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Market and Non-Market Mechanisms for the Optimal Allocation of Scarce Resources

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Abstract

Both market (e.g. auctions) and non-market mechanisms (e.g. lotteries and priority lists) are used to allocate a large amount of scarce public resources that produce large private benefits and small consumption externalities. I study a model in which the use of both market and non-market mechanisms can be rationalized. Agents are risk neutral and heterogeneous in terms of their monetary value for a good and their opportunity cost of money, which are both private information. The designer wants to allocate a set of identical goods to the agents with the highest values. To achieve her goal, she can screen agents on the basis of their observable characteristics, and on the basis of information on their willingness to pay that she can extract using market mechanisms. In contrast to models where willingness to pay and value coincide, a first best cannot be achieved. My main result is that both market and non-market mechanisms, or hybrid mechanisms, can be optimal depending on the prior information available to the designer. In particular, non-market mechanisms may be optimal if the value is positively correlated with the opportunity cost of money.

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1 Introduction

Governments play a key role in the initial allocation of property rights for a large number of scarce public resources that produce large private benefits and small consumption externalities. Examples include, but are not limited to, the assignment of broadcasting licences, oil drilling rights, access to education, draft exemption in war time, subsidized housing, and scarce medical resources. The allocation methods used to distribute these types of public resources can be classified into two broad groups: *market mechanisms* (e.g. auctions or posted prices), where the final allocation is based upon the claimants' willingness to pay for the good; and *non-market mechanisms* (e.g. lotteries and priority lists) that do not exploit information on the willingness to pay. The main aim of this paper is to provide a framework in which the existence of both groups of mechanisms can be rationalized and the two systems compared.[1]

Although there are several papers that study market and non market mechanisms independently, few efforts have been devoted to explaining the operation of both types of mechanisms in the same economic environment.[2] In particular, the study of market mechanisms dominates the literature, and the papers that consider non-market mechanisms motivate this restriction exogenously, outside the model. This is attributable to the fact that in a model where utility is perfectly transferable through monetary exchange, assignment of a set of private goods to those consumers with the highest willingness to pay is the only Pareto efficient outcome.[3] Therefore, if the designer maximizes welfare, and the agents are not budget constrained, there is no room for non-market mechanisms.[4]

[1]Hybrid systems combining market and non market mechanisms are also possible and are considered in the analysis (see e.g. Evans, Vossler and Flores (2009)).

[2]For our purposes, the analysis of market mechanisms under incomplete information can be summarized as auction theory (see e.g. Milgrom (2004), Klemperer (2004) or Krishna (2002)). The main branch of the literature that examines non-market mechanisms is the literature on two sided matching, based on Gale and Shapley's (1962)original study (see Roth and Sotomayor (1990) for a textbook treatment). The papers that compare market and non-market mechanisms are discussed in the section on related literature.

[3]Otherwise a mutually beneficial possibility for trade would exist between the recipient of the good and someone with a higher willingness to pay. Note that distributional objectives can be achieved by distributing the proceedings from a market mechanism rather than by implementing a different type of allocation.

[4]The traditional rationale for interfering with markets, that is externalities, is not applicable to the case of private goods (for a classical discussion see Pigou (1932), pp.115-116). For why non-market

In order to provide a framework where both market and non market mechanisms can be compared, I assume that individuals are heterogeneous in terms of both, the *value* that they can enjoy by obtaining one good (e.g. return from education in the case of a school admissions process), and the *opportunity cost of money* that they face, which for simplicity can be thought of as the interest rate payable in an imperfect capital market. Further, I assume that the designer allocates the goods in order to maximize, not the welfare of agents, which would require implementing a Pareto efficient outcome, but rather total value.[5] Finally, I maintain that the information held by agents is not available to the designer, which can only control the allocation of the goods and asks for payments.

The main contribution of this paper is to investigate the above mechanism design problem. A non-standard problem arises because *willingness to pay*, which is the potentially observable information, is only a noisy signal of value, which is the piece of private information which the designer is interested in acquiring. The main insight that emerges from this study is that the use of non-market mechanisms may be optimal when informational asymmetries impede the extraction of information on value that is facilitated by the acquisition of information on the willingness to pay.

In the example of the schools admission process, the argument would be as follows. Assume that the returns from education for a given individual are determined by some set of observable characteristics and by some unobservable characteristics (e.g. individual motivation). Furthermore, suppose that individuals are all the same in terms of their observable characteristics.[6] If, on average, students with lower levels of motivation display a higher willingness to pay, an admission committee would prefer, *ceteris paribus*, to assign places to those that show the lowest willingness to pay. However, because this latter information is private, the designer will not be able to implement such an allocation.[7] Therefore collecting information on the willingness to pay is not useful, and a lottery or a priority list based on observable characteristics, is preferable. The flip side to this is that if, on average, agents with higher values display a higher willingness to pay, then it is optimal to adopt a standard market mechanism (or a hybrid mechanism).

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mechanisms can be useful to implement a Pareto efficient allocation if agents are budget constrained see Che and Gale (2007).

[5]Equality of treatment as a motivation for non-market mechanisms can be seen as an extreme case of this idea: the designer maximizes welfare under the constraint that every individual receives equal treatment (see Young (1994), p.20).

[6]Note that my analysis will allow for heterogeneity among agents.

[7]This is so because the desired allocation is not monotone increasing in the willingness to pay.

A practical implication of my analysis is that, in general, when the objective of the designer is not welfare maximization, a case by case evaluation is needed to establish whether a market or a non-market mechanism is optimal in a given environment. As another possible application of my results consider the problem of allocating scarce medical resources. The medical profession appears to be strongly against the idea of assigning scarce medical resources to the highest bidder.[8] Instead, different criteria are used to select among applicants for the same scarce medical resource. For example, saving the highest number of lives is a classic and long-standing rationing principle.[9] This implies also that scarce life-saving medical resources should be assigned to patients with the highest chances of survival from receiving the resource.[10] My analysis suggests that, in some cases, using information on the willingness to pay might be beneficial, even if the goal set for allocation is saving the highest number of lives. The empirical question in this case is: do we expect that people with a higher willingness to pay will be likely to benefit more from a given resource? If the answer to the question is positive, but policy makers insist on refusing to consider the willingness to pay as a possible allocation method, then a rethinking of the main goal of the allocation would seem necessary.

There are three other studies that appear closely related to the present one: Che and Gale (2007), Esteban and Ray (2006) and Fernandez and Gali (1999). Che and Gale (2007) compare market and non-market allocation methods for the efficient initial assignment of ownership, to a set of wealth constrained agents. In their model, if a good is sold at its market clearing price, it might not be acquired by the individual with the highest willingness to pay for it. In fact, it might be acquired by a wealthy individual with a lower willingness to pay, rather than by someone that would be willing to pay

[8] E.g., in a recent paper surveying methods of allocation for scarce medical interventions, Persad, Wertheimer, and Emanuel (2009) state: "we do not regard ability to pay as a plausible option for the scarce life-saving interventions we discuss". Resistance to the introduction of monetary markets for organs for transplant is also documented in Becker and Elias (2007) and in Roth (2007). A discussion of the reasons behind these ethical judgments are beyond the scope of this paper.

[9] This principle is the motivation for policies related to the allocation of influenza vaccine (see Emanuel and Wertheimer (2006)) and responses to bioterrorism (see Phillips (2006)).

[10] My model would apply for example to the case where a limited number of vaccines is available, for a non easily transmissible disease that could be fatal if contracted. All the assumptions in my model are satisfied: (i) increasing the probability of surviving a disease is a private good; (ii) the effectiveness of a vaccine may depend on some private information held by the patient (e.g. sexual, or alcohol consumption behaviour); and (iii) people may have different opportunity costs for money.

more, but is unable to do so. Therefore, a lottery outperforms a market based allocation, if the recipient of the good is allowed to resell. In contrast with my model, in Che and Gale's setting the ultimate goal of the designer remains that of allocating the goods to the agents with the highest willingness to pay.

Esteban and Ray (2006) analyse how wealth inequality may distort the allocation of public resources. In their model a government seeks to allocate limited resources to productive sectors. However both sectoral productivity and wealth are privately known. The government, even if it seeks to assign the resources to the most productive sectors, may be confounded by the possibility that both high wealth and true economic desirability create loud lobbying. While this argument is strongly related to my main insight, they work within a very specific lobbying game and do not address the underlying mechanism design problem. Moreover, they assume that wealth and productivity are uncorrelated variables, which in my general analysis is treated as a special case.

Fernandez and Gali (1999) compare the performance of markets and tournaments (i.e. non-market mechanisms where agents engage in costly signalling activity) as allocation mechanisms in an economy with borrowing constraints. They study a model where a continuum of individuals, with different skills and different access to capital markets, is matched to a continuum of inputs with different productivity (following Becker (1973)). While tournaments induce a costly effort, which ultimately is wasted, they might provide a better match between skilled individuals and inputs. Therefore, tournaments might be preferred to market mechanisms (where the best inputs go to those that can pay the most for them), if their signalling ability is substantial. While, again, the insight is related to mine, Fernandez and Gali's analysis is quite different. In particular, they consider different types of non-market mechanisms, where signalling is performed via a costly effort.[11]

The paper is organized as follows: the following section discussing the model, section 4 presents the main results and section 5 concludes the paper.

[11] Finally, my paper is also related to the literature on auctions and mechanism design with financial constraints. The most closely related papers are probably Che and Gale (2000) and (1998). The study in the first paper examines revenue maximizing mechanisms in a one seller/one buyer model. My treatment of financial constraints is less general . However, their results do not easily generalize to a multi-agent setting. In the second paper they study standard auctions focusing on a model where agents are budget constrained. In both of these papers, the performance of market allocation mechanisms is evaluated in terms of revenue and welfare maximization, while my analysis focuses on a mechanism designer with other objectives.

2 The model

Environment. A risk neutral *designer* has $k \geq 1$ units of an indivisible good to allocate, which she values at zero. There are $n > k$ risk neutral *agents*, who have unitary demand and private *monetary valuation* for the goods. The valuation is the amount of money that an agent would be willing to pay if he could borrow at zero interest rate.[12] Valuation differs from the willingness to pay because agents are heterogeneous in terms also of the *opportunity cost of making their payments* (e.g. the interest rate in an imperfect capital market). If an agent with value $v_i \geq 0$, is faced with a private interest rate of $r_i \geq 0$, and obtains a good with probability $0 \leq p_i \leq 1$ and pays an amount of money m_i, his *vN-M* utility is $u_i(p_i, m_i, v_i, r_i) = v_i p_i - (1+r_i)m_i$. The type of an agent is a pair (v_i, r_i).[13] Types are independently distributed across agents, according to commonly known distributions $F_{v_i,r_i}(\cdot,\cdot)$ with $i = 1, \ldots, n$. For each i, I assume that the density f_{v_i,r_i} is strictly positive in some convex $X_i \subseteq \Re^2_+$, and zero elsewhere.

I assume explicitly that, in trying to implement her objectives, the designer can control only the allocation of the goods and require payments. Furthermore, I assume that all observable information about values and opportunity costs is incorporated in the prior distribution. Finally, I assume that the designer is not able to perform any test that would reveal information about the valuation or the interest rate, other than that already contained in the observable characteristics of the agent.

Characterization of the type space. Suppose that agents are required to choose within a *possibility set*, in which each element is a pair (p, m) (i.e. each element of the possibility set is a couple formed by a probability of obtaining the good and the monetary transfer). The observable behaviour of two bidders of types (v, r) and (v', r'), such that $\frac{v}{1+r} = \frac{v'}{1+r'}$, is indistinguishable. Their utility functions are the same up to a scale-normalization and, therefore, they represent the same preferences.[14]

[12] E.g., in the case of a scarce medical resource, the idea would be that, net of the heterogeneous opportunity cost of money, different monetary evaluations would reflect the different benefits from using the resource, that the different agents might expect.

[13] The restriction that $0 \leq p_i \leq 1$ is harmless because agents have unitary demand.

[14] To see the point, assume that type (v, r) weakly prefers allocation (p, m) to allocation (p', m'), while type (v', r') weakly prefers p', m' over (p, m). For this to be the case we must have: $pv - (1+r)m \geq vp' - (1+r)m'$ and $p'v' - (1+r')m' \geq v'p - (1+r')m$. Next, divide the first inequality by $(1+r)$ and the second by $(1+r')$. It is easy to verify that, whenever $\frac{v}{1+r} = \frac{v'}{1+r'}$, both types must be exactly indifferent between the two allocations, that is, $pv - (1+r)m = vp' - (1+r)m'$ and $p'v' - (1+r')m' = v'p - (1+r')m$.

Let us denote the *willingness to pay* of an agent i by $w_i = \frac{v_i}{(1+r_i)}$. In other words, w_i is the maximum amount of money that i would be willing to spend in order to obtain the good, given that the value is v_i and he faces an interest rate of r_i. For all i, w_i has a well defined prior distribution F_{w_i} (hereafter denoted Z_i), supported in $W_i = (\underline{w}_i, \overline{w}_i)$, where $\underline{w}_i = \inf_{(v_i,r_i)\in X_i} \frac{v_i}{(1+r_i)}$ and $\overline{w}_i = \sup_{(v_i,r_i)\in X_i} \frac{v_i}{(1+r_i)}$.[15] In particular:

$$Z_i(w) = \int_0^w \left[\int_0^{+\infty} (1+x) f_{v_i,r_i}(y(1+x), x) dx \right] dy$$

It is a consequence of the independence across i of vectors (v_i, r_i), that the willingness to pay is distributed independently across agents. If one sees the type space as a Cartesian plane where the coordinates are v_i and $1 + r_i$, then all types with the same willingness to pay w_i will lie along the same straight line from the origin. Figure 2 is an illustration of this fact.

Mechanism design. We perform a standard static mechanism design exercise in which the designer wants to implement an outcome that maximizes her objective function (see next section for details). Consistent with the empirical evidence, I assume that the designer (i.e. a government or some other institution) can ban resale.[16] The revelation principle states that, for any possible mechanism that can be designed, there exists an incentive compatible direct mechanism that achieves the same equilibrium outcome. Therefore, in the search for an optimal mechanism (e.g. welfare or revenue maximizing), the designer can restrict attention to direct revelation mechanisms in which each agent has an incentive to report his private information truthfully to the designer.

The alignment of preferences for types with same willingness to pay greatly simplifies the mechanism design problem. In fact, in this setting there is no hope of obtaining full revelation of both values and interest rates, unless everyone with the same willingness to pay obtains the same outcome (p, m).[17] Therefore, I can restrict attention, without loss of generality, to incentive compatible direct mechanisms that assign the same pair (p, m)

[15] The distributions $Z_i(\cdot)$ are defined in intervals because X_i is a convex subset of $\Re^2_+$. For simplicity, the interval is left open.

[16] In this setting it can never be in the interests of the designer to allow resale. This assumption is not innocuous, unless in the allocation that the designer wants to implement the agents with the highest willingness to pay obtain the goods (see Zheng (2002))

[17] Several studies examine settings with multidimensional signal spaces and lower dimensional policy spaces. See, e.g., McAfee and McMillan (1988), Armstrong (1996), Jehiel and Moldovanu (2001) and, more recently, Deneckere and Severinov (2009).

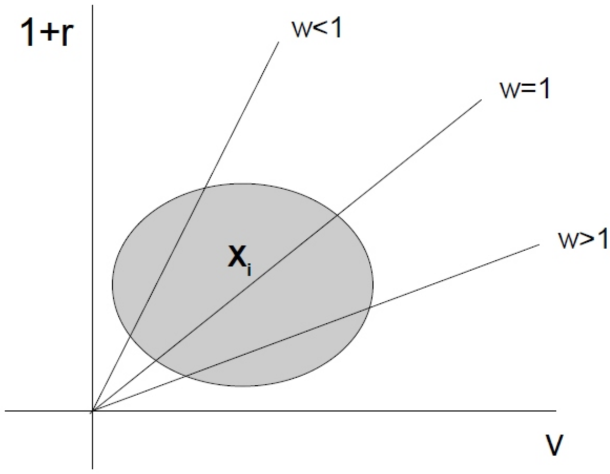

Figure 1: Typical Type Space

to bidders with the same willingness to pay.[18]

Thus, the content of the remainder of this section is standard. Let $\boldsymbol{w} = w_1, \ldots, w_n$, let $\boldsymbol{W} = W_1 \times \ldots W_n$, and finally let $\boldsymbol{w_{-i}}$ indicate the vector of the $n-1$ willingness to pay other than that of player i. The direct mechanism $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ specifies the probability for each agent to obtain the object, $p_1(\boldsymbol{w}), \ldots, p_n(\boldsymbol{w}) \in [0,1]^n$ and the payment that is

[18]More precisely, I focus on the mechanism whereby $p_i(v_i, r_i, \boldsymbol{v_{-i}}, \boldsymbol{r_{-i}}) = p_i(v_i', r_i', \boldsymbol{v_{-i}}, \boldsymbol{r_{-i}})$ and $m_i(v_i, r_i, \boldsymbol{v_{-i}}, \boldsymbol{r_{-i}}) = m_i(v_i', r_i', \boldsymbol{v_{-i}}, \boldsymbol{r_{-i}})$ for any (v_i, r_i) and (v_i', r_i') such that $v(1+r') = v'(1+r)$. While incentive compatibility requires only that types with the same willingness to pay obtain the same utility, there is no loss of generality in looking at mechanisms where the allocation and the payment rules coincide for such types.

required $m_1(\boldsymbol{w}), \dots, m_n(\boldsymbol{w}) \in \mathbb{R}^n$ as a function of the profile of reported willingness to pay. A mechanism is *feasible* if and only if:

$$\sum_{i=1}^{n} p_i(\boldsymbol{w}) \leq k \quad \forall \boldsymbol{w} \in \boldsymbol{W} \tag{1}$$

I require that the designer cannot support an ex-post deficit, that is, I impose the following *no-deficit constraint*:

$$\sum_{i=1}^{n} m_i(\boldsymbol{w}) \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W} \tag{2}$$

The expected utility for agent i from reporting w_i' in $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ when there is a true willingness to pay w_i, assuming that all other agents report truthfully, is $U_i^{\langle \boldsymbol{p}, \boldsymbol{m} \rangle}(w_i, w_i') = \mathrm{E}_{\boldsymbol{w_{-i}}}[p_i(w_i', \boldsymbol{w_{-i}})w_i - m_i(w_i', \boldsymbol{w_{-i}})]$. For a mechanism to be *incentive compatible*, bidders must have an incentive to report their information truthfully, if the other bidders are doing the same. Formally, Bayesian incentive compatibility requires that for all $w_i \in W_i$

$$w_i = \arg\max_{x \in W_i} U_i^{\langle \boldsymbol{p}, \boldsymbol{m} \rangle}(w_i, x) \tag{3}$$

Finally, I assume that agents can opt out of the mechanism and obtain zero utility before reporting any information to the designer. That is, *participation constraints* require that every type of every agent must always obtain a utility higher than zero: for all i and $w_i \in W_i$

$$U_i^{\langle \boldsymbol{p}, \boldsymbol{m} \rangle}(w_i, w_i) \geq 0 \tag{4}$$

To conclude this section, I can state without proof two well known results from the theory of mechanism design. The first Lemma simplifies the designer's problem by showing that an allocation rule can be implemented if and only if $\mathrm{E}_{\boldsymbol{w_{-i}}}[p_i(w_i, \boldsymbol{w_{-i}})]$ is increasing. The second Lemma states that restricting attention to ex-post incentive compatible mechanisms that satisfy an ex-post participation constraint, comes without loss of generality.

Lemma 1. *Let* $P_i(w_i) = \mathrm{E}_{\boldsymbol{w_{-i}}}[p_i(w_i, \boldsymbol{w_{-i}})]$ *and* $M_i(w_i) = \mathrm{E}_{\boldsymbol{w_{-i}}}[m_i(w_i, \boldsymbol{w_{-i}})]$ *(we call these, the reduced form allocation and payment schedules). A direct mechanism* $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ *is incentive compatible, satisfies participation constraints and does not run a deficit if and only if:*

$$\forall w, w' : w \geq w' \quad P_i(w) \geq P_i(w') \tag{5}$$

$$\forall w \quad M_i(w) = wP_i(w) - \int_0^w P_i(x)dx \tag{6}$$

Lemma 2. *Take any feasible (1) and incentive compatible (3) mechanism that satisfies participation (4) and no deficit (2) constraints. There is a feasible mechanism that satisfies the no-deficit constraint, has the same allocation rule and generates the same reduced form payment schedule, satisfies ex-post participation constraints and is ex-post incentive compatible.*[19]

3 Results

The designer maximizes the following objective function under incentive compatibility, feasibility, no deficit and participation constraints:

$$E_{\boldsymbol{v},\boldsymbol{r}}\left[\sum_{i=0}^{n} p_i\left(\frac{v_1}{1+r_1},\ldots,\frac{v_n}{1+r_n}\right)v_i\right] \tag{7}$$

Call *first best* the mechanism whose outcome maximizes the above function without incentive constraints. A first best mechanism, which serves as a benchmark for the analysis of the problem under incomplete information, allocates the goods to the k agents with the highest values. Clearly, if the designer knows the values of the agents, implementation of a first best outcome is trivial.

Next, I consider the case where the designer knows the prevailing interest rates, but not the values. It is not difficult to see that the first best is attainable in this case. In fact, for any given willingness to pay w_i and known interest rate r_i, there is a unique value $v_i = (1 + r_i)w_i$. Moreover, v_i increases with the willingness to pay w_i. Therefore, building on the fact that, according to Lemma 1, every monotonic increasing allocation is implementable, it is possible to construct a mechanism to achieve the desired outcome. I have the following proposition (see appendix A for a formal statement, and for the proof).

Proposition 1. *Assume that the designer knows the interest rates. A mechanism* $\langle \boldsymbol{p}, \boldsymbol{m}\rangle$ *that allocates the goods to the agents with the highest values, breaking ties using fair lotteries, and determines payments according to* (6) *is feasible, incentive compatible, satisfies participation constraints and generates no-deficit.*

[19]More formally, let $\langle \boldsymbol{p}, \boldsymbol{m}\rangle$ be a mechanism that satisfies (1)-(4). There exists $\langle \boldsymbol{p'}, \boldsymbol{m'}\rangle$ such that $\boldsymbol{p'} = \boldsymbol{p}$, and for all i,w_i, w_i' and w_{-i} we have $U_i^{\langle \boldsymbol{p},\boldsymbol{m}\rangle}(w_i, w_i) = U_i^{\langle \boldsymbol{p'},\boldsymbol{m'}\rangle}(w_i, w_i)$, $p_i(w_i, \boldsymbol{w_{-i}})w_i - m_i(w_i, \boldsymbol{w_{-i}}) \geq 0$ and $p_i(w_i, \boldsymbol{w_{-i}})w_i - m_i(w_i, \boldsymbol{w_{-i}}) \geq p_i(w_i', \boldsymbol{w_{-i}})w_i - m_i(w_i', \boldsymbol{w_{-i}})$

This is rather a special case. In general, the presence of uncertainty about the opportunity cost of money impairs the ability of the designer to achieve an optimal outcome. As shown in the next proposition, if there is uncertainty, conditional on knowing the willingness of the agents to pay, as to which agent has the highest value, then a first best cannot be attained.[20] The proof is straightforward and shows that full knowledge of the interest rates is (almost always) necessary to achieve a first best.

Proposition 2. *If there is a player i with (v_i, r_i) and (v'_i, r'_i) in X_i such that $\frac{v_i}{1+r_i} = \frac{v'_i}{1+r'_i}$, and a player j with (v_j, r_j) in X_j such that $v_i > v_j > v'_i$, and there are less than k players such that for each player s within them $v_s > v_i$ for each (v_s, r_s) in X_s, a first best cannot be achieved.*

Proof. The last condition ensures that no set of players exists to which the designer would want to assign the good for sure, when player i is around. When no such set of players exists, sometimes the designer will want to assign a good to player i. However, she cannot discriminate between types (v_i, r_i) and (v'_i, r'_i) of player i. The fact that a player j exists such that the designer might prefer j over i whenever i is of type (v_i, r_i), implies that a first best cannot be achieved. In fact, in any case, either assigning the good to i or to j there is a positive probability that the allocation is not optimal. Figure 3 depicts this situation. □

As first best is not attainable under incomplete information, we are interested in constrained optimal mechanisms. In this case the designer maximizes the following objective function:[21]

$$\mathrm{E}_{\boldsymbol{w}}\left[\sum_{i=0}^{n} p_i(\boldsymbol{w})\,\mathrm{E}[v_i \mid w_i]\right] \tag{8}$$

The maximization is subject to feasibility, incentive compatibility, participation and no-deficit constraints. Because the objective function is linear in the allocation rule, I can characterize a solution to this problem using Myerson's ironing technique. The next proposition achieves this. The formal statement and proof are presented in appendix C.

Proposition 3. *Define, for all $x \in [0,1]$:*

$$H_i(x) = \int_0^x \mathrm{E}[v_i \mid w_i = Z_i^{-1}(y)]dy \quad G_i(x) = \mathrm{conv}\,\langle H_i(x)\rangle \quad g_i(x) = G'_i(x)$$

[20] This must be contrasted with the standard private value case where the designer maximizes welfare. In that case, a first best outcome can be achieved by means of a Vickrey auction (see Vicrey (1961)).

[21] Appendix B shows how this objective function is formally obtained.

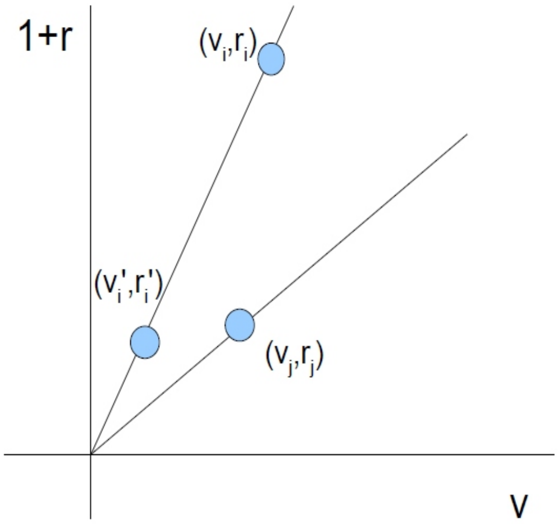

Figure 2: Impossibility of Achieving a First Best

Here, conv $\langle\cdot\rangle$ *denotes the convex hull of the function and* Z_i^{-1} *indicates the inverse function of* Z_i*.*[22] *Where the derivative of* $G_i(v)$ *is not defined, we extend it using the right or left derivative. Define the priority function* λ_i *for agent* i *as:*

$$\lambda_i(w_i) = g_i(Z_i(w_i))$$

In an optimal mechanism, the k *agents with a willingness to pay that achieves the highest priority level, obtain the goods and ties are broken by a lottery. The payment for*

[22] $G_i(x)$ is the highest convex function in [0,1] such that $G_i(x) \leq H_i(x)$ $\forall x$. The fact that X_i is a convex set ensures that Z_i is strictly increasing (when it is different from 0 or 1) and therefore invertible.

an agent who obtains a good is equal to the minimum willingness to pay that would have allowed the agent to obtain the good, given the mechanism in place:

$$m_i(\boldsymbol{w}) = p_i(\boldsymbol{w})w_i - \int_0^{w_i} p_i(x_i, \boldsymbol{w_{-i}})dx_i \tag{9}$$

To make the solution more intelligible, observe that if $E[v_i|w_i]$ is increasing in w_i then $\lambda_i(w_i) = E[v_i|w_i]$ (because $G \equiv H$). Instead, if $E[v_i|w_i]$ is decreasing, then $\lambda_i(w_i) = E[v_i]$ (because $H(x)$ is a straight line going from 0 to $E[v_i]$, when x goes from 0 to 1). Some further explanation is required.

Remark 1. If $\mathrm{E}[v_i|w_i]$ is non-decreasing for all i, then a mechanism that allocates goods to the bidders reporting the highest conditional expected values, is an optimal mechanism. Therefore, information on the willingness to pay is extracted and payments are requested from agents.

Remark 2. When agents are ex-ante symmetric and $\mathrm{E}[v_i|w_i]$ is non-decreasing any standard auction without a reserve price (which allocates goods to the agents with the highest willingness to pay) is an optimal mechanism. An important consequence of this fact is that the objectives of Pareto efficiency (within the coalition formed by all agents) and value maximization are not in contrast. Therefore, the designer need not prohibit resale of the goods.[23]

Remark 3. If $\mathrm{E}[v_i|w_i]$ is a non-increasing function of w_i for all i, then the optimal mechanism assigns the goods to the agents with the highest unconditional expected values $E[v_i]$. That is, no information about the willingness to pay is extracted by the designer. No one needs to make a payment to the designer. If agents are heterogeneous the optimal mechanism is a priority list, where agents are ranked only on the basis of their observable characteristics. If agents are all alike, all goods are assigned through an equal chance lottery. In order to implement the outcome of these mechanisms the designer needs to ban resale of the goods.

In the next proposition I show that extracting *some* information on the willingness to pay is always beneficial to the designer in the case where agents are symmetric and the set X_i is the Cartesian product of two intervals: $[\underline{v}_i, \overline{v}_i]$ and $[\underline{r}_i, \overline{r}_i]$. In particular, I

[23]The same reasoning applies to the optimal auction problem: banning resale is not necessary if, whenever the good is allocated, it is allocated efficiently (see Myerson (1981) or Riley and Samuelson (1981)).

show that $\mathrm{E}[v_i|w_i]$ cannot be non-increasing if it is defined, regardless of the correlation between v_i and r_i.

Proposition 4. *Suppose that $X_i = [\underline{v}_i, \overline{v}_i] \times [\underline{r}_i, \overline{r}_i]$ for each i. Then, $\mathrm{E}[v|w]$ is not decreasing in $W_i = [\frac{\underline{v}_i}{1+\overline{r}_i}, \frac{\overline{v}_i}{1+\underline{r}_i}]$ for any density function $f_{v_i,r_i}(\cdot,\cdot)$ which is strictly positive in X_i and zero elsewhere.*

Proof. It suffices to show that there are $x > x'$ in the interval $[\frac{\underline{v}_i}{1+\overline{r}_i}, \frac{\overline{v}_i}{1+\underline{r}_i}]$, such that $\mathrm{E}[v_i|w_i = x] > \mathrm{E}[v_i|w_i = x']$. First, take $x' = \frac{\overline{v}_i+\underline{v}_i}{2(1+\overline{r}_i)}$. The support for $f_{v_i|w_i}(v_i|w_i = x')$ is a subset of the interval $[v_l, \frac{\overline{v}_i+\underline{v}_i}{2}]$. Next, take $x > \frac{\overline{v}_i+\underline{v}_i}{2(1+\underline{r}_i)}$. The support for $f_{v_i|w_i}(v_i|w_i = x)$ is a subset of $(\frac{\overline{v}_i+\underline{v}_i}{2(1+\underline{r}_i)}, \overline{v}_i]$. Therefore, because the supports do not overlap, we have that $\mathrm{E}[v_i|w_i = x] > \mathrm{E}[v_i|w_i = x']$. □

While the above result holds when X_i is a Cartesian product between two intervals, in general, however, the function $\mathrm{E}[v_i|w_i]$ can also be decreasing in the entire interval where it is defined. This is illustrated by the following example, where high valuations are correlated to high interest rates.

Example 1. Let $X_i \subseteq \Re^2_+$ be the triangle defined by $1 \leq \frac{v_i}{1+r_i} \leq 2$ and $1+r_i \geq \frac{3}{2}v_i - 2$ (see Figure 3). Furthermore, let $f_{v_i,r_i} = 1$ be X_i and zero elsewhere. After some computations I have that:

$$\mathrm{E}[v_i \mid w_i] = \frac{16 - 64w_i + 76w_i^2}{12 - 48w_i + 45w_i^2}$$

This function is strictly decreasing for $1 \leq w_i \leq 2$. Therefore, if all agents' values are drawn from X_i according to $f_{v_i,r_i}(\cdot,\cdot)$, then the optimal mechanism is a lottery. No information about the willingness to pay can be exploited.[24]

In general, the optimal mechanism may be a combination of market and non-market mechanisms. The following mechanisms are all possible outcomes depending on the distributions of prior information. For example:

- Allocating the goods to all the members of a group A and selling the remaining goods to the highest bidders in the second group B (e.g. if the minimum value for an agent in group A is above the maximum for an agent in group B and $E(v_i \mid w_i)$ is increasing for agents in B).

[24]Note that if the designer could extract information about the willingness to pay she would want to allocate the goods to those with the lowest willingness to pay. This is the reason why no information extraction is the best incentive compatible outcome.

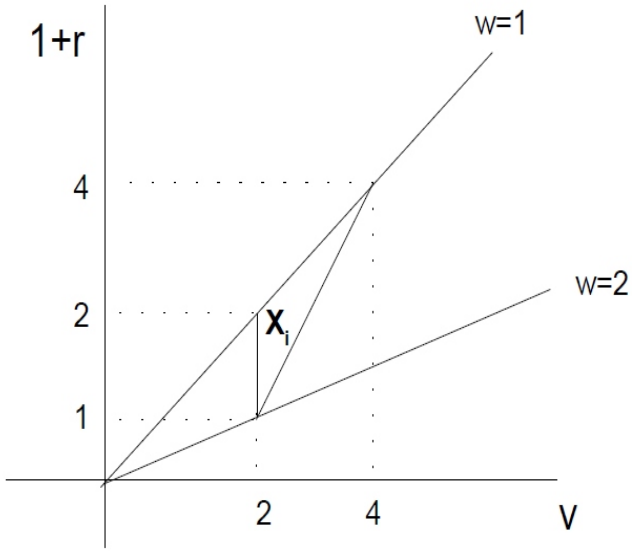

Figure 3: Type Space in Example 1

- Allocating the goods to the highest bidder up to a maximum bid, after which everyone has the same probability of obtaining one good (e.g. if agents are symmetric and $E(v_i \mid w_i)$ is first increasing and then decreasing).
- Allocating the goods to the highest bidders, for bids above a certain threshold, but randomizing the (possibly) remaining goods among those that did not bid above the threshold (e.g. if agents are symmetric and $E(v_i \mid w_i)$ is first decreasing and then increasing).

Proposition 3 shows how to construct an optimal mechanism using as inputs the functions $\mathrm{E}[v_i|w_i]$ for $i = 1, \ldots, n$. It is interesting to try to characterize $\mathrm{E}[v_i|w_i]$ in terms of the primitive model, that is the joint density of v_i and r_i. However, it appears that the behaviour of $\mathrm{E}[v_i \mid w_i]$ is difficult to characterize a priori. The next example shows that even in the case where v_i and r_i are independently distributed $\mathrm{E}[v_i|w_i]$ can be non monotonic.

Example 2. Assume that v and r are independently distributed and consider two cases: (i) $f_v(v) = 1$ for $v \in [0, 1]$ and zero otherwise, and $f_r(r) = 1$ for $r \in [0, 1]$ and zero otherwise; (ii) $f_v(v) = 1$ for $v \in [0, 1]$ and zero otherwise, and $f_r(r) = 1/2$ for $0 \leq r \leq \frac{1}{2}$, $f_r(r) = 3/2$ for $1/2 < r \leq 1$ and zero otherwise. It can be shown that in case (i) $\mathrm{E}[v_i|w_i]$ is increasing and then constant. Therefore, if agents are symmetric a market mechanism is optimal in this case. However, in case (ii) $\mathrm{E}[v_i|w_i]$ is first increasing, then decreasing, and then increasing again. Therefore, in this second case, a hybrid mechanism (combining features of both market and non-market mechanisms) may be optimal.

4 Conclusions

In this paper, I study a model for the allocation of scarce resources, where agents are heterogeneous in terms of both their value for the goods and their opportunity costs for money. Both variables are the private information of the agents. The designer would like to maximize total value, but she can only extract information about the willingness to pay, which is a noisy signal of value.

The solution to the outlined multidimensional mechanism design problem provides two main insights. First in contrast to models where willingness to pay and value coincide, an ex-post unconstrained optimal outcome cannot be achieved generically. Second, both market and non-market allocations can be optimal, depending on the joint distribution of prior information. That is, lotteries, priority lists and other hybrid mechanisms dominate pure market mechanisms in cases where high values are positively correlated to high opportunity costs for money.

I briefly discuss the implications of my results in the context of two examples: allocation of school places and allocation of scarce medical resources. However, there is another application that is worthy of mention: the design of an efficient auction for the allocation of licences to operate in a regulated market (e.g. mobile telecommunications). In fact, due to capital market imperfections, firms may have different expected operating profits from obtaining a licence and also different opportunity costs for their investments. For example, the costs of capital may be lower for large firms than smaller ones, but the latter may be more efficient. My analysis shows that if the regulator has reason to believe that this is the case, a non-market allocation may dominate an auction on the grounds of maximizing social welfare (i.e. assuming that expected operating profits reflect social welfare more than overall profits).[25]

Concluding, an important topic for further research would be to a model that incorporates time preferences with heterogeneous opportunity costs for money. This would allow the evaluation, within the same framework, of queues (and tournaments, as in Fernandez and Gali (1999)) as resource allocation methods.[26]

References

Armstrong, Mark, "Multiproduct Nonlinear Pricing," *Econometrica*, January 1996, *64* (1), 51–75.

Becker, Gary S., "A Theory of Marriage: Part I," *Journal of Political Economy*, July-Aug. 1973, *81* (4), 813–46.

___ **and Julio Jorge Elias**, "Introducing Incentives in the Market for Live and Cadaveric Organ Donations," *Journal of Economic Perspectives*, Summer 2007, *21* (3), 3–24.

Che, Yeon-Koo and Ian Gale, "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, January 1998, *65* (1), 1–21.

___ **and** ___ , "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Journal of Economic Theory*, June 2000, *92* (2), 198–233.

___ **and** ___ , "Market versus Non-Market Assignment of Ownership," *mimeo*, 2007.

Condorelli, Daniele, "What money can't buy: allocations with priority lists, lotteries and queues," *mimeo, University College London*, 2009.

[25]Papers that analyse financial constraints in auctions include: Che and Gale (1998), Zheng (2001) and Pai and Vohra (2009).

[26]Another related paper of mine Condorelli (2009) compares queuing with other non-market mechanisms such as lotteries and priority lists.

Deneckere, Raymond and Sergei Severinov, "Multi-Dimensional Screening with a One-Dimensional Allocation Space," mimeo University of Wisconsin 2009.

Emanuel, Ezekiel and Alan Wertheimer, "Who should get influenza vaccine when not all can?," *Science*, 2006, *312*, 854–855.

Esteban, Joan and Debraj Ray, "Inequality, Lobbying, and Resource Allocation," *American Economic Review*, March 2006, *96* (1), 257–279.

Evans, Mary F., Christian A. Vossler, and Nicholas E. Flores, "Hybrid allocation mechanisms for publicly provided goods," *Journal of Public Economics*, February 2009, *93* (1-2), 311–325.

Fernandez, Raquel and Jordi Gali, "To Each According to . . . ? Markets, Tournaments, and the Matching Problem with Borrowing Constraints," *Review of Economic Studies*, October 1999, *66* (4), 799–824.

Gale, D. and L. S. Shapley, "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 1962, *69* (1), 9–15.

Jehiel, Philippe and Benny Moldovanu, "Efficient Design with Interdependent Valuations," *Econometrica*, September 2001, *69* (5), 1237–59.

Klemperer, Paul, *Auctions: Theory and Practice*, Princeton University Press, 2004.

Krishna, Vijay, *Auction Theory*, Academic Press, 2002.

McAfee, R. Preston and John McMillan, "Multidimensional incentive compatibility and mechanism design," *Journal of Economic Theory*, December 1988, *46* (2), 335–354.

Milgrom, Paul, *Putting Auction Theory to Work*, Cambridge University Press, 2004.

Myerson, Roger, "Optimal Auction Design," *Mathematics of Operations Research*, 1981, *6* (1), 58–73.

Pai, Mallesh and Rakesh Vohra, "Optimal Auctions with Financially Constrained Bidders," mimeo Northwestern University 2009.

Persad, Goving, Alan Wertheimer, and Ezekiel Emanuel, "Principles for allocation of scarce medical interventions," *Lancet*, 2009, *373*, 423–431.

Phillips, S, "Current status of surge research," *Acad. Emerg. Med.*, 2006, *13*, 11031108.

Pigou, Arthur C., *The Economics of Welfare*, Macmillan and Co., 1932.

Riley, John G and William F Samuelson, "Optimal Auctions," *American Economic Review*, June 1981, *71* (3), 381–92.

Roth, Alvin, "Repugnance as a Constraint on Markets," *Journal of Economic Perspectives*, 2007, *21* (3), 37–58.

Roth, Alvin E. and Marilda A. Sotomayor, *Two Sided Matching*, Cambridge University Press, 1990.

Vickrey, William, "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 1961, *XVI*, 8–37.

Young, Peyton, *Equity: in Theory and Practice*, Princeton University Press, 1994.

Zheng, Charles Z., "High Bids and Broke Winners," *Journal of Economic Theory*, September 2001, *100* (1), 129–171.

Zheng, Charles Zhoucheng, "Optimal Auction with Resale," *Econometrica*, November 2002, *70* (6), 2197–2224.

5 Appendix A

Proposition 5. *Assume that interest rates are known. For each* $\boldsymbol{w}$*, the set of agents is partitioned in a chain of ordered sets,* $M_1(\boldsymbol{w}), M_2(\boldsymbol{w}), \dots$ *according to their values as implied by the willingness to pay. Formally, set* $M_0(\boldsymbol{w}) \equiv \emptyset$ *and define* $M_x(w)$ *recursively as follows:*

$$M_{x+1}(\boldsymbol{w}) \equiv \left\{ i \in N \setminus \bigcup_{z \leq x} M_z \mid w_i(1+r_i) = \max_{j \in \{N \setminus \cup_{z \leq x} M_z\}} w_j(1+r_j) \right\}$$

Define $I_j(\boldsymbol{w})$ *as the set of agents with the highest priority levels, up to those included in* $M_j(\boldsymbol{w})$*:*

$$I_j(\boldsymbol{w}) = \{i \in \bigcup_{z \leq j} M_z(\boldsymbol{w})\}$$

Let $|X|$ denote the cardinality of an arbitrary set X. Pick the highest natural number s such that $|I_s(\boldsymbol{w})| \leq k$. Call $m = k - |I_s(\boldsymbol{w})|$, and $r = |M_{s+1}(\boldsymbol{w})|$.

An incentive compatible symmetric [27] *direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{m} \rangle$ maximizes* (7) *if, and only if, $\forall i \in N$, $\forall \boldsymbol{w} \in \mathbb{R}^n$:*

$$p_i(\boldsymbol{w}) = \begin{cases} 1 & \text{if } i \in I_s(\boldsymbol{w}) \\ m/r & \text{if } i \in M_{s+1}(\boldsymbol{w}) \\ 0 & \text{otherwise} \end{cases}$$

$$m_i(\boldsymbol{w}) = p_i(\boldsymbol{w})w_i - \int_0^{w_i} p_i(x_i, \boldsymbol{w_{-i}})dx_i$$

Proof. The allocation rule clearly maximizes (7) because it puts full weight on highest value agents. The allocation rule is feasible because $\sum_{i=1}^n p_i(\boldsymbol{w}) = k$ for each $\boldsymbol{w} \in \boldsymbol{W}$. Participation constraint are satisfied because: $U_i(\boldsymbol{w}) = \int_0^{w_i} P_i(x)dx \geq 0$. The mechanism runs no deficit as $m_i(\boldsymbol{w}) \geq 0$. Given that the payment rule satisfy (6), in order to show that the mechanism is incentive compatible, we need to show that $P_i(w_i)$ is increasing. As the willingness to pay are uncorrelated, this is an immediate consequence of the fact that for each $\boldsymbol{w_{-i}}$, $p_i(\boldsymbol{w})$ is increasing in w_i. □

6 Appendix B

The ex-ante surplus is:

$$E_{\boldsymbol{v},\boldsymbol{r}}\left[\sum_{i=0}^n p_i\left(\frac{v_1}{1+r_1}, \ldots, \frac{v_n}{1+r_n}\right) v_i\right]$$

That is:

$$\sum_{i=0}^n \left\{\mathrm{E}_{v_i,r_i}\left[\mathrm{E}_{\boldsymbol{v_{-i}},\boldsymbol{r_{-i}}}\left[p_i\left(\frac{v_1}{1+r_1}, \ldots, \frac{v_n}{1+r_n}\right)\right] v_i\right]\right\} =$$

[27] I restrict attention to symmetric mechanisms. This is without loss of generality here because considering asymmetric mechanisms will not improve on the symmetric solution obtained.

$$= \sum_{i=0}^{n} \left\{ \mathrm{E}_{v_i,r_i} \left[P_i \left(\frac{v_i}{1+r_i} \right) v_i \right] \right\}$$

Now take the expression inside the brackets for each i. Omitting the subscript:

$$\mathrm{E}_{v,r} \left[P \left(\frac{v}{1+r} \right) v \right] = \int_0^\infty \int_0^\infty P \left(\frac{v}{1+r} \right) v f(v, 1+r) dv \; d(1+r)$$

By performing a change of variable we get:

$$\int_0^\infty \int_0^\infty P \left(\frac{v}{1+r} \right) (1+r)^2 f(v, 1+r) dv \; d\frac{v}{1+r}$$

We can now substitute $w = \frac{v}{1+r}$ and get:

$$\int_0^\infty \int_0^\infty P(w) \left(\frac{v}{w} \right)^2 f \left(v, \frac{v}{w} \right) dv dw$$

Now, define $f_{v,w}(\cdot, \cdot)$ as the joint density between v and w. We have:

$$f_{v|w}(v \mid w) = \frac{f_{v,w}(v, w)}{z(w)} = \frac{f \left(v, \frac{v}{w} \right)}{z(w)} \frac{v}{w^2}$$

It follows that we can write:

$$\int_0^\infty \int_0^\infty P(w) \left(\frac{v}{w} \right)^2 f \left(v, \frac{v}{w} \right) dv_i dw_i =$$

$$= \int_0^\infty \int_0^\infty P(w) v f_{v|w}(v \mid w) z(w) dv dw =$$

$$= \mathrm{E}_w \left[P(w) E[v \mid w] \right]$$

Therefore we get the desired transformation, that is:

$$\mathrm{E}_{\boldsymbol{v},\boldsymbol{r}} \left[\sum_{i=0}^{n} p_i \left(\frac{v_1}{1+r_1}, \ldots, \frac{v_n}{1+r_n} \right) v_i \right] = \mathrm{E}_{\boldsymbol{w}} \left[\sum_{i=0}^{n} p_i(w_1, \ldots, w_n) E[v_i \mid w_i] \right]$$

7 Appendix C

I use the ironing technique as developed in Myerson (1981). First, I will recall the definitions in the text:

Definition 1. *Define, for all $x \in [0,1]$:*

$$H_i(x) = \int_0^x \mathrm{E}[v_i \mid Z^{-1}(y)]dy \quad G_i(x) = \mathrm{conv}\,\langle H_i(x)\rangle \quad g_i(x) = G_i'(x)$$

Here, conv $\langle\cdot\rangle$ *stands for the convex hull of the function.*[28] *Where the derivative of $G_i(v)$ is not defined, we extend it using the right or left derivative. We define the* priority function λ_i *for agent i as:*

$$\lambda_i(w_i) = g_i(Z_i(w_i))$$

Proposition 6. *For each $\boldsymbol{w}$, the set of agents is partitioned in a chain of ordered sets, $M_1(\boldsymbol{w}), M_2(\boldsymbol{w}), \ldots$ according to their priority levels. Formally, set $M_0(\boldsymbol{w}) \equiv \emptyset$ and define $M_x(w)$ recursively as follows:*

$$M_{x+1}(\boldsymbol{w}) \equiv \left\{ i \in N \setminus \bigcup_{z\leq x} M_z \mid \lambda_i(w_i) = \max_{j\in\{N\setminus\cup_{z\leq x}M_z\}} \lambda_j(v_j) \right\}$$

Define $I_j(\boldsymbol{w})$ as the set of agents with the highest priority levels, up to those included in $M_j(\boldsymbol{w})$:

$$I_j(\boldsymbol{w}) = \{i \in \bigcup_{z\leq j} M_z(\boldsymbol{w})\}$$

Let $|X|$ denote the cardinality of an arbitrary set X. Pick the highest natural number s such that $|I_s(\boldsymbol{w})| \leq k$. Call $m = k - |I_s(\boldsymbol{w})|$, and $r = |M_{s+1}(\boldsymbol{w})|$.

An incentive compatible symmetric [29] *direct allocation mechanism $\langle \boldsymbol{p}, \boldsymbol{m}\rangle$ maximizes* (8) *if, and only if, $\forall i \in N,\ \forall \boldsymbol{w} \in \mathbb{R}^n$:*

$$p_i(\boldsymbol{w}) = \begin{cases} 1 & \text{if } i \in I_s(\boldsymbol{w}) \\ m/r & \text{if } i \in M_{s+1}(\boldsymbol{w}) \\ 0 & \text{otherwise} \end{cases}$$

[28] $G_i(x)$ is the highest convex function such that $G_i(x) \leq H_i(x)\ \forall x$.

[29] I restrict attention to symmetric mechanisms. This is without loss of generality here because considering asymmetric mechanisms will not improve on the symmetric solution obtained.

The payment rule can be any set of functions $\boldsymbol{m}$ *such that for all* i *and* $w \in \mathbb{R}$ *satisfies (6). In particular, the mechanism is ex-post incentive compatible if:*

$$m_i(\boldsymbol{w}) = p_i(\boldsymbol{w})w_i - \int_0^{w_i} p_i(x_i, \boldsymbol{w_{-i}})dx_i \tag{10}$$

Proof. The designer's problem has already been reduced to the following:

$$\max_{p_i:\mathbb{R}\to[0,1]\ i=1,\dots,n} \mathrm{E}_{\boldsymbol{w}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{w}) \, \mathrm{E}[v_i \mid w_i] \right]$$

subject to:

$$\sum_{i=1}^{n} p_i(\boldsymbol{w}) \leq k \ \forall \boldsymbol{w} \in \mathbb{R}^n$$

$$P_i(w) \geq P_i(w^*) \ \forall i \in N, \quad \forall w, w^* \in \mathbb{R} : v \geq v^*$$

It can readily be seen that the candidate solution satisfies the first constraint above, and that $\boldsymbol{m}$ satisfies (6). To prove that (5) is also satisfied, note that $\lambda_i(v)$ is the derivative of a convex function and therefore it is monotonically increasing. Then, $\forall \boldsymbol{w_{-i}}\ p(\boldsymbol{w})$ is increasing in w_i, which implies that $P_i(\cdot)$ is also increasing.

Now, let us sum and subtract $\lambda_i(w_i)$ inside the objective function and rewrite it to obtain:

$$\sum_{i=1}^{n} \mathrm{E}_{w_i} \left\{ P_i(w_i)\lambda_i(w_i) + P_i(w_i) \left[\mathrm{E}[v_i \mid w_i] - \lambda_i(w_i) \right] \right\}$$

Consider the second term of this expression for every i:

$$\int_0^{u_i} P_i(w_i) \left[\mathrm{E}[v_i \mid w_i] - g_i(Z_i(w_i)) \right] z(w_i) dv_i$$

Integrating by parts:

$$P_i(w_i) \left[H_i(Z_i(w_i)) - G_i(Z_i(w_i)) \right] \, |_0^{\infty} - \int_0^{\infty} \left[H_i(Z_i(w_i)) - G_i(Z_i(w_i)) \right] dP_i(w_i)) \tag{11}$$

Consider the first term of (9). It is equal to zero: $H_i(0) = G_i(0)$ and $H_i(1) = G_i(1)$, because G_i is the convex hull of the continuous function H_i and thus they coincide at endpoints (the continuity of H_i follows from assuming an atomless Z_i). To summarize, the objective function becomes:

$$\sum_{i=1}^{n} \mathrm{E}_{\boldsymbol{w}} \left[p_i(\boldsymbol{w})\lambda_i(w_i)\right] - \sum_{i=1}^{n} \int_0^\infty \left[H_i(Z_i(w_i)) - G_i(Z_i(w_i))\right] dP_i(w_i)$$

It is easy to see that the candidate solution $\langle \boldsymbol{p}, \boldsymbol{c} \rangle$ maximizes the first sum as it puts all the probability on the players for whom $\lambda_i(w_i)$ is maximal. To conclude the proof, we can show that the second term is equal to zero. It must always be non negative, as $\forall w_i \ H_i \geq G_i$. That it is equal to zero, follows because G_i is the convex hull of H_i and so, whenever $H_i(Z_i(w_i)) > G_i(Z_i(w_i))$, then G_i must be linear. That is, if $G(x) < H(x)$, $G_i''(x) = g_i'(x) = 0$. Therefore, to conclude, $\lambda_i(w_i)$ will be a constant in a neighborhood of w_i, which implies that $P_i(w_i)$ will also be a constant.

□