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Abstract

Investment tournament is a type of decision problem introduced and studied in this paper. These problems involve allocation of investments among several alternatives whose values are subject to exogenous shocks. The payoff to the decision maker is a weighted sum of final values of each alternatives with weights convex in final values. (1) For the case of constant returns to scale it is optimal to allocate all resources to the most promising alternative.(2) In tournaments for a promotion the agents would rationally choose to put forth more effort in the early stage of the tournament in a bid to capture a larger share of mentoring resources.

key words: *matching, maximal problem, optimal search, personnel economics, promotions, rat race, research and development, tournament theory.*

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1 Introduction

This paper introduces and investigates a class of decision problems referred to here as *investment tournaments*. In an investment tournament problem, the decision maker and nature take alternating turns (or act simultaneously). At each decision node the decision maker selects an investment level for each of the N alternatives, and the value of each alternative increases in the amount of the investment. The values of alternatives also change due to random shocks (actions of nature). The payoff in an investment tournament depends only on the realization of the most valuable alternative. (Only investments into the winning alternative are useful. The other investments are wasted from an ex-post perspective).[1] This setting is a generalization of the auction environment introduced in Schwarz and Sonin (2001).

For example of an investment tournament, let us turn to the process of new product development. Suppose a few prototypes are being developed simultaneously. The firm (decision maker) has to allocate investment dollars among prototypes (alternatives) before the performance (value) of prototypes becomes known. The expected performance of a prototype is increasing in the amount of resources committed to it. The new product is based on the best prototype. Consequently, the profit of the firm depends only on the realization of the highest value alternative.[2]

Career choice can also be viewed as an investment tournament. A

[1] Optimal search problems first investigated by Manning and Moragan (1985) and Weitzman (1979) are also maximal problems and thus related to investment tournaments considered herein. Neither problem is a special case of the other. In the optimal search literature there is a cost of obtaining an extra draw. In other words, there is a cost of obtaining information about a particular alternative. Loosely speaking, the optimal search literature studies the optimal strategy for investment into information acquisition. In contrast, in investment tournament problem the information about the value of each alternative is revealed over time at no cost to the decision maker and investment into alternatives plays no role in information acquisition.

[2] It is worth noting that many real life decision problems share features of both investment tournaments and optimal search problems. For instance, both investment tournaments and search models are relevant for understanding different aspects of R&D investment decisions. For a recent example of applications of optimal search results to R&D experimentation, see Dahan and Mendelson (2001) who study optimal prototyping strategy. In particular, they investigate the optimal number of prototypes and the optimal combination between parallel and sequential prototyping. Note that in the context of optimal search literature, the amount of resources invested in each prototype is assumed to be exogenously fixed and all prototypes are assumed equally promising. In contrast, investment tournament model considers a world where a firm can adjust its investment level into each prototype depending on preliminary (noisy) evaluations of the potential of each prototype. Thus, investment tournament model isolates an aspect of the optimal prototyping problem that has not been previously investigated.

person hesitating between accounting and engineering majors (alternatives) might choose to take some courses in each filed. Taking courses can be viewed as investment into alternatives. From an ex-post perspective the courses taken in accounting are a waste of time for someone who eventually chooses the engineering profession.[3] Investment tournaments have received very little previous attention. It suffices to say that this paper is the first to use the term investment tournament for lack of an existing term. Yet investment tournaments are ubiquitous. Applications of investment tournaments range from decisions as diverse as choosing a mate to making investments in firm-specific human capital.

We propose a model of investment tournaments and characterize the optimal investment strategy. The simplest model of investment tournament is a three period decision problem. In the first and third periods the decision maker receives signals about the value of each alternative, in the second period the decision maker chooses the level of investment into each alternative (by dividing a fixed budget among competing alternatives). In the third period the final value of each alternative is realized; it is equal to the sum of the first and the third period signals plus the amount of the investment received by that alternative in the second period. The payoff of the decision maker is the final value of the most valuable alternative. The class of investment tournaments analyzed in this paper is far more general than a simple three period model. In Section 2 we consider investment tournaments with a large number of periods where the decision maker and nature are alternating taking actions or act simultaneously. A multi-period model captures a possibility that the decision maker learns over time about the value of each alternative and uses this information to adjust the share of investment into each alternative. Proposition 1 characterizes the optimal investment strategy in a multi-period setting. Optimal strategy calls for investing all resources into the leading alternative[4] at every decision node (thus, magnifying the lead of the leading alternative). Note that due to information revelation the leading alternative at one decision node need not remain in the lead at the next decision node. Thus, we can interpret Proposition 1 as a statement that the optimal strategy is invariably to

[3] A basic investment tournament model highlights the aspects of the career choice steaming from a necessity to make invests into learning trades (investing into alternatives) before the information regarding which trade fits a person best (has the highest value) is revealed. Obviously, the complexity of educational choices can not be fully captured by a simple investment tournament model. A person may take courses in various fields in order to learn what career suits him best or in order to satisfy intellectual curiosity.

[4] The leading alternative is the one with the highest sum of all past investments into that alternative and the signals about its value.

put all the eggs in the "favorite basket," although the "favorite basket" may change over time. Proposition 3 establishes that this result continues to hold if the payoff to the decision maker is not the final value of the highest alternative but a weighted sum of final values of all alternatives with convex weights. Section 4 farther generalizes the benchmark model of investment tournaments by allowing the returns to investment into an alternative to be decreasing. In this case more than one alternative may receive positive investment. However, we show that the difference in the amount of resources invested into the leading alternative and the alternative with the second highest current value does not converge to zero even as the difference between the current values of these alternatives converges to zero. Thus, even though the optimal strategy in this case is less extreme than "putting all eggs in one basket," it is still optimal to "substantially" favor the leading alternative (even if the lead is infinitesimal). This can explain a seemingly myopic behavior of agents in investment tournaments. For instance, a firm working on several prototypes of a new product rationally invests substantially more in the development of the prototype that is slightly more promising at the moment than other prototypes. A student selecting between two majors may rationally devote a lot of time to a major that seems slightly more promising. A very small amount of new information can reverse the order of the alternatives causing a student to dramatically change the amount of time she invests in each alternative. Thus, a seemingly irrational jumping back and forth from one major to another is consistent with expected utility maximization.

Section 5 applies investment tournaments to personnel economics. It builds upon the literature on incentive aspects of tournaments. The study of incentive tournaments was pioneered by Lazear and Rosen (1981) and quickly developed into a flourishing literature.[5] In an incentive tournament workers (alternatives) are competing for a prize (usually a promotion). The performance (value) of each worker depends on his effort and luck. The best performing worker wins a promotion.[6] Within the incentive tournament framework the firm's profit depends primarily on the average performance of workers in a tournament. The value of the worker who wins the tournament has no special significance (aside from being a term in the summation). We argue that in many contexts,

[5] See, for example, Bhattacharya and Guasch (1988), Bull et al. (1987), Ehrenberg and Bognannon(1990), Eriksson (1999), Ferrall (1996), Green and Stokey (1983), Nalebuff and Stiglitz (1983), and Taylor (1995).

[6] The workers are motivated to exert costly effort because performance, and hence the probability of winning the tournament increases in effort.

firm-specific capital of worker who wins the tournament has special significance. For instance, in an up or out tournament[7] investments into firm-specific human capital of losing contenders are wasted from an expost perspective. Workers accumulate firm-specific human capital over the course of the promotion contest. The value of worker's human capital at the end of the contest depends on a number of factors including investments into worker's firm-specific human capital undertaken during the contest by the decision maker within the firm. In Proposition 7 we show that other things being equal, the greater is the investment component of a tournament the more effort workers exert in the beginning of the tournament. In the context of an up or out contest for a partnership in a law firm, the greater is the role of mentoring in formation of firm-specific human capital the harder the associates will work in the beginning of the tournament in a bid to "get ahead" and rip a larger share of mentoring resources.[8]

The rest of the paper is organized as follows, Section 2 introduces and solves the benchmark model of investment tournament. Section 3 considers investment tournaments where the payoff is a weighted sum of final values of some or all alternatives. Section 4 considers investment tournaments where returns to investments are decreasing. Section 5 combines investment tournament model with the models of tournament theory and applies the results to personnel economics. Section 6 concludes.

2 Benchmark Model of Investment Tournament

This section formulates and solves the benchmark model of an investment tournament. The subsequent sections generalize and build upon the benchmark model. Consider a T period decision problem. There are N alternatives and each period the decision maker chooses how to distribute a budget of size B of scarce investment resources among these alternatives. The amount of resources invested into alternative i at period t is some non-negative quantity b_{ti}, and the sum of all investments in each period must not exceed the amount of available investment resources B.[9, 10] More formally, the action of the decision maker in period

[7] In an up or out tournament workers are either promoted (winners) or laid off (losers). Obviously the firm specific human capital of losers is unimportant.

[8] Mentoring is a scarce resource that can take form of meeting important clients or being assigned to projects that develop firm specific human capital.

[9] For simplicity of notations we assume that the amount of resources available for investment is the same each period. Results and proofs do not change if the investment budget is also a random variable that changes from period to period.

[10] When investments into alternatives are intangible it may be appropriate to assume that there is a fixed investment budget. For example, a partner in a law firm

t is represented by a vector $\mathbf{b}_t = (b_{t1}, b_{t2}, ...b_{tN})$ and the action space of the decision maker is represented by $A = \{\mathbf{b} \in R^N : b_{ti} \geq 0$ and $\sum_{i=1}^{N} b_{ti} = B\}$. Each period nature draws a random shock to each alternative from an atomless distribution $F(\cdot)$. The shock to the value of alternative i in period t is denoted by s_{ti}. The shocks are independent across alternatives and across time. The support of $F(\cdot)$ is bounded from below. The action of nature at time t is denoted by vector $\mathbf{s}_t = (s_{t1}, s_{t2}, ...s_{tN})$, drawn from a distribution $F(\cdot) \times ... \times F(\cdot)$. We assume here that the decision maker and nature act simultaneously, but note that the problem is identical if they take alternating turns with the decision maker taking the first turn. The value of alternative i at the terminal node is given by $V_i = \sum_{t=1}^{T} s_{ti} + \sum_{t=1}^{T} b_{ti}$. In other words, the final value of alternative i is the sum of all shocks and the sum of all investments into alternative i. The winner of a tournament is the alternative with the highest terminal node value. The decision maker's payoff is equal the value of the winning alternative, i.e. $\max\{V_1, ...V_N\}$. Unless otherwise indicated, the decision maker is assumed to be risk neutral. We would like to find an optimal strategy for this decision problem.

First let us introduce some useful notation. The history at time t is denoted by $\mathbf{h}_t = (\mathbf{s}_1, \mathbf{b}_1, \mathbf{s}_2, \mathbf{b}_2, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1})$. Information about the values of alternatives contained in a history $\mathbf{h}_t$ can be summarized by a vector $\mathbf{V}_t = \mathbf{V}_t(\mathbf{h}_t) = (V_{t1}, V_{t2}...V_{tN})$, where $V_{ti} = \sum_{\tau=1}^{t-1} s_{\tau,i} + \sum_{\tau=1}^{t-1} b_{\tau,i}$.[11] We will say that an alternative i is a favorite at time t if for any j we have $V_{ti} \geq V_{tj}$ (note that in principle there may be more than one favorite). The winner of the investment tournament is the favorite at the terminal node.[12]

Proposition 1 *In the benchmark model of an investment tournament the following strategy is optimal: at every decision node the decision maker allocates all investment resources to an alternative that is a favorite at that decision node. The remaining $N-1$ alternatives receive zero amount of investment. (If there is more than one favorite alternativee at a decision node all resources are allocated to one of the favorites).*

The proof is provided in the appendix. Let us discuss the intuition behind the proof. We first note that the expected payoff under a strategy σ can be expressed as a function of current values of alternatives (or extended values defined as $\widetilde{V}_{ti} = V_{ti} + b_{ti}$). Thus, we can express the

may invest into human capital of associates by bringing an associate alone for a meeting with an important client.

[11] The value at the terminal node is $V_{(T+1)i} \equiv V_i$.

[12] If there is more than one favorite at the terminal node some tie braking rule is used to determine the winner.

expected payoff as $\Pi(V_{t1}...V_{tN}, \sigma)$ or $\widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN}, \sigma)$. Essentially, values of alternatives at node t contain all relevant information about the history prior to time t, and extended values of alternatives at node t contain all relevant information regarding both the decision maker's action at time t and the history prior to time t. In Lemma 9 we prove using the envelope theorem that under an optimal strategy, say σ^*, $\frac{d\widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN},\sigma^*)}{\partial \widetilde{V}_{ti}}$ equals the probability that alternative i wins the tournament. The proof of this lemma is based on the fact that under the optimal strategy a small change in the value (or extended value) of a particular alternative has a second order effect on the probability that this alternative wins the tournament. Lemma 9 shows that at the decision node where an alternative receives positive investment, the probability of winning for this alternative must be at least as high as for any other alternative. It is intuitive but not obvious that the favorite alternative is most likely to win. The proof by induction is provided in Lemma 10.

Both the result and the proof of Proposition 1 are more general than it might appear. Results similar to Proposition 1 continue to hold in a variety of settings.

For example, if the decision maker and nature alternate their turns in any way the result of the proposition still holds. The proof remains virtually unchanged. Furthermore, it is worth noting that the assumption of risk neutrality is not absolutely necessary. Indeed, if the decision maker is infinitely risk averse, then investing all resources into the favorite alternative remains the optimal strategy. Consider a decision problem with the same decision tree and the same space of possible histories as in the benchmark tournament model. The only difference is that now the agent is infinitely risk averse. For an infinitely risk averse decision maker the expected payoff from some strategy σ conditional on history $\mathbf{h}_t$ equals to $\min_{\mathbf{s}_t, \mathbf{s}_{(t+1)}...\mathbf{s}_T}[\max\{V_1, V_2...V_N|\sigma, \mathbf{h}_t\}]$, which is the lowest possible value of the winning alternative.

Proposition 2 *An infinitely risk averse decision maker will choose to invest all resources into the favorite alternative.*

Interestingly, infinitely risk averse and risk neutral decision makers select the same strategy for completely different reasons. Unlike Proposition 1, Proposition 2 is essentially obvious. Because the value of random shocks $s_{\tau i}$ is bounded from below, investing everything into the favorite alternative maximizes the payoff in the worst possible case.[13] Proposition 1 and Proposition 2 establish that investing all resources into the

[13] Conditional on history $\mathbf{V}_t$, in the worst possible state of the world all future shocks are $s_{t+j,i} = \underline{s}$ for all $i = 1...N$ and $j = 1...(T-t)$, where $\underline{s}$ is the lower bound for a random shock. In this case the value of alternative i at the terminal node is

favorite alternative maximizes both the expected payoff and the worst possible payoff.

3 Investment Tournament with Convex Weights

The benchmark model assumes that the decision maker cares only about realization of the highest value alternative. In a number of environments this is an uncontroversial assumption. Consider, for example, a firm selecting a blue print for a new product, a student selecting a major, or a young person choosing a mate. However, in a number of situations this assumption holds only approximately or not at all. In fact, the tournament theory makes exactly the opposite assumption. Models of incentive tournaments investigate worker's choice of effort, under the underlying assumption that a promotion is merely a prize for the past performance. This is an appropriate assumption whenever a promotion does not entail a change in the amount of responsibility (as in promotion from an assistant to an associate professor). In many instances, a combination of incentive tournament theory and investment tournament models is most appropriate. For example, the firm-specific human capital (value) of the tournament winner is of particular importance when promotion entails increased responsibility.[14] In this case developing firm-specific human capital of any worker may contribute to profits but not as much as the firm-specific human capital of the tournament winner. This suggest another generalization of the benchmark model. Suppose the action space of the decision maker and the actions of nature are exactly the same as in the benchmark model. The only modification is a mapping from terminal histories into payoffs. Now we assume that the payoff to the decision maker is the weighted sum of realizations of all alternatives. We assume that the payoff of the decision maker is $\sum_{k=1}^{N} \mu(V_k)$ where $\mu(\cdot)$ is increasing and convex.

Proposition 3 *Consider a modification of the benchmark model. The*

given by $V_i = V_{ti} + (T-t)\underline{s} + \sum_{\tau=t}^{T} b_{\tau i}$. The optimal strategy of the decision maker solves the following program.

$$\max_{b_{\tau i}}[\max\{V_{t1} + (T-t)\underline{s} + \sum_{\tau=t}^{T} b_{\tau 1}, ..., V_{tN} + (T-t)\underline{s} + \sum_{\tau=t}^{T} b_{\tau N}\}]$$

subject to $\sum_{i=1}^{N} b_{\tau i} = B$ for $\tau = t...T$.

The $(T-t)\underline{s}$ term is the same for all alternatives so it is optimal for the decision maker to invest all resources into the alternative with the highest V_{ti}.

[14] The human capital of a person advancing to the next level of management may enter the profit equation as a multiplier of the combined value (performance) of subordinates.

structure of the decision problem remains unchanged, except that the payoff of the decision maker is now equal to $\sum_{k=1}^{N} \mu(V_k)$ *where* $\mu'(\cdot) > 0$ *and* $\mu''(\cdot) > 0$*. Then the optimal strategy is to invest all resources into the favorite alternative.*

Note that Proposition 1 is not a particular case of Proposition 3 because in the context of Proposition 3 the weight of an alternative is increasing in the value of the alternative, not in its rank. We can also consider a generalization of the benchmark model where the weight of an alternative is an increasing function of it's rank. In this case we assume that the decision-maker's payoff is given by $\sum_{k=1}^{N} \lambda_k V_{r(k)}$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_N$ and $V_{r(k)}$ denotes the final value of the alternative ranked k at the end of the tournament; $r(1)$ and $r(N)$ are the alternatives with the lowest and the highest final values respectively. The benchmark model is a special case of this "rank-weighted" model with $\lambda_1 = 1$ and $\lambda_j = 0$ for $j = 2...N$. The following proposition establishes that the optimal strategy for rank-weighted model is the same as for the benchmark model.

Proposition 4 *Consider a modification of the benchmark model. The structure of the decision problem remains unchanged, except that the payoff of the decision maker is now given by* $\sum_{k=1}^{N} \lambda_k V_{r(k)}$ *where*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_N \geq 0$$

and $r(k)$ *denotes the alternative with k-th highest final value. Then, investing all resources into the favorite alternative at every decision node is an optimal strategy.*

Note that for $\lambda_1 = \lambda_2 = \lambda_3 ... = \lambda_N$ any investment strategy is optimal. However, if at least one inequality is strict then any optimal strategy calls for investing all resources into a favorite alternative.

4 Investment Tournaments with Decreasing Returns to Investment

In the benchmark model we assumed that returns to investment are constant and that investment budget is fixed exogenously. These assumptions are released in the present section. First, we consider a case where returns to investment into an alternative are decreasing in the amount of investment into that particular alternative. This tends to be the case whenever investments are of financial nature. Then, we consider a somewhat less prevalent case of returns to investment decreasing in the total amount of investment into all alternatives (in this case the cost of investment necessary for increasing the values of alternative i and j by one unit each is the same as the cost of increasing the value of one alternative by two units. But the cost of increasing the value of alternative i and j is more than double the cost of increasing only the value of alternative i by one unit). This possibility is relevant if investment takes form of the decision maker's time or effort.[15]

Let us consider the case of diminishing returns to investment in individual alternatives. In the benchmark model the action each period was represented by an N dimensional vector $\mathbf{b}_t$, where each element of $\mathbf{b}_t$ represented an increase in the value of an alternative, and the total increase in values of all alternatives due to investment in period t was constrained to be smaller or equal to an exogenously fixed constant B_t. Let us relax this constraint. Instead, assume that the decision maker can choose to increase the level of each of N alternatives by any non-negative amount. The action space at node t is $A_t = \{\mathbf{b}_t : b_{ti} \geq 0$ for all $i = 1...N\}$, and the total cost of investment at period t is equal to $\sum_{i=1}^{N} G(b_{ti})$, where $G'(\cdot) > 0$ and $G''(\cdot) > 0$. The decision maker's payoff is equal to $\max\{V_1, V_2...V_N\} - \sum_{t=1}^{T}\sum_{i=1}^{N} G(b_{ti})$. It is straightforward to generalize Lemma 8 to show that there exists an optimal strategy of the form $b_{ti} = b_{ti}(V_{t1}...V_{tN})$. The following proposition characterizes this optimal strategy.

Proposition 5 *Consider a modification of the benchmark tournament model where the action space at each decision node is*

$$A = \{\mathbf{b} \in R^N : b_{ti} \geq 0\}.$$

The decision maker's payoff is

$$\max\{V_1, V_2...V_N\} - \sum_{t=1}^{T}\sum_{i=1}^{N} G(b_{ti}),$$

where $G'(\cdot) > 0$ and $G''(\cdot) > 0$. Then,

[15] For example, consider a student hesitating between accounting and engineering professions. One extra course in accounting brings a student one course closer to becoming an accountant, regardless if it is a second or fifth course in accounting. Thus, as a first order approximation we can assume that marginal increase in the value of the accounting alternative is the same regardless of the number of accounting courses that a student takes in a given semester. However, the cost of effort associated with taking an additional course in accounting depends on the total number of courses that a student takes. The cost of effort in this case depends only on the number of courses that a student takes. The breakdown of courses by field may be unimportant for assessing the cost of effort. In such a case the returns to investment are decreasing in the total amount of investment.

(1) for any two alternatives i and j such that $V_{ti} > V_{tj}$ any optimal strategy σ calls for investments $b_{ti} \geq b_{tj}$ and if $b_{ti}, b_{tj} > 0$ then $b_{ti} > b_{tj}$.
(2) Consider a history $\mathbf{V}_t = (V_{t1}...V_{tk}...V_{tm}...V_{tN})$ where alternative k is the unique favorite and at least one alternative receives positive investment under the optimal strategy. Then under any optimal strategy,

$$\lim_{V_{tm} \to V_{tk}^{-}} (b_{tk}(V_{t1}...V_{tN}) - b_{tm}(V_{t1}...V_{tN})) > 0$$

for any $m \neq k$.
(3) $\frac{db_{ti}(V_{t1}...V_{tN})}{dV_{ti}} \geq 0$ for any alternative i at any decision node t.

This proposition states that under an optimal strategy the amount of resources invested into an alternative is increasing in the rank of the alternative (if two alternatives receive positive amounts of investment the higher ranked alternative receives strictly more investment.) Also, as long as at least one alternative receives positive amount of investment the difference between investment in the favorite alternative and the alternative ranked second does not converge to zero as the value of the alternative ranked second approaches the value of the favorite alternative. In other words, if for some history $\mathbf{V}_t = (V_{t1}...V_{tk}...V'_{tm}...V_{tN})$ alternative k is such that $V_{tk} \geq V_{ti}$ for all $i \neq m$ then function $b_{tk}(V_{t1}...V_{tN})$ representing optimal strategy at decision node t is discontinuous in $m-$th argument at the point where $V'_{tm} = V_{tk}$. Finally, other things being equal, the amount of investment into an alternative at some decision node t is increasing in the value of that alternative at decision node t.

Above we considered a case of decreasing returns to scale in investment in individual alternatives. Now we proceed to a case where returns to investment are decreasing in total investment in all alternatives.[16]. We assume that the decision maker can choose the amount by which he increases the value of each alternative. The cost of investment at period t is given by increasing and convex function $\sum_{t=1}^{T} C(B_t)$, where $B_t = \sum_{i=1}^{N} b_{ti}$. The following proposition characterizes the optimal strategy for this case.

Proposition 6 *Consider a modification of the benchmark tournament model where action space at each decision node is*

$$A = \{\mathbf{b} \in R^N : b_{ti} \geq 0\}$$

[16]Both types of diminishing returns to scale can be combined in a single model, with the decision maker's payoff $\max\{V_1, V_2...V_N\} - \sum_{t=1}^{T} C(\sum_{i=1}^{N} G(b_{ti}))$, where $C'(\cdot) > 0, C''(\cdot) > 0, G'(\cdot) > 0, G''(\cdot) > 0$. The results and the proof of Proposition 5 continue to holds for this setup. For the sake of expositional clarity Proposition 5 is formulated and proved in this paper rather than this slightly more general, but notation-heavy result.

and the payoff of the decision maker is

$$\max\{V_1, V_2 ... V_N\} - \sum_{t=1}^{T} C(B_t),$$

where $C'(\cdot) > 0$ *and* $C''(\cdot) > 0$ *and* $B_t = \sum_{i=1}^{N} b_{ti}$*. Then,*
(1) the optimal strategy requires that at each decision node only one favorite alternative receives positive investment, and
(2) the amount of investment at decision node t *is increasing in the value of the favorite alternative at decision node* t*.*

5 Promotion Tournaments: Mix of Incentive and Investment Tournaments

In the earlier sections we considered a situation where "competing alternatives" are not players in the tournament game; we assumed that an alternative cannot take actions intended to influence its value. This assumption is appropriate if "competing alternatives" are non-conscious or otherwise indifferent to the outcome of the investment tournament (for example, competing alternatives can be occupations). In this section we consider tournaments where "alternatives" are able to influence the outcome of the tournament by choosing their actions (as in a tournament for a promotion among associates in a law firm.) This adds an additional layer of complexity to the benchmark tournament model of Section 2.

In order to emphasize that alternatives are now players in the game, we will use the word "contender" instead of the word "alternative". For example, in an up or out tournament "contenders" may be associates in a law firm. At every decision node associates select the level of effort that they invest into acquiring firm-specific human capital. The decision maker is a senior partner of a firm. His investment decision is how to divide scarce mentoring resources among contenders. Essentially, the partner selects levels of investment into contenders' firm-specific human capital. For example, the partner selects whom he brings along for a meeting with an important client. For simplicity, we assume that in an up or out tournament the investments into firm-specific human capital of associates who do not become partners are wasted from an ex-post perspective.[17]

The model offered in this section combines benchmark investment tournaments introduced above with elements of an incentive tournaments model (see for example Lazear and Rosen (1981)). We refer to

[17] For most results it is enough to assume that an investment in the winning alternative is more useful than that into a losing alternative.

tournaments where the only players are contenders and nature as incentive tournaments. In contrast, in the benchmark investment tournament model, the only players are the decision maker and nature.

Let us consider a T period model of a promotion tournament that combines features of investment and incentive tournaments. Nature takes an action in period zero. In subsequent periods nature, the decision maker and contenders move simultaneously. Each period contenders select the level of investment in their firm-specific human capital (investments are in the form of effort). The action space of each contender is $A^c = \{e_{ti} : e_{ti} > 0\}$, where e_{ti} is the amount of effort contender i invests in building firm-specific human capital in period t. The decision maker selects investment level into each contender $A^d = \{\mathbf{b}_t \in R^N : b_{ti} \geq 0$ and $\sum_{i=1}^{N} b_{ti} = B\}$. A random shock to the value of contender i at time period t is denoted by s_{ti}. We continue to assume that each period nature independently draws a random shock to value of each contender from an atomless distribution $F(\cdot)$. The support of $F(\cdot)$ is bounded from below as before. The value (the amount of firm-specific human capital) of a contender at the terminal node is a sum of his investments, investments by the decision maker and random shocks $V_i = \sum_{t=0}^{T} s_{ti} + \sum_{t=1}^{T} b_{ti} + \sum_{t=1}^{T} e_{ti}$. The contender with the highest value at the terminal node is the winner of the tournament. The history of the game at period τ is summarized by $\left(V_{\tau i} = \sum_{t=0}^{\tau-1} s_{t,i} + \sum_{t=1}^{\tau-1} b_{t,i} + \sum_{t=1}^{\tau-1} e_{ti}\right)_{i=1}^{N}$. We refer to $V_{\tau i}$ as the intermediate value of contender i (at decision node τ). We will say that contender i is a favorite at decision node τ if $V_{\tau i} \geq V_{\tau j}$ for all j. The utility payoff of the decision maker is the value of the winner of the tournament $\max\{V_1, ...V_N\}$. The utility payoff of a contender i who loses the tournament is equal to the cost of effort that he invested $-\sum_{t=1}^{T} \theta(e_{ti})$ where $\theta'(\cdot) > 0$, $\theta''(\cdot) > 0$. If contender i wins the tournament his payoff is $R - \sum_{t=1}^{T} \theta(e_{ti})$, where R can be interpreted as the amount of rents associated with winning a promotion. The game has the following information structure. At any decision node the decision maker knows the intermediate value of each contender. The contenders do not observe random shocks or investments by the decision maker (at each decision node, each contender's information set contains only his or her effort level from earlier periods). The following result characterizes equilibria of the promotion tournament game.[18]

[18] We model decreasing returns to effort by assuming that per period cost of effort is convex. Consider a modified version of promotion tournament game, where all elements of the game are unchanged except the payoff of the winning contender which is now given by $R - Q(e_{1,i} + e_{2,i})$ and $Q'(\cdot) > 0$, $Q''(\cdot) > 0$ (in this case the cost of effort depends only on $e_{1,i} + e_{2,i}$). In this game the decision maker invests all resources into the favorite contender, and the contenders invest all effort in the first

Proposition 7 *In any pure strategy equilibrium of a promotion tournament the following is true:*
(1) at every decision node the decision maker chooses to invest all resources into one of the favorite contenders,
(2) the effort of each contender is decreasing each period,
(3) contender's effort is weakly increasing in the decision maker's investment budget B.

Part (1) of Proposition 7 is a generalization of Proposition 1. The investment strategy of the decision maker is essentially unaffected by the incentive tournament component of the tournament because from decision maker's perspective there is no difference between actions of nature and of contenders, since the contenders cannot condition their behavior on that of the decision maker. Contenders do have an incentive to influence the actions of the decision maker, however. Contenders invest more effort into improving their position in the tournament in the early stages of the contest because early effort can attract "free" investments from the decision maker (if an employee shows unusual promise in the early stages of his career he may find himself on the fast track being groomed for a senior managerial position). The larger is the amount of the investment resources in the hands of the decision maker the more effort the contenders put forth at each decision node prior to node T. We can interpret the size of investment budget B available to the decision maker as a measure of investment tournament component of a promotion tournament (if $B = 0$ the promotion tournament considered above reduces to a pure incentive tournament). Thus, part (3) of Proposition 7 can be interpreted as following: the larger is the investment tournament component of the promotion tournament the more effort agents put forth in the early stages of the tournament.

Consequently, we should expect that first year graduate students work harder than second year students or that first year associates in a law firm work longer hours than second year associates. For the same reason we predict that employees are less likely to get married in the beginning of the promotion tournament.

6 Concluding Remarks

In a wide class of situations, a contender with a small lead tends to enjoy a substantially better chances of winning the tournament than

period.

The intuition is straightforward. We assumed that the cost of effort depends only on $(e_{1,i} + e_{2,i})$ and contenders receive no additional information after the end of the first period. Consequently, they shift all effort to the first period in a bid to receive more mentoring (investment) from the decision maker.

other contenders. Consequently, as long as contenders are motivated to win the tournament, all contenders put forth more effort in the early stages of the tournament. Casual conversations about career development include expressions like "fast track" that young managers aspire to and "rat race" that young professionals presumably had to endure (perhaps because they are trying to get on the fast track).[19] Evidence that this is a reality at least in some occupations can be found in Landers, Rebitzer and Taylor (1996). They offer an adverse selection based explanation of rat race and provide evidence that associates in law firms log in an inefficiently high number of hours.[20] Holmstrom (1999) model predicts that effort is relatively high early in a worker's career because of signaling. The present paper generates the same prediction without relying on asymmetric information or bounded rationality. The model developed herein implies that effort declines with time most rapidly in careers where mentoring plays an important role in accumulation of human capital. This prediction is distinct from aforementioned literature.

Applications of investment tournaments are not limited to career development and product design examples. The investment tournament model is a metaphor for many decisions involving choice. The domain of applicability of the model stretches beyond the purely economic realm. For instance, investment tournaments may help to explain why people tend to date one person at a time. Dating (in an innocent interpretation of the word) amounts to spending time with a potential mate. This can be viewed as an investment that increases the value of the mate. As we have learn from Proposition 7, even if it is highly uncertain which mate will be chosen it is still optimal to invest disproportionately into the most promising alternative. Proposition 7 predicts that the effort invested into relationship by suitors is the largest in the early stages of dating. It is worth noting that modeling dating as an investment tournament is by any standard a very crude approximation. For investment tournament model to be applicable a member of one gender must act as a decision maker while several members of the other gender act as contenders.[21] The limitations of investment tournament model when applied to dating suggest a few promising directions for future work. It may be challenging but useful to extend investment tourna-

[19] Meyer (1991) considers a model of boundedly rational decision-makers who can bias noisy rank-order contests sequentially, thereby changing the information they convey. He shows that the optimal final-period bias favors the leader. This may be interpreted as a bounded rationality based explanation of fast track. The present paper provides an explanation of why an early leader may be favored even if the decision maker is fully rational.

[20] The first adverse selection model of rat race can be found in Akerlof (1976).

[21] This assumption may fit medieval times better than it fits the modern era.

ment model to two-sided matching, where both sides of the market can make investments into individual contenders on the other side of the market. Also, investment tournament model neglects the search aspect of dating. In this and in many other contexts optimal search models and investment tournament models are complementary– each highlights one important aspects of choice while neglecting all others. The models of search focus on information acquisition while assuming away a possibility of investment into improving the quality of a match. In contrast, the investment tournament model assumes away a possibility of strategic information acquisition while explicitly modeling investments into relationships. Combining search and matching models with investment tournament models is likely to be another fruitful direction for future work.

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A Appendix

A.1 Proof of Proposition 1.

We start with a list of notations and definitions used in the proof. Then we formulate and prove three lemmas that combined establish the result of the proposition.

Notations and definitions:

Memoryless strategy: a strategy is memoryless if at each decision node the action depends only on the summary of the history contained in vector $\mathbf{V}_t$. When considering a memoryless strategy we can abuse notation and refer to $\mathbf{V}_t$ as history.

Extended history: $\widetilde{\mathbf{h}}_t = (\mathbf{h}_t, \mathbf{b}_t) = (\mathbf{s}_1, \mathbf{b}_1, \mathbf{s}_2, \mathbf{b}_2, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1}, \mathbf{b}_t)$

Extended value: $\widetilde{V}_{ti} = V_{ti} + b_{ti}$; $\widetilde{\mathbf{V}}_t = \widetilde{\mathbf{V}}_t(\widetilde{\mathbf{h}}_t) = (\widetilde{V}_{t1}, \widetilde{V}_{t2}...\widetilde{V}_{tN})$

Extended favorite: we will say that an alternative i is an extended favorite at time t if for any j we have $\widetilde{V}_{ti} \geq \widetilde{V}_{tj}$.

Allocated investment: $b_{ti}(\mathbf{h}_t, \sigma)$ (or $b_{ti}(\mathbf{V}_t, \sigma)$) is the investment allocated to alternative i by a pure strategy σ conditional on history $\mathbf{h}_t$ or $\mathbf{V}_t$.

Expected payoff: $\Pi(\mathbf{h}_t, \sigma)$ (or $\Pi(\mathbf{V}_t, \sigma)$) is the expected payoff from strategy σ conditional on $\mathbf{h}_t$, (or $\mathbf{V}_t$); $\widetilde{\Pi}(\widetilde{\mathbf{h}}_t, \sigma)$ (or $\widetilde{\Pi}(\widetilde{\mathbf{V}}_t, \sigma)$) is the expected payoff from strategy σ conditional on $\widetilde{\mathbf{h}}_t$, (or $\widetilde{\mathbf{V}}_t$).

Hybrid history: $f(\mathbf{h}_t, \mathbf{h}'_\tau) \equiv (\mathbf{s}_1, \mathbf{b}_1, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1}, \mathbf{s}'_t, \mathbf{b}'_t, ...\mathbf{s}'_{\tau-1}, \mathbf{b}'_{\tau-1})$ where $\mathbf{h}_t = (\mathbf{s}_1, \mathbf{b}_1, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1})$ and $\mathbf{h}'_\tau = (\mathbf{s}'_1, \mathbf{b}'_1, ...\mathbf{s}'_{\tau-1}, \mathbf{b}'_{\tau-1})$ and $\tau \geq t$.

Continuation strategy: $\sigma(\mathbf{h}_t)$; according to $\sigma(\mathbf{h}_t)$ decision maker acts at decision node τ conditional on history $\mathbf{h}'_\tau$ as if he plays strategy σ at a decision node τ conditional on history $f(\mathbf{h}_t, \mathbf{h}'_\tau)$. For a memoryless strategy we can write $\sigma(\mathbf{V}_t)$ instead of $\sigma(\mathbf{h}_t)$.

Equivalence: $\mathbf{h}_t$ is equivalent to $\mathbf{h}'_t$ if and only if $\mathbf{V}_t(\mathbf{h}_t) = \mathbf{V}_t(\mathbf{h}'_t)$

Probability of winning: $P_{ti}(\mathbf{V}_t, \sigma)$ represents the probability that alternative i wins the tournament conditional on $\mathbf{V}_t$ and memoryless strategy σ; $\mathbf{P}_t = (P_{t1}...P_{tN})$; $\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)$ represents the probability that alternative i wins the tournament conditional on $\widetilde{\mathbf{V}}_t$ and memoryless strategy σ; $\widetilde{\mathbf{P}}_t = (\widetilde{P}_{t1}...\widetilde{P}_{tN})$.

Modified value: $\mathbf{V}_{t|i}(\delta) = (V_{t1}, ..., V'_{ti} = V_{ti} + \delta, ...V_{tN})$ and $\widetilde{\mathbf{V}}_{t|i}(\delta) = (\widetilde{V}_{t1}, ..., \widetilde{V}'_{ti} = \widetilde{V}_{ti} + \delta, ...\widetilde{V}_{tN})$

Lemma 8 *There exists a memoryless optimal strategy σ. (This is equivalent to a claim that an optimal strategy can be represented by a function of the form $b_{ti}(\mathbf{V}_t, \sigma) = b_{ti}$.)*

Proof. The above lemma is almost obvious. The proof is provided for the sake of completeness. First, we note that for any optimal strategy σ and for any two equivalent histories $\mathbf{h}_t$ and $\mathbf{h}'_t$ we have $\Pi(\mathbf{h}'_t, \sigma) = \Pi(\mathbf{h}_t, \sigma)$. This follows from the fact that the expected payoff under strategy σ conditional on history $\mathbf{h}_t$ can be achieved after history $\mathbf{h}'_t$ by following continuation strategy $\sigma(\mathbf{h}_t)$. More formally, by assumption $\mathbf{V}_t(\mathbf{h}_t) = \mathbf{V}_t(\mathbf{h}'_t)$ and by construction after decision node t both random shocks and investments are identical under strategy $\sigma(\mathbf{h}_t)$ following history $\mathbf{h}'_t$ and under strategy σ after history $\mathbf{h}_t$. Thus,

$$\Pi(\mathbf{h}'_t, \sigma) = \Pi(\mathbf{h}_t, \sigma(\mathbf{h}'_t)) \leq \Pi(\mathbf{h}_t, \sigma)$$

and

$$\Pi(\mathbf{h}_t, \sigma) = \Pi(\mathbf{h}'_t, \sigma(\mathbf{h}_t)) \leq \Pi(\mathbf{h}'_t, \sigma)$$

Hence, $\Pi(\mathbf{h}'_t, \sigma) = \Pi(\mathbf{h}_t, \sigma)$.
Consequently, based on any optimal non-memoryless strategy we can construct a memoryless strategy with the same expected payoff by dividing histories into equivalency classes and using the same course of action for each history in an equivalence class. ■

Lemma 9 *If $P_{ti}(\mathbf{V}_t, \sigma)$ is continuous at $\mathbf{V}_t$ then $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}} = P_{ti}(\mathbf{V}_t, \sigma)$ and $\frac{d\widetilde{\Pi}(\widetilde{\mathbf{V}}_t,\sigma)}{d\widetilde{V}_{ti}} = \widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)$ where σ is an optimal strategy.*

Proof. Let us first show that for any strategy σ the following is true

$$\frac{d\Pi(\mathbf{V}_t, \sigma(\mathbf{V}_t))}{dV_{ti}} = P_{ti}(\mathbf{V}_t, \sigma) \tag{1}$$

Consider $\Pi(\mathbf{V}_{t|i}(\delta), \sigma(\mathbf{V}_t)) - \Pi(\mathbf{V}_t, \sigma(\mathbf{V}_t))$. Note that the difference between these payoffs is zero if alternative other than alternative i wins the tournament for both histories $\mathbf{V}_{t|i}(\delta)$ and $\mathbf{V}_t$. There are two remaining contingencies:
(1) With probability $P_{ti}(\mathbf{V}_t, \sigma(P_{ti}(\mathbf{V}_t, \sigma))$ alternative i wins the tournament under both histories $\mathbf{V}_{t|i}(\delta)$ and $\mathbf{V}_t$. In this case the difference in expected payoffs is exactly δ.
(2) With probability proportional to δ alternative i wins the tournament conditional on history $\mathbf{V}_{t|i}(\delta)$ but not on history $\mathbf{V}_t$. In this case the difference between expected payoffs is less than δ.

Thus, $\Pi(\mathbf{V}_{t|i}(\delta), \sigma(\mathbf{V}_t)) - \Pi(\mathbf{V}_t, \sigma(\mathbf{V}_t)) = P_{ti}(\mathbf{V}_t, \sigma)\delta + o(\delta)$[22]. Consequently, Equation 1 holds.
It remains to show that if σ is an optimal strategy then

$$\frac{d\Pi(\mathbf{V}_t, \sigma(\mathbf{V}_t))}{dV_{ti}} = \frac{d\Pi(\mathbf{V}_t, \sigma)}{dV_{ti}}.$$

First note that we can write:

$$\frac{d\Pi(\mathbf{V}_t, \sigma)}{dV_{ti}} = \frac{d\Pi(\mathbf{V}_{t|i}(\delta), \sigma(\mathbf{V}_{t|i}(\delta)))}{d\delta}|_{\delta=0}$$

Now we can apply the envelope theorem.

$$\frac{d\Pi(\mathbf{V}_t, \sigma)}{dV_{ti}} = \frac{\partial\Pi(\mathbf{V}_{t|i}(\delta), \sigma(\mathbf{V}_t))}{\partial\delta}|_{\delta=0} + \frac{\partial\Pi(\mathbf{V}_t, \sigma(\mathbf{V}_{t|i}(\delta)))}{\partial\delta}|_{\delta=0}.$$

Since σ is an optimal strategy the second term is zero. The first term is $P_{ti}(\mathbf{V}_t, \sigma)$ according to Equation 1. Thus, $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}} = P_{ti}(\mathbf{V}_t, \sigma)$.

The proof that $\frac{d\widetilde{\Pi}(\widetilde{\mathbf{V}}_t,\sigma(\widetilde{\mathbf{V}}_t))}{d\widetilde{V}_{ti}} = \widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)$ is analogous. ■

Lemma 10 *Assumption A: suppose at decision node K the value of $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}}$ is the highest for the favorite alternative and at all subsequent nodes the optimal strategy allocates all investment to one favorite alternative. If assumption A holds then the optimal strategy must allocate all resources to the favorite at the decision node $K-1$.*

Proof. The proof consists of three steps. First, we show that only an extended favorite alternatives may receive positive amount of investment. Second, we show that only one extended favorite receives positive investment. Finally, we show that the extended favorite alternative that receives all investment is also a favorite.
Throughout the proof we assume that assumption A holds. It is easy to see that if assumption A holds, then an extended favorite alternative at decision node $K-1$ is most likely to win the tournament under the optimal strategy. Combining this fact and Lemma 9 we conclude that under the optimal strategy all resources at decision node $K-1$ are received by extended favorite alternative(s). (There may be more than one).
Now let us show that all investment at node $K-1$ is allocated to one extended favorite (we already know that investment is allocated to extended favorites; now we would like to show that investment is not divided among a few extended favorites). We use proof by contradiction.

[22]Note that this argument relies on continuity of $P_{ti}(\mathbf{V}_t, \sigma)$ at V_{ti}. As we will see later $P_{ti}(\mathbf{V}_t, \sigma)$ is not continuous at V_{ti} if and only if there is a tie such that $V_{ti} = V_{tj}$ for some j.

Suppose alternatives i and j receive positive investments b_{Ki} and b_{Kj}. In this case "redistributing δ" from i to j does not increase expected profit, i.e.

$$\widetilde{\Pi}((\widetilde{V}_{(K-1)1}, ..., \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma) \geq$$

$$\widetilde{\Pi}((\widetilde{V}_{(K-1)1}, ... \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)}))$$

and by construction

$$\widetilde{\Pi}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)})) \leq$$

$$\widetilde{\Pi}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma)$$

Consequently,

$$\arg\max_{\delta}[\widetilde{\Pi}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)}))] = 0 \tag{2}$$

We would like to check if Equation 2 holds. Towards this end we need to write down first and second order conditions. Applying Lemma 9 we obtain the equation for the first order condition

$$\frac{\partial \widetilde{\Pi}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)}))}{\partial \delta} =$$

$$\begin{aligned} &\widetilde{P}_{(K-1)j}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)})) \\ &- \widetilde{P}_{(K-1)i}((\widetilde{V}_{(K-1)1}, \widetilde{V}_{(K-1)i} - \delta ... \widetilde{V}_{(K-1)j} + \delta ... \widetilde{V}_{(K-1)N}), \sigma(\widetilde{\mathbf{V}}_{(K-1)})) = 0 \end{aligned} \tag{3}$$

Consequently, the first order condition is satisfied as long as

$$\widetilde{P}_{(K-1)i}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)})) = \widetilde{P}_{(K-1)j}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)})).$$

Let us show that the second order condition is violated. Differentiating Equation 3 we obtain the expression for the second derivative of Equation 2.

$$\begin{aligned} &-\frac{d\widetilde{P}_{(K-1)j}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)}))}{V_{(K-1)i}} + \frac{d\widetilde{P}_{(K-1)j}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)}))}{V_{(K-1)j}} + \\ &+\frac{d\widetilde{P}_{(K-1)i}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)}))}{V_{(K-1)i}} - \frac{d\widetilde{P}_{(K-1)i}(\widetilde{\mathbf{V}}_{(K-1)}, \sigma(\widetilde{\mathbf{V}}_{(K-1)}))}{V_{(K-1)j}} \end{aligned} \tag{4}$$

It follows immediately from assumption A that $\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma(\widetilde{\mathbf{V}}_t))$ is strictly increasing in $\widetilde{V}_{ti}$ and decreasing in $\widetilde{V}_{tj}$. Consequently, Expression 4 is strictly positive. Thus the second order condition is violated and Equation 2 can not be true. Thus, all investment at node $K-1$ is allocated to one extended favorite.

To complete the proof we need to show that an extended favorite is also a favorite at decision node $K-1$. Suppose, to the contrary, that at the decision node $K-1$ alternative i is a favorite and alternative j is an extended favorite but not a current favorite. Thus, $V_{K-1,i} > V_{K-1,j}$ and $b_{K-1,j} = B, b_{K-1,i} = 0$. Consequently, $\widetilde{V}_{K-1,j} = V_{K-1,j} + B > \widetilde{V}_{K-1,i} = V_{K-1,i}$. Note that the same expected payoff may be achieved with investments $b'_{K-1,j} = V_{K-1,i} - V_{K-1,j}$ and $b_{K-1,j} = B - V_{K-1,i} + V_{K-1,j}$.[23] Thus, if an alternative other than the favorite receives investment under optimal strategy, then there exists another optimal strategy where more than one alternative receives positive investment. But we've already showed that cannot be true. Hence, we reach a contradiction. ∎

Proof. The inductive proof of Proposition 1 follows from Lemma 10 and the fact that assumption A automatically holds at the terminal node. ∎

A.2 Proof of Proposition 3

Proof. Introducing a few modifications to the proof of Proposition 1 allows us to establish the result of Proposition 3. Here we only discuss the steps of the proof of Proposition 1 that require modification. First we note that there exists a memoryless optimal strategy: the proof is identical to that of Lemma 8. In order to establish an analogue of Lemma 9 for the present setting we need to introduce some notation. Let $\rho_i(V_i, \mathbf{V}_t, \sigma)$ represent the probability density function of the final value of alternative i conditional on history $\mathbf{V}_t$ and strategy σ. Repeating the proof of Lemma 9 with obvious modification yields that if $\rho_i(V_i, \mathbf{V}_t, \sigma)$ is continuous at $\mathbf{V}_t$ then $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}} = \int_0^{+\infty} \rho_i(v, \mathbf{V}_t, \sigma)\mu'(v)dv$ where σ is an optimal strategy.[24] The intuition is simple: $\mu'(v)$ represents the change in the payoff if the final value of alternative i is increased provided that $V_i = v$

[23] Let us illustrate this by means of a numerical example. Suppose that alternative j with $V_{K-1,j} = 4$ receives all of the resources, and the favorite is alternative i with $V_{K-1,i} = 6$. Suppose $B = 4$. Then, entering next period the alternative j is the extended favorite with $\widetilde{V}_{K-1,j} = V_{K-1,j} + 4 = 8$ and $V_{K-1,i} = V_{K-1,i} + 0 = 6$. However, exactly the same outcome (up to relabelling) can be achieved by dividing investment between alternatives i and j. Indeed if $b_{K-1,i} = 2$ and $b_{K-1,i} = 2$ then $V_{K-1,i} = V_{K-2,i} + 2 = 6$ and $V_{K-1,j} = V_{K-1,j} + 2 = 8$.

[24] Of course, the analogue of the second part of Lemma 9 holds as well.

and $\rho_i(v, \mathbf{V}_t, \sigma)$ represents the likelihood that $V_i = v$.[25]. The rest of the proof very closely follows the proof of Proposition 1. The claim of Lemma 10 remains true in the context of the present proposition. The proof of Lemma 10 remains valid if the words 'alternative most likely to win the tournament' are replaced with the words 'alternative with the the highest value of $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}}$'. To complete the proof we need to show that assumption A holds in the context of the present theorem at the last node. To see that it is true we note that if for any function $\mu(x)$ such that $\mu'(x) > 0$ and $\mu''(x) > 0$ and any two positive random variables X and Y (with p.d.f's $\rho_X(\cdot)$ and $\rho_Y(\cdot)$, respectively) such that X stochastically dominates Y we have $\int_0^{+\infty} \rho_X(x)\mu'(x)dx \geq \int_0^{+\infty} \rho_Y(x)\mu'(x)dx$. Combining this with the fact that at the last node the final value of the extended favorite stochastically dominates the final value of any other alternative establishes that assumption A holds at the final node. ■

A.3 Proof of Proposition 4.

Proof. The proof of Proposition 1 readily generalizes for the present setting. Lemma 9 is the first place where the proof of Proposition 1 needs modification. The analogue of Lemma 9 states that if $P_{ti}^k(\mathbf{V}_t, \sigma)$ is continuous at $\mathbf{V}_t$ then $\frac{d\Pi(\mathbf{V}_t,\sigma)}{dV_{ti}} = \sum_{k=1}^N P_{ti}^k(\mathbf{V}_t, \sigma)\lambda_k$ where $P_{ti}^k(\mathbf{V}_t, \sigma)$ denotes the probability that alternative i will have final value ranked k conditional on a history $\mathbf{V}_t$ and optimal strategy σ. The remaining modifications are essentially identical to these contained in the proof of Proposition 3. ■

A.4 Proof of Proposition 5

Proof.
Part (1): Lemma 8 and Lemma 9 and their proofs hold for the modified model essentially without changes. Let us formulate an analogue of Lemma 10. First we need to modify assumption A of Lemma 10.
Assumption B: Suppose that for any optimal strategy σ and for some node K the following is true: for any $t = K+1, ...T+1$ and any i, j if $V_{ti} > V_{tj}$ then $P_{ti}(\mathbf{V}_t, \sigma) > P_{tj}(\mathbf{V}_t, \sigma)$ and $b_{ti}(\mathbf{V}_t, \sigma) \geq b_{tj}(\mathbf{V}_t, \sigma)$ and for $b_{tj}(\mathbf{V}_t, \sigma) \neq 0$ we have $b_{ti}(\mathbf{V}_t, \sigma) > b_{tj}(\mathbf{V}_t, \sigma)$.
The analogue of Lemma 10 is: provided assumption B is satisfied then for any i and j if $V_{Ki} > V_{Kj}$

$$P_{Ki}(\mathbf{V}_K, \sigma) > P_{Kj}(\mathbf{V}_K, \sigma) \quad (5)$$

[25] This intuition is not a complete proof, because it remains to show that a small change in the value of V_{ti} at time t has only a second order effect on the subsequent strategy. We refer the reader to the proof of Lemma 9 for this detail.

and

$$b_{Ki}(\mathbf{V}_K,\sigma) \geq b_{Kj}(\mathbf{V}_K,\sigma) \quad (6)$$
$$\text{if } b_{Kj}(\mathbf{V}_K,\sigma) \neq 0 \text{ then } b_{Ki}(\mathbf{V}_K,\sigma) > b_{Kj}(\mathbf{V}_K,\sigma)$$

First let us show that if $\widetilde{V}_{Ki} > \widetilde{V}_{Kj}$ then

$$\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma) > \widetilde{P}_{Kj}(\widetilde{\mathbf{V}}_K,\sigma) \quad (7)$$

and

$$b_{Ki}(\mathbf{V}_K,\sigma) \geq b_{Kj}(\mathbf{V}_K,\sigma) \quad (8)$$
$$b_{tj}(\mathbf{V}_K,\sigma) \neq 0 \text{ then } b_{Ki}(\mathbf{V}_K,\sigma) > b_{Kj}(\mathbf{V}_K,\sigma)$$

Suppose $\widetilde{V}_{Ki} > \widetilde{V}_{Kj}$. Note that for any two optimal strategies σ and σ', if V_{Ki} (or $\widetilde{V}_{Ki}$) is not equal to value (extended value) of any other alternative then $\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma) = \widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma')$ and $b_{Ki}(\mathbf{V}_K,\sigma) = b_{Ki}(\mathbf{V}_K,\sigma')$. Also, note that the optimal strategy must be invariant with regard to relabeling alternatives. Hence, if $\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma)$ is increasing in V_{Ki} then Equation 7 holds. It is straightforward that $\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma(\widetilde{\mathbf{V}}))$ is increasing in V_{Ki} because as V_{Ki} increases the terminal value of alternative i increases while the terminal values of all other alternatives are the same for all states of the world. But, by the envelope theorem $\frac{d\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma)}{d\widetilde{V}_{ti}} = \frac{d\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma(\widetilde{\mathbf{V}}))}{d\widetilde{V}_{ti}}$. Hence, $\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma)$ is increasing in V_{Ki}. We have thus established that Equation 7 must hold. Combining Equation 7 and Lemma 9 yields that $\frac{d\widetilde{\Pi}(\widetilde{\mathbf{V}}_t,\sigma)}{d\widetilde{V}_{Ki}} > \frac{d\widetilde{\Pi}(\widetilde{\mathbf{V}}_t,\sigma)}{d\widetilde{V}_{Kj}}$ when $\widetilde{V}_{Ki} > \widetilde{V}_{Kj}$. Let us show that this implies that Inequality 8 must hold as well. Note that $G'(b_{Ki})$ can be interpreted as the cost of increasing $\widetilde{V}_{Ki}$ by one unit and $\frac{d\widetilde{\Pi}(\widetilde{\mathbf{V}}_t,\sigma)}{d\widetilde{V}_{Ki}} = \widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma)$ can be interpreted as the corresponding increase in the expected final value of the favorite at the terminal node. For an interior solution the following FOC must hold

$$\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma) - G'(b_{Ki}) = \widetilde{P}_{Kj}(\widetilde{\mathbf{V}}_K,\sigma) - G'(b_{Kj}) \quad (9)$$

for any $b_{Ki} > 0, b_{Kj} > 0$. Combining this FOC with Equation 7 and the fact that $G(b)$ is convex shows that Inequality 8 must hold. In order to show that $\widetilde{P}_{Ki}(\widetilde{\mathbf{V}}_K,\sigma) > \widetilde{P}_{Kj}(\widetilde{\mathbf{V}}_K,\sigma) \Longrightarrow P_{Ki}(\mathbf{V}_K,\sigma) > P_{Kj}(\mathbf{V}_K,\sigma)$ it only remains to show that if $V_{Ki} > V_{Kj}$ then $\widetilde{V}_{Ki} > \widetilde{V}_{Kj}$. The proof of this fact is identical to that contained in Lemma 10 and thus we omit it here. We have established the analogue of Lemma 10. Part (1) follows from the above lemma by induction since assumption B automatically holds at the terminal node.

Part (2): It is easy to see that $b_{ti}(V_{t1}...V_{tN}, \sigma)$ is continuous in V_{ti} at any point where $V_{ti} \neq V_{tj}$ for any $j \neq i$. Let us show that there is a discontinuity at $V_{ti} = V_{tj}$ whenever $b_{ti}(V_{t1}...V_{tN}, \sigma) > 0$. From Part (1) it follows that if $b_{ti}(V_{t1}...V_{tN}, \sigma)$ were continuous at $V_{ti} = V_{tj}$ then

$$b_{ti}(V_{t1}...V_{ti} = V'...V_{tj} = V'...V_{tN}, \sigma) = b_{tj}(V_{t1}...V_{ti} = V'...V_{tj} = V'...V_{tN}, \sigma).$$

We would like to show that this is not consistent with σ being an optimal strategy. In other words, we show that $\delta = 0$ does not maximize

$$\Pi((V_{t1}...V_{ti} = V' - \delta...V_{tj} = V' + \delta...V_{tN}, \sigma(\mathbf{V})))$$

It is straightforward to show that the second order condition fails (to check the second order condition we combine Lemma 9 and the result that the probability of winning is increasing in the value of the alternative). Hence, there is a discontinuity in the size of investment into an alternative at the value where the alternative changes rank.

Part (3): In the proof of Part (1) we have established that the probability that an alternative wins is increasing in its value. Combining this fact with Equation 9 and taking into account that $G(\cdot)$ is convex completes the proof of Part (3). ■

A.5 Proof of Proposition 6

Proof. The proof of the first statement of the Proposition 6 is a straightforward corollary of Proposition 1. To prove the second part of Proposition 6 we can use an argument virtually identical to that of Lemma 9 to show that in the present setup

$$\frac{\widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN}, \sigma)}{\partial \widetilde{V}_{ti}} = \widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma) \tag{10}$$

Because the favorite alternative receives all investment under the optimal strategy we have

$$\frac{\widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN}, \sigma)}{\partial V_{ti}} = \frac{dC(B_t)}{dB_t} \tag{11}$$

Combining Equations 10 and 11 we have

$$\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma) = \frac{dC(B_t)}{dB_t} \tag{12}$$

We also can obtain from the first part of the proposition that

$$\frac{d\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)}{d\widetilde{V}_{ti}} > 0 \tag{13}$$

(the probability that the favorite alternative wins is increasing in the value of the favorite). Taking into account that $C'(\cdot) > 0$ and $C''(\cdot) > 0$ and combining Equations 12 and 13 completes the proof. ■

A.6 Proof of Proposition 7

Proof.
Part (1): The result follows directly from Proposition 1. Suppose that the strategy of contender i is to select effort levels $e_{1i}...e_{Ti}$. Then, as far as the decision maker is concerned this is equivalent to the benchmark model where contenders do not take actions and the constant equal to $\sum_{t=1}^{T} e_{ti}$ is added to the first shock received by contender i.
Part (2): Suppose that for some contender i we have $e_{ti} \geq e_{t'i}$ with $t > t'$. Then, by part (1), switching the effort levels in periods t and t' will increase the chances that contender i wins the tournament while leaving the cost of effort unchanged. Hence, we must have $e_{t'i} > e_{ti}$ when $t > t'$.
Part (3): First we need to show that the symmetric pure strategy equilibrium is unique. Note that in all symmetric pure strategy equilibria the distribution of the sum of $\sum_{t=1}^{T} b_{ti}$ is the same. Consequently, in all symmetric equilibria the last period tournaments are identical. Thus, it is straightforward to see that there is at most one symmetric equilibrium in the last stage of the game and by inductive argument any promotion tournament has at most one pure strategy symmetric equilibrium. Now, suppose that in some period t, the unique symmetric equilibrium calls for each contender to exert e_t when the budget is B and e'_t when the budget is B' with $B' > B$. We need to show $e'_t \geq e_t$. Suppose to the contrary that $e'_t < e_t$. Consider some contender i. Given that e_t was optimal when budget was B, he has an incentive to deviate for two reasons: (i) since $\theta''(\cdot) > 0$, the cost of the marginal increase in effort is lower, and (ii) since $B' > B$, the benefit of the marginal increase in effort is higher. Hence e'_t cannot be an equilibrium. ■