with multiple attributes such that agents perceive attributes to be separable, agents measure distance on these separable attributes by aggregating the distance in each attribute. Therefore, if attributes are separable, geometric models should use the city block or l_1 metric rather than the Euclidean or l_2 metric.

I make a direct theoretical argument in favor of the city block metric, irrespective of how agents cognitively perceive and measure distance. I provide an axiomatic foundation for utility functions that depend on the city block metric, finding conditions on preferences over policy bundles such that the utility function that represents these preferences must be a function of the city block metric. First I consider a set of alternatives that is a subset of a Euclidean space. Then I take a step back toward more remote primitives, studying preferences over a more abstract set of alternatives that is not a subset of a Euclidean space.

Given a set of alternatives that is a subset of a Euclidean space, I show that if the preferences of an agent are representable by a utility function and the agent is risk neutral in the given Euclidean space, then the utility function that represents the preferences is linearly decreasing in the city block distance to the ideal policy bundle. Risk neutrality is a strong and intuitively problematic assumption in an economic environment in which the dimensions corresponds to different goods that the agent can consume. Diminishing marginal utility of consumption is then an argument to assume that the agent is risk averse. However, as noted by Osborne [18], preferences over ideological issues are conceptually different than preferences over consumption of goods, and the evidence of risk aversion over consumption is not relevant to the question of risk attitudes over ideological issues. For instance, it is highly unclear that enjoying a second unit of civil liberties provides less extra utility than enjoying a first one. It depends not only on the attitude toward risk of the agent, but also on the chosen representation of units of civil liberty in a space. Using data from the American National Election Studies, Berinsky and Lewis [4] find that US voters are risk neutral. They reach this finding using data on the self-placement of the citizen on a seven-point ideology scale, the location of the candidates as perceived by the voter, the uncertainty about this perception, and the candidate preferred by the voter.

If agents are risk neutral given a set of alternatives defined as a subset of a Euclidean space, the utility function that represents the preferences of the agent is linearly decreasing in the l_1 distance to the ideal policy bundle of the agent. It follows that theoretical models ought to discard the Euclidean distance and use the city block distance instead as the standard to construct specific utility functions in any application where the findings of Berinsky and Lewis are robust and agents are risk neutral in the chosen spatial representation of alternatives.

In many applications, there is no natural spatial representation of the policy alternatives. Rather, the spatial representation is an abstract construction. Alternatives have multiple attributes and each attribute is endowed with a natural order, but the exact location of each alternative in a Euclidean space and the distance between alternatives is an object of choice for the theorist who arbitrarily chooses to endow the set of alternatives with a spatial representation by mapping alternatives into a Euclidean space. For instance, Freedom House classifies countries according to the civil rights they allow, dividing them into seven tiers, more freedom corresponding to a lower tier. While we may accept the partial order of countries given by these tiers, the one-to-seven scale is arbitrary and it is difficult to accept that the distance in rights from tier one to tier two is equal to the distance from tier two to tier three in any objective sense.

In the second part of the paper I consider abstract alternatives that have multiple attributes. In the political economy application, an alternative is a policy bundle, and each attribute corresponds to a given political issue. Although policies on a given issue are endowed with a natural order, there is no exogenously given spatial representation of policies. Instead, I take as a primitive a preference relation on the abstract set of policy bundles and I assume that this preference is representable by a utility function. I seek conditions on preferences such that there exists a mapping of the set of policy bundles into a Euclidean space such that each dimension in this space corresponds to an issue, the location of policies along each dimension in the space is monotonic in the exogenous order of policies within each issue, and the preferences over points in the Euclidean space are representable by a utility function that is linearly decreasing in the city block distance to the ideal point of the agent.

I find that the necessary and sufficient conditions on primitives for there to be a spatial representation such that in this space, preferences can be represented by a utility function that is decreasing in the p power of the l_p distance to an ideal point are the same for any of Minkowski's [17] family of l_p norms, $l_p(x) = \left(\sum_k |x_k|^p\right)^{1/p}$. Linear city block preferences or quadratic Euclidean preferences are only specific cases of this general result. The preferences over the original set of policy bundles must be multi-attribute single peaked and modular. Multi-attribute single peakedness is an extension of the standard notion of single peakedness, so that preferences are single peaked on any given dimension. Modularity is a separability condition consistent with the arguments of the psychology literature I mentioned above, such as Shepard [23] and Arabie [1]. Preferences are modular if an agent evaluates attributes or policies independently of each other, so that her preference over one policy is invariant with changes in other policies.

To my knowledge, the directly related literature is scant. Kannai [9] and Richter and

Wong [22] find conditions such that preferences in a given space can be represented by a concave utility function, but they do not consider a mapping into a new space in which the utility function could be concave. The closest reference is by Bogomolnaia and Laslier [6], who seek to find how many dimensions must be used to represent any ordinal preference profile over a finite number of alternatives using Euclidean preferences. They do not require that the spatial representation of alternatives respect a natural order within each attribute, and they are interested only in a finite number of alternatives, disregarding, for instance, lotteries. Hence, for a single individual, their problem is trivial. Any preference can be represented in just one dimension by assigning alternatives to natural numbers according to the preference order of the agent. By contrast, I consider an infinite number of alternatives by studying lotteries over alternatives, and I seek to find a spatial representation in K dimensions that is consistent in each dimension with the natural order of values within each of exogenously given K attributes. Since the problem I address has more restrictions, not every preference relation is representable in any space using the city block distance or Euclidean distances, even if there is a single agent. I find axiomatic conditions on the preference relation under which, in some space, it is representable by a utility function that depends on any desired l_p norm.

For a society with multiple agents I find further necessary and sufficient conditions to guarantee that there exists a common space that satisfies the restriction on the number of dimensions and the monotonicity with respect to the natural order of values within each attribute, and is such that preferences over points in this common space can be represented by means of utility functions that are linearly decreasing in the city block distance to the ideal point of each agent. I also find a different set of conditions such that there exists a space in which preferences can be represented by a utility function that is linearly decreasing in the square of the Euclidean distance. As noted above, the result on city block preferences has very important implications for political competition over multiple dimensions: If preferences can be represented by the city block distance, for open sets of distributions of ideal policies there exists a policy bundle in the core of the majority voting rule so that it is a stable outcome, which contrasts sharply with the generic inexistence of stable policies in the majority rule core if preferences are smooth.

The main contribution of this paper is to provide theoretical foundations for the assumption of utility functions that are decreasing in the city block distance to an ideal policy in a multidimensional policy space, parting from abstract primitives in which alternatives have multiple attributes and are endowed with a natural order within each attribute.

An Exogenous Spatial Representation

Let $X \subset \mathbb{R}^K$ be a convex set of alternatives with a non empty interior. Let ΔX be the set of all simple lotteries defined over X. For any given lottery $p \in \Delta X$, let $p(x)$ denote the probability that p assigns to $x \in X$. For any $p \in \Delta X$, the support of p is the set $\{x : p(x) > 0\}$. Slightly abusing notation, let $x, y, z, w \in X$ denote as well degenerate lotteries, so they belong to ΔX. Let x_k denote the $k - th$ coordinate of x and let x_{-k} denote the vector of $K - 1$ dimensions that contains all the coordinates of x except x_k. Then we can write x as $x = (x_k, x_{-k})$.

Let $\succsim$ be a complete and transitive binary relation on ΔX representing the weak preferences of agent i over lotteries on X. Let $x \succ y$ denote ($x \succsim y$, not $y \succsim x$) and let $x \sim y$ denote ($x \succsim y$, $y \succsim x$). Let $\succsim$ satisfy the independence and archimedean axioms due to Von Neumann and Morgenstern [25].

Axiom 1 *(Archimedean): If $p, q, r \in \Delta X$ such that $p \succ q \succ r$, then there is an $\alpha \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \sim q$.*

Axiom 2 *(Independence): For all $p, q, r \in \Delta X$ and any $\alpha \in (0, 1)$, then $p \succsim q$ if and only if $\alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r$.*

Then the preferences over lotteries can be represented by a utility function $u : X \longrightarrow \mathbb{R}$ such that for any $p, q \in \Delta X$, $p \succsim q$ if and only if $\sum_X p(x)u(x) \geq \sum_X q(x)u(x)$. This is part of the celebrated expected utility theorem by Von Neumann and Morgenstern.

Assume that $\succsim$ has a unique maximal element $x^* \in X$ such that $x^* \succ p$ for any $p \in \Delta X$, $p \neq x^*$. The degenerate lottery x^* is the most preferred alternative of the agent. For convenience, relocate the origin of coordinates so that $x^* = (0, ..., 0)$. The set X is divided into 2^K orthants. Each orthant is one of the subsets composed of the points that do not contain both points that are strictly positive and points that are strictly negative in any given dimension; the analog of a quadrant on $\mathbb{R}^2$ or an octant on $\mathbb{R}^3$. Let O^j denote an arbitrary one of them, for $j \in \{1, 2, 3, ..., 2^K\}$

An agent is risk neutral if for any $p \in \Delta X$, $p \sim \sum_{x \in X} p(x)x$. Note that $\sum_{x \in X} p(x)x \in X$ is the expectation value of lottery p. A risk neutral agent is indifferent between a lottery, or the expected value of the lottery. This is the standard condition in economics, with non satiated agents who always want more good to consume. However, with ideological preferences, that are satiated at x^*, this condition is violated by all agents: For any $p \in \Delta X$ such that $\sum_{x \in X} p(x)x = x^*$ and $p(x^*) < 1$, the agent prefers her ideal point for sure over the lottery. This preference is not really indicative of an attitude towards risk, but rather, of a preference for attaining the best outcome for certain, over variance over outcomes that are either too much or too little with respect to the optimum.

A weaker risk neutrality condition is appropriate. I call it orthant risk aversion. The intuition simply tries to pin back the notion of risk in an environment with satiated preferences to the standard definition of risk with monotonic preferences. Consider the orthant that is non positive in all dimensions. Within that orthant, the agent wants more of everything. This is the standard economic environment. Hence the standard risk neutrality concept applies, *within this orthant*. A risk neutral agent is indifferent between a lottery that assigns positive probability to outcomes in this orthant and its expected value. I impose the same condition within any other orthant: A risk neutral agent is indifferent between any lottery that assigns positive probability to outcomes in only one orthant and the expected value of this lottery.

Axiom 3 *(Orthant risk neutrality)* $p \sim \sum_{x \in X} p(x)x$ *for any* p *whose support is contained in a single orthant.*

Similarly, $\succsim$ is orthant risk averse if $\sum_{x \in X} p(x)x \succ p$ for any p whose support is contained in a single orthant and contains x, y such that $x \succ y$. That is, the preferences of the agent are orthant risk averse if given any lottery that introduces uncertainty about the desirability of the outcome, but the outcome is sure to be within an orthant, the agent strictly prefers the expected value of the lottery for sure over the lottery.

I show that if preferences over lotteries satisfy orthant risk neutrality, along with completeness, transitivity, independence and the archimedean axiom, then the preferences over sure outcomes can be represented by a utility function that is linear in the distance measured by a generalization of the l_1 or city bloc norm. To state the result formally, I first need to define the generalized weighted city block norm.

Definition 1 *For any $\lambda \in \mathbb{R}^{2K}_{+}$, let $\|\cdot\|_{\lambda}$ be the generalized weighted city block norm:*

$$\|\cdot\|_{\lambda} = \sum_{k=1}^{K} \lambda_k(x_k)|x_k|, \text{ with } \lambda_k(x_k) = \left\{ \begin{array}{c} \lambda_{k+} > 0 \text{ if } x_k > 0 \\ \lambda_{k-} > 0 \text{ if } x_k < 0 \end{array} \right\}, \ k \in \{1, ..., K\}.$$

Let $(X, \|\cdot\|_{\lambda})$ be the metric space with set of elements X and distance $d(x, y) = \|x - y\|_{\lambda}$.

The standard l_1 norm, which I denote $\|\cdot\|_1$ has equal weights in every dimension, and on each side of the origin, that is, $\lambda_{k+} = \lambda_{k-} = \lambda_{j+} = \lambda_{j-}$ for any $j, k \in \{1, ..., K\}$. A standard weighted city block norm has different weights for each dimension, but equal weights along each dimension, $\lambda_{k+} = \lambda_{k-}$ for any $k \in \{1, ..., K\}$. It corresponds to the intuition of computing time of driving in Manhattan, were driving north-south along an avenue is faster than driving east-west along a street. The generalized weighted city bloc norm allows for weights to be different to each side of the origin along the same dimension. For an intuition, this captures the time of driving where not only north-south avenues are faster, but the east and west sides of the city have different qualities of pavement on the streets, so that it is easier to transit in some quarters than others, even while moving in the same direction.

Now I state the first result of this paper.

Theorem 1 *Let $\succsim$ be orthant risk neutral, have a unique maximal element x^* and be representable by the expected utility of a function $u : X \longrightarrow \mathbb{R}$. Then $\exists \lambda \in \mathbb{R}^{2K}_{+}$ such that u is linearly decreasing in distance to x^* in the space $(X, \|\cdot\|_{\lambda})$.*

Proof. Recall $x^* = \{0\}^K$ and O^j denotes an arbitrary orthant j, for any $j \in \{1, ..., 2^K\}$. Let $int(X)$ be the interior of X, let $N(x, \varepsilon)$ be the neighborhood of radius ε around x, and let O^1 be the non negative orthant. Let $Y \subseteq \mathbb{R}^K$ be the smallest Cartesian product $Y = Y_1 \times Y_2 ... \times Y_K$ such that $Y_k \subseteq \mathbb{R}$ for each $k \in \{1, ..., K\}$ and $X \subseteq Y$. For notational simplicity, let $u(x^*) = 0$.

Since $int(X) \neq \emptyset$, $\exists j \in \{1, ..., 2^K\}$ such that $\exists x \in int(X) \cap O^j$ with $x_k \neq 0$ for every $k \in \{1, ..., K\}$. Without loss of generality, assume $j = 1$ and choose a point $x \in int(X) \cap O^1$, such that $x_k > 0$ for every $k \in \{1, ..., K\}$. For each dimension $k \in \{1, ..., K\}$ and a sufficiently small ε such that $N(x, \varepsilon) \in O^1$, choose an $x^k \in N(x, \varepsilon)$ such that $x^k = \alpha_k x + (1 - \alpha_k) y^k$ for some $\alpha_k \in [0, 1)$ and some y^k such that $y^k_k > 0$ and $y^k_i = 0$ for all $i \neq k$, $i \in \{1, ..., K\}$. That is, each x^k is a convex combination of x and a point in the non negative orthant that differs from x^* only on coordinate k. I construct an extended preference relation $\succsim_S$

that is orthant risk neutral in Y and is such that for any $x, y \in X$, $x \succsim_S y \Longleftrightarrow x \succsim y$. I also extend the domain of u from X to Y. In order to satisfy orthant risk neutrality in Y, $u(x^k) = \alpha_k u(x) + (1-\alpha_k)u(y^k)$, or $u(y^k) = \frac{u(x^k)-\alpha u(x)}{(1-\alpha)}$. By orthant risk neutrality, for any $\delta \geq 0$, $u(\delta y^k) = \delta u(y^k)$. Let $(y_k, 0)$ denote an arbitrary point y that assigns value y_k to coordinate k and value 0 to any other coordinate. For any $\gamma < 0$, and any $k \in \{1, ..., K\}$, let $y^{k,\gamma}$ denote the point $(y_k^{k,\gamma}, 0) \in Y$ such that $u(y^{k,\gamma}) = \gamma$, if such point exists. Arbitrarily fix $\lambda_{1+} = 1$. For any $k \in \{2, ..., K\}$, and $\gamma < 0$ close enough to zero such that $y_k^{k,\gamma}$ exists for every k, let $\lambda_{k+} = \frac{y_1^{1,\gamma}}{y_k^{k,\gamma}}$. For this γ, let $Z = \{x \in O^1 \cap X : x = \sum_{k=1}^{K} p_k y^{k,\gamma}$ with $p_k \geq 0$ for all k and $\sum_{k=1}^{K} p_k = 1\}$. That is, Z is the set of points in the non negative orthant and in X that can be constructed as a linear combination of points in the axis of Y that generate utility γ. By orthant risk neutrality, $u(z) = \gamma$ for any $z \in Z$.

Any $x \in O^1 \cap X$ can be expressed as a linear transformation αz for some $\alpha \geq 0$ and some $z \in Z$, or, rather, there exists a function $z(x) : O^1 \cap X \longrightarrow Z$, in particular, $z(x) = \frac{x}{\alpha}$. By orthant risk neutrality, $u(x) = \alpha\gamma$. The distance to x^* from such x in the space $(X, ||\cdot||_\lambda)$ is

$$\sum_{k=1}^{K} \lambda_{k+} x_k = \sum_{k=1}^{K} \lambda_{k+} \alpha z_k = \sum_{k=1}^{K} \lambda_{k+} \alpha p_k y_k^{k,\gamma} = \sum_{k=1}^{K} \frac{y_1^{1,\gamma}}{y_k^{k,\gamma}} \alpha p_k y_k^{k,\gamma} = \alpha y_1^{1,\gamma}.$$

The first equality holds because x is a transformation of z. The second holds because z is a linear combination of $\{y^{1,\gamma}, ..., y^{K,\gamma}\}$. The third equality simply expands the formula of λ_{k+} and the last is algebra. Distance $||\cdot||_\lambda$ is linearly increasing in α and utility is linearly decreasing in α. So utility is linearly decreasing in distance $||\cdot||_\lambda$, within the non negative orthant.

The non negative orthant was arbitrarily chosen; the same result holds in any other orthant. Specifically, for each dimension k, repeating the above procedure on orthant j, we find weights λ_k^j that correspond to the weight on dimension k found in the proof constructed for orthant j. Consider any two orthants O^1 and O^2 that differ on the sign on dimension k but coincide on the sign on all other dimensions, and the sign in at least one dimension is positive. Take $x \in O^1 \cap O^2$ such that $x_i > 0$ and $x_j = 0$ for all $j \neq i$. That is, take a point that is positive in only one dimension and belongs to both O^1 and O^2. Since x belongs to the same indifferent curve when measured as a point in orthant O^1 or orthant O^2, if λ_i^1 and λ_i^2 respectively denote the weight of dimension i in the norm used in orthants O^1 and O^2, it must be $\lambda_i^1 = \lambda_i^2$. Orthants O^1 and O^2 were arbitrarily chosen among the class of orthants that are positive on dimension i and differ on the sign only on dimension k. So for any two orthants that are positive on dimension i and differ on only one dimension, the weight on dimension i is the same. By induction, it follows that the weight on dimension i is the same for all orthants that are positive on dimension i. But i positive was arbitrary, so it follows that for any $i \in \{1, ..., K\}$, the weight on dimension i in the metric used on all orthants that are positive in dimension i is a common parameter λ_{i+}, and, similarly, the weight on dimension i is λ_{i-} for all orthants that are negative on dimension i. ■

The presence of a maximal element captures the satiated preferences typical in a space of ideological issues. The theorem then says that if the agent is risk neutral in each orthant, then only a generalization of the city block distance represents preferences.

A common assumption in spatial models is that preferences are symmetric on each dimension, taking the maximal element as the origin of coordinates.

Axiom 4 *(Spatial symmetry) For any $x \in X$ and any $k \in \{1, 2, ..., K\}$, if $(-x_k, x_{-k}) \in X$ then $(x_k, x_{-k}) \sim (-x_k, x_{-k})$.*

I call this property *spatial symmetry* to distinguish it from the unrelated definition of symmetry of binary relations.

Corollary 1 *Let $\succsim$ be orthant risk neutral, spatially symmetric, contain a degenerate maximal element x^* and be representable by the expected utility of a function $u : X \longrightarrow \mathbb{R}$. Then $\exists(\lambda_1, \lambda_2, ..., \lambda_K) \in \mathbb{R}^K_+$ such that u is linearly decreasing in distance to x^* in the space $(X, \|\cdot\|_\lambda)$, where $\|\cdot\|_\lambda$ is a weighted city block metric so that $\lambda_{k+} = \lambda_{k-}$ for all $k \in \{1, 2, ..., K\}$.*

If preferences are spatially symmetric, along each dimension utility loses are equal in each direction away from the origin, so the generalized weighted city block metric reduces to a standard weighted city block metric where the weights correspond to the importance of each dimension, without distinctions between each of the two half-spaces along each dimension.

Utility functions that depend on the Euclidean distance do not represent risk neutral agents. It is well known that concave utility functions represent risk averse individuals. It is perhaps less obvious that utility functions that are linear on Euclidean distance are neither orthant risk neutral, nor orthant risk averse. Linear Euclidean preferences are not separable once we consider lotteries over alternatives and the risk attitude of the agent varies depending on the lottery under consideration. An agent with linear Euclidean preferences is risk neutral about lotteries that involve outcomes that all lay in a ray away from the ideal point, but she is risk averse about any other lotteries. For instance, an agent with ideal policy $(0,0)$ and linear Euclidean preferences is indifferent between $(1,0)$ for sure or an even lottery between $(0,0)$ and $(2,0)$. However, if we change the value of the second dimension, and make the same comparison, the agent prefers $(1,y)$ to an even lottery between $(0,y)$ and $(2,y)$ for any $y \neq 0$.

A practical implication is that the preferences of agents that are risk neutral, or risk averse, should not be represented by a utility function that is linear on Euclidean distance, choosing instead a utility representation that is concave in some distance if the agents are risk averse, or linear on the city block metric if the agents are risk neutral. Risk aversion is a more frequent assumption, but, as noted by Osborne [18], risk aversion over economic decisions does not imply risk aversion over ideological issues and "in the absence of any convincing empirical evidence, it is not clear which of the assumptions [concavity, linearity, or convexity] is more appropriate." However, recent empirical work by Berinsky and Lewis [4] finds an application where US voters are risk neutral.

In summary, recent empirical work has found applications in which agents are risk neutral on their preferences defined over the spatial representation of the policy space used in these applications. If these findings are robust, theoretical models of political competition in these applications should assume that utility is linear in the city block distance.

An Endogenous Spatial Representation

Theorem 1 shows that if agents have risk neutral preferences over points in a multidimensional space that are representable by a utility function, then this utility function is linear in distance to the ideal point, where distance is measured according to a generalization of the l_1 norm. It is important to note that the multidimensional space is itself an abstract representation of the set of policy alternatives. The units of measurement of the ideological dimensions and the mapping from specific policies to their location on the spatial representation may be objects of design. In some applications, the spatial representation of alternatives may be exogenously given by the available data, as it is, for instance, in the empirical work on US voters by Berinsky and Lewis [4], where voters identify the point where they subjectively locate the candidates. In other applications, there may not be a clear way to represent alternatives on a Euclidean space, and the theorist may choose among competing mappings of the set of alternatives into the Euclidean space. In this environment, risk neutrality is an assumption on risk attitudes *given the chosen spatial representation*, or alternatively, we can interpret the assumption of risk neutrality as a joint assumption on the mapping of the set of alternatives to the space and the risk attitude of the agent. A now consider a more abstract model that take as primitive an ordinal preference relation over a set of policies with multiple attributes, without any spatial representation.

Let the set of attributes, denoted A, be of size K. For each attribute $k \in A = \{1, ..., K\}$, let X_k be the set of possible values on attribute k. This set can be finite, countable or uncountable. Let the elements of X_k be ordered by a linear order $\geq_k$ and let this order have a unique maximal and minimal element. Given the possible policies on each issue, let the set of alternatives be the Cartesian product $X = X_1 \times X_2 \times ... \times X_K$ and let ΔX be the set of simple lotteries on X. In a political economy application, each attribute $k \in A$ is a policy issue and X is the set of alternative policy bundles.

The primitive on preferences is a complete and transitive binary relation $\succsim$ on ΔX that satisfies the archimedean and independence axiom, so that $\succsim$ is representable by the expected utility of a utility function defined over X.

A spatial representation of X is a vector valued function $f = (f_1, f_2, ..., f_K)$ such that $f_k : X_k \longrightarrow \mathbb{R}$ is strictly increasing in $\geq_k$ for each $k \in A$ and $f(x) \in \mathbb{R}^K$ represents alternative $x \in X$. The motivating question is under what conditions on $\succsim$ there exists a spatial representation f such that the preferences over $f(X) \subseteq \mathbb{R}^K$ are risk neutral. Under these conditions, the preferences $\succsim$ over the abstract set X can be represented by a utility function that is linearly decreasing in a generalized l_1 distance to an ideal point in a Euclidean space. Let $L(x, y)$ be a lottery that assigns equal probability to x and y. Let $x \vee y = (\max\{x_1, y_1\}, ..., \max\{x_K, y_K\}$ and $x \vee y = (\min\{x_1, y_1\}, ..., \min\{x_K, y_K\}$ be the join and the meet of x and y

Axiom 5 *(Modularity) For all $x, y \in X$, $L(x, y) \sim L(x \vee y, x \wedge y)$.*

Modular preferences are such that the agent evaluates changes in one attribute in the same manner, regardless of the values in other attributes. For added intuition, consider an example with two issues and let x, y lie in the non positive quadrant with respect to the ideal policy, so that among all four options in the lotteries, $x \vee y$ is the best outcome in both issues, $x \wedge y$ is the worst outcome in both issues, and x and y are each good in one issue and bad in the other. If the outcome is determined by two lotteries, one on each issue, and these lotteries assign equal probability to the good and bad outcome on their respective issue, an agent with modular preferences is indifferent about the correlation of the two lotteries. Birkhoff [5] calls a function f satisfying $f(x)+f(y)=f(x \vee y)+f(x \wedge y)$ a *valuation*. See Kreps [10], Milgrom and Shannon [16] and Topkis [24] for related ordinal and cardinal definitions of modularity.

Axiom 6 *(Multi-attribute single peakedness)* $\exists! x^* \in X$ *such that for each* $k \in \{1,2,...,K\}$, *and any* $x_k^1, x_k^2, x_k^3, x_k^4 \in X_k$:

$$x_k^1 \leq_k x_k^2 \leq_k x_k^* \leq_k x_k^3 \leq_k x_k^4 \Longrightarrow (x_k^2, x_{-k}^*) \succ (x_k^1, x_{-k}^*) \text{ and } (x_k^3, x_{-k}^*) \succ (x_k^4, x_{-k}^*).$$

A multi-attribute single peaked preference relation has a best policy such that, moving away from the peak on any given attribute, preferences decrease, as in a unidimensional single peaked relation. This condition of single-peakedness is weaker than the multidimensional single peakedness used by Barberà, Gul and Stacchetti [2], but together with modularity, it suffices to guarantee that their stricter restriction is also satisfied, and that alternatives and preferences can be represented in a vector space such that the utility of the agent is a decreasing function of any desired l_p norm. Let $||\cdot||_p = \left(\sum_{k=1}^{K} |f_k(x_k)|^p\right)^{\frac{1}{p}}$ be the l_p norm.

Theorem 2 *Suppose* $\succsim$ *is representable by the expected utility of* $u : X \longrightarrow \mathbb{R}$. *For any* $p \in [1, \infty)$, *a spatial representation* $f^p = (f_1^p, ..., f_K^p)$ *such that*

i) $f_k^p : X_k \longrightarrow \mathbb{R}$ *is strictly increasing in* $\geq_k$ *for each* $k \in \{1, ..., K\}$, *and*

ii) $u(x) = -(\|f^p(x)\|_p)^p$,

exists if and only if $\succsim$ *is multi-attribute single peaked and modular.*

Proof. (only if). Suppose preference $\succsim$ is not multi-attribute single peaked. Then, there exists $k \in \{1, ..., K\}$ and x^1, x^2 such that either $x_k^1 \leq x_k^2 \leq x_k^*$ or $x_k^* \leq x_k^2 \leq x_k^1$, and $x^1 = (x_k^1, x_{-k}^*) \succ (x_k^2, x_{-k}^*) = x^2$, so $u((x_k^1, x_{-k}^*)) > u((x_k^2, x_{-k}^*))$. Note that for any f,

$\left|\left|f(x^1)\right|\right|_1 - \left|\left|f(x^2)\right|\right|_1 = f_k(x_k^1) - f_k(x_k^2)$ and $||f(x^*)||_1 - \left|\left|f(x^2)\right|\right|_1 = f_k(x_k^*) - f_k(x_k^2)$. Since f_k is strictly increasing in $\geq_k$, $\min\{f_k(x_k^*), f_k(x_k^1)\} < f_k(x_k^2) < \max\{f_k(x_k^*), f_k(x_k^1)\}$. But $u(x^2) < \min\{u(x^1), u(x^*)\}$. Hence $u(x)$ is not decreasing in $||f(x)||_1$. Suppose (absurd) $u(x) = -(\|f^p(x)\|_p)^p$ for some $p > 1$ and some f^p. Then

$$u(x) = -\left(\left(\sum_{k=1}^{K} \left|f_k^p(x_k)\right|^p\right)^{\frac{1}{p}}\right)^p = -\sum_{k=1}^{K} \left|f_k^p(x_k)\right|^p.$$

For each k, let $g_k(x_k)$ be such that $|g_k(x_k)| = \left|f_k^p(x_k)\right|^p$ and $g_k(x_k) \geq 0 \iff f_k^p(x_k) \geq 0$. Note that under the spatial representation $g = (g_1, ..., g_K)$, $u(x) = -\sum\limits_{k=1}^{K} |g_k(x_k)| = -\|g(x)\|_1$. But $u(x)$ is not decreasing in $||f(x)||_1$ for any f, hence this is a contradiction.

Suppose $\succsim$ is not modular. Then $\exists x, y \in X$ s.t. $L(x, y) \nsim L(x \vee y, x \wedge y)$, which implies $u(x) + u(y) \neq u(x \vee y) + u(x \wedge y)$. However, $||f(x)||_1 + ||f(y)||_1 = ||f(x \vee y)||_1 + ||f(x \wedge y)||_1$. So $u(x)$ is not linearly decreasing in $||f(x)||_1$. The same transformation as in the previous paragraph extends this proof to any p.

(if). Suppose $\succsim$ is modular and multi-attribute single peaked. Let (x_k^*, x_{-k}^*) denote x^*. For each $k \in \{1, 2, ..., K\}$, construct $f_k^1 : X_k \longrightarrow \mathbb{R}$ as follows: $f_k^1(x_k^*) = 0$; for any $x_k^j \in X_k$ such that $x_k^j \leq_k x_k^*$, $f_k^1(x_k^j) = u(x_k^j, x_{-k}^*) - u(x_k^*, x_{-k}^*)$; and for any $x_k^j \in X_k$ such that $x_k^j \geq_k x_k^*$, $f_k^1(x_k^j) = u(x_k^*, x_{-k}^*) - u(x_k^j, x_{-k}^*)$. By multi-attribute single peakedness, these functions f_k^1 are strictly increasing in $\geq_k$. By construction, $u(x)$ is linearly decreasing in distance to $f^1(x^*)$ in the space $(f^1(X), ||\cdot||_1)$ for any x such that $x_k = x_k^*$ for any $k \in A_{K-1}$, where $A_{k-1} \subset A$ and $|A_{k-1}| = K - 1$.

Suppose (proof by induction) that $u(x)$ is linearly decreasing in distance to $f^1(x^*)$ in the space $(f^1(X), ||\cdot||_1)$ for any x such that $x_k = x_k^*$ for any $k \in A_m$, where $A_m \subset A$. I want to show that $u(z)$ is also linearly decreasing in distance for any $z \in X$ be such that $z_k = z_k^*$ for any $k \in A_{m-1}$, where $A_{m-1} \subset A_m$ and $|A_{m-1}| = |A_m| - 1$. Let i be a dimension such that $z_i = x_i \neq x_i^*$. Let j be the dimension such that $z_j \neq x_j = x_j^*$. Let $w \in X$ be such that $w_i = x_i^*; w_j = x_j^*$ and $w_k = z_k$ for $k \notin \{i, j\}$. Let $y \in X$ be such that $y_i = x_i^*$ and $y_k = z_k$ for $k \neq i$. Then $\{w, y, x, z\}$ is a lattice, where $w \succsim x, y \succsim z$. By the inductive hypothesis, $u(w)$ is linearly decreasing in distance to $f^1(x^*)$ in the space $(f^1(X), ||\cdot||_1)$,

$$u(w) = -\sum_{k=1}^{K} |f_k^1(w_k) - f_k^1(x_k^*)| = -\sum_{k \notin \{i,j\}} |f_k^1(w_k) - f_k^1(x_k^*)|$$

where the second equality follows from $w_k = x_k^*$ for $k \in \{i, j\}$. Since, again by the inductive hypothesis, $u(x)$ and $u(y)$ are also linearly decreasing in distance, $u(x) = u(w) - |f_i^1(x_i) - f_i^1(x_i^*)|$ and $u(y) = u(w) - |f_j^1(y_j) - f_j^1(x_j^*)|$.

By modularity,

$$\begin{aligned}
u(z) &= u(y) + u(x) - u(w) = u(w) - |f_i^1(x_i) - f_i^1(x_i^*)| - |f_j^1(y_j) - f_j^1(x_j^*)| \\
&= u(w) - |f_i^1(z_i) - f_i^1(x_i^*)| - |f_j^1(z_j) - f_j^1(x_j^*)| \\
&= -\sum_{k \notin \{i,j\}} |f_k^1(w_k) - f_k^1(x_k^*)| - \sum_{k \in \{i,j\}} |f_k^1(z_k) - f_k^1(x_k^*)| \\
&= -\sum_{k=1}^{K} |f_k^1(z_k) - f_k^1(x_k^*)|.
\end{aligned}$$

Since the inductive hypothesis is true, as shown, for $|A_m| \geq K - 1$, it is true by the inductive argument for any size of A_m, and therefore, for any $z \in X$, $u(z)$ is linearly decreasing in distance to $f^1(x^*)$ in the space $(f^1(X), ||\cdot||_1)$. For any $p > 1$, let f_k^p be such that $\left|f_k^p(x_k)\right| = \left|f_k^1(x_k)\right|^{1/p}$ and $f_k^p(x_k) \geq 0 \iff f_k^1(x_k) \geq 0$. Then

$$u(x) = -\left|\left|f^1(x)\right|\right|_1 = -\sum_{k=1}^{K} \left|f_k^1(x_k)\right| = -\sum_{k=1}^{K} \left|f_k^p(x_k)\right|^p = -\left(\|f^p(x)\|_p\right)^p.$$

■

In particular, and most relevant in applications, theorem 2 says that if preferences are modular and multi-attribute single peaked, we can represent alternatives and preferences in a specific vector space using a utility function that is linear in the l_1 norm, or we can represent them in a different space using a utility function that is quadratic in the l_2 norm. What we cannot do is represent them in any space using a utility function that is linear in the l_2 norm, or a utility function that exponential in the l_2 norm, such as the one used in the celebrated D-NOMINATE method to estimate the location of the ideal policy in two dimensions of US legislators devised by Poole and Rosenthal [20]. Euclidean utility functions that are not quadratic in the Euclidean distance are inconsistent with preferences satisfying the modularity assumption. Modularity is a separability assumption that requires agents to treat issues independently, assessing their preferences over policies on one issue (or over lotteries over policies on one issue) in the same manner regardless of the policies in any other issue. Whether preferences are separable across issues is an empirical question. Lacy [11] searches for evidence of non separability across pairs of issues that seem to be related, such as taxes and spending, pollution regulation and cleaning up of the environment, or the status of English as an official language and immigration laws. He finds mixed evidence: Many respondents to surveys report non separable preferences in some pairs of related issues, such as income tax and anti-crime spending, but almost no respondents report non separable preferences in other pairs of close issues, such as English and immigration laws. While outside his study, I conjecture that most agents have separable preferences across issues that do not seem to be related, such as the status of English as an official language and environmental protection.

Many issues, such as abortion, gay rights, civil rights or environmental policy among others, do not have a natural mapping from policy alternatives to the real line, so the set of alternative policy bundles is not endowed with a spatial representation as a primitive and any spatial representation is only one of many possible representations. Theorem 1 showed that using the city block metric to represent preferences implies assuming that the agent is risk neutral in a given spatial representation. To the extent that the spatial representation is an object of choice for theorists and not a primitive object, any assumption over preferences on the space is difficult to interpret. It is preferable to make assumptions on the primitives of the choice problem: on the original set of alternatives, which is not exogenously endowed with any spatial representation, and the preferences over this set. Theorem 2 makes explicit the restriction on primitives implied by the use of the city block or quadratic Euclidean utilities. Interestingly, the implicit restriction is the same for these two commonly used utility functions: Preferences over the primitive set of alternatives must satisfy separability in the sense of modularity, and single peakedness.

To my knowledge, there is no parallel characterization of the set of preference profiles over the primitive set of alternatives that are consistent with a utility function that is linear or exponential in the Euclidean distance. Bogomolnaia and Laslier [6] show that any preference relation can be represented by Euclidean preferences in a space with a sufficiently large number of dimensions, but the assumption of linear Euclidean preferences in a given space with a fixed, small number of dimensions, while ubiquitous in the literature, implies unknown and possibly unwarranted restrictions on the admissible preferences defined over the primitive set of alternatives.

It may seem surprising that the same preference relation can be represented using a city block utility function, or using a quadratic Euclidean utility function, particularly in light of the result by Plott [19] on generic inexistence of majority voting core outcomes if preferences are Euclidean (or more generally, if they preferences are smooth), and the more positive results on the existence of core outcomes under majority voting with city block preferences by Rae and Taylor [21], Wendell and Thorson [26], McKelvey and Wendell [15] and Humphreys and Laver [8]. The explanation of these divergent results on existence of core outcomes depending on whether utility functions are smooth or city block, when it is possible to map a spatial representation in which an agent has quadratic Euclidean preferences into another space in which the agent has city block preferences is that the results on existence of core outcomes rely on a common space for all agents in a society with at least three agents. It does not suffice for each agent to have her own spatial representation of the set of alternatives such that according to this subjective representation, her preferences are city block. The need for a common spatial representation imposes further restrictions that I detail in the next section.

A Common Spatial Representation for Multiple Agents

In the previous section, the primitive on preferences is a complete and transitive binary relation $\succsim$ on ΔX that satisfies the archimedean and independence axiom, so that $\succsim$ is representable by the expected utility of a utility function defined over X.

In a society N with n agents, the new primitive are n such binary relations defined on ΔX that satisfy the archimedean and independence axiom, so that $\succsim_i$ is representable by the expected utility of a utility function u_i defined over X for any $i \in N = \{1, ..., n\}$. For any $p \in \Delta X$, let support of p be $supp(p) = \{x \in X : p(x) > 0\}$. This is the subset of alternatives to which lottery p assigns positive probability.

In this section I extend theorem 2 to a society with multiple agents for the cases of the l_1 norm in theorem 3, and for the l_2 norm in theorem 4. The additional necessary and sufficient conditions to find a common spatial representation for all agents such that all their utility functions are linear in the city block distance to their respective ideal points, while heavy on notation, have a simple interpretation that I detail after the formal statement of theorem 3. The conditions for representability by means of an l_2 norm in a common space are more complex.

Let $(f(X), \|f(x)\|_1)$ be the metric space given by the spatial representation $f : X \longrightarrow \mathbb{R}^K$ and the metric based on the l_1 norm, so that the distance between two points $f(x)$,

$f(y) \in \mathbb{R}^K$ is $\|f(x) - f(y)\|_1$. Let $\mathcal{F}$ be the set of spatial representations $f = (f_1, ..., f_K)$ such that $f_k : X_k \longrightarrow \mathbb{R}$ is strictly increasing in $\geq_k$ for each $k \in A$. For each attribute k, let $x_k^{\max}$ and $x_k^{\min}$ be such that $x_k^{\min} \leq_k x_k \leq_k x_k^{\max}$ for any $x_k \in X_k$. Given any two agents i, j with preferred alternatives x^i and x^j, for each k, relabel the agents according to a function $\sigma_k \colon \{i, j\} \longrightarrow \{h, l\}$ such h is the agent with higher ideal value in attribute k, labeled x_k^h and l is the agent with a lower ideal value x_k^l in attribute k. With this notation, I state the result.

Theorem 3 *Assume $\succsim_i$ is multi-attribute single peaked and modular for every $i \in N$. A spatial representation $f \in \mathcal{F}$ such that*

$$u_i(x) = -\left\|f(x) - f(x^i)\right\|_1 \text{ for every } i \in N$$

exists if and only if, for any $i, j \in N$, the following conditions hold.

1. *For any $k \in A$, $\forall x_k^1, x_k^2, x_k^3 \in X_k$ such that $x_k^1 \leq_k x_k^l \leq_k x_k^2 \leq_k x_k^h \leq_k x_k^3$ and $\forall \alpha \in [0,1]$, given $p^1, p^2, p^3 \in \Delta X$ such that $p^1(x_k^{\min}, x_{-k}^l) = \alpha$, $p^1(x^l) = 1 - \alpha$, $p^2(x_k^h, x_{-k}^l) = \alpha$, $p^2(x^l) = 1 - \alpha$, $p^3(x_k^{\max}, x_{-k}^l) = \alpha$ and $p^3(x_k^h, x_{-k}^l) = 1 - \alpha$,*

$$p^z \sim_h (x_k^z, x_{-k}^l) \Longleftrightarrow p^z \sim_l (x_k^z, x_{-k}^l) \text{ for any } z \in \{1, 2, 3\}.$$

2. *For any $k \in A$ such that $x_k^i \leq_k x_k^j$ and any x_k^1 such that $x_k^1 \leq_k x_k^i$, and for any $k \in A$ such that $x_k^i \geq_k x_k^j$ and any $x_k^1 \geq_k x_k^i$, $\exists \alpha \in (0, 1]$ and $\exists \delta > 0$ such that:*

   - *$p \sim_i q$ for $p, q \in \Delta X$ such that* $\left\{ \begin{array}{c} p(x_k^1, x_{-k}^i) = \delta\alpha \\ p(x^i) = 1 - \delta\alpha \end{array} \right\}$ *and* $\left\{ \begin{array}{c} q(x_k^j, x_{-k}^i) = \delta \\ q(x^i) = 1 - \delta \end{array} \right\}$,
   - *and $r \sim_j x^i$ for $r \in \Delta X$ such that* $\left\{ \begin{array}{c} r(x_k^1, x_{-k}^i) = \frac{\alpha}{1+\alpha} \\ r(x_k^j, x_{-k}^i) = \frac{1}{1+\alpha} \end{array} \right\}$.

Proof. (only if). Suppose (absurd) that $u_i(x) = -\left\|f(x) - f(x^i)\right\|_1$ for every $i \in N$, but condition 1 fails for $z = 1$. Then, $\exists x_k^1$ such that, for α such that $p \sim_l (x_k^1, x_{-k}^l)$, $p \nsim_h (x_k^1, x_{-k}^l)$. In utility terms, $p \sim_l (x_k^1, x_{-k}^l)$ implies

$$\alpha u_l((x_k^{\min}, x_{-k}^l)) + (1-\alpha) u_l((x_k^l, x_{-k}^l)) = u_l((x_k^1, x_{-k}^l)),$$

which, since $u_l(x)$ is linearly decreasing in

$$\left\|f(x) - f(x^l)\right\|_1 = |f_k(x_k) - f_k(x_k^l)| + \sum_{m \neq k} |f_m(x_m) - f_m(x_m^l)|,$$

implies

$$
\begin{aligned}
\alpha|f_k(x_k^{\min}) - f_k(x_k^l)| &= |f_k(x_k^1) - f_k(x_k^l)|. \\
\alpha(f_k(x_k^l) - f_k(x_k^{\min}) &= f_k(x_k^l) - f_k(x_k^1).
\end{aligned}
$$

In utility terms, $p \nsim_j (x_k^1, x_{-k}^l)$ implies

$$
\begin{aligned}
\alpha|f_k(x_k^{\min}) - f_k(x_k^h)| + (1-\alpha)|f_k(x_k^l) - f_k(x_k^h)| &\neq |f_k(x_k^1) - f_k(x_k^h)| \\
\alpha f_k(x_k^h) - \alpha f_k(x_k^{\min}) + (1-\alpha) f_k(x_k^h) - (1-\alpha) f_k(x_k^l) &\neq f_k(x_k^h) - f_k(x_k^1) \\
\alpha(f_k(x_k^l) - f_k(x_k^{\min})) &\neq f_k(x_k^l) - f_k(x_k^1),
\end{aligned}
$$

a contradiction. The cases for $z = 2$ and $z = 3$ follow an analogous argument.

Suppose (absurd) that condition 2 does not hold. Without loss of generality assume that there exists k, x_k^1, α and δ such that $x_k^1 \leq_k x_k^i \leq_k x_k^j$ and

$$
\begin{aligned}
\delta\alpha u_i((x_k^1, x_{-k}^i)) + (1-\delta\alpha) u_i(x^i) &= \delta u_i((x_k^j, x_{-k}^i)) + (1-\delta) u_i(x^i) \\
\delta\alpha|f_k(x_k^1) - f_k(x_k^i)| &= \delta|f_k(x_k^j) - f_k(x_k^i)| \\
\alpha f_k(x_k^i) - \alpha f_k(x_k^1) &= f_k(x_k^j) - f_k(x_k^i),
\end{aligned}
$$

while

$$
\begin{aligned}
\frac{\alpha}{1+\alpha} u_j((x_k^1, x_{-k}^i)) + \frac{1}{1+\alpha} u_j((x_k^j, x_{-k}^i)) &\neq u_j(x^i) \\
\frac{\alpha}{1+\alpha}|f_k(x_k^1) - f_k(x_k^j)| &\neq |f_k(x_k^i) - f_k(x_k^j)| \\
\alpha f_k(x_k^j) - \alpha f_k(x_k^1) &\neq (1+\alpha)(f_k(x_k^j) - f_k(x_k^i)) \\
\alpha f_k(x_k^i) - \alpha f_k(x_k^1) &\neq f_k(x_k^j) - f_k(x_k^i),
\end{aligned}
$$

a contradiction.

(if). By theorem 2, $\succsim_i$ can be represented by a utility function that is linearly decreasing in the l_1 norm in the space given by the spatial representation f. For any dimension k, assume $x_k^j \geq_k x_k^i$ (the $x_k^i \geq_k x_k^j$ case follows an analogous argument). If $x_k^j \neq x_k^{\max}$, since u_i is rescalable, let $f_k(x_k^{\max}) - f_k(x_k^j) = 1$. For any $x_k^3 \in [x_k^j, x_k^{\max}]$, find α_3 such that $p^3 \sim_i (x_k^3, x_{-k}^i)$ for $p^3 \in \Delta X$ such that $p^3(x_k^j, x_{-k}^i) = 1 - \alpha_3$ and $p^3(x_k^3, x_{-k}^*) = \alpha_3$, so $f_k(x_k^3) - f_k(x_k^j) = \alpha_3$. By condition 1, case $z = 3$, $p^3 \sim_j (x_k^3, x_{-k}^i)$, so, if we independently construct the spatial representation f^j under which $\succsim_j$ is representable by u_j linearly decreasing in the l_1 distance to x^j, and we fix $f_k^j(x_k^{\max}) - f_k^j(x_k^j) = 1$, we find that $f_k(x_k^3) - f_k(x_k^j) =$

$f_k^j(x_k^3) - f_k^j(x_k^3)$. A symmetric argument applies to any $x_k^1 \leq_k x_k^i$, and a very similar one for any $x_k^2 \in (x_k^i, x_k^j)$. With just condition 1, we can find a common spatial representation such that, in each dimension, the utility of agents i and j is piecewise linearly decreasing in the distance to their respective ideal points, with three different pieces corresponding to the set of points with a higher value in the given dimension than both ideal points, lower than both ideal points, and in between both ideal points. Condition 2 is necessary to guarantee that utility is linear in distance across all three intervals. Take any x_k^1 such that $x_k^1 \leq_k x_k^i$. Choose α and δ such that

$$\begin{aligned} \alpha\delta u_i((x_k^1, x_{-k}^i)) + (1-\alpha\delta)u_i(x^i) &= \delta u_i((x_k^j, x_{-k}^i)) + (1-\delta)u_i(x^i) \\ \alpha f_k(x_k^i) - \alpha f_k(x_k^1) &= f_k(x_k^j) - f_k(x_k^i). \end{aligned}$$

By condition 2,

$$\begin{aligned} \frac{\alpha}{1+\alpha}u_j((x_k^1, x_{-k}^i)) + \frac{1}{1+\alpha}u_j((x_k^j, x_{-k}^i)) &= u_j(x^i) \\ \alpha f_k^j(x_k^i) - \alpha f_k^j(x_k^1) &= f_k^j(x_k^j) - f_k^j(x_k^i), \end{aligned}$$

where f^j is the spatial representation such that u_j is linearly decreasing in the l_1 distance to x^j. Notice that f_k^j coincides with f_k in evaluating the distance $|f_k^j(x_k^i) - f_k^j(x_k^1)|$ as exactly $(1/\alpha)|f_k^j(x_k^j) - f_k^j(x_k^i)|$. Then, both spatial representations coincide for any $x_k \leq_k x_k^j$. An analogous argument for the symmetric case with $x_k^i \leq_k x_k^j \leq_k x_k^3$ shows that f_k^j also coincides with f_k for any $x_k \geq_k x_k^i$ completes the proof for $n = 2$. But j was arbitrary, so every $j \in N$ shares the common spatial representation f and has a utility representation that is linearly decreasing in the l_1 distance to the ideal point of the agent in the space $f(X)$. ■

Condition 1 has a very simple interpretation. Fixing the value of all attributes except k, and evaluating lotteries that assign different values to attribute k, if i and j agree in their ordinal preference among all the possible outcomes of the lotteries, then they agree on their ranking of the lotteries as well. If i has a lower ideal value on attribute k than j, then i and j share the same ranking among all lotteries on dimension k that assign positive probability only to values that are no greater than the ideal value of i. Similarly, for lotteries that are in any event above both ideal policies, agents agree that they want less of attribute k, and by condition 1 they agree on their ranking of these lotteries. In the intermediate interval between their two ideal policies, the agents have opposite rankings over sure outcomes: One agent wants less, the other one wants more. Condition 1 states that if agent i is indifferent between a lottery in this interval and a sure outcome, agent j must be indifferent as well. An intuition is that in this region the agents are in a zero-sum game: Whatever i gains, j loses, so if i is indifferent between two lotteries, j must be indifferent as well.

Condition 2 is easier to interpret if X is convex. In this case it can be restated simply as:

Given $x_k^1 \leq_k x_k^i \leq_k x_k^2 \leq_k x_k^j$ such that $(x_k^1, x_k^i) \sim_i (x_k^2, x_k^j)$, and $p \in \Delta X$ such that $p(x_k^1, x_k^i) = p(x_k^2, x_k^i) = \frac{1}{2}$, then $x^i \sim_j p$.

If agent i finds that lowering the value of attribute k from her ideal x_k^i to x_k^1 is as bad as increasing it to x_k^2, then j considers that increasing it from x_k^1 to x_k^i is as good as increasing it from x_k^i to x_k^2. Or, if agent i subjectively considers her ideal value the midpoint between x_k^1 to x_k^2, then j agrees and also considers x_k^i to be the midpoint between x_k^1 to x_k^2. If X is not convex, we need the richer notation to express the same intuition: If agent i places her ideal point at a fraction $\frac{\delta}{1+\delta}$ of the way from x_k^1 to x_k^j, then j also subjectively places x_k^i at a fraction $\frac{\delta}{1+\delta}$ from x_k^1 to x_k^j. Agents agree on the importance of a change in value in their interval of agreement, relative to a change in value in their interval of disagreement. That is, if agent i is willing to shift up to a certain amount of probability from value x_k^1 in an interval of agreement to a less preferred value x_k^2 in order to change the outcome from x_k^3 to x_k^4 in the region of disagreement, then agent j is willing to cede exactly the same amount of probability to avoid this change from x_k^3 to x_k^4.

Succinctly, and a bit informally, if agents agree on lotteries when they agree on sure outcomes, if they have exactly opposite preferences over lotteries when they have exactly opposite preferences over sure outcomes, and they concede the same importance to the region of agreement vis a vis the region of disagreement, then their ordinal preferences over multiatribute objects can be represented in a common space such that these preferences can all be represented by utility functions that are linearly decreasing in the city block distance to the respective ideal points.

Theorem 3 has very important consequences in political competition over policy bundles with multiple policy dimensions: If agents have city block preferences over a common space, then under certain conditions that are not non-generic, there exist policy bundles that are in the majority voting rule core, so they cannot be defeated by any other policy, as shown by Rae and Taylor [21], Wendell and Thorson [26], McKelvey and Wendell [15] and Humphreys and Laver [8].

With regards to representability by the l_2 distance, Bogomolnaia and Laslier [6] show that any profile of preference relations can be represented by Euclidean preferences in a space with a sufficiently large number of dimensions. With a fixed number of dimensions, the goal of representing preferences by means of a Euclidean utility function in a common space becomes a much more difficult task, and the conditions on preferences become very restrictive.

Theorem 4 *Assume $\succsim_i$ is multi-attribute single peaked and modular for every $i \in N$. A spatial representation $f \in \mathcal{F}$ such that*

$$u_i(x) = -(\left\|f(x) - f(x^i)\right\|_2)^2 \text{ for every } i \in N$$

exists if and only if, for any $i, j \in N$, the following conditions hold.

3. *For any $k \in A$ and any $x_k^1, x_k^2, x_k^3 \in X_k$ such that $x_k^1 \leq_k x_k^l \leq_k x_k^2 \leq_k x_k^h \leq_k x_k^3$:*
   - *Let $\gamma > 0$, $\delta \in (0,1]$ and $r, s \in \Delta X$ with $s(x_k^{\min}, x_{-k}^l) = \delta$, $s(x^l) = 1 - \delta$, $r(x_k^h, x_{-k}^l) = \frac{\delta}{\gamma^2}$ and $r(x^l) = 1 - \frac{\delta}{\gamma^2}$ be such that $s \sim_l r$. For any $p, q \in \Delta X$ such that $p(x_k^{\min}, x_{-k}^l) = \alpha_l, p(x^l) = 1 - \alpha_l$, $q(x_k^{\min}, x_{-k}^l) = \alpha_h, q(x^l) = 1 - \alpha_h$, and $p \sim_l (x_k^1, x_{-k}^l)$,*

   $$q \sim_h (x_k^1, x_{-k}^l) \iff \alpha_h = \sqrt{\alpha_l} + \frac{\alpha_l - \sqrt{\alpha_l}}{1 + 2\gamma}.$$

   - *For any $p, q \in \Delta X$ such that $p(x_k^h, x_{-k}^l) = \alpha_l, p(x^l) = 1 - \alpha_l$, $q(x_k^h, x_{-k}^l) = \alpha_h, q(x^l) = 1 - \alpha_h$, and $p \sim_l (x_k^2, x_{-k}^l)$,*

   $$q \sim_h (x_k^2, x_{-k}^l) \iff \alpha_h = 2\sqrt{\alpha_l} - \alpha_l;$$

   - *Let $\gamma > 0$ and $r \in \Delta X$ with $r(x_k^{\max}, x_{-k}^l) = \left(\frac{\gamma}{1+\gamma}\right)^2$ and $r(x_k^l, x_{-k}^l) = 1 - \left(\frac{\gamma}{1+\gamma}\right)^2$ be such that $(x_k^h, x_{-k}^l) \sim_l r$. For any $p, q \in \Delta X$ such that $p(x_k^{\max}, x_{-k}^l) = \alpha_l, p(x_k^h, x_{-k}^l) = 1 - \alpha_l$, $q(x_k^{\max}, x_{-k}^l) = \alpha_h, q(x_k^h, x_{-k}^l) = 1 - \alpha_h$, and $p \sim_l (x_k^3, x_{-k}^l)$,*

   $$q \sim_h (x_k^3, x_{-k}^l) \iff \alpha_h = 2\gamma^2 + 2\gamma\alpha_l + \alpha_l - 2\gamma(\gamma^2 + 2\gamma\alpha_l + \gamma)^{1/2}.$$

4. *For any $k \in A$ such that $x_k^i \leq_k x_k^j$ and any x_k^1 such that $x_k^1 \leq_k x_k^i$, and for any $k \in A$ such that $x_k^i \geq_k x_k^j$ and any $x_k^1 \geq_k x_k^i$, $\exists \alpha \in (0,1]$ and $\exists \delta > 0$ such that:*

- *$p \sim_i q$ for $p, q \in \Delta X$ such that* $\left\{ \begin{array}{c} p(x_k^1, x_{-k}^i) = \delta\alpha \\ p(x^i) = 1 - \delta\alpha \end{array} \right\}$ *and* $\left\{ \begin{array}{c} q(x_k^j, x_{-k}^i) = \delta \\ q(x^i) = 1 - \delta \end{array} \right\}$,
- *and $r \sim_j x^i$ for $r \in \Delta X$ such that* $\left\{ \begin{array}{c} r(x_k^1, x_{-k}^i) = \frac{\alpha}{(1+\sqrt{\alpha})^2} \\ r(x_k^j, x_{-k}^i) = \frac{1+2\sqrt{\alpha}}{(1+\sqrt{\alpha})^2} \end{array} \right\}$.

Proof. (only if). Suppose that $u_i(x) = -(\|f(x) - f(x^*)\|_2)^2$ for every $i \in N$. In the first case of condition 3, in utility terms, $p \sim_l (x_k^1, x_{-k}^l)$ implies

$$\alpha_l u_l((x_k^{\min}, x_{-k}^l)) + (1 - \alpha_l) u_l(x^l) = u_l((x_k^1, x_{-k}^l)),$$

which, since $u_l(x)$ is linearly decreasing in $(\left\|f(x) - f(x^l)\right\|_2)^2 = (f_k(x_k) - f_k(x_k^l))^2 + \sum_{m \neq k} (f_m(x_m) - f_m(x_m^l))^2$, implies

$$\alpha_l (f_k(x_k^{\min}) - f_k(x_k^l))^2 = (f_k(x_k^1) - f_k(x_k^l))^2.$$

Arbitrarily rescale $u_l(x)$ so that $(f_k(x_k^l) - f_k(x_k^{\min}) = 1$. Then, $f_k(x_k^l) - f_k(x_k^1) = \sqrt{\alpha_l}$. Let $f_k(x_k^h) - f_k(x_k^l) = d$.

We want to show that $q \sim_h (x_k^1, x_{-k}^l) \iff \alpha_h = \sqrt{\alpha_l} + \frac{\alpha_l - \sqrt{\alpha_l}}{1+2\gamma}$. In utility terms, $q \sim_h (x_k^1, x_{-k}^l)$ if and only if

$$\begin{aligned} \alpha_h (f_k(x_k^{\min}) - f_k(x_k^h))^2 + (1 - \alpha_h)(f_k(x_k^l) - f_k(x_k^h))^2 &= (f_k(x_k^1) - f_k(x_k^h))^2 \\ \alpha_h (d+1)^2 + (1 - \alpha_h) d^2 &= (d + \sqrt{\alpha_l})^2 \\ \alpha_h + 2d\alpha_h &= \alpha_l + 2d\sqrt{\alpha_l} \\ \alpha_h &= \frac{\alpha_l + 2d\sqrt{\alpha_l}}{1 + 2d} = \sqrt{\alpha_l} + \frac{\alpha_l - \sqrt{\alpha_l}}{1 + 2d} \end{aligned}$$

Note that if $u_l(x) = -(\left\|f(x) - f(x^l)\right\|_2)^2$, then $s \sim_l r$ if and only if $d = \gamma$. The expression of α_h as a function of α_l in the third case of condition 3 is the same as solving for α_l in terms of α_h in the first case of the condition. For the remaining second case, rescaling u_l to let $f(x_k^h) - f(x_k^l) = 1$, $p \sim_l (x_k^2, x_{-k}^l)$ implies that $f(x_k^2) - f(x_k^l) = \sqrt{\alpha_l}$, so $q \sim_h (x_k^1, x_{-k}^l)$ if and only if

$$\begin{aligned} (1 - \alpha_h) &= (1 - \sqrt{\alpha_l})^2 \\ \alpha_h &= 2\sqrt{\alpha_l} - \alpha_l. \end{aligned}$$

Suppose (absurd) that condition 4 does not hold. Without loss of generality assume that there exists k, x_k^1, α and δ such that $x_k^1 \leq_k x_k^i \leq_k x_k^j$ and

$$\begin{aligned}
\delta\alpha u_i((x_k^1, x_{-k}^i)) + (1-\delta\alpha)u_i(x^i) &= \delta u_i((x_k^j, x_{-k}^i)) + (1-\delta)u_i(x^i) \\
\alpha(f_k(x_k^i) - f_k(x_k^1))^2 &= (f_k(x_k^j) - f_k(x_k^i))^2,
\end{aligned}$$

while

$$\frac{\alpha}{(1+\sqrt{\alpha})^2} u_j((x_k^1, x_{-k}^i)) + \frac{1+2\sqrt{\alpha}}{(1+\sqrt{\alpha})^2} u_j((x_k^j, x_{-k}^i)) \neq u_j(x^i).$$

Rescale u_l such that $f_k(x_k^j) - f_k(x_k^i) = 1$, so $f_k(x_k^i) - f_k(x_k^1) = \frac{1}{\sqrt{\alpha}}$. Then

$$\frac{\alpha}{(1+\sqrt{\alpha})^2} \left(1 + \frac{1}{\sqrt{\alpha}}\right)^2 \neq 1,$$

a contradiction.

(if). By theorem 2, $\succsim_i$ can be represented by a utility function that is decreasing in square of the l_2 distance to x^i in the space given by the spatial representation f. For any dimension k, assume $x_k^i \leq_k x_k^j$ (we only need to relabel the agents to do without this assumption). If $x_k^i \neq x_k^{\min}$, since u_i is rescalable, let $f_k(x_k^i) - f_k(x_k^{\min}) = 1$. For any $x_k^1 \in [x_k^{\min}, x_k^i]$, find α_i such that $p^1 \sim_i (x_k^1, x_{-k}^i)$ for $p^1 \in \Delta X$ such that $p^1(x_k^{\min}, x_{-k}^i) = \alpha_1$ and $p^1(x^i) = 1 - \alpha_1$, so $f_k(x_k^i) - f_k(x_k^1) = \sqrt{\alpha_1}$. If we construct the spatial representation f^j under which $\succsim_j$ is representable by u_j decreasing in the square of the l_2 distance to x^j, we fix $f_k^j(x_k^i) - f_k^j(x_k^{\min}) = f_k(x_k^j) - f_k(x_k^i) = 1$ and we let $f_k^j(x_k^j) - f_k^j(x_k^i) = d$ and $f_k(x_k^j) - f_k(x_k^i) = \gamma$, by taking $x_k^1 = x_k^{\min}$ in condition 4, we find that

$$\begin{aligned}
\left(\frac{\alpha}{(1+\sqrt{\alpha})^2}\right)(1+d)^2 &= d^2, \text{ where} \\
\delta\alpha &= \delta\gamma^2, \text{ so} \\
\frac{\gamma^2}{(1+\gamma)^2}(1+d)^2 &= d^2; \\
\gamma &= d.
\end{aligned}$$

For any $x_k^1 \leq_k x_k^i$, find α_j such that

$$\begin{aligned}
\alpha_j u_j(x_k^{\min}, x_{-k}^i) + (1-\alpha_j)\, u_j(x^i) &= u_j(x_k^1, x_k^i). \\
\alpha_j(1+\gamma)^2 + (1-\alpha_j)\,\gamma^2 &= (\gamma+\lambda)^2, \text{ where} \\
\lambda &= f_k^j(x_k^i) - f_k^j(x_k^1), \text{ so} \\
\alpha_j + 2\gamma\alpha_j &= \lambda^2 + 2\gamma\lambda.
\end{aligned}$$

By condition 3, $\alpha_j = \frac{2\gamma\sqrt{\alpha_i}+\alpha_i}{1+2\gamma}$. Therefore,

$$\begin{aligned}\frac{2\gamma\sqrt{\alpha_i}+\alpha_i}{1+2\gamma}(1+2\gamma) &= \lambda^2 + 2\gamma\lambda \\ \lambda &= -\gamma + \sqrt{\gamma^2 + 2\gamma\sqrt{\alpha_i}+\alpha_i} = \sqrt{\alpha_i}.\end{aligned}$$

Therefore, arbitrarily setting $f_k(x_k^{\min}) = f_k^j(x_k^{\min}) = 0$, then $f_k^j(x_k^1) = f_k(x_k)$ for any $x_k^1 \leq x_k^i$.

For any x_k^2 such that $x_k^i \leq_k x_k^2 \leq_k x_k^j$, $p \in \Delta X$ such that $p(x_k^j, x_{-k}^i) = \alpha_i, p(x^i) = 1-\alpha_i$, $p \sim_i (x_k^2, x_{-k}^i)$ implies

$$\begin{aligned}\alpha_i(f_k(x_k^j) - f_k(x_k^i))^2 &= (f_k(x_k^2) - f_k(x_k^i))^2 \\ f_k(x_k^2) - f_k(x_k^i) &= \sqrt{\alpha_i}\gamma.\end{aligned}$$

Similarly, $q(x_k^j, x_{-k}^i) = \alpha_j, q(x^i) = 1-\alpha_j$ and $q \sim_h (x_k^2, x_{-k}^l)$ if and only if

$$\begin{aligned}(1-\alpha_j)(f_k^j(x_k^j) - f_k^j(x_k^i))^2 &= (f_k^j(x_k^j) - f_k^j(x_k^2))^2 \\ \gamma - f_k^j(x_k^2) + f_k^j(x_k^i) &= \sqrt{(1-\alpha_j)}\gamma \\ f_k^j(x_k^2) - f_k^j(x_k^i) &= \gamma\left(1 - \sqrt{(1-\alpha_j)}\right),\end{aligned}$$

which, since $\alpha_h = 2\sqrt{\alpha_l} - \alpha_l$, leads to

$$f_k^j(x_k^2) - f_k^j(x_k^i) = \gamma\left(1 - \sqrt{(1-2\sqrt{\alpha_i}+\alpha_i)}\right) = \gamma(1-(1-\sqrt{\alpha_i}) = \sqrt{\alpha_i}\gamma.$$

Therefore, $f_k^j(x_k^2) = f_k(x_k)$ for any x_k^2 such that $x_k^i \leq_k x_k^2 \leq_k x_k^j$. Finally, for any $x_k^3 \geq x_k^j$, starting with $f_k^j(x_k^3)$ such that u_j is quadratic in the Euclidean distance in $f^i(X)$, the construction of $f_k^i = f_k$ for which u_i is quadratic Euclidean is a symmetric case to the finding that $f_k^j(x_k^1) = f_k(x_k^1)$ for $x_k^1 \leq_k x_k^i$. Altogether, $f_k^j = f_k$. Since i and j were arbitrary, the same representation f serves as a common space for all n agents. ■

While these conditions for representability using a quadratic Euclidean utility function lack a transparent interpretation, it is useful to compare them to the simpler conditions 1 and 2 in theorem 3. Condition 1 in theorem 3 can be rewritten in terms that follow the structure of condition 3, and then, condition 1 holds if $q^z \sim_h (x_k^z, x_{-k}^l) \iff \alpha_h = \alpha_l$ for $z \in \{1,2,3\}$. Under condition 1, agents i and j evaluate the sure outcome (x_k^z, x_{-k}^l) in the same way in terms of weights to the best and worst alternative in each of the three intervals under consideration, so they are both indifferent between p, q and (x_k^1, x_{-k}^l) if and only if

$p = q$, or equivalently, $\alpha_h = \alpha_l$. In order for preferences to satisfy condition 3, agents must instead differ in their preferences over lotteries even in the interval where they agree on their ordinal preferences over sure outcomes. The differences between conditions 1 and 3 can be related to notions of risk attitude in the population. Condition 1 can be interpreted as risk neutrality by all agents, while condition 3 can be interpreted as a specific common degree of risk aversion by all agents.

For the purpose of a clearer intuition, let X be convex and either restrict the number of dimensions to one, or assume that $x^i_{-k} = x^j_{-k}$ so that the ideal value of i and j is distinct only on attribute k, where $x^i_k \leq_k x^j_k$. Since X is convex and preferences are continuous, we can find a point between the ideal policy of i and j such that both agents are indifferent between this point and a lottery that grants them their ideal point with probability α and the ideal point of the other player with probability $1-\alpha$. Formally, there exists a point $x^2_k \in (x^i_k, x^j_k)$ such that if $p(x^i) = \alpha$, $p(x^j) = 1-\alpha$, $q(x^j) = \alpha$ and $q(x^i) = 1-\alpha$, then $(x^2_k, x^i_{-k}) \sim_i p$ and $(x^2_k, x^i_{-k}) \sim_j q$. A representation that locates x^2_k as the midpoint between x^i_k and x^j_k generates the same risk attitude for agents i and j. Condition 1 for a representation by means of an l_1 norm requires that $\alpha = 0.5$, which I interpret as risk neutrality, while condition 3 for a representation by means of an l_2 norm requires $\alpha = 0.25$, which I interpret as risk aversion.

For lotteries over values below x^i_k or values above x^j_k, condition 1 requires agents i and j to agree on their preferences over such lotteries. Condition 3 is more cumbersome. Consider the first case, $x^1_k \leq_k x^i_k$ and set $f(x^i_k) - f(x^{\min}_k) = 1$. Then the parameter γ is such that $f(x^j_k) - f(x^i_k) = \gamma$ in the spatial representation f such that $\succsim_i$ can be represented by u_i quadratic Euclidean in $f(X)$. From the perspective of agent i, x^j is at a distance γ of x^i in the space where i has quadratic Euclidean preferences and the distance from $x^{\min}_k$ to x^i_k is chosen as the unit of measure. In order for $\succsim_j$ to also be representable by a quadratic Euclidean u_2 in the same space, it must be that the preferences over lotteries of agent j depend on the distance γ. If we take a sequence of preference relations of agent i such that $\gamma \longrightarrow 0$, the preferences over lotteries of i and j must converge, so $\alpha_h \longrightarrow \alpha_l$. If instead we consider a different sequence so that γ increases toward infinity, the preferences over lotteries of i and j differ in such way that $\alpha_j \longrightarrow \sqrt{\alpha_i}$. As $\succsim_i$ changes so that γ increases toward infinity, j must become closer to risk neutral on lotteries over values below x^i_k in order for her preferences to be quadratic Euclidean in the spatial representation dictated by the preferences of agent i. While this condition may appear unduly restrictive, it is implicit in the formulation of any spatial model that uses the standard assumption of quadratic Euclidean preferences.

Whether preferences over multi-attribute alternatives are such that $\alpha = 0.5$, and $\alpha_j = \alpha_i$ in the lotteries discussed in the previous two paragraphs, or whether $\alpha < 0.5$ or in fact $\alpha = 0.25$, and $\alpha_j = \sqrt{\alpha_i} + \frac{\alpha_i - \sqrt{\alpha_i}}{1+2\gamma}$ is a testable empirical question. Evidence that $\alpha \approx 0.25$, and $\alpha_j \approx \sqrt{\alpha_i} + \frac{\alpha_i - \sqrt{\alpha_i}}{1+2\gamma}$ would support the assumption of quadratic Euclidean preferences. On the other hand, evidence that $\alpha \approx 0.5$, and $\alpha_j \approx \alpha_i$ would suggest that, albeit standard, the assumption of quadratic Euclidean preferences is unwarranted and it is appropriate to assume instead linear city block preferences, with positive implications for existence of majority core outcomes in multidimensional policy competition.

Discussion

It is a standard in spatial models of political competition to assume that utilities are decreasing –linearly, exponentially or in quadratic form- in the Euclidean distance from an ideal point in the space of policies.

The first theoretical contribution of this paper is to prove that given a spatial representation of the set of alternatives, under standard assumptions, risk neutrality on this space implies necessarily a utility representation that is linear in a generalized city block distance, not the Euclidean distance.

In many applications, the primitive set of alternatives is not a subset of a vector space, and any spatial representation is subjective, arbitrary, or made for convenience. Any assumption on preferences over alternatives in a vector space is not an assumption on primitives, such as preferences over alternatives; it is a joint assumption on preferences over alternatives, and on the chosen spatial representation of the preferences. The second theoretical contribution of this paper is to find simple and intuitive necessary and sufficient conditions on the preference relation over the primitive set of alternatives, such that for any $p \geq 1$, there exists a spatial representation of these alternatives under which preferences can be represented by a utility function that is decreasing in the p power of the l_p distance to an ideal point in the space. This result includes representations by a utility function that is linear in the city block distance or a utility function that is quadratic in the Euclidean distance as special cases. The conditions amount to separability across attributes, and single peakedness within each attribute.

The third contribution of this paper extends the second result to a society with multiple agents for the case of linear city block and quadratic Euclidean preferences, finding additional conditions under which there exists a spatial representation common to all agents such that the preference profile of every agent is representable by a utility function that is linearly decreasing in the l_1 distance or quadratic decreasing in the l_2 to the ideal point of the agent in the common space. These conditions are simpler and more intuitive for the case of city block preferences.

Recent empirical research by Berinsky and Lewis [4] finds that agents are risk neutral in their political preferences given their subjective spatial representation of the policy space in the US. In applications with a given spatial representation of the space of alternatives where the results of Berinsky and Lewis [4] are robust and agents are risk neutral, political economy theories of spatial competition on ideological issues should discard utility representations that are a function of Euclidean distance, and accept instead as the benchmark a utility function that is linear in the city block metric.

In any application without a given exogenous spatial representation, it is standard to represent preferences over alternatives with multiple attributes as a linear, quadratic, or exponential function of the Euclidean distance to an ideal point. I have shown that if preferences are separable and single peaked, utility functions that are linear or exponential on the Euclidean distance are untenable. Under these assumptions, individual preferences can be represented by either a linear city block utility function or a quadratic Euclidean utility function. While either utility representation requires stringent restrictions on preference profiles in a society with multiple agents, I have shown that the conditions such that the preferences of every agent can be represented in some space common to all agents by a city block utility function are simpler and more intuitive than the analogous conditions for a quadratic Euclidean utility function.

An implication of the results in this paper that some received wisdom that relies on the Euclidean distance perhaps should be reevaluated. For instance, it is well known that the conditions for existence of an equilibrium in multidimensional policy competition detailed by Plott [19] and generalized by McKelvey and Schofield [14] hold non-generically if agents have Euclidean preferences, but these conditions hold more generally if agents have city block preferences; if Plott´s conditions are not satisfied and agents have Euclidean preferences, McKelvey [12] shows that majority preferences are intransitive under very weak assumptions, but McKelvey [13] shows that this so-called "chaos result" holds only under stronger assumptions for non-Euclidean preferences. At the same time, theoretical contributions that work with the city block metric, such as results on the existence of core outcomes by Rae and Taylor [21], Wendell and Thorson [26], McKelvey and Wendell [15] and Humphreys and Laver [8], or that make assumptions on preferences consistent with city block preferences, such as the results on the existence of strategy proof outcomes by Barberà, Sonnenschein and Zhou [3], become more relevant.

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