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# Competing Auctions

by

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*This paper studies the conditions under which two competing and otherwise identical markets or auction sites of different sizes can co-exist in equilibrium, without the larger one attracting all of the smaller one's patrons. We find that the range of equilibrium market sizes depends on the aggregate buyer-seller ratio, and also whether the markets are especially "thin."*

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## 1. Introduction

Activity in prominent auction markets seems to be concentrated at a few sites. For example, Sotheby's and Christie's have dominated the traditional fine art auction business for a century or more, with relatively equal market shares over that period.[1] The combined market share of the two firms is now estimated to be greater than 90%.[2] In the online auction market, eBay refused to cut commission rates in the face of highly-touted lower priced competition following the September 1998 opening of Yahoo! Auctions and the March 1999 opening of Amazon Auctions. Despite eBay's higher fees and tenfold growth of online auctions since Yahoo's entry, eBay has maintained a dominant position.[3] While eBay has (often via acquisition) achieved a large share of online auctions in many countries, a notable exception is Japan, where Yahoo entered before eBay and is now reported to have a 95% market share.

This paper develops a very simple model of competing auction sites to analyze some forces that may lead auction markets to be concentrated. Specifically, we study when two competing markets or auction sites of different sizes co-exist in equilibrium,

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[1] Sotheby's is descended from the book auction firm founded by Samuel Baker in 1745 and Christie's from a general art, furniture, etc. business founded by James Christie in 1766. The two firms achieved prominent positions by the early 19th century. See Learmount (1985). Each firm had about $2.25 billion in gross merchandise sales in 1999.

[2] The U.S. Department of Justice uses the 90% figure in press releases related to the recent price-fixing case. Bonhams (founded 1793) and Phillips (founded 1796) recently merged their UK operations to form what is said to be the third largest traditional fine art auction house with gross merchandise sales of about $200m. The third largest fine art auction house in the U.S., Butterfield and Butterfield, had about $100m in annual merchandise sales prior to its acquisition by eBay. The dominant position of the two firms is particularly striking given that they were accused of collusion following fee increases in 1975 and 1992, and were convicted of price-fixing in connection with their joint adoption in 1995 of a nonnegotiable scale for sellers' commissions.

[3] About $8 billion in merchandise will be auctioned on eBay this year. This is perhaps twenty times the volume of trade on Yahoo or Amazon. The second largest "auction" site for consumer products is actually Ubid.com which sells computers, electronics and a variety of other goods (often refurbished and/or surplus merchandise) directly to consumers and mostly uses "no reserve" auctions rather than posted prices. It also lists items for sale by other firms.

and when this is impossible because the larger auction will attract all of the smaller one's patrons. It is clear that two markets can co-exist if they are geographically distinct and clients face sufficiently large transport costs, and conversely it is clear that otherwise identical markets of different sizes cannot co-exist if larger markets offer a greater variety of goods and buyers have a sufficiently large preference for more diverse markets. This paper abstracts away from both of these issues by examining an extremely stark model with a single good that is traded in a single period.

In our model, there are B ex-ante identical buyers, each with unit demand, and S sellers, each with a single unit of the good to sell. At the start of the period a population of B buyers and S sellers simultaneously choose between two possible locations. Buyers then learn their private values and a uniform price auction is held at each location. This is a very stark model, but we hope that some of the insights it provides will be useful, and that it can provide a benchmark case for richer and more realistic models.

Our model always has an equilibrium in which only a single market is active: With such extreme coordination a player who switched to the other market would find that there is no one to trade with. When the numbers of buyers and sellers are even, there is also an equilibrium where the two markets are exactly the same size. An important question is whether the equal-size market is an unstable "knife-edge" case, or whether in fact there is a range of equilibrium allocations of agents to markets.

To begin, we show in Section 2 that larger markets are more efficient than smaller ones, holding the seller/buyer ratio constant. This efficiency effect makes it harder for the smaller market to survive, but on its own it does not rule out equilibria with two active markets, because the efficiency advantage (on a per-trader basis) of a larger market is on the order of the inverse of the number of traders. Hence, it can be offset the impact that any individual trader has on the market price when he or she switches to the other market, which tends to be of the same magnitude.  Because of this  "crowding" or "market impact" effect,   the equal-sizes configuration mentioned above is not only an equilibrium, but a strict equilibrium: If a buyer or seller were to switch to the other market he or she would find that there were now more participants on his or her side of the market and no more on the other, which would make it strictly less attractive.

Our goal in this paper is to determine the range of equilibrium market sizes that is permitted by these two conflicting effects.  Proposition 3 presents a general but correspondingly weak first result on this question: It is impossible for a market with a finite number of buyers and sellers to coexist with market with a continuum of participants.  Intuitively, there is no crowding from moving into an infinite market, so equilibrium requires that both the buyers and the sellers do as well in the small market as in the large one, but this is inconsistent with the fact that the deterministic outcome in the infinite market gives each buyer the highest possible expected surplus for a given expected utility of the sellers.

Section 3 contains some simple general results.  The results here serve two purposes: they provide intuition for why the equilibrium set has the form it does, and they are used as lemmas in later sections.  This section simplifies the analysis in two ways. First, the section looks at "quasi-equilibria," which drop the constraint that the numbers of buyers and sellers in each market must be integers.  Second, instead of deriving the utility functions for the buyers and sellers in each market from the auction game, section 3 treats the utility functions as exogenous primitives, and simply imposes a set of conditions on them. (The conditions are satisfied for the two distributions that we have analyzed, the uniform and exponential.) Notably, we strengthen the conclusion of Proposition 3 by assuming that either buyers or sellers are worse off in a finite market than in a small market than in a larger one, even if the larger market is not infinite. Also, we assume that the amount that either buyers or sellers prefer the larger market is bounded by a term proportional to the square of the difference in sizes, divided by the total number of buyers. It is intuitive that this difference should go to zero as the difference in sizes shrinks; the quadratic bound is sufficient for a range of equilibrium sizes to persist as the total number of participants grows.

Sections 4 and 5 specialize the model to two particular distributions. In section 4 we assume that buyers' valuations are uniformly distributed on [0, 1]; this might be a reasonable assumption for thinking about Pokemon cards, Beanie Babies, and other ordinary items being sold on eBay. We verify that the induced market utilities satisfy the assumptions of Section 3, and use those results to get an exact characterization of the set of quasi-equilibria. It is typically possible to have a stable two-site equilibrium where the small site is one-third or one-fourth as large as the larger site; just how unequal the two markets can be depends on the buyer-seller ratio, for reasons explained in section 4. We note that this qualitative conclusion is robust in a couple ways: it would suffice for agents on one side of the market rather than both to recognize that they have a market impact; and similar results still obtain if we restrict attention to full equilibria with integer numbers of agents in each market.

Section 5 examines the model with exponentially distributed valuations. This might be a reasonable assumption for thinking about fine art items. We note some differences in how the model works out, but the general pattern of the results is the same. There are equilibria with somewhat unequal market sizes, and there again is a cutoff such that a market with less than that fraction of all agents can not be viable. Our take on the results is that the efficiency effect can lead agents to concentrate, but does not seem sufficiently powerful to account for all of the concentration we observe.

Section 6 examines another factor that may support concentration – market "thinness". One aspect of eBay listings that we found striking when we looked at a random sample of items is that most of them seemed to be unique items. This may be an important common trait of fine art and online auctions. One way to model the sale of unique items would be to modify our basic model so that each seller only has a chance q of having an object for sale. When q is sufficiently small (relative to S) it is easy to see that this model predicts complete concentration (except for a knife-edge equilibrium with a 50-50 split). When q is small enough sellers can essentially ignore the possibility that they will be competing with other sellers. If one market has more buyers, then all sellers will go to that market, and hence it must have all buyers. While this is a powerful argument, we note that it requires a fairly extreme thinness. If there are only three sellers, each of whom is certain to have the good (so we return to q = 1) there is a split equilibrium with two sellers going to one market and one seller to the other.

The fact that a size ratio of 3:1 or 4:1 is possible is consistent with the long-term coexistence of Christie's and Sotheby's; it is also consistent with the lack of successful large-scale entry into this market. In contrast, eBay is too much larger than its competitors to be consistent with the bounds we derive; from the viewpoint of our model this market looks more like one that has "tipped" to a single auction than a stable coexistence of multiple auction sites.

Our conclusion that a range of size ratios is possible differs from that of past analyzes of the choice of location or market inn cases where the efficient outcome would be for everyone to go to the same location. We discuss this in greater detail in section 7, but let us note here two important features of our model. First of all, previous analyzes have argued that that the "crowding" or "competitive" effect of one agent moving on equilibrium outcomes in each location is small enough to be ignored; in our model this effect is as large as the efficiency force that pushes in the opposite direction. Second, in some past work, such as Pagano [1989], there is a single group of agents, rather than the distinct sets of buyers and sellers here, so that there is no "market impact effect" to counterbalance the efficiency advantage of larger markets.[4]

## 2. The Efficiency of Large Markets

### A. Welfare in a Single Market

Consider a single market with $S$ sellers and $B > S$ buyers. The efficient outcome here is for the buyers with the $S$ highest values to receive the good, so the maximum total surplus that can be achieved is $B \bullet \Pr(v \geq v^{S:B}) \bullet E(v \mid v \geq v^{S:B}) = S \bullet E(v \mid v \geq v^{S:B})$. The maximized surplus per seller (or per item sold) is $E(v \mid v \geq v^{S:B}) = E(v \mid v > v^{S+1:B})$, which we define to be $w(S,B)$. This can be written as

[4] In Pagano's model, players do not know ex-ante whether they will be buyers or sellers.

$$w(S,B)=\int_0^{\bar{b}}\left(\int_{v^{S+1:B}}^{\bar{b}} vf(v \mid v>v^{S+1:B})dv\right)f(v^{S+1:B})dv^{S+1:B}$$

where the *f*s are the relevant densities. Note that as the market grows, holding the buyer/seller ratio fixed, the efficient outcome converges to a deterministic limit, with the good given to all buyers whose value exceeds the market-clearing price $p^{S:B}$ defined by $1-F(p^{S:B})=\frac{S}{B}$. Put differently, in the continuum limit the good is allocated to all buyers with $v \geq \bar{v} \equiv F^{-1}\left(1-\frac{S}{B}\right)$. Thus welfare per seller converges to the average of the buyer values on the range where value is at least the market price, $E(v \mid v \geq \bar{v})$.

The following propositions show that our model has the intuitive property that larger markets are more efficient. The first result compares a finite market to the continuum limit discussed above; the second extends this to a comparison of finite markets of different sizes.

<u>Proposition 1</u>

The maximal expected surplus per buyer in a finite market with *S* sellers and B buyers is strictly less than the maximal surplus in a market with continua of buyers and sellers and the same seller-buyer ratio.

*Remark:* The idea of the proof is simply that the distribution of realized utility as a function of buyer's value in the large market first-order stochastically dominates the distribution in the small one. In both markets, buyers have the same probability $\frac{S}{B}$ of receiving the good, but in the small market, buyers "win" in a less efficient way, as they sometimes win when $v < \overline{v}$.

*Proof:* Let $c(v)$ be the buyer's probability of consuming the good in the small market when his value is *v*. Then the buyer's expected utility in the large market is $E(v \mid v \geq \overline{v}) \cdot \Pr(v \geq \overline{v})$, and her expected utility in the small market is $E(v \cdot c(v))$. And so the difference in utility between the large and small market is

$$E(v \mid v \geq \overline{v}) \cdot \Pr(v \geq \overline{v}) - E(v \cdot c(v)) =$$

$$E(v \mid v \geq \overline{v}) \cdot \Pr(v \geq \overline{v}) - E(v \cdot c(v) \mid v \geq \overline{v}) \cdot \Pr(v \geq \overline{v}) - E(v \cdot c(v) \mid v < \overline{v}) \cdot \Pr(v < \overline{v}) =$$

$$E\left(v(1-c(v)) \mid v \geq \overline{v}\right) \cdot \Pr(v \geq \overline{v}) - E(vc(v) \mid v \leq \overline{v}) \Pr(v \leq \overline{v}) >$$

$$E\left(\overline{v}(1-c(v)) \mid v \geq \overline{v}\right) \cdot \Pr(v \geq \overline{v}) - E(\overline{v}c(v) \mid v \leq \overline{v}) \Pr(v \leq \overline{v}) =$$

$$\overline{v}\left[E\left(1-c(v) \mid v \geq \overline{v}\right) \cdot \Pr(v \geq \overline{v}) - E(c(v) \mid v \leq \overline{v}) \Pr(v \leq \overline{v})\right] =$$

$$\overline{v}\left[\Pr(v \geq \overline{v}) - E(c(v))\right] = \overline{v}\left[\frac{S}{B} - \frac{S}{B}\right] = 0;$$

where the strict inequality follows from the fact that the trade in the small market is not deterministic (because S<B and the distribution of types is strictly monotone.)

QED.

The next result extends the previous one by showing that efficiency is in fact monotone in market size. It implies the first, but has slightly more complex proof that views the smaller market as a subsample of the larger one.

<u>Proposition 2</u>

If *m* and *n* are integers with *m*<*n*, then holding *S* and *B* fixed we have

$w(mS, mB) < w(nS, nB)$.

*Proof:* See appendix.

*Remark:* We conjecture that this convergence is at rate *1/n*. We can show that this is the case for the normal approximations to the distribution of the order statistics; a complete proof along this line would require an analysis of the rate of convergence of the distributions of the order statistics to the approximating distributions.

B. Equilibrium Prices in an Auction

To model price formation, we suppose there is a uniform-price sealed bid auction. Thus if the market doesn't have excess supply, the price will be at the (S+1)st highest of the B buyer values, that is $v^{S+1:B}$, which we will also denote as $p$. We assume that $S+1 < B$ so that if all agents go to a single market there is excess demand and so with probability 1 the market price is strictly positive. Since sellers have 0 reservation value and are risk neutral, their expected utility in this market is just the expected price, which we denote by $\overline{p}$ ,or $\overline{p}(S, B)$ when we need to track the dependence on the number of buyers and sellers. The expected utility of a buyer in a market with S sellers and B buyers (including himself) is

$u_b(S, B) = E(v - v^{S+1:B} \mid v \geq v^{S:B})$.

We also have that $u_b(S, B) = \dfrac{S\left(w(S, B) - \overline{p}(S, B)\right)}{B}$.

C. A Finite Market Cannot Co-exist with an Infinitely Large One

We can use the fact that large markets are more efficient that small ones to get a general impossibility result: A market of fixed finite size cannot coexist with a market that is infinitely large. Let the finite market be market 1, with numbers $S_1, B_1$ of sellers and buyers, and let market 2 be deterministic with seller-buyer ratio $\frac{S_2}{B_2}$, and the deterministic price $\overline{p}^2$ in market 2 defined by $1-F(\overline{p}^2)=\frac{S_2}{B_2}$. A buyer or seller moving into the large market will have no effect on the price there, so for both markets to coexist it is necessary and sufficient that the expected price in market 1 satisfies $\overline{p}^1 \geq \overline{p}^2$ (or else sellers move to market 2) and that $u_b^1 \geq u_b^2$ (or else the buyers move.)

Proposition 3

There is no equilibrium in which trade takes place in both a continuum market and a finite one.

*Proof:* As before let $c(v)$ be the buyer's probability of consuming the good in the small market when her type is $v$. Then $u_b^2 = E(v-\overline{p}^2 \mid v \geq \overline{p}^2)\Pr(v \geq \overline{p}^2)$ is the buyer's expected utility in the large market, and $u_b^1 = E\left((v-p^1)\bullet c(v)\right)$ is the buyer's expected utility in the small market. Since $E\left(p^1 \bullet c(v)\right)$ is the buyer's expected payment, and (from symmetry) each buyer purchases a unit with probability $\frac{B_1}{S_1}$, we know that

$E\left(p^1 \bullet c(v)\right) = \overline{p}^1 \frac{B_1}{S_1} = \overline{p}^1 E(c(v)) = E\left(\overline{p}^1 \bullet c(v)\right)$. Hence

$u_b^1 = E\left((v - p^1) \bullet c(v)\right) = E\left((v - \overline{p}^1) \bullet c(v)\right)$, and so $\overline{p}^1 \geq \overline{p}^2$ implies that

$u_b^1 = E\left((v - \overline{p}^1) \bullet c(v)\right) \leq E\left((v - \overline{p}^2) \bullet c(v)\right)$.

The expression on the right hand side is the buyer's expected utility when he pays price $\overline{p}^2$ and purchases according to $c(v)$. This expression is maximized when the buyer purchases exactly when $v \geq \overline{p}^2$, and the maximized utility is exactly the buyer's utility in the large market. Any other specification of $c$ leads to strictly lower utility, so $u_b^1 < u_b^2$ because with a strictly monotone cdf F, equilibrium trade in the small market cannot follow the deterministic rule "consume exactly when $v \geq \overline{p}^2$".

QED

## 3. Two Finite Markets

Proposition 3 suggests that there must be a bound on the ratio of the sizes of two markets for both of them to remain active. In this section, we present some general results on the coexistence of two finite markets. The results illustrate and provide some intuition for our main observation – that markets of somewhat different sizes can coexist. We will also use them in subsequent sections, where we assume specific distributions of buyer valuations to study how the maximum inequality in equilibrium market size relates to the size of the overall "economy" (that is the combined sizes of the two markets) and to the aggregate buyer-seller ratio.

Let $u_s(S,B)$ and $u_b(S,B)$ be the seller and buyer utility in a market with $S$ sellers and $B$ buyers. While these functions would normally be defined only on the nonnegative integers, for the purposes of this section we will think of them as continuous functions defined on $[0,\infty)\times[0,\infty)$. Assume that these functions satisfy:

$$\text{(A1)} \begin{array}{l} u_s(S,B) > 0 \text{ if } B > S,\ u_s(S,B) = 0 \text{ if } B \leq S \\ u_b(S,B) > 0 \text{ if } S > 0,\ u_b(0,B) = 0 \end{array}$$

For $B > S$ assume that the functions are differentiable and satisfy the natural monotonicity properties:

$$\text{(A2)} \quad \frac{\partial u_s}{\partial S} < 0, \frac{\partial u_s}{\partial B} > 0, \frac{\partial u_b}{\partial S} > 0, \text{ and } \frac{\partial u_b}{\partial B} < 0$$

We showed in Proposition 2 that holding the buyer-seller ratio constant larger markets are more efficient. In Proposition 3 we showed that either buyers or sellers are worse off in a finite market than in a market with a continuum of sellers, regardless of whether the buyer-seller ratio is different. In this section, we'll assume that in a comparison of two finite markets, the smaller market is less efficient in a similar sense.

(A3) If $B_1 < B_2$ then $u_s(S_1,B_1) < u_s(S_2,B_2)$ or $u_b(S_1,B_1) < u_b(S_2,B_2)$.

This condition is satisfied in the uniform and exponential cases we consider in the next two sections, and we conjecture that it holds under fairly general conditions.

In our analysis we will often present simpler results ignoring integer constraints. Our basic definitions are

Definition. An allocation $(S_1, S_2, B_1, B_2)$ is a *quasi-equilibrium* if it satisfies the following four constraints:

(S1) $u_s(S_1, B_1) \geq u_s(S_2+1, B_2)$

(S2) $u_s(S_2, B_2) \geq u_s(S_1+1, B_1)$

(B1) $u_b(S_1, B_1) \geq u_b(S_2, B_2+1)$

(B2) $u_b(S_2, B_2) \geq u_b(S_1, B_1+1)$

<u>Definition</u>. An allocation $(S_1, S_2, B_1, B_2)$ is an *equilibrium* if it is a quasi-equilibrium and $S_1, S_2, B_1$ and $B_2$ are all integers.

Checking whether an allocation is an equilibrium in principle requires checking four constraints. However, we will show that only the constraints ensuring that agents do not want to leave the smaller market are relevant for determining the smallest possible number of buyers in an active market.

To make the argument, it is helpful to define the loci where the incentive constraints in the small market are satisfied with equality. For fixed values of $S$ and $B$, let $S_1^{B1}(B_1)$ be the value of $S_1$ such that such that (B1) holds with equality at $(S_1, S-S_1, B_1, B-B_1)$. Let $S_1^{S1}(B_1)$ the value of $S_1$ such that (S1) holds with equality at $(S_1, S-S_1, B_1, B-B_1)$.

<u>Lemma 1:</u> Given assumptions (A1)–(A3), $S_1^{B1}$ is a well-defined, increasing, differentiable function on the domain $[0, B]$. There exist constants $B^{\min}$ and $B^{\max}$ with $0 < B^{\min} < B/2 < B^{\max} < B$ such that $S_1^{S1}$ is a well-defined, increasing, differentiable

function on the domain $[B^{\min}, B^{\max}]$ with $S_1^{S1}(B^{\min}) = 0$ and $S_1^{S1}(B^{\max}) = S$. Moreover, $S_1^{S1}(B_1) = S/2 + 1/2$ and $S_1^{B1}(B_1) < S/2$.

*Proof*: See appendix.

Proposition 4 is a precise statement of our observation that one need only consider whether the "small market" constraints can be satisfied in order to determine whether there is a quasi-equilibrium with specified numbers of buyers in the two markets.

<u>Proposition 4</u>

Fix $S$ and $B$ with $S+1 < B$. Assume (A1) – (A3) and that $B_1 \leq B/2$. Then, there exists an $S_1$ such that $(S_1, S-S_1, B_1, B-B_1)$ is a quasi-equilibrium if and only if there exists an $S_1$ such that $(S_1, S-S_1, B_1, B-B_1)$ satisfies the (B1) and (S1) constraints.

*Proof*: The "only if" direction is trivial. To establish the "if" result, suppose that $B_1$ and $\tilde{S}_1$ are such that $(\tilde{S}_1, S-\tilde{S}_1, B_1, B-B_1)$ satisfies the (B1) and (S1) constraints.

By analogy to the construction in the lemma, let $S_1^{B2}(B_1)$ be the (unique) value of $S_1$ such that (B2) holds with equality, and let $S_1^{S2}(B_1)$ be such that (S2) holds with equality. Considering the payoffs as $S_1$ approaches 0 and $S$ it is easy to see that the former is well defined. For $B_1$ less than $B - B^{\max}$ there is no $S_1$ that makes (S2) hold with equality. If this is the case, condition (S2) always holds and can be ignored (or one can regard all of the equations below as applying with $S_1^{S2}(B_1) = 0.$)

(B1) is satisfied if and only if $S_1 \geq S_1^{B1}(B_1)$. (S1) is satisfied if and only if $S_1 \leq S_1^{S1}(B_1)$. (B2) is satisfied if and only if $S_1 \leq S_1^{B2}(B_1)$. (S2) is satisfied if and only if $S_1 \geq S_1^{S2}(B_1)$. The fact that (B1) and (S1) are both satisfied at $\tilde{S}_1$ implies that $S_1^{S1}(B_1) \leq S_1^{B1}(B_1)$. Assumption (A2) immediately implies that $S_1^{B1}(B_1) < S_1^{B2}(B_1)$ and $S_1^{S2}(B_1) < S_1^{S1}(B_1)$, e.g.

$$u_b(S_1^{B1}(B_1), B_1) = u_b(S - S_1^{B1}(B_1), B_2 + 1) \Rightarrow u_b(S_1^{B1}(B_1), B_1 + 1) < u_b(S - S_1^{B1}(B_1), B_2).$$

Finally, using (A2), (A3), and then (A2) again we can see that $S_1^{S2}(B_1) < S_1^{B2}(B_1)$:

$$\begin{aligned} u_s(S_1^{S2}(B_1)+1, B_1) = u_s(S - S_1^{S2}(B_1), B_2) &\Rightarrow u_s(S_1^{S2}(B_1), B_1) > u_s(S - S_1^{S2}(B_1), B_2) \\ &\Rightarrow u_b(S_1^{S2}(B_1), B_1) < u_b(S - S_1^{S2}(B_1), B_2) \\ &\Rightarrow u_b(S_1^{S2}(B_1), B_1 + 1) < u_b(S - S_1^{S2}(B_1), B_2). \end{aligned}$$

Combining these inequalities gives $\max(S_1^{S2}(B_1), S_1^{B1}(B_1)) \leq \min(S_1^{S1}(B_1), S_1^{B2}(B_1))$.

Any $S_1$ between these bounds will satisfy (B1), (B2), (S1) and (S2).

QED

The following result on the possibility of very unequal or almost equal market splits follows easily from Proposition 4.

<u>Proposition 5</u>

Fix $S$ and $B$ with $S+1<B$. Assume (A1) – (A3). Then,

(a) There exist quasi-equilibria for all $B_1$ in some neighborhood of $B/2$.

(b) There do not exist quasi-equilibria for positive $B_1$ in some neighborhood of 0.

*Proof*: For part (a) note that (S1) and (B1) are satisfied with strict inequalities for $B_1 = B/2$ and $S_1 = S/2$. By continuity, they are also satisfied for nearby values of $B_1$. For $B_1 < B^{\min}$ (A2) implies that (S1) cannot be satisfied and hence there is no quasi-equilibrium. QED

*Remark*: Proposition 5 shows that the quasi-equilibrium with a 50-50 split is not a knife-edge for any fixed market size. This leaves open the possibility that the set of two-market equilibria shrinks to a 50-50 split as the market size grows. Proposition 7 below provides a much stronger result on the coexistence of small and large markets.

The following result provides a little more detail on the structure of the quasi-equilibrium set.

<u>Proposition 6</u>

Assume (A1)–(A3). Then there exists a $\underline{B}_1 \in [0, B/2]$ and an $\underline{S}_1 \in [0, S/2]$ for which (S1) and (B1) both hold with equality. If there is a unique such $\underline{B}_1$, then there exists an $S_1$ such that $(S_1, S - S_1, B_1, B - B_1)$ is a quasi-equilibrium if and only if $B_1 \in [\underline{B}_1, B - \underline{B}_1]$.

*Proof*: Assume w.l.o.g. that $B_1 \le B/2$. We noted in Lemma 1 that $S_1^{S1}(B^{\min}) = 0$. The assumption in (A1) that buyers receive zero utility when there are no sellers implies that $S_1^{B1}(B^{\min}) > 0$. Lemma 1 also shows that $S_1^{S1}(B/2) = S/2 + 1/2$ and $S_1^{B1}(B/2) < S/2$. By the intermediate value theorem there exists a $\underline{B}_1 \in [0, B/2]$ for which $S_1^{B1}(\underline{B}_1) = S_1^{S1}(\underline{B}_1)$. For $B_1 = \underline{B}_1$ and $\underline{S}_1 = S_1^{B1}(\underline{B}_1)$, (S1) and (B1) hold with equality.

We know from Proposition 4 that a quasi-equilibrium exists with $B_1$ buyers in the smaller market if and only if $B_1 \ge B^{\min}$ and $S_1^{B1}(\underline{B}_1) \le S_1^{S1}(\underline{B}_1)$. If the two curves have a unique intersection, then the fact that they are continuous and that $S_1^{B1}(B/2) < S_1^{S1}(B/2)$ implies that $S_1^{B1}(\underline{B}_1) \le S_1^{S1}(\underline{B}_1)$ if and only if $B_1 \in \left[\underline{B}_1, B/2\right]$.

QED

*Remark*: We will sometimes refer to the assumption in the second part of Proposition 6 as the single crossing condition for the $S_1^{B1}$ and $S_1^{S1}$ curves. It will be straightforward to show that it holds for the distributions of valuations considered in sections 4 and 5. A major component of our characterizations of the quasi-equilibrium set will be explicitly solving for $\underline{B}_1$.

Figures 1 and 2 illustrate the structure of the equilibrium and quasi-equilibrium sets in a typical case. (The $u_b$ and $u_s$ functions were taken to be the "natural" extensions to the reals of the functions one would derive in an example with ten buyers with valuations drawn from a uniform distribution on $[0,1]$ and five sellers.)

Figure 1 graphs the fractions of sellers in market 1 that make buyers and sellers exactly indifferent between the two markets against the fraction of buyers in market 1. The functions satisfy the assumptions (A1)-(A3). As a result, the solid $S_1^{B1}$ curve lies above the dotted $S_1^{S1}$ curve when $B_1 < B/2$. (For $S_1 = S_1^{S1}(B_1)$ sellers are indifferent between the small and large market. (A3) implies that buyers prefer the small market for this $S_1$. (A2) then implies that buyers are indifferent only with a higher $S_1$.) The unique intersection of the curves is at $B_1 = B/2$. If buyers and sellers did not adversely affect prices when moving to the other market, the only quasi-equilibrium with split markets would be an unstable equilibrium with an exactly 50-50 split between the two markets.

Figure 2 graphs the values of $S_1 / S$ for which the (B1), (B2), (S1) and (S2) constraints hold with equality for the same utility functions as in Figure 1. The quasi-equilibrium set is the parallelogram-shaped region in the center of the figure below the curves where (S1) and (B2) hold with equality and above the curves where (S2) and (B1) hold with equality. In this example, quasi-equilibria exist whenever the smaller market has at least 11% of the buyers (meaning $B_1$ is at least 1.1).

We have placed small stars in the figure at points within the quasi-equilibrium set where the numbers of buyers and sellers are both integers. These are the equilibria. In an equilibrium the smaller market can have two buyers and one seller or four buyers and two sellers. There is no equilibrium with three or five buyers in the smaller market. With three buyers in the smaller market, for example, then there are a range of values of $S_1$ near one-and-a-half which satisfy the quasi-equilibrium conditions. None of these allocations, however, satisfy the integer constraints – sellers would be unwilling to stay in small market if there were two sellers, while buyers would be unwilling to stay in a market if there was only one seller.

Unequal sized markets are only possible in our model because buyers and sellers have an adverse "market impact" if they switch markets. The market impact is small when the number of buyers and sellers is large, but so are the efficiency differences. For example, while there is a very big difference between having two and four buyers in a market, a market with two hundred sellers is already close to efficient and there is little difference between such a market and a market with four hundred sellers.

To illustrate how markedly the efficiency effect declines with the size of the market, Figure 3 graphs the equal-buyer-utility and equal-seller-utility curves which apply to a model with 30 buyers and 15 sellers (and a uniform distribution of seller valuations as in Figure 1). The curves are much closer together than the curves in Figure 1. Sellers are indifferent when prices are equal in the two markets. The closeness of the two curves reflects that efficiency differences are fairly small and can only offset a small difference in price.

Figure 4 graphs the four curves that bound the quasi-equilibrium set in this case. One interesting thing to note is that the range of market sizes in the quasi-equilibrium set is very similar to that in Figure 2: Here a quasi-equilibrium exists whenever at least 12% of the buyers are in the small market, as compared to the 11% in Figure 2. The quasi-equilibrium set looks much flatter in the *S*-dimension. This reflects that the "market impact" is much smaller and hence buyer and seller utility (the latter of which is equal to the price) have to be more nearly equal in the two markets in equilibrium. The stars in the figure illustrate that there are nonetheless a substantial number of true equilibria. (Recall that the y-axis graphs $S_1 / S$ and hence the integers are closer together.) Recall that Proposition 5 did not rule out the possibility that the fraction of buyers necessary to make the small market viable in a quasi-equilibrium might be converging to one-half as the number of buyers and sellers grows. In the examples discussed in sections 4 and 5 the degree of asymmetry between markets that is possible in a quasi-equilibrium is roughly independent of the total market size (for a given buyer-seller ratio). To provide some intuition for this, we offer a more general sufficient condition under which there is a set of quasi-equilibria with a non-vanishing interval of market sizes even in the limit as market size goes to infinity.

Assume that $B_1 < B_2$. Note that conditions (B1) and (S1) can be rewritten as

(B1′) $u_b(S_2, B_2) - u_b(S_2, B_2 + 1) \geq u_b(S_2, B_2) - u_b(S_1, B_1)$

(S1′) $u_s(S_2, B_2) - u_s(S_2 + 1, B_2) \geq u_s(S_2, B_2) - u_s(S_1, B_1)$.

The left-hand sides of each of these conditions measure the "market impact" that buyers and sellers, respectively, have when they move to market 2. The right-hand sides measure the degree to which buyers and sellers, respectively, find the larger market more attractive. Assumption (A3) implies that the RHS of at least one of these two equations is positive.

In the examples we examine in the following two sections, when the buyer-seller ratio in each of the two markets is held fixed at $\gamma$, the market impact and large-market efficiency effects will satisfy the following condition:

There exists an $x \in (0,1/2)$ and positive constants $k_1, k_2, k_3$ and $k_4$ such that

$$\text{(A4)} \quad \begin{aligned} & u_b(S_2,B_2) - u_b(S_2,B_2+1) \geq \frac{k_1}{B} \\ & u_s(S_2,B_2) - u_s(S_2+1,B_2) \geq \frac{k_2}{B} \\ & u_b(S_2,B_2) - u_b(S_1,B_1) \leq \frac{B_2 - B_1}{B}\frac{k_3}{B} \\ & u_s(S_2,B_2) - u_s(S_1,B_1) \leq \frac{B_2 - B_1}{B}\frac{k_4}{B} \end{aligned}$$

when $B_1 / B \in [x,1/2], B_2 = B - B_1, S_1 = \gamma B_1$ and $S_2 = \gamma B_2$.

Roughly, what this means is that the "market impact" of a buyer or seller is a $1/B$ effect and that the "efficiency advantage" of a large market goes to zero at rate $1/B$ as $B$ gets large and also goes to zero linearly in the difference $\frac{B_2 - B_1}{B}$ between the sizes of the two markets (expressed as a fraction of the total number of buyers).

It is easy to show that when the market impact and efficiency advantages of the smaller market are of this magnitude, then the range of $B_1$'s for which a quasi-equilibrium exists (with equal buyer-seller ratios in the two markets) is a nonvanishing fraction of the total market size.

<u>Proposition 7</u>

Suppose $u_s$ and $u_b$ satisfy (A1)–(A3). Suppose that for a given value of $\gamma$, (A4) is satisfied and the inequalities in (A4) hold for constants $x, k_1, k_2, k_3,$ and $k_4$. Then for any $B$ and any $B_1$ with $\frac{B_1}{B} \in [\frac{1}{2} - \frac{1}{2}\min(k_1/k_3, k_2/k_4), \frac{1}{2} + \frac{1}{2}\min(k_1/k_3, k_2/k_4)]$ and $\frac{B_1}{B} \in [x, 1-x]$, the model with $B$ buyers and $\gamma B$ sellers has a quasi-equilibrium with $B_1$ buyers in market 1.

*Proof*: We know from Proposition 4 that it suffices to show that for a fixed $B$ and $S$, (B1′) and (S1′) are satisfied for some $S_1$ whenever $B_1 \in [x, 1/2]$ and $(B_2 - B_1)/B < \min(k_1/k_3, k_2/k_4)$. That (B1′) holds for $S_1 = \gamma B_1$ follows immediately from (A4) and $(B_2 - B_1)/B < k_1/k_3$

$$u_b(S_2, B_2) - u_b(S_2, B_2 + 1) \geq \frac{k_1}{B} \geq \frac{B_2 - B_1}{B} \frac{k_3}{B} \geq u_b(S_2, B_2) - u_b(S_1, B_1).$$

That (S1′) also holds for this $S_1$ follows just as easily from (A4) and $(B_2 - B_1)/B < k_2/k_4$

$$u_s(S_2, B_2) - u_s(S_2 + 1, B_2) \geq \frac{k_2}{B} \geq \frac{B_2 - B_1}{B} \frac{k_4}{B} \geq u_s(S_2, B_2) - u_s(S_1, B_1).$$

QED

In the following two sections we will consider two particular specifications of the value distribution, verify that they satisfy each of the assumptions used in this section, and then perform some explicit calculations to provide a clearer view of the quasi-equilibrium and equilibrium sets.

## 4. Two Finite Markets: The Uniform Case

To learn more about when two markets can co-exist, and to verify that the assumptions in Section 3 are reasonable, we will consider two tractable distributions in some detail. This section analyzes the uniform distribution, which has a bounded support, and the next section analyzes the exponential case, where the distribution of values is unbounded. The bounded-support case may be a better description of markets for goods where there is a readily available close substitute that effectively caps the maximum willingness to pay, while the unbounded support may be a better description of models for one-of-a-kind objects like paintings or rare collectibles. On the technical side, the distinction between bounded and unbounded support is relevant to our analysis because with a bounded support, the expected value of the highest value is less sensitive to adding more buyers.

A. Preliminaries

Under the uniform distribution, the expected value of the ith lowest order statistic out of n draws is distributed Beta(*i, n-i+1*) and has expectation $\frac{i}{n+1}$ (see e.g. David [1970].) The seller's expected utility is equal to the expectation of the price, which we will denote by $\overline{p}$. The price is given by the $S$+1$^{\text{th}}$ highest buyer value, which is the $B-S^{\text{th}}$ lowest. Hence, $u_s(S,B)=\overline{p}=\frac{B-S}{B+1}$, and $1-\overline{p}$ is $\frac{S+1}{B+1}$. We can also compute $f(v\,|\,v^{S+1:B})=\frac{S}{B(1-v^{S+1:B})}$ for $v>v^{S+1:B}$. Because the buyers valuation conditional on being greater than $p$ is uniform on $(p,1]$, the buyer's expected utility conditional on winning the good at $p$ is $(1-p)/2$. Each buyer wins the good with probability $S/B$, so the buyer's expected utility is $u_b(S,B)=\frac{S}{B}\frac{\left(1-\overline{p}(S,B)\right)}{2}=\frac{S(1+S)}{2B(1+B)}$. Note that holding $\frac{S}{B}$ constant, buyers are actually better off in smaller markets, even though the larger market is more efficient.

Adding a seller to a market causes the price to fall by $\frac{1}{B+1}$, irrespective of the number of sellers in the market, while adding a buyer reduces buyer utility by $\frac{S(S+1)}{B(B+1)(B+2)}$. Thus the "crowding" or "market impact" effect of adding another buyer is strongest when $\frac{S}{B}$ is near to 1; this will allow more unequal size ratios to be equilibria. Although the derivation of the utility functions assumes that the numbers of uyers and sellers are integers, the utility functions given above are well-defined for all non-negative real numbers. Moreover, it is clear from inspection that these functions satisfy (A1) and (A2). For (A3) (at least one side is better off in a larger market) note that we can rewrite the buyer utility as $u_b = \frac{(1-\overline{p})}{2}(1-\overline{p}(1+\frac{1}{B}))$; [5] thus if prices are higher in market 1 and $B_1$ is smaller than $B_2$, then buyers must be better off is market 2. (A4) requires that there be lower bounds on the market impact and upper bounds on the efficiency advantage of larger markets of a particular form when $B_1 / B$ is in some interval $[x, 1-x]$.

The market impact effect of adding a seller is at least $1/B$, while the market impact of adding a buyer is greater than $\frac{\gamma^2}{B}$. As we remarked above, holding $\frac{S}{B}$ fixed, buyers are worse off in larger markets, so we can take $k_3 = 0$. Finally, we compute

$$u_s(S_2, B_2) - u_s(S_1, B_1) = \frac{S_1 B_2 + S_1 + B_2 - S_2 B_1 - S_2 - B_1}{(B_1+1)(B_2+1)} = \frac{(B_2 - B_1)(1-\gamma)}{(B_1+1)(B_2+1)}.$$

If we restrict $\frac{B_1}{B}$ to lie in the interval $[x, 1-x]$, this is less than $\frac{(B_2 - B_1)(1-\gamma)}{B^2}\frac{2}{x}$.

B. Necessary And Sufficient Conditions for Quasi-Equilibrium

Proposition 8

When buyer values have the uniform distribution, there is an unique $\underline{B}_1 \in [0, B/2]$ for which (S1) and (B1) both hold with equality. Moreover, $\frac{\underline{B}_1}{B} > \frac{1}{4}\left(1 - \frac{S+5}{B}\right)$ and $\lim_{B\to\infty} \frac{\underline{B}_1(B)}{B} = \frac{1}{4} - \frac{1}{4}\lim_{B\to\infty}\frac{S}{B}$.

Corollary

There exists an $S_1$ such that $(S_1, S - S_1, B_1, B - B_1)$ is a quasi-equilibrium if and only if $B_1 \in [\underline{B}_1, B - \underline{B}_1]$. There are no equilibria in which both markets are active and

$$\frac{B_1}{B} \leq \frac{1}{4} - \frac{S+5}{4B} = \frac{1}{4}\left(1 - \frac{S+5}{B}\right).$$

*Proof of Proposition 8:* Set $S_2 = S - S_1$ and $B_2 = B - B_1$. The (S1) constraint can be rewritten as $S_1 + 1 \leq c(B_1 + 1)$, where $c = \frac{S+3}{B+2}$.

When (S1) holds with equality, the (B1) constraint, which is

$\frac{S_1(1+S_1)}{B_1(1+B_1)} \geq \frac{S_2(1+S_2)}{(B_2+1)(B_2+2)}$, becomes $\frac{c(B-B_1)(B-B_1+1)}{B_1} \geq \frac{(S-S_1)(S-S_1+1)}{S_1}$.

Further algebra shows that if we define $z = \frac{B_1+1}{B+2}$ and $f = \frac{1}{B+2}$, this can be rewritten as

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[5] To show this, note that S/B = (S+1)/(B+1) + (S/B - (S+1)/(B+1)).

$$c^3z^3-\left(cf+c^2(1+f)\right)z^2+\left(cf+c^2\right)(1+f)z-cf(1+f)\geq$$
$$c^2z^3+\left(-(cf+c-2f)c-c(c-f)\right)z^2+\left(f)c-2f)c+(cf+c-2f)(c-f)\right)z-$$
$$f(c-2f)(c-f)$$

Subtracting the right-hand side from the left yields the quadratic equation

$$-4cz^2+(5c-c^2+4cf-2f)z+2f^2-c-4cf+c^2\geq 0.$$

Rearranging terms gives

(a") $\quad \left[-4cz^2+(5c-c^2)z-c+c^2\right]+\left[2f(f+2cz-z-2c)\right]\geq 0.$

The quadratic expression in the first set of square brackets has two roots, $z=\dfrac{1-c}{4}$ and $z=1$. It is positive between these roots. Because $f=\dfrac{c}{S+3}=\dfrac{1}{B+2}<\dfrac{S+3}{4(B+2)}=\dfrac{c}{4}$, and $z<1/2$, the term in the second set of square brackets is negative. Hence, the full expression must either have two roots in the interval $[(1-c)/4,1]$ or no roots at all. The fact that (B1) is satisfied at $(B_1,S_1)=(B/2,S/2+1/2)$ (where (S1) holds with equality) implies that the expression (a") is positive at $z=1/2$. Hence, there is a unique solution with $B_1<1/2$ and it satisfies $\dfrac{B_1+1}{B+2}>\dfrac{1-c}{4}=\dfrac{B-S-1}{4(B+2)}$ which is equivalent to

$\dfrac{B_1}{B}>\dfrac{1}{4}\left(1-\dfrac{S+5}{B}\right).$

When $B$ goes to infinity, $f$ converges to zero, and the smaller solution to (a") converges to the smaller solution to the first quadratic in that equation. As noted above this is $\dfrac{B_1}{B}=\dfrac{1}{4}\left(1-\dfrac{S+5}{B}\right).$

QED

Here is an intuition for the role of the aggregate seller/buyer ratio $\frac{S}{B} = \gamma$ in the limiting value of $\underline{B}_1$. When both markets are large, the crowding effects are small, so the seller/buyer ratios in each market must be about the same. We saw above that with equal seller/buyer ratios buyers are actually better off in the smaller market, while the seller's advantage in the larger market is proportional to $(1-\gamma)\frac{(B_2 - B_1)}{(B_1 + 1)(B_2 + 1)}$. To have a quasi-equilibrium, the efficiency advantage must be offset by the market impact that a seller moving to the larger market would have. The market impact is $1/(B_2 + 1)$. When $\gamma$ is larger, the efficiency advantage is reduced, while the market impact is unchanged. Hence, the (S1) constraint can be satisfied for larger values of $\frac{(B_2 - B_1)}{(B_1 + 1)}$.

C. One-sided Market Impact

One feature of the model that some may find unintuitive is that buyers and sellers consider their market impact even when they are very small relative to the total market size. For example on eBay, where most buyers are casual consumers and most sellers are small and not-so-small businesses, it might be more plausible that sellers would consider the market impact effect than that buyers would.[6] Possible reasons for this would be that the market impact of a typical buyer is so small that the buyer might round it off to zero,

[6] A recent New York Times article (Guernsey, 2000) reports that one-quarter of one percent of eBay's registered users are responsible for over two-thirds of the items listed. These sellers had an average of seventy items each in the process of being auctioned at any point in time.

or that buyers do not think about things enough or have enough experience to learn about the effect.

In this subsection, we note that it is not necessary to assume that both buyers and sellers recognize that they have a market impact to obtain our conclusions. It would suffice for one side to do so. The proposition below establishes that there are still quasi-equilibria with substantially different market sizes if we add the restriction that sellers must be exactly indifferent between the two markets (as one would want to if only buyers recognized the market impact effect.) The minimum possible fraction of agents in the smaller market is increased from about $\frac{1}{4}-\frac{S}{4B}$ to $\frac{1}{2}-\frac{S}{2B}$ by this change. We have chosen to add a seller indifference condition rather than a buyer indifference condition only because the algebra is simpler that way. Which side of the market recognizes that there is a market impact is not important.

<u>Proposition 9</u>

For fixed total numbers of buyers and sellers *B* and *S* with $B > S + 2$, for every partition $(B_1, B_2)$ such that for i=1,2, $\frac{B_i}{B} \in [\frac{1}{2}-\frac{S}{2B}, \frac{1}{2}+\frac{S}{2B}]$, there is a quasi-equilibrium $(S_1, S_2, B_1, B_2)$ with $u_s(S_1, B_1) = u_s(S_2, B_2)$.

*Proof*: See appendix.

## <u>D. Integer-valued Equilibrium</u>

So far we have been ignoring the constraint that the numbers of buyers and sellers in each marker should be an integer. With a small number of traders it may be that only a few ratios of markets sizes are possible. However, one would expect these integer problems to become less important in large markets, and the next result shows that the any ratio of market sizes in the interval given in Proposition 9 can be approximated by an integer-valued equilibrium when the number of traders is sufficiently large. Since given "target ratios" of $B_1$ to $B$ and $B$ to $S$ can only be approximated by integers, the statement of the result uses $\alpha$ as the "target level" of $\frac{B_1}{B}$ and $\gamma$ as the "target level" of $S/B$.

Proposition 10

For any "target market ratios" $\alpha, \gamma > 0$ with $\alpha \in \left( \frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2} \right)$ and any $\varepsilon > 0$ there exists $\underline{B}$ such that for all $B > \underline{B}$ there is an equilibrium $(S_1, S_2, B_1, B_2)$ with $B_1 + B_2 = B$, $|B_1 / B - \alpha| < \varepsilon$, and $|S / B - \gamma| < \varepsilon$.

*Proof*: See Appendix. The proof first constructs a quasi-equilibrium with equal prices that approximates the target ratios, but where only $B_1$ and $B_2$ are guaranteed to be integers; we then use this partition to construct an integer-valued partition where all of the incentive constraints are satisfied but prices are only approximately equal.

Proposition 10 proves that when the number of buyers is large there exist equilibria throughout the range of market sizes for which Proposition 9 shows that quasi-equilibria with equal seller utility exist. We present this result rather than trying to show that equilibria exist throughout the full range of market sizes for which Proposition 8 shows that quasi-equilibria exist because accounting for the integer constraints is cumbersome, and the algebra characterizing the equal-seller-utility equilibria is much simpler.

## 5. Exponentially Distributed Values

To test the robustness of our results we now consider another tractable distribution, the exponential with $f(v) = \exp(-v)$ for $v \geq 0$. As we remarked earlier, the exponential has an unbounded support, which might be appropriate for thinking about rare art objects.

Because of this unbounded support, we would expect that adding more buyers has a greater effect on the size of the highest order statistic, and this is indeed the case. Here the mean of the $r^{\text{th}}$ highest of $n$ draws is $\mu^{r:n} = \sum_{i=r}^{n} i^{-1}$, so the expected price in a market with $B$ buyers and $S$ sellers is $\overline{p}(S,B) = \sum_{i=S+1}^{B} i^{-1} = \sum_{i=1}^{B} i^{-1} - \sum_{i=1}^{S} i^{-1}$.

The buyer's expected utility is

$$\frac{1}{B}\sum_{k=1}^{S} E\left(\mu^{k:B} - \mu^{S+1:B}\right) = \frac{1}{B}\sum_{k=1}^{S}\left[\sum_{i=k}^{B} i^{-1} - \sum_{i=S+1}^{B} i^{-1}\right] = \frac{1}{B}\sum_{k=1}^{S}\sum_{i=k}^{S} i^{-1} = \frac{1}{B}\sum_{i=1}^{S}\sum_{k=1}^{i} i^{-1} = \frac{S}{B}.$$

The buyer's utility function naturally extends to the positive reals by setting $u_b(S,B) = S/B$ if $B > S$ and $u_b(S,B) = 1$ if $B \leq S$. The seller's utility function can be extended to noninteger values of $B$ and $S$ by setting $u_s(S,B) = 0$ for $B \leq S$ and $u_s(S,B) = \Psi(B+1) - \Psi(S+1)$ for $B > S$, where $\Psi(x)$ is the digamma function. The property of the digamma function that makes this a natural extension is that

$\Psi(x) = -\eta + \sum_{i=1}^{x-1} i^{-1}$ when $x$ is an integer, where $\eta \approx 0.577$ is Euler's constant.[7] The digamma function has an asymptotic expansion of the form $\Psi(x+1) = \ln(x) + \frac{1}{2x} - \sum_{k=1}^{\infty} C_{2k} \frac{1}{x^{2k}}$. When $B$ and $S$ are large this gives the approximation $u_s(S,B) \approx \ln\left(\frac{B}{S}\right) - \frac{1}{2}\left(\frac{1}{S} - \frac{1}{B}\right) + o(1/S) - o(1/B)$.[8] Note that the seller's utility increases without bound as $B$ increases for a given $S$.

The $u_b$ and $u_s$ functions clearly satisfy (A1). (A2) is also satisfied , as $u_b$ has the desired monotonicity properties, and $u_s$ does because the digamma function is monotone increasing.[9] The fact that larger markets are more efficient implies that (A3) is satisfied on the integers. If the seller-buyer ratio is lower in the smaller market, then buyers will prefer the larger market. If the two markets have the same seller-buyer ratio, then buyers are indifferent, and efficiency (Proposition 2) implies that the sellers prefer the larger

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[7] The standard definition of the digamma function is $\Psi(x) = \Gamma'(x)/\Gamma(x)$ where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ is the gamma function.

[8] The coefficient $C_{2k}$ is the $2k^{\text{th}}$ Bernoulli number divided by $2k$. The first two values for these coefficients are $C_2 = -1/12$ and $C_4 = 1/120$. As a result, the approximation is quite accurate even for relatively small values of $B$ and $S$. For example, the error is approximating $u_s(S,B)$ is less than 0.01 for any $B$ if $S$ is at least 3.

[9] An alternate definition of the digamma function is $\Psi(z) = -\eta + \sum_{k=1}^{\infty} \frac{z-1}{k(k+z-1)}$. Differentiating this expression gives $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2}$, which is clearly positive.

one. By monotonicity, sellers also prefer the larger market if the seller-buyer ratio is higher in the small market. We believe that (A3) is satisfied for all noninteger $B$ and $S$ as well, but have only been able to show that it holds for sufficiently large $B$ (which is all that is required for the proposition we give below.) To see this, note that differentiating the asymptotic approximation to the digamma function gives

$$\frac{d}{dB}u_s(\gamma B, B) = \frac{\gamma(1-\gamma)}{2}\frac{1}{(\gamma B)^2} + \sum_{k=1}^{\infty} 2kC_{2k}\gamma(\gamma^{2k}-1)\frac{1}{(\gamma B)^{2k+1}}.$$

This will be positive for all $B$ greater than some cutoff $\underline{B}$.

For (A4), the market impact effect of adding a buyer to market 2 is $\frac{S_2}{B_2(B_2+1)} = \frac{\gamma}{B_2+1} > \frac{\gamma}{B}$, and the market impact of adding a seller is $\frac{1}{S_2+1}$.[10] Holding $\frac{S}{B}$ fixed, buyers are exactly indifferent between a smaller and a larger market. When $B_1/B > x$, the seller efficiency effect is bounded by

$$u_s(\gamma B_2, B_2) - u_s(\gamma B_1, B_1) \le \Psi(B_2+1) - \Psi(B_1+1) = \sum_{k=1}^{B_2-B_1}\frac{1}{B_1+k} \le \frac{B_2-B_1}{B_1+1} \le \frac{B_2-B_1}{xB}.$$

When $B$ is large enough so that (A3) is satisfied, Proposition 5 implies that we can find the range of values of $B_1$ for which a quasi-equilibrium exists by finding the intersection of the curves where (B1) and (S1) are exactly satisfied. In our calculations we use the approximation to the expected price given above, and thereby find approximate bounds on the quasi-equilibrium set for large $B$.

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[10] The digamma function always satisfies $\Psi(z+1) - \Psi(z) = 1/z$, so this expression is exact even for nonintegral values of $B$ and $S$.

Proposition 11

For any $\varepsilon > 0$ there exists a $\underline{B}$ such that for all $B > \underline{B}$ the model with $B$ buyers, $\gamma B$ sellers and exponentially distributed values satisfies

(a) If $\frac{B_1}{B} \in \left[\frac{1-\gamma}{4} + \varepsilon, \frac{3+\gamma}{4} - \varepsilon\right]$, then there exists an $S_1$ for which $(S_1, \gamma B - S_1, B_1, B - B_1)$ is a quasi-equilibrium; and

(b) If $\frac{B_1}{B} < \frac{1-\gamma}{4} - \varepsilon$ or $\frac{B_1}{B} > \frac{3+\gamma}{4} + \varepsilon$ then there is no $S_1$ for which $(S_1, \gamma B - S_1, B_1, B - B_1)$ is a quasi-equilibrium.

*Remark:* As with the uniform case, we can get intuition for the role of $\gamma = \frac{S}{B}$ in the range of equilibrium market sizes by comparing the seller's market impact and efficiency effects. (Buyers are indifferent between the two markets when the buyer-seller ratios are equal.) The price impact of adding a seller is proportional to $1/(S_2 + 1)$. Using the approximation $u_b(\gamma B, B) \approx \ln(1/\gamma) + (1/\lambda B - 1/B)/2$, the efficiency advantage of the large market is approximately

$$u_s(\gamma B_2, B_2) - u_s(\gamma B_1, B_1) \approx -\frac{1}{2}\left(\frac{1}{\gamma B_2} - \frac{1}{B_2}\right) + \frac{1}{2}\left(\frac{1}{\gamma B_2} - \frac{1}{B_2}\right) = \frac{1-\gamma}{2\gamma}\frac{B_2 - B_1}{B_1 B_2}.$$

When $\gamma$ is larger, the efficiency advantage is smaller for fixed $B_1$ and $B_2$, and thus $B_2 - B_1$ can be increased without violating the constraint that the efficiency effect must be smaller than the market impact.

*Remark*: Proposition 11 says that when $B$ is large quasi-equilibrium requires that the fraction of buyers in the small market be at least about $(1-\gamma)/4$. The number of buyers need not be very large for the "about" in this statement to be practically unimportant. For example, we've examined this numerically and found that when $\gamma = 0.2$ conclusions (a) and (b) of the proposition will be true for $\varepsilon = 0.01$ if $B$ is at least eleven. For $\gamma = 0.8$, they will be true for $\varepsilon = 0.01$ if $B$ is at least eighteen.

*Proof:* It suffices to show that for $B$ sufficiently large, all intersections of the $S_1^{B1}$ and $S_1^{S_1}$ curves with $B_1 < B/2$ have $B_1/B$ within $\varepsilon$ of $(1-\gamma)/4$. (B1) holds with equality if and only if $\frac{S_1}{B_1} = \frac{S_2}{B_2+1}$. Hence, $S_1^{B1}(B_1) = \frac{B_1}{B+1}\gamma B$.

The set of $B_1$ with $S_1^{B1}(B_1) = S_1^{S1}(B_1)$ is thus the set of solutions to

$$u_s\left(\frac{B_1}{B+1}\gamma B, B_1\right) - u_s\left(\gamma B - \frac{B_1}{B+1}\gamma B + 1, B - B_1\right) = 0.$$

Suppose that the LHS of this equation can be approximated by a function $f(B_1/B)$ in the sense that

$$\lim_{B\to\infty} B\left[ f(z) - u_s\left(\frac{zB}{B+1}\gamma B, zB\right) + u_s\left(\gamma B - \frac{zB}{B+1}\gamma B + 1, B - zB\right) \right] = 0$$

for all $z \in (0,1/2]$. If $\{\underline{B}_1(B)\}$ is a sequence of solutions to $S_1^{B1}(\underline{B}_1(B)) = S_1^{S1}(\underline{B}_1(B))$ for $B$=1, 2, … and $\underline{z}$ is a subsequential limit point of $\underline{B}_1(B)/B$ then $f(\underline{z}) = 0$. Hence, it suffices to show that for some choice of the approximating function $f$ the unique solution to $f(z) = 0$ in $(0,1/2]$ is $(1-\gamma)/4$.

We approximate the equation for (S1) holding with equality using the formula noted earlier:

$$u_s(S_1,B_1)-u_s(S_2+1,B_2)=\ln\left(\frac{B_1}{S_1}\right)-\ln\left(\frac{B_2}{S_2+1}\right)-\frac{1}{2}\left(\frac{1}{S_1}-\frac{1}{B_1}\right)+\frac{1}{2}\left(\frac{1}{S_2+1}-\frac{1}{B_2}\right)+o\left(\frac{1}{B}\right).$$

For $S_1=\frac{B_1}{B+1}\gamma B$ we have

$\ln\left(\frac{B_1}{S_1}\right)-\ln\left(\frac{B_2}{S_2+1}\right)=\ln\left(\frac{B+1}{\gamma B}\right)-\ln\left(\frac{B_2}{B_2+1}\frac{B+1}{\gamma B}\frac{S_2}{S_2+1}\right)=\frac{1}{B_2}+\frac{1}{S_2}+o\left(\frac{1}{B}\right)$. Plugging in $zB$ for $B_1$, $(1-z)B$ for $B_2$, $\gamma zB^2/(B+1)$ for $S_1$, etc. and approximating to first order, e.g. $B(1/B_1)=1/z$, $B(1/S_1)=B(B+1)/(\gamma zB^2)=1/\gamma z+o(1)$, etc. gives

$$2B(u_s(\frac{zB}{B+1}\gamma B,zB)-u_s(\gamma B-\frac{zB}{B+1}\gamma B,B-zB))=\frac{2}{(1-z)}+\frac{2}{\gamma(1-z)}-\frac{1}{\gamma z}+\frac{1}{z}+\frac{1}{\gamma(1-z)}-\frac{1}{(1-z)}+o(1).$$

Setting $f(z)$ equal to the function on the RHS of this expression and multiplying through by $\gamma z(1-z)$ we find $f(z)=0$ if and only if

$$2\gamma z+2z-(1-z)+\gamma(1-z)+z-\gamma z=0,$$

which reduces to $z=(1-\gamma)/4$.

QED

## 6 Thin Markets

We have seen that a substantial range of market sizes is possible in both the uniform and exponential cases. We now test the robustness of that conclusion to the possibility that markets are "thin," in the sense of there being very few items available for trade. Specifically, we suppose that each seller only has the good with a probability q<1, and investigate the effect of varying *q*. We suppose that buyers and sellers choose markets before either the buyer's uncertainty or the seller's is resolved; we think of $q$ as the probability that the seller has a good of the appropriate type to sell in the "current period." Thus when *S* is the number of potential sellers who enter a market, the number of actual units for sale is a random variable $\tilde{S}$ that is distributed binomial (q, S).

The exponential model is more tractable when this sort of uncertainty is considered, so this is the only case we analyze here. As noted above, the expected utility of a buyer in a market with *B* buyers and *S* units for sale is simply $\frac{S}{B}$. Thus when supply is random, if we let $\tilde{S}$ be the random number of units that are available for sale, we have that buyer expected utility is $E\left(\frac{\tilde{S}}{B}\right)=\frac{qS}{B}$. Since sellers only care about the price in the event that they have a unit of the good to sell, their expected utility conditional on having a good to sell from being in a market with *B* buyers and *S-1* other sellers is $\sum_{i=1}^{B} i^{-1} - 1 - E\left(\sum_{i=2}^{\tilde{S}} i^{-1}\right)$. If $qS \to 0$, the probability that any other seller has a unit for sale also goes to 0, and so all sellers prefer to be in the market with more buyers. More precisely, for large *B* there cannot be an equilibrium with two active markets and $B_1 \neq B_2$ if $\frac{1}{B} > qS$. This is intuitive: when q is very small, each seller expects to be a monopolist, and so prefers to be in the larger market; even if it has more sellers.

However, when $qS \to 0$ there are vanishingly few objects offered for sale, so this is a fairly extreme version of a thin market. We next we consider a somewhat less extreme version, with exactly three sellers, each of whom has an object to sell (so we go back to $q=1$.) Here there can be two active markets even when the number of buyers is very large. To see this in the exponential case, let $S_1 = 1$, $S_2 = 2$, $B_1 = B/3$ and $B_2 = 2B/3$. Then buyers receive exactly the same utility in both markets and strictly prefer not to switch. The seller's payoff in market 1 is $\sum_{i=1}^{B/3} i^{-1} - 1$, which is approximately

$$\ln\left(\frac{B}{3}\right) + \gamma - 1;$$

the seller's payoff in market 2 is $\sum_{i=1}^{2B/3} i^{-1} - 3/2$, which is approximately

$$\ln\left(\frac{2B}{3}\right) + \gamma - 3/2.$$

So the sellers in the large market get a higher utility by approximately $\ln 2 - 1/2 > 0$. Despite this difference in utility, the partition is still an equilibrium: because of the "crowding effect," the seller in the small market does not wish to move to the large one. If she did, the number of sellers there would increase to 3, and she would receive approximately $\ln\left(\frac{2B}{3}\right) + \gamma - 11/6$, which is less than her current utility because $\ln 2 \approx .693 < 5/6$.

A similar exercise shows that $S_1 = 1$, $S_2 = 2$, $B_1 = \frac{2(B+2)}{5} - 1$ and $B_2 = \frac{3(B+2)}{5} - 1$ is an equilibrium of the model with uniformly distributed buyer values for any B.

## 7. Related Work

There are several sets of papers that develop other sorts of models of multiple markets. One set is on consumers having a preference for shopping at markets with a more diverse range of products, as in Gehrig [1998]. In this model, firms choose between locations; all firms at given location are equally spaced on a circle that represents product characteristics, as in Salop [1979], and in equilibrium all firms at a given location charge the same prices. Consumers are located on the line a la Hotelling and pay a transportation cost to visit the markets, which are located at the endpoints. Additionally, as in Stahl [1982] and Wolinsky [1983], consumers prefer larger markets because they have a finer grid of available "varieties;" in Gehrig's model the externality arises because a finer grid of varieties reduces the expected distance between the closest available good and the consumer's preferred type. Our model abstracts away both the preference for more varieties at a site and the exogenous difference in market "locations."

A second set of papers is the finance literature on competing exchanges, whose starting point is the relationship between the volume of trade and the elasticity of demand, which is interpreted as "asset liquidity."[11] Pagano [1989] explores the implications of this relationship in a model of competing asset markets, where agents have identical CARA preferences, and are driven to trade because they have differing endowments of a risky asset. As in Kyle [1989], Pagano assumes that players submit demand functions as opposed to making simple bids, with the market outcome determined by an implicit "auctioneer." He finds that if transaction costs in the two markets are the same, the two markets can co-exist only if they are identical, but that the

[11] See Kyle [1985, 89] and Admati and Pfleiderer [1988].

two markets can co-exist if the choice of market influences the variance of endowment and if transaction costs differ across markets.[12] [13]

Then there is a literature on competing auctions in "large" markets, for example McAfee [1993] and Peters and Severinov [1997]. In these papers, as in our model, all goods are identical, and all agents are risk-neutral. However, these models assume that each market has a single seller, and they study only equilibria where each market has the same size. [14]

We should also mention the literature on the asymptotic efficiency of double auctions and other exchange mechanisms, e.g. Gresik and Satterthwaite [1989] and Satterthwaite and Williams [1989]. These papers study trade in settings of two-sided incomplete information, where we know from the Myerson-Satterthwaite theorem that no (incentive-compatible) mechanism can be ex-post efficient, and derive bounds on the rate at which the ex-post inefficiency disappears as the economy grows. Our welfare results concern a different sort of inefficiency: trade in a given market is always ex-post efficient in that market, but as Proposition 2 shows, ex-ante welfare is increasing in market size.

## *8.* Ideas for Future work

We would like to develop a model that incorporates adverse selection in the market-participation decision. Our casual empiricism suggests that a major reason that

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[12] Pagano calls the identical-market outcome for zero transaction costs a knife-edge because all traders would prefer market A if they expect it to be even slightly larger. However, the analysis throughout ignores terms of order $1/N^2$, which may matter for the existence and/or robustness of the two-market equilibria.
[13] Cuny [1993] develops a related model in which each market trades a single, possibly different, security.
[14] In Peters and Severinov, each seller runs a separate second-price auction with a reserve price; buyers observe the reserve prices and then choose a single auction in which to bid. The paper looks at symmetric equilibria in which all sellers attract an equal number of buyers. McAfee analyzes a related model that

the Amazon and Yahoo auction sites have struggled is that they tried to compete by having zero listing fees. This led to their listings being filled up with products being offered by non-serious sellers with very high reserve prices. If we suppose that there is a cost to reading web pages, or to investigating the quality of a good and/or its seller, then buyers will prefer to frequent sites with a high percentage of "good" listings- listings by reputable sellers who have high-quality goods and are willing to sell them at a reasonable price. In this case, a market with too many "bad" sellers might collapse. However, two markets might be able to co-exist if sites have some background flow of captive traffic from people who click in from Yahoo or Amazon without considering another auction site.

The issue of reserve prices poses a problem for a would-be new market site: On the one hand, when the market is new, sellers may not expect to get competitive bids, and so be unwilling to participate unless they can protect themselves with a reserve price. However, while the imposition of a uniform reserve price in a market can increase the payoff of sellers for any fixed buyer-seller ratio, it lowers the overall efficiency of the market, and so we would expect it to reduce the viability of the new market.

The main point of the paper is that the "market impact effect" of switching markets is of the same order as the "efficiency effect" favoring larger markets, so that some analysis is needed to determine which effect dominates. We explored this idea in the context of $k+1^{st}$ price sealed-bid auctions, but the same intuition should apply to other settings as well. Consider for example "matching markets," where the sole objective of both A's and B's is to be matched with an agent from the other group. Here too once can

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supposes players ignore the fact that a change in one seller's mechanism may change the distribution of show that larger markets are more efficient, but an A moving from market 1 to market 2 lowers the probability of A's being matched in market 2. Moreover, while we have not done the computations necessary to verify this, we suspect that both effects are again of order 1/n. So here too it seems premature to conclude that only the one market outcome is stable. Another example is Krugman's [1991] "Marshallian" model of location choice by firms and workers. This model differs from ours in that each firm hire multiple workers, but it is similar in that the larger market is more efficient, and that there is a market impact effect (which Krugman ignores) in finite markets.

Finally we should note that while the paper has analyzed competition between two markets, its analysis also applies to the study of 2M markets, M "smaller" and M "larger:" Such a configuration will be an equilibrium provided that it is an equilibrium for M= 1. One reason why such configurations may be less common in practice is that our model suggests they could be quite fragile –if two or more of the markets merge, the merged entity may be sufficiently large relative to the others so as to attract all of the patrons of every small market. This shows that any tendency to have only two markets, as opposed to more, must be due either to "relatively small" numbers of participants, or to agglomerative forces not captured by our model.

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buyers participating in other mechanisms, and thus alter buyers' incentives to participate in them.

<u>Appendix</u>

*Proof of Proposition 2*: One way to generate a sample $Y^M$ of *mB* draws from $F$ is to first generate a sample $Y^N$ of *nB* i.i.d. draws, and then randomly select a subset of *mB* elements; we will use this method to relate the distributions of the order statistics of the two samples. Let $q_i$ be the probability that the ith highest draw from $Y^N$ is one of the $mS$ highest elements of $S^M$. This probability is independent of the realized values of the order statistics; it depends only on which elements of $Y^N$ are chosen. In particular, for $i=1$ to $mS$, $q_i$ is simply the probability that the element in question is chosen, namely $\frac{m}{n}$; for $i = mS+1$ and thereafter each subsequent $q_i$ is strictly less than the preceding one since this $i = mS + j$ will only be one of the *mS* highest elements if it is chosen and at least *j* of the higher realizations are not.

Then

$$w(mS,mB) = E(v \mid v \geq v^{mS:mB}) = \frac{\sum_{i=1}^{nB} q_i E(v^{i:nB})}{mS},$$

while

$$w(nS,nB) = E(v \mid v \geq v^{nS:nB}) = \frac{\sum_{i=1}^{nS} E(v^{i:nB})}{nS} = \frac{\sum_{i=1}^{nB} Q_i E(v^{i:nB})}{nS},$$

where $Q_i$ is an indicator function that equals 1 for $i =$ 1 to *nS* and 0 otherwise.

So

$w(mS, mB) - w(nS, nB) = E(v \mid v \geq v^{mS:mB}) - E(v \mid v \geq v^{nS:nB})$

$$= \frac{\sum_{i=1}^{nB} q_i E(v^{i:nB})}{mS} - \frac{\sum_{i=1}^{nB} Q_i E(v^{i:nB})}{nS},$$
$$= \sum_{i=1}^{nB} c_i E(v^{i:nB})$$

where $c_i = \dfrac{nq_i - mQ_i}{nmS}$.

Now for $i =$ 1 to *m,* $c_i = 0$, for $i = m+1$ to n, $c_i$ is negative, and for $i > n$ $c_i$ is positive, and $\sum_{i=1}^{n} c_i$ = 1. Since the $E(v^{i:nB})$ are monotone decreasing in *i,* it follows that $\sum_{i=1}^{nB} c_i E(v^{i:nB}) < 0$.

QED

*Proof of lemma 1:* $S_1^{B1}(B_1)$ is well-defined because $u_B(S_1, B_1) - u_B(S - S_1, B - B_1 + 1)$ is continuous and monotone increasing in $S_1$ with

$u_B(0, B_1) - u_B(S, B - B_1 + 1) = -u_B(S, B - B_1 + 1) < 0$

and $u_B(S, B_1) - u_B(0, B - B_1 + 1) = u_B(S, B_1) > 0$.

Let $B^{\min}$ be the solution to $u_S(0, B^{\min}) = u_S(S+1, B - B^{\min})$. That an unique solution exists and is in $(0, B/2)$ follows from monotonicity and the boundary conditions $\lim_{B_1 \to 0} u_S(0, B_1) - u_S(S+1, B - B_1) = -u_S(S+1, B) < 0$ and $u_S(0, B/2) - u_S(S+1, B/2) > 0$. Let $B^{\max}$ be the solution to $u_S(S, B^{\max}) = u_S(1, B - B^{\max})$. That an unique solution exists and is in $(B/2, 1)$ follows similarly.

$S_1^{S1}$ is well-defined on $[B^{\min}, B^{\max}]$ because $u_S(S_1, B_1) - u_S(S - S_1 + 1, B - B_1)$ is continuous and monotone decreasing in $S_1$ with

$u_S(0, B_1) - u_S(S+1, B - B_1) \geq u_S(0, B^{\min}) - u_S(S+1, B - B^{\min}) = 0,$

and $u_S(S, B_1) - u_S(1, B - B_1) \leq u_S(S, B^{\max}) - u_S(1, B - B^{\max}) = 0$.

That $S_1^{B1}$ and $S_1^{S1}$ are monotone increasing and differentiable follows immediately from the implicit function theorem.

$S_1^{B1}(B_1) < S/2$ follows from $u_B(S/2, B/2) - u_B(S/2, B - B/2 + 1) > 0$.

$S_1^{S1}(B_1) = S/2 + 1/2$ follows immediately from the definition of $S_1^{S1}(B_1)$.

QED

*Proof of Proposition 9:* When prices are equal, both seller constraints are satisfied. Equal prices also imply that $\gamma = \dfrac{S_i + 1}{B_i + 1}$ is the same in both markets, so $(S_1 + 1)(B_2 + 1) = (S_2 + 1)(B_1 + 1)$. Then by canceling terms equal to $\gamma$ we can rewrite the buyer constraints as

$$(a') \quad \frac{S_1}{B_1} > \frac{S_2}{(B_2 + 2)}$$

$$(b') \quad \frac{S_2}{B_2} > \frac{S_1}{(B_1 + 2)}$$

Rewrite (a’) and (b’) as

$S_i(B_j + 2) \geq S_j B_i$.

Add and subtract terms to obtain

$(S_i + 1)(B_j + 1) + S_i - (B_j + 1) \leq (S_j + 1)(B_j + 1) - (S_j + B_j + 1)$

Divide both sides by $(B_i+1)(B_j+1)$

(*) $$\frac{S_i+1}{B_i+1} \geq \frac{S_j+1}{B_j+1} - \frac{B_j-(S_i+S_j+B_i)}{(B_i+1)(B_j+1)}$$

Using the fact that prices are equal, (*) is equivalent to

(**) $B_j \leq S_i+S_j+B_i$,

and this is equivalent to

$$\frac{B_i}{B} \leq \frac{1}{2}+\frac{S}{2B} \text{ or}$$

(***) $$\frac{B_i}{B} \in [\frac{1}{2}-\frac{S}{2B}, \frac{1}{2}+\frac{S}{2B}]$$

For any $B_1, B_2$ that satisfy (***), the buyer constraints are satisfied for the $S_1$ and $S_2$ that equate the expected prices in the two markets. The last step of the proof is to show that under (***) there must exist a pair $S_1, S_2$ that does equate the expected prices. Holding $B_1, B_2$ fixed, and setting $S_2 = S - S_1$, the difference in expected prices is

$$\overline{p}(S_1,B_1)-\overline{p}(S_2,B_2) = \frac{(B_1-S_1)(B_2+1)-(B_2-(S-S_1))(B_1+1)}{(B_1+1)(B_2+1)} = \frac{(B_1+1)(S-S_1)-(B_2+1)S_1+(B_1-B_2)}{(B_1+1)(B_2+1)}$$

which is a linearly decreasing function of $S_1$. When $S_1 = 0$ the difference is proportional to $S(B_1+1)+(B_1-B_2)$; from (***) this is at least $SB_1 > 0$. Similarly when $S_1 = S$ the difference is proportional to $-S(B_2+1)+(B_1-B_2) < -SB_2 < 0$. So there is a solution with $0 < S_1, S_2 < S$.

QED

*Proof of Proposition 10:* We will first construct an equal-price partition $(B_1, B_2, \hat{S}_1^j, \hat{S}_2^j)$ that approximates the target ratios, but where only $B_1$ and $B_2$ are guaranteed to be integers; we will then use this partition to construct an integer-valued partition $(B_1, B_2, S_1^*, S_2^*)$ where all of the incentive constraints are satisfied but prices are only approximately equal.

Assume that $\alpha < 1/2$, let $\gamma^* = \frac{\lfloor \gamma(B+2) \rfloor}{B+2}$, and $\alpha^* = \frac{\lceil \alpha(B+2) \rceil}{B+2}$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.[15] Note that $\gamma^* \le \gamma$ ,and $1-2\alpha^* \ge 1-2\alpha$; since we have already assumed that $\gamma > 1-2\alpha$, we know that $\gamma^* > 1-2\alpha^*$ for $B > \frac{2}{\gamma-(1-2\alpha)}$. Note also that for $B$ sufficiently large we have $\alpha^* < 1/2$.

Let $v^* = \min\{\alpha^*, 1-2\alpha^*\}$, and let $k = \left\lceil \frac{1}{v^*} \right\rceil$. Define $B_1 = \alpha^*(B+2)-1$, $B_2 = B - B_1$. For any non-negative integer *j,* define $\gamma^j = \gamma^* + \frac{2j}{B+2}$ and $\tilde{\gamma}^j = \gamma^* + \frac{j}{B+2}$.

If $\alpha^* \ge 1-2\alpha^* = v^*$, set $S^j = \gamma^j(B+2)-2$, $\hat{S}_1^j = \gamma^j(B_1+1)-1$, and $\hat{S}_2^j = \gamma^j(B_2+1)-1$.

If $\alpha^* < 1-2\alpha^*$, define

$S^j = \tilde{\gamma}^j(B+2)-2$, $\hat{S}_1^j = \tilde{\gamma}^j(B_1+1)-1$, and $\hat{S}_2^j = \tilde{\gamma}^j(B_2+1)-1$.

In either case, by construction $B_1, B_2$ and $S^j$ are integers, $\hat{S}_1^j + \hat{S}_2^j = S^j$, and $\frac{\hat{S}_1^j+1}{B_1+1} = \frac{\hat{S}_2^j+1}{B_2+1}$.

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[15] The case $\alpha > 1/2$ is symmetric. A separate argument is needed for $\alpha = 1/2$; we omit this argument here but will provide it on request.

If $\alpha^* \geq 1-2\alpha^* = \nu^*$,

$\frac{S^j}{B} = \frac{\gamma^j(B+2)-2}{B} = \frac{\gamma^*(B+2)-2}{B} + \frac{2j}{B} \in [\gamma^* - \frac{2(1-\gamma^*)}{B}, \gamma^* + \frac{2(k-(1-\gamma^*))}{B}]$, which is within $\varepsilon$ of $\gamma$ if $B > \frac{2k}{\varepsilon}$. If $\alpha^* < 1-2\alpha^*$, a similar calculation shows

$\frac{S^j}{B} \in [\gamma^* - \frac{1(1-\gamma^*)}{B}, \gamma^* + \frac{k-(1-\gamma^*)}{B}]$ ; this is also within $\varepsilon$ of $\gamma$ if $B > \frac{2k}{\varepsilon}$.

Note also that $\hat{S}_1^{j+1} - \hat{S}_1^j = \gamma^{j+1}(B_1+1) - \gamma^j(B_1+1) = (\gamma^{j+1}-\gamma^j)\alpha^*(B+2) = 2\alpha^*$.

If $\alpha^* < 1-2\alpha^*$,

$\hat{S}_1^{j+1} - \hat{S}_1^j = \tilde{\gamma}^{j+1}(B_1+1) - \tilde{\gamma}^j(B_1+1) = \alpha^*$.

The assumption that $\alpha \in \left(\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2}\right)$ and the bound on $\underline{B}$ implies that,

$\frac{B_1+1}{B+2} = \alpha^* \in [\frac{1}{2} - \frac{\gamma^j}{2}, \frac{1}{2}]$. Thus Proposition 4 implies that each partition $(B_1, B_2, \hat{S}_1^j, \hat{S}_2^j)$ satisfies all four incentive constraints. And

If $\hat{S}_1^j$ is an integer for any $j \in \{0,1,...,k\}$ we are done. If not, and if $\alpha^* \geq 1-2\alpha^* = \nu^*$, let m be the smallest integer j with $\left\lfloor \hat{S}_1^j \right\rfloor = \left\lfloor \hat{S}_1^{j-1} \right\rfloor$. We know that $\hat{S}_1^m - \left\lfloor \hat{S}_1^m \right\rfloor \geq 2\alpha^*$, so $\left\lceil \hat{S}_1^m \right\rceil - \hat{S}_1^m \leq 1-2\alpha^* = \nu^*$. If $\alpha^* < 1-2\alpha^*$, let m be the largest j integer with $\hat{S}_1^j < \left\lceil \hat{S}_1^0 \right\rceil$, then $\left\lceil \hat{S}_1^m \right\rceil - \hat{S}_1^m \leq \alpha^* = \nu^*$. In either case let $S_1^* = \left\lceil \hat{S}_1^m \right\rceil$ and $S_2^* = S - S_1^*$. We will now show that $\left(B_1, B_2, S_1^*, S_2^*\right)$ is an equilibrium; that is, the deviation of the partition from exactly equal prices (which is necessary to satisfy the integer constraint) is small enough that the incentive constraints are still satisfied.

Since we set the number of sellers in market 1 to be slightly higher than the number needed for equal prices, the constraints (a) - that buyers are willing to stay in market 1- and (d)- that sellers stay in market 2- will be the easiest to check. For (a), note that by Proposition 4, $(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$ satisfies the constraint; the fact that $S_1^* > \hat{S}_1^m$ implies it is satisfied by $\left(B_1, B_2, S_1^*, S_2^*\right)$. For (d), note that note that since $(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$ has equal prices, the fact that $S_1^* > \hat{S}_1^m$

$$\frac{S_2^*+1}{B_2+1} < \frac{\hat{S}_2^m+1}{B_2+1} = \frac{\hat{S}_1^m+1}{B_1+1} < \frac{S_1^*+1}{B_1+1} < \frac{S_1^*+2}{B_1+1}.$$

The constraint ( c) that sellers are willing to stay in market 1 requires that

$\dfrac{\hat{S}_1^m+1}{B_1+1}\dfrac{S_1^*+1}{\hat{S}_1^m+1} \le \dfrac{S_2^*+2}{B_2+1} = \dfrac{\hat{S}_2^m+1}{B_2+1}\dfrac{S_2^*+2}{\hat{S}_2^m+1}$ or, using the facts that prices are equal at

$(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$, $\dfrac{S_1^*+1}{\hat{S}_1^m+1} \le \dfrac{S_2^*+2}{\hat{S}_2^m+1}$. Since $S_1^* - \hat{S}_1^m = \hat{S}_2^m - S_2^*$, we can rewrite this as

$\dfrac{S_1^* - \hat{S}_1^m}{\hat{S}_1^m+1} \le \dfrac{S_1^m - S_1^* + 1}{\hat{S}_2^m+1}$, or $S_1^* - \hat{S}_1^m \le \dfrac{\hat{S}_1^m}{S}$. By construction $S_1^* - \hat{S}_1^m \le \nu^*$, and

$$\frac{\hat{S}_1^m}{S} \ge \frac{\hat{S}_1^0}{S} = \frac{\gamma^*\alpha^*(B+2)}{S} \ge \frac{\gamma B\alpha^*}{S} = \alpha^*.$$

Finally we come to the constraint that buyers be willing to stay in market 2. This is $\dfrac{S_2^*(S_2^*+1)}{B_2(B_2+1)} \ge \dfrac{S_1^*(S_1^*+1)}{(B_1+1)(B_1+2)}$, which we can write as

$\frac{S_2^*}{B_2}\left(\frac{S_2^*+1}{B_2+1} \bullet \frac{B_1+1}{S_1^*+1}\right) \geq \frac{S_1^*}{B_1+2}$. When we ignored the integer constraint, the term in the brackets was equal to 1; the issue now is whether the integer partition keeps this term close enough to 1. Note that $\frac{S_2^*+1}{B_2+1} = \frac{\hat{S}_2^m+1}{B_2+1}\frac{S_2^*+1}{\hat{S}_2^m+1} \geq \frac{\hat{S}_2^m+1}{B_2+1}\left(1-\frac{\nu^*}{\hat{S}_2^m+1}\right)$.

Also $\frac{B_1+1}{S_1^*+1} = \frac{\hat{B}_1+1}{\hat{S}_1^m+1}\frac{S_1^m+1}{S_1^*+1} \geq \frac{\hat{B}_1+1}{\hat{S}_1^m+1}(1-\frac{\nu^*}{S_1^*+1})$.

Using the fact that $\frac{\hat{S}_2^m+1}{B_2+1} = \frac{\hat{S}_1^m+1}{B_1+1}$, we conclude that the incentive constraint is satisfied if $\frac{S_2^*}{B_2}\left(1-\frac{\nu^*}{\hat{S}_2^m+1}\right)\left(1-\frac{\nu^*}{\hat{S}_1^*+1}\right) \geq \frac{S_1^*}{B_1+2}$.

If we rewrite this as

$\frac{S_2^*}{B_2}(1-x)(1-y) \geq \frac{S_1^*}{B_1+2}$, where $x$ and $y$ are defined as the two fractions insides of the large brackets, we see that a sufficient condition for incentive compatibility for large $B$ is

$\frac{S_2^*}{B_2}(1-\delta) > \frac{S_1^*}{B_1+2}$ for

$\delta = \frac{\nu^*}{\hat{S}_2^m+1} + \frac{\nu^*}{S_1^*+1}$. Moreover,

(*) $$\delta = \frac{\nu^*}{(1-\alpha)\gamma\alpha(B+2)} + O(\varepsilon / B)$$

By algebra similar to the proof of Proposition 4, the condition $\frac{S_2^*}{B_2}(1-\delta) \geq \frac{S_1^*}{B_1+2}$ is equivalent to

$$\text{(**)} \qquad \frac{S_2^*+1}{B_2+1} - \frac{S_1^*+1}{B_1+1} \geq \delta\left(\frac{S_2^*+1}{B_2+1}\right) + \frac{B_1 - S_2^*+1}{(B_1+1)(B_2+1)}(1-\delta) - \frac{S_1^*+B_2+1}{(B_1+1)(B_2+1)}.$$

We claim that the left hand side is at most $-\frac{v^*}{\alpha(1-\alpha)(B^*+2)}$ as shown by:

$$\frac{S_2^*+1}{B_2+1} - \frac{S_1^*+1}{B_1+1} = \frac{S_2^*+1}{B_2+1} - \frac{\hat{S}_2^m+1}{B_2+1} + \frac{\hat{S}_1^m+1}{B_1+1} - \frac{S_1^*+1}{B_1+1} \geq -v^*\left(\frac{1}{B_2+1} + \frac{1}{B_1+1}\right) \geq -\frac{v^*}{\alpha(1-\alpha)(B+2)}.$$

We also know that the term $\left(\frac{S_2^*+1}{B_2+1}\right)$ in the right-hand side of (**) is less than $\gamma$.

Therefore it will be sufficient to show that

$$-\frac{v^*}{\alpha(1-\alpha)(B^*+2)} \geq \delta\gamma + \frac{B_1 - S_2^*+1}{(B_1+1)(B_2+1)}(1-\delta) - \frac{S_1^*+B_2+1}{(B_1+1)(B_2+1)}.$$

Substituting the approximation for $\delta$ from (*) into the expression $\delta\gamma$, and using the fact that $\frac{\delta}{B} = o(1/B)$ to replace $1-\delta$ by 1, it is sufficient to show that

$$\frac{2v^*}{\alpha(1-\alpha)(B^*+2)} \leq \frac{S_1^*+B_2+1}{(B_1+1)(B_2+1)} - \frac{B_1 - S_2^*+1}{(B_1+1)(B_2+1)} + O(\varepsilon/B) + o(1/B).$$

The right hand side of this expression is

$$\frac{S_1^*+1}{(B_1+1)(B_2+1)} + \frac{B_2+1}{(B_1+1)(B_2+1)} - \frac{B_1+1}{(B_1+1)(B_2+1)} + \frac{S_2^*+1}{(B_1+1)(B_2+1)} + O(\varepsilon/B) + o(1/B) =$$

$$\frac{\gamma}{(1-\alpha)(B+2)}+\frac{1}{\alpha(B+2)}-\frac{1}{(1-\alpha)(B+2)}+\frac{\gamma}{\alpha(B+2)}+O(\varepsilon/B)+o(1/B).$$

Multiplying through by $\alpha(1-\alpha)B$ and collecting terms, we see that for the constraint is satisfied if

$$2v^{*}\leq\gamma+1-2\alpha+O(\varepsilon/B)B+Bo(1/B).$$

Since $v^{*}\leq 1-2\alpha$ and $v^{*}<\gamma$, we conclude that for all sufficiently small $\varepsilon$ (the necessary value depending on $\gamma-v^{*}$) the incentive constraint is satisfied for all sufficiently large *B*. This is sufficient to complete the proof, since we can always choose to carry out the above construction with $|S/B-\gamma|<\varepsilon'<\varepsilon$ if the exogenous $\varepsilon$ in the hypothesis of the proof is too large for this last step to be valid.

QED

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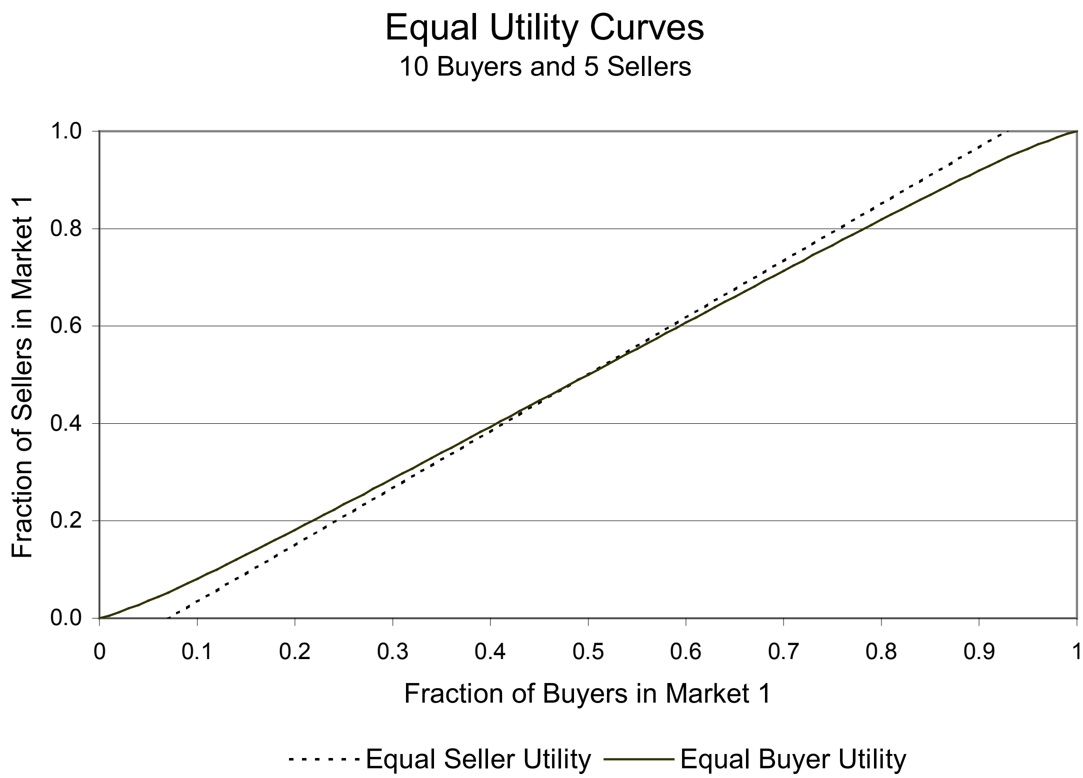
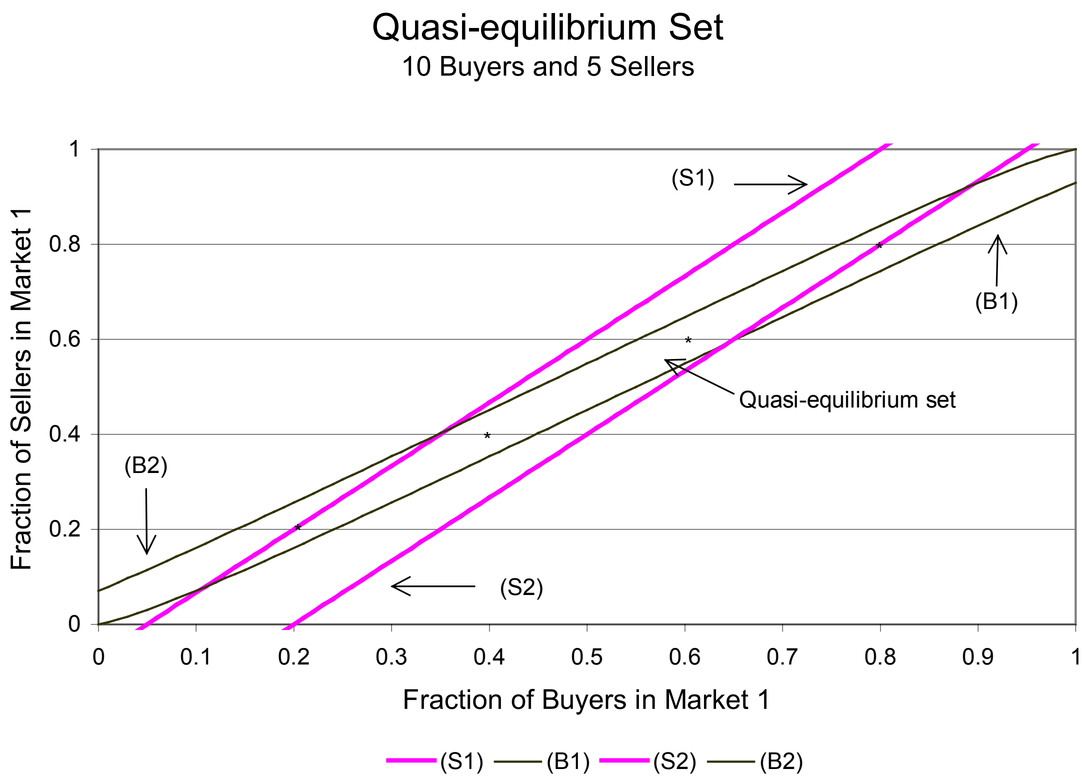

Equal Utility Curves
10 Buyers and 5 Sellers
Fraction of Sellers in Market 1
1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fraction of Buyers in Market 1
Equal Seller Utility
Equal Buyer Utility

**Figure 1**



Quasi-equilibrium Set
10 Buyers and 5 Sellers
Fraction of Sellers in Market 1
Fraction of Buyers in Market 1
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(S1)
(B1)
(B2)
(S2)
Quasi-equilibrium set
(S1) (B1) (S2) (B2)


**Figure 2**

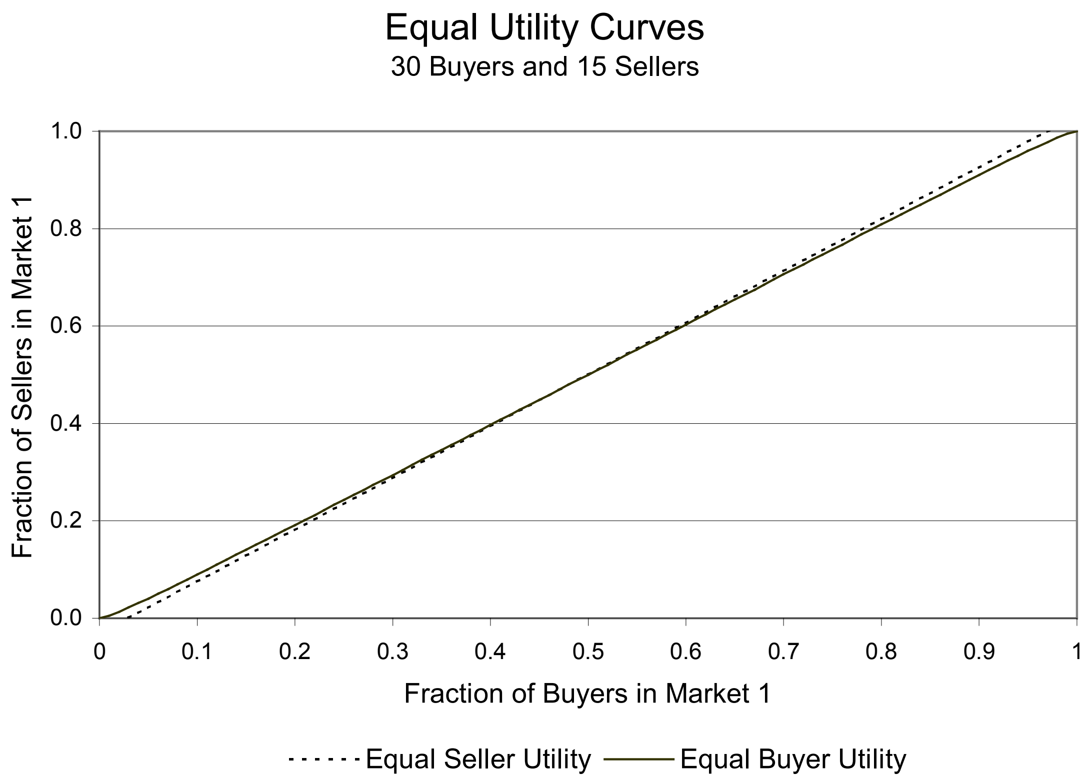

Equal Utility Curves
30 Buyers and 15 Sellers
Fraction of Sellers in Market 1
1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fraction of Buyers in Market 1
Equal Seller Utility
Equal Buyer Utility

**Figure 3**

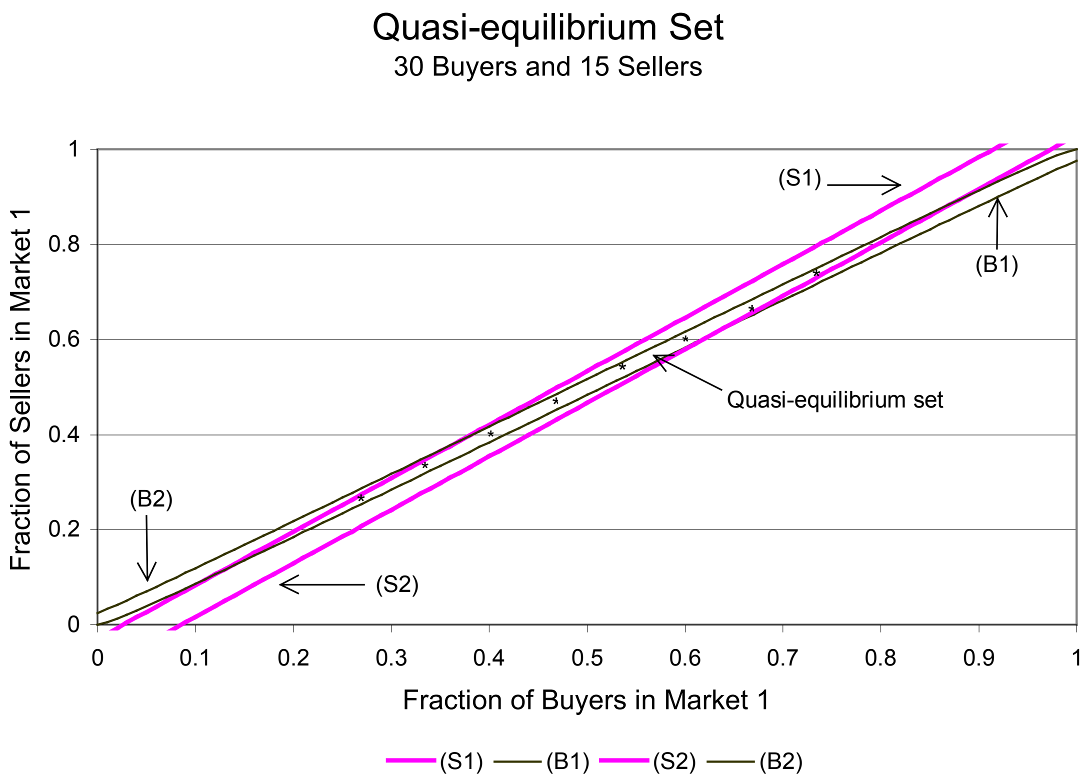

Quasi-equilibrium Set
30 Buyers and 15 Sellers
Fraction of Sellers in Market 1
Fraction of Buyers in Market 1
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(S1)
(B1)
(B2)
(S2)
Quasi-equilibrium set
(S1) (B1) (S2) (B2)

**Figure 4**