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Abstract

We show the existence of a pure strategy, symmetric, increasing equilibrium in double auction markets with correlated private valuations and many participants. The equilibrium we find is arbitrarily close to fully revealing as the market size grows. Our results provide strategic foundations for price-taking behavior in large markets.

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1 Introduction

This paper establishes the existence of pure-strategy equilibria for large double auctions markets with correlated private values. In these equilibria, bids are very close to valuations, and so can be interpreted as approximately truthful reports of the agents' information. Thus the equilibrium we find approximates price-taking behavior in large markets.

The idea of our proof is to use perturbations to ensure that best responses exist, and then use a fixed point theorem for increasing functions. With correlated private values, best responses need not be increasing in small auctions; the main difficulty in our argument is to show that best responses are indeed increasing when the auction is sufficiently large. We do this by first showing that, because each agent is rarely pivotal in large auctions, best responses to strategies that are approximately truthful are themselves approximately truthful. We then note that the only reason why best responses may be non-monotonic is that the agent's valuation conveys information about opponent values and bids. We prove that the size of this learning effect is proportional to the misrepresentation of an agent's bid, because the effect operates through changing the likelihood that the agent is pivotal. Since in large auctions bids are approximately truthful, the learning effect is small, and monotonicity obtains. So we get existence of equilibria in large perturbed auctions, and sending the perturbations to zero completes the proof. [1]

Jackson and Swinkels (2001) prove the existence of a (non-trivial) mixed-strategy equilibrium in a variety of auctions by taking the limits of equilibria in auctions with a discretized space of bids. Reny and Perry (2003), in an affiliated interdependent values setting, show that all sufficiently large, discretized auctions have an equilibrium with non-decreasing bid functions; this equilibrium approximates the rational expectations equilibrium of the continuum limit. This paper was inspired by an early version of Reny and Perry (2003); we obtain a somewhat stronger form of monotonicity without the use of grids and with a much shorter proof. Rustichini, Satterthwaithe, and Williams (1994) showed that in the independent private values case a symmetric equilibrium, if exists, must be close to truth telling, while Cripps and Swinkels (2003) showed that in a broad class of private value auctions all non-trivial equilibria are asymptotically efficient. Thus the main contribution of our paper is the existence result; we also extend Rustichini, Satterthwaithe, and Williams (1994) to correlated values.

Of course there are trade-offs between our methodology and the alternatives in the literature, and issues that are taken for granted in the finite case, such as continuity of best response, require more attention. On the other hand, the approach based on grids may only lead to mixed equilibria (as in the case of

[1] Athey (2001)'s existence theorem for one-sided auctions also makes use of monotonicity to appeal to a fixed point theorem; she obtains monotonicity from a single-crossing assumption.

Jackson and Swinkels (2001)) and needs to take extra care of ties.[2] Indeed, one main contribution of the paper by Jackson and Swinkels is their clever approach to dealing with ties. In our case, tie breaking is resolved with a small perturbation.

Our fixed point argument requires that the best responses exist, are unique, and are increasing. The first requirement is easier to deal with, as the only reason that best responses may fail to exist is that ties can have positive probability. To take care of ties, we perturb the model so that the distribution of the realized price conditional on any bid profile is a smooth random variable. This makes the payoffs into continuous functions of the bids, so that best responses exist. We ensure that best responses will be unique and increasing by restricting attention to large auctions, and to strategies that are approximately truthful. This allows us to apply a fixed point theorem to the compact space of increasing functions.

There are also a number of technical difficulties. Prominent among these is that even if best responses stay close to truth-telling, they need not be as close as the opponents' strategies, so that the best response map need not send a small neighborhood of the diagonal (that is, a set of approximately truthful strategies) into itself. To deal with this issue, we introduce a truncated best response map that forces the best response bid function of any player to remain sufficiently close to the diagonal. A related issue is that a best response does not need to be close to truth-telling when relevant bidder has zero probability of trading. To take care of this, we introduce a second small perturbation, namely that with small probability ε the price is drawn from an exogenous distribution. This guarantees that otherwise indifferent traders bid their true valuation.[3]

Given these modifications, we are able construct a fixed point of the "truncated (ε, φ)-perturbed best response map." Once we have the fixed point, we take the limit as φ, which measures the extent of the first perturbation, goes to zero. Then we make use of the argument in Rustichini, Satterthwaithe, and Williams (1994) to show that the limiting profile is in fact so close to the diagonal that we were not truncating at all. Finally, we send the magnitude of the second perturbation, ε to zero, and this gives a symmetric, pure strategy, increasing equilibrium of the κ-double auction.

The κ-double auction was introduced to the literature by Chatterjee and Samuelson (1983), and by Wilson (1985) for the multilateral case. The pure, symmetric, increasing equilibria of the auction we consider here were studied by Rustichini, Satterthwaithe, and Williams (1994) in the independent private values case. They do not prove existence, but they do show that any symmetric equilibrium has to

[2] This may not be exclusively for technical reasons. The nature of tie-breaking can matter for equilibrium existence in certain auctions, as pointed out by Maskin and Riley (2000). Moreover, the McAdams (2003) example of an equilibrium with non-monotonic bids in a one-sided auction also depends on the form of the tie-breaking rule.

[3] Jackson and Swinkels (2001) use a similar perturbation to avoid the no trade equilibrium.

be arbitrarily close to truth-telling if the number of participants is large enough. We extend their argument to the correlated private-values setting in this paper. Williams (1991) settles the existence question with independent private values in the the buyer's bid double auction. In that auction, sellers always bid their valuations, which makes the existence argument easier. Finally, the Reny and Perry (2003) and Jackson and Swinkels (2001) papers mentioned above provided existence results for more general distributions.

2 The Model

Consider the κ-double auction with m buyers and n sellers. The auction mechanism is defined as follows. Each seller has a single unit of the indivisible good, and each buyer wishes to purchase one unit. Sellers and buyers have correlated valuations that are private information. The joint distribution of uncertainty, as well as the structure of the game is common knowledge. Given her realized value, each actor submits a bid to the market. These bids are then ordered from highest to lowest. The market price is determined to be a weighted average of the n-th and $n+1$-th bids, with weights κ and $1-\kappa$ respectively. Buyers whose bids are above and sellers whose bids are below, this market price, buy or sell respectively one unit at the prevailing price. In the case of a tie (i.e., if a bid is equal to the price) some feasible tie-breaking mechanism is applied; the exact nature of tie breaking will not be relevant.[4] For a more detailed discussion of the κ-double auction please see Rustichini, Satterthwaithe, and Williams (1994).

We make a number of assumptions about the joint distribution of valuations. We focus on correlated private values. We will let s be a random variable that captures the common component of the valuations. The distribution of s is $G(s)$, which is assumed to be concentrated on the unit interval (the substantive part of this assumption is compact support) and absolutely continuous with respect to the Lebesgue measure. Conditional on s, the valuations of all buyers and sellers are independent. Let the conditional probability distribution of the valuation of a buyer i be $F_B(v_i|s)$, that of a seller j be $F_S(v_j|s)$ and let $f_B(v_i|s)$ and $f_S(v_j|s)$ be the corresponding densities. All valuations are concentrated on the unit interval, with density uniformly bounded away from zero for all s. From these distributions one can calculate the "inverse conditionals." Define $H_i(s|v_i)$ to be the conditional distribution of s given v_i for player i (who can be either a buyer or a seller) and let $h_i(s|v_i)$ be the corresponding density. We assume that $H_i(s|v_i)$ is absolutely continuous with respect to the Lebesgue measure, with full support on the set of values s assumes. Moreover, the density $h_i(s|v_i)$ is assumed to be uniformly Lipschitz in

[4]This mechanism is equivalent to constructing the piecewise linear demand and supply curves from buyers' and sellers' bids and then finding one of the many possible market clearing prices.

v_i and bounded away from zero. Note that this specification of correlated values includes the independent private values case (see Rustichini, Satterthwaithe, and Williams (1994) and Williams (1991)).

In the rest of the paper, we would like to prove that for a large enough number of participants, this auction game has a symmetric Bayesian equilibrium in pure strategies. We need to formalize the notion of large enough auctions. The idea is to increase n and m simultaneously, such that the ratio $\gamma = n/(n+m)$ remains bounded away from both zero and one. For convenience, we will use the notation $N = n+m$. Whenever we use the term "for large enough auctions", what we have in mind is increasing N while keeping γ bounded away from zero and one. Just how large the auction needs to be depends on how tight these bounds are.

We attack the problem by first finding equilibria of slightly different games. Fix $1 > \varepsilon, \varphi \geq 0$, and let us introduce the (ε, φ)-perturbed auction. In that game, payoffs are defined as follows: with probability ε the price $\tilde{p}$ is independent of the bids, and is drawn from a uniform distribution on $[0,1]$. In this event, the expected payoff of a buyer making bid b_i is $\int_0^{b_i} (v_i - p)\, dp$ and the expected payoff of a seller is $\int_{b_i}^1 (p - v_i)\, dp$. With remaining probability $1-\varepsilon$ the mechanism is as follows. The preliminary price p is determined by the standard κ-double auction but the actual price $\tilde{p}$ will be a smooth random variable which is φ-close to the preliminary price. Formally, the perturbed price $\tilde{p}$ is defined as

$$\tilde{p}(p, \omega_\varphi) = \begin{cases} 2p(\omega_\varphi - \frac{1}{2}) + p & \text{if } p < \varphi \\ 2\varphi(\omega_\varphi - \frac{1}{2}) + p & \text{if } \varphi \leq p < 1 - \varphi \\ 2(1-p)(\omega_\varphi - \frac{1}{2}) + p & \text{if } 1 - \varphi \leq p < 1 \end{cases} \tag{1}$$

where ω_φ is a uniform random variable on the unit interval. The price perturbation has the following properties: (1) the random variable $\tilde{p}$ is smooth (absolutely continuous) conditional on the preliminary price p and is always in the unit interval; (2) the function $\tilde{p}(p, \omega_\varphi)$ is continuous in p; (3) $\tilde{p}(p, \frac{1}{2}) = p$; (4) the perturbed price $\tilde{p}(p, \omega_\varphi)$ is strictly increasing in both p and ω_φ except when $p = 0$ or $p = 1$; (5) the perturbed price is φ-close to the preliminary price, i.e., $|\tilde{p}(p, \omega_\varphi) - p| < \varphi$ always holds. Any other perturbation which satisfies these five properties is suitable for our purposes. Properties (1), (2) and (4) make the expected payoffs a continuous function of the bids. Property (3) guarantees says that the median perturbation is zero. Property (4) also ensures that the event $\{\omega_\varphi | \tilde{p}(p, \omega_\varphi) < b\}$ is shrinking as p increases. Property (5) lets the perturbed game converge to the undisturbed game when we later first take φ to zero and then ε to zero. Note that all agents are allowed to make their desired trades at the actual price which need not result in a feasible outcome of the unperturbed game.

The ε perturbation ensures that bidding above one's private value is strictly dominated for a buyer because there is a positive probability that the actual price lies between the private value and the bid in which case the buyer suffers a loss. Likewise, sellers will never submit offers below their private value.

The way the uncertainty is structured in the perturbed auction is as follows. There is a probability space (Ω_s, μ_s) which generates the common signal s. For each agent i there is an independent uniform draw $(\Omega_{0,i}, \mu_{0,i})$ from the unit interval which generates the value $v_i = v_i(\omega_s, \omega_{0,i})$ of agent i. There is another probability space $(\Omega_\varepsilon, \mu_\varepsilon)$ that generates the ε probability event and the uniform price draw. Finally, $(\Omega_\varphi, \mu_\varphi)$ is a uniform draw from the unit interval, that generates the φ-perturbation and the perturbed price $\tilde{p} = \tilde{p}(p, \omega_\varphi)$. The complete probability space is denoted with (Ω, μ) where $\Omega = \prod_i \Omega_{0,i} \times \Omega_s \times \Omega_\varphi \times \Omega_\varepsilon$ and $\mu = \prod_i \mu_{0,i} \times \mu_s \times \mu_\varphi \times \mu_\varepsilon$. Elements of Ω are referred to with ω. We will frequently use the *conditional* measure μ^{v_i} of agent i who knows her private value v_i and which is defined as

$$\mu^{v_i}(A) = \frac{\mu\left(A \cap \{\omega | v_i = v_i(\omega)\}\right)}{\mu\left(\{\omega | v_i = v_i(\omega)\}\right)}. \tag{2}$$

We will denote the probability of some event A conditional on v_i with $P^{v_i}(A) = \int_A \mathrm{d}\mu^{v_i}(\omega)$ and the conditional expectation over some random variable $X(\omega)$ with $E^{v_i}(X) = \int X(\omega)\mathrm{d}\mu^{v_i}(\omega)$.

Formally, the strategy of agent i (buyer or seller) with private value v_i is referred to as $x_i(.)$ where $b_i = x_i(v_i)$ is the bid of agent i. The vector of all agents' strategies is denoted with $x(.)$ and $x_{-i}(.)$ refers to the strategies of all players except player i. Given the realization of uncertainty ω an agent i who makes bid b_i faces a price $p(b_i, x_{-i}(.), \omega)$. To simplify notation we will sometimes suppress the dependence of the price on other players' bidding strategies and write $p(b_i, \omega)$ or simply $p(b_i)$.

Finally, throughout the analysis we require that buyers play strategies that are not greater than their valuations, and similarly, we require that sellers play strategies that are not smaller than their valuations. This restriction is used to ensure that a best response exists for any strategy profile of the opponents. Since the strategies thus ruled out are weakly dominated for both buyers and sellers by bidding their true valuations, the equilibrium we find under the restriction will also be an equilibrium without the restriction.

3 Results

Our goal is to establish the following theorem:

Theorem 1 *The κ-double auction with correlated private values has a symmetric equilibrium in increasing pure strategies for all N large enough. In particular, there exists an equilibrium profile where each bid is of order $1/N$ close to being fully revealing.*

We begin by summarizing the main argument.

Fixed point. In order to use Schauder's fixed point theorem for Banach spaces, we need to use a continuous map defined from a compact, convex set of a Banach space to the same set. We choose the space to be $L^1[0,1]$. Compactness is ensured if all relevant functions are increasing and the φ-perturbation implies that the best response map is continuous. However, the best-response map need not map a given set of increasing strategies into itself, because a player's best response can be further away from truth telling then the opponents' strategies are. In order to deal with this difficulty, we introduce the truncated best response map; Schauder's fixed point theorem gives a fixed point of that map.

Symmetry. We now take φ to zero and consider a sequence of fixed points. This sequence has a limit by compactness. The limit turns out to be strictly increasing everywhere for two reasons. In the regions where trade takes place with positive probability, the limit is strictly increasing because of the symmetry of the profile. In the no-trade regions the limit is strictly increasing because of the ε-perturbation. Because of this strict monotonicity no ties are possible. It follows that the best response property is preserved under taking the limit. Therefore we can find a fixed point of the truncated best response mapping of the ε-perturbed auction.

Relaxing the truncation. Given a symmetric fixed point of the truncated best response mapping, we show that in the fixed point truncation is in fact never used. The argument applied here is borrowed from Rustichini, Satterthwaithe, and Williams (1994), who prove that any symmetric equilibrium stays close enough to truth telling (in the independent private values case). Using the logic of that paper, we show that our truncated fixed point never gets far enough from the diagonal for the truncation to become effective.

Taking ε to zero. Given a symmetric equilibrium for all ε, we can select a convergent subsequence. Because of symmetry, the limit will continue to be strictly increasing at all points where trade takes place with positive probability. At values where trade has zero probability, this need not be so. To check whether a limit profile is an equilibrium, we only need to verify the best response property at points where trade has positive probability; and at those points it is guaranteed by the no ties condition.

3.1 Best response of the perturbed game

Lemma 1 *In the (ε, φ)-perturbed κ-double auction, for $\varphi > 0$ a best response to any opponent profile exists.*

Proof: Fix a buyer i with valuation v_i, and consider the function $W(b,c) = \Pr(p(b) < c)$ where $b = b_i$ is the bid of buyer i. We begin by showing that $W(b,c)$ is continuous in both b and c. Note that

$$W(b,c) = \varepsilon c + (1-\varepsilon) \int_{p_0 \in [0,1]} \Pr(\tilde{p}(p_0, \omega_\varphi) < c) \; \mathrm{d}Q_0(p_0; b)$$

where p_0 is the preliminary price, $Q_0(.;b)$ is the distribution of p_0 given that buyer i bids b (and all opponents use their strategies) and $p = \tilde{p}(.,.)$ is the φ-perturbed price used to calculate payoffs.

The function $\tilde{p}(.,.)$ is by definition continuous and strictly increasing in both arguments, except when p_0 equals zero or one. It follows that $\Pr(\tilde{p}(p_0, \omega_\varphi) < c)$ is continuous in c for each $p_0 \neq 0, 1$. But the preliminary price is almost surely different from zero and one by our assumption that all opponent buyers and sellers bid weakly below, respectively above, their valuations. Hence the integral is also continuous in c. To check continuity in b, note that the map $b \to Q_0(.;b)$ from bids to distribution functions is continuous in the weak topology, because the preliminary price is a continuous function of bids. Because $\Pr(\tilde{p}(p_0, \omega_\varphi) < c)$ is a bounded, continuous function (in p_0), it follows that the integral is continuous in b.

Since $W(b,c)$ is continuous in both arguments and increasing in c, it is easy to see that for $b_k \to b$, we have $W(b_k,.) \to W(b,.)$ in the uniform topology. The payoff of i from bidding b equals

$$\int_{p \in [0,1]} v_i - p \; \mathrm{d}W(b,p)$$

which is easily shown to be continuous in b, by the uniform convergence of the functions $W(b,.)$. A symmetric argument applies for a seller j.[5] $\diamondsuit$

This lemma has shown the importance of the φ-perturbation: it guarantees that a best response exists. We now show that a best response to "almost truthful" strategies is also "almost truthful", where the "modulus of continuity" depends on the number of agents.

Theorem 2 *In the (ε, φ)-perturbed auction, with $\varepsilon, \varphi > 0$, for any $M > 0$ there exists $M' > 0$ such that in a large enough auction, if φ and ε are small enough and the opponents' strategies are M' close to the diagonal, then the best response of any buyer is at least M close to the diagonal.*

Proof: See appendix A.

[5] In the following we will only present proofs for buyers. All seller proofs are analogous.

The only reason why a player would bid differently from her valuation is to try to influence the price. If all players' bid functions are close to the diagonal and there are many players, then any bid is unlikely to be pivotal, so the best response must be close to the diagonal as well, though perhaps not as close as the original profile was. We also need that the best response not only stays close but is also increasing. The proof of this requires a simple lemma about the relation between the conditional expectations of a function given a value v_i and the conditional expectation given a different value v_i'.

Lemma 2 *There exists a constant K such that for any positive function $u(s)$*

$$\int_s u(s)\,|h(s|v_i') - h(s|v_i)|\ \ ds \leq K \cdot \Delta v \cdot \int_s u(s)\ \ h(s|v_i)\ \ ds. \tag{3}$$

Proof: We have

$$|h(s|v_i') - h(s|v_i)| \leq K \cdot \Delta v_i \cdot\ \ h(s|v_i)$$

for any s and v_i, v_i' because $h(.|.)$ is uniformly Lipschitz in the second argument and bounded away from zero. This completes the proof. $\diamondsuit$

Theorem 3 *In the (ε, φ)-perturbed auction, with $\varepsilon, \varphi > 0$, if $v_i - x_i(v_i) < 1/4K(K+1)$, then the best response to any opponent strategy profile is weakly increasing.*

Proof: See appendix B.

To get some intuition for this theorem we consider the first-order condition of a buyer i

$$T\left(b_i, v_i\right) = \left(v_i - b_i\right) r\left(b_i, v_i\right) - \kappa R\left(b_i, v_i\right) = 0, \tag{4}$$

where $r(b_i, v_i)$ is the density of the n-th highest bid at b_i of all other agents except i, and $R(b_i, v_i)$ is the probability that i has the n-th highest bid (and is therefore pivotal). The first terms captures the gain from raising a bid by a small amount (and becoming pivotal in the process) and the second term captures the cost of doing so. We next replace b_i with $x_i(v_i)$ and take the first derivative of the first order condition with respect to v_i (that is, we use the Implicit Function Theorem) to obtain

$$\frac{\partial T}{\partial b_i} x_i'(v_i) + \frac{\partial T}{\partial v_i} = 0. \tag{5}$$

We know that $\frac{\partial T}{\partial b_i} \leq 0$ because this is the second-order condition for a local maximum. Therefore, we only have to show that $\frac{\partial T}{\partial v_i} > 0$ to get increasingness of i's best response $x_i(v_i)$. We can calculate

$$\frac{\partial T}{\partial v_i} = \underbrace{r\left(b_i, v_i\right)}_{\text{Term I}} + \underbrace{\left(v_i - b_i\right)\frac{\partial r}{\partial v_i}}_{\text{Term II}} - \underbrace{\kappa\frac{\partial R}{\partial v_i}}_{\text{Term III}}\ . \tag{6}$$

Also, note that

$$\begin{aligned} r(b_i, v_i) &= \int_0^1 r(b_i, s) h(s|v_i) ds \\ R(b_i, v_i) &= \int_0^1 R(b_i, s) h(s|v_i) ds \end{aligned} \tag{7}$$

where $r(b_i, s)$ is the conditional density for some fixed s and $R(b_i, s)$ is the conditional pivotal probability. Lemma 2 implies that the partial derivatives $\frac{\partial r}{\partial v_i}$ and $\frac{\partial R}{\partial v_i}$ are of the same order of magnitude as $r(b_i, v_i)$ and $R(b_i, v_i)$. By the first order condition (4) $R(b_i, v_i)$ is the same order of magnitude as $(v_i - b_i)\, r\,(b_i, v_i)$. Therefore, as long as $v_i - b_i$ is small, both terms II and III are small while term I is large and determines the sign of the partial derivative $\frac{\partial T}{\partial v_i}$ which is indeed positive.[6]

Given this result and Theorem 2, we can choose C such for n large enough, if all opponents' strategies are at most C away from the diagonal, then any best response is weakly increasing. Fix such a C.

Theorem 4 *For n large enough, the best response to any profile that is at most C far from the diagonal is unique and increasing.*

Proof: We have just seen increasingness. For uniqueness, suppose that a buyer i has two best response functions, $x_i(v_i)$ and $x'_i(v_i)$. If these differ on a set of positive measure, then there is a point of continuity v_0 of both x_i and x'_i inside the unit interval where they differ (because both are increasing), say $x_i(v_0) > x'_i(v_0)$. But then there is a neighborhood of v_0 where this inequality continues to hold. Define x''_i to be equal to x'_i to the left of v_0, and equal to x_i to the right of v_0. Clearly x''_i is a best response, since it is a best response for almost every valuation v_i. However, x''_i is not increasing; to the left of v_0 it approaches $x_i(v_0)$, and to the right of v_0 it approaches $x'_i(v_0)$. Thus x''_i is a non-increasing best response. This is a contradiction. $\diamondsuit$

3.2 Fixed point

We now introduce the truncated best response mapping of the perturbed game. For the C fixed above, let X be the set of weakly increasing functions defined on the unit interval with values in the C wide corridor around the diagonal. Formally,

$$X = \{x_i : [0,1] \to [0,1] \mid x_i(v_i) \leq x_i(v'_i) \text{ if } v_i \leq v'_i, \text{ and } |x_i(v_i) - v_i| \leq C \; \forall v_i, v'_i\}\,.$$

[6]In the special case of independent private values terms II and III are zero - therefore the best response of player i is always increasing in v_i.

Furthermore, define X_B and X_S to be the subsets of X corresponding to profiles that are (weakly) below, respectively above, the diagonal, that is

$$X_B = X \cap \{x_i : [0,1] \to [0,1] \mid x_i(v_i) \leq v_i\}$$

and

$$X_S = X \cap \{x_i : [0,1] \to [0,1] \mid x_i(v_i) \geq v_i\}\,.$$

When all opponents play strategies in X the best response of any player is unique and increasing, but it need not be in X. The truncated best response map just truncates this best response by setting it equal to the bound of the corridor at points v_i where it is outside. Formally, if the best response of a buyer is $x_i(v_i)$, the truncated best response at v_i is equal to $\max(x_i(v), v_i - C, 0)$. Importantly, the truncated best response is still increasing, since it is the max (min in the case of a seller) of increasing functions.

Lemma 3 *X, X_B and X_S are compact, convex subsets of the Banach space $L^1[0,1]$.*

Proof: The L^1 topology restricted to X is the same as convergence in measure. Indeed, a set of uniformly bounded functions converge to a limit in L^1 if and only if they converge in measure. In addition X is also compact: By Helly's theorem (see Billinglsley (1999)), X is compact in the weak topology, so from any sequence of functions in X we can select a subsequence that is converging to some limit function in all of its points of continuity. But this implies almost everywhere convergence, and that implies convergence in L^1 by Lebesgue's dominated convergence theorem, since all function involved are in X. Since X_B and X_S are convex and closed subsets of X, the conclusion follows for them too. $\diamondsuit$.

In the rest of the paper we will focus on symmetric profiles, that is, all buyers respectively all sellers using the same bidding function. Therefore we only have to keep track of the pair $x = (x_B(.), x_S(.))$ of the buyer's and the seller's strategy. Then we have $x \in X_B \times X_S$.

It is clear that $X_B \times X_S$ is a compact, convex subset of the product Banach space $L^1[0,1] \times L^1[0,1]$. Now for any positive (ε, φ) we have the truncated best response map $TBR^{\varepsilon,\varphi}(.) : X_B \times X_S \to X_B \times X_S$. Then $TBR^{\varepsilon,\varphi}(x_B, x_S)$ is the pair of truncated best responses for any buyer respectively seller, when all opponent buyers play $x_B(.)$, and all opponent sellers play $x_S(.)$. We are interested in a fixed point of this map. To get one, first we need continuity.

We will now consider a sequence of perturbed games for $k = 1, 2, ...$ Game k will have perturbations $(\varepsilon^k, \varphi^k)$. Suppose the strategies played in game k are $x^k = (x^k_B, x^k_S)$, and assume that $y^k = (y^k_B, y^k_S)$ is a best response to x^k in game k. We are interested in whether the best response property is preserved as k goes to infinity. Note that this framework incorporates continuity of the best response map if the sequence $(\varepsilon^k, \varphi^k)$ is constant.

Fix a player i with value v_i. If the player bids b_i in the k-th game the price is denoted with $p^k(b_i)$ which is a random variable. The price in the limit game is denoted by $p(b_i)$.

Lemma 4 *Suppose $x^k \to x$ in measure, and $y^k \in BR^k(x^k)$ converges in measure to y. Assume for all players i with value v_i and bid $b_i = x_i^k(v_i)$ (1) opponents play strategies $x_{-i}^k(.)$; (2) the price $p^k(b_i)$ converges in probability to $p(b_i)$; (3) the distribution of $p(b_i)$ has no atom at b_i. Then $y \in BR(x)$.*

Proof: See appendix C.

The intuition for this result is the following. Suppose player i is a buyer, and denote the payoff to this buyer with value v_i and bid b_i in the k-th game by $\Pi^k(v_i, b_i)$. If $p^k(b_i)$ converges in probability, it also converges in distribution to $p(b_i)$. Hence in particular the distribution function of $p^k(b_i)$ converges to that of $p(b_i)$ at all points of continuity of the latter, in particular at b_i. In addition, the payoff function to bidding b_i equals

$$\Pi^k(v_i, b_i) = \int_{p<b_i} v_i - p \; \mathrm{d}R^k(p(b_i))$$

where $R^k(p(b_i))$ is the price distribution in game k where our buyer bids b_i. By weak convergence, this payoff converges to the limit payoff

$$\Pi(v_i, b_i) = \int_{p<b_i} v_i - p \; \mathrm{d}R(p(b_i)).$$

Note that the integrand is not everywhere continuous, so we cannot directly apply the weak convergence result; however, the limit distribution function is continuous at b_i, so the point of discontinuity of the integrand at b_i does not cause a problem.

The lemma will have a number of applications regarding continuity of the best response map and taking the limit as φ and ε are going to zero. In order to state these applications, we need to introduce a concept that we call " positive probability of trade." We say that a player i in the (ε, φ)-perturbed game with opponent profile x_{-i} has positive probability of trade at value v_i and bid b_i, if in the $1-\varepsilon$ probability event when the price is not drawn from a uniform distribution, there is positive probability that the player gets a positive payoff (gets to trade). This is equivalent to her expected payoff being strictly larger than $\varepsilon \cdot \int_0^{b_i} (v_i - p)\, dp$ in case of of a buyer, or strictly larger than $\varepsilon \cdot \int_{b_i}^1 (p - v_i)\, dp$ in the case of a seller.

Corollary 1 *Suppose that either*

(a) $\varphi^k = \varphi > 0$ and $\varepsilon^k = \varepsilon > 0$ fixed, or

(b) $\varphi^k \to 0$, $\varepsilon^k = \varepsilon > 0$ is fixed, and x is strictly increasing, or

(c) $\varphi^k = 0$, $\varepsilon^k \to 0$, $y^k = x^k$ and x is strictly increasing for each player i at all values v where there is positive probability of trade.

Then $y \in BR(x)$.

Proof: We will use the previous lemma. The convergence in probability of the conditional prices is obvious in all three cases, given that the bid functions converge in measure, and that the perturbations are continuous. Fixing a buyer, we only need to check whether the limiting price $p(b_i)$ has a distribution that is atomless at b_i. This is obvious in case (a) because the price distribution is completely atomless by the perturbation. In case (b), it follows from the fact that all opponent profiles are strictly increasing. Indeed, any atom that $p(b_i)$ may have at b_i has to come from some opponent bidding b_i with positive probability; this is ruled out.

In case (c), suppose b_i^k converges to b_i, and b_i is not a best response in the limit game, so b_i' does better. We can assume that no opponent bids b_i' with positive probability, otherwise we could have chosen b_i'' that is a little bit larger than b_i', and still get a higher payoff than that earned by b_i. Therefore, by the argument of the lemma, $\Pi^k(v_i, b_i') \to \Pi(v_i, b_i')$. Now if no other player bids b with positive probability, then by the argument of the lemma we also have that $\Pi^k(v_i, b_i^k) \to \Pi(v_i, b_i)$, which yields a contradiction. Thus the only problem we may have is that some opponent bids b_i with positive probability in the limit game. If this opponent is a buyer, then by assumption (c) no trade takes place for a buyer with bid b in the limit game. But then as k goes to infinity, the payoff to bidding b_i^k must be vanishingly small. If the payoff to bidding b_i' is positive in the limit game, then by $\Pi^k(v_i, b_i') \to \Pi(v_i, b_i')$ bidding b_i' for k large enough is better than bidding b_i^k. This is a contradiction.

If the opponent who bids b_i with positive probability is a seller, then that seller faces no trade in the limit game. Thus all buyers have to bid below b_i with probability one. But note that $b_i = y_B(v_i) = x_B(v_i)$. By assumption, $x_B(v_i)$ is increasing here, because buyers trade with positive probability at this stage. So $x_B(\frac{v_i+1}{2}) > x_B(v_i) = b_i$, but then our seller who bids b should get to trade with positive probability. This is a contradiction.

Note how this final argument hinges on the fact that the x^k profiles were already equilibria of game k. $\diamondsuit$

Corollary 2 *In the (ε, φ)-perturbed auction, with $\varepsilon, \varphi > 0$, the truncated best-response map is continuous.*

It follows that $TBR^{\varepsilon,\varphi}(.)$ is a continuous self-map of a compact, convex subset of a Banach space.

Theorem 5 *The truncated best response map $TBR^{\varepsilon,\varphi}(.)$ for $\varepsilon, \varphi > 0$ small enough and all large enough auctions has a fixed point.*

Proof: Immediate from Schauder's fixed point theorem. $\diamondsuit$

3.3 Taking φ to zero

Lemma 5 *Let z^k be a fixed point of $TBR^{\varepsilon_k,\varphi_k}$ and suppose $z^k \to z$. If $(\varepsilon_k, \varphi_k)$ are such that either*

(a) $\varepsilon_k = \varepsilon$ and $\varphi_k \to 0$, or

(b) $\varepsilon_k \to 0$ and $\varphi_k = 0$.

then z is strictly increasing at all points where there is positive probability of trade in the limiting game.

Proof: Suppose not, and let $[v_i, v_i']$ be an interval where the buyer's bid function is constant b_i. Then this has to be a region where the truncated and the non-truncated best responses are the same. Then for k large, the k-th bid function will be δ close to this plateau. Because there is positive probability of trade, for k large enough, with probability bounded away from zero there is a seller's bid below $b_i^k(v_i)$. Then it is an event with probability bounded away from zero that all buyers have values in $[v_i, v_i']$ and all sellers bid below $b_i^k(v_i)$.

But then by increasing her bid by 2δ, a buyer of value v_i could get an incremental probability of winning that is bounded away from zero. For δ small, the cost of this bid increment in terms of price impact is arbitrarily small. Thus for k large bidding $b_i^k(v_i)$ cannot be optimal for a buyer of value v_i. This is a contradiction. $\diamondsuit$

The firs part of the Lemma allows us to show that the $(\varepsilon, 0)$-perturbed double auction has a (truncated) fixed point.

Corollary 3 *In a large enough auction, the truncated best response map $TBR^{\varepsilon,\varphi}(.)$ with $\phi = 0$ and ε small enough has a fixed point.*

Proof: Now fix $\varepsilon > 0$ and pick a sequence $\varphi^k \to 0$, and let z^k be a fixed point of TBR^k. By compactness, the sequence z^k has a convergent subsequence. By relabelling, we can assume that z^k converges to z. Now at all points where there is positive probability of trade, z is strictly increasing. At a point where there is zero probability of trade, as k goes to infinity there had to be vanishingly small probability of trade. But then the ε probability event would gradually overrule any other consideration. Thus at points with zero probability of trade, bidders have to bid their own value. It follows that z is everywhere strictly increasing.

Then by the above lemma, we have that $z \in TBR(z)$. To see why, note that $w^k = BR^k(z^k)$ also has a convergent subsequence. The limit of that sequence, w, has to be a best response to z because z is everywhere strictly increasing. But then the limit of $TBR^k(z^k) = T(w^k)$ has to be $T(w)$ as truncation is a continuous operation. On the other hand, $TBR^k(z^k) = T(w^k) = z^k$, thus the limit of z^k, which is z, is equal to $T(w)$. It follows that $z = T(w)$, in other words, z is indeed a truncated best response to z. $\diamondsuit$

3.4 Relaxing the Truncation

Theorem 6 *In the $(\varepsilon, 0)$-perturbed double auction for all sufficiently large N any fixed point of the truncated best response map is a Bayesian Nash equilibrium.*

We prove this result by showing that for all sufficiently large auctions the truncation $|v_i - x_i(v_i)| \leq C$ does not bind.

Consider a fixed point of the ε-perturbed auction. We denote $b_i = x_B(v_i) = x_S(v_S)$. The first-order condition of a buyer i with valuation v_i becomes

$$(1-\varepsilon)\int_s \left[(v_i - b_i)\left(nK^s_{n,m}(b_i)\frac{f_S(v_S|s)}{x'_S(v_S)} + (m-1)L^s_{n,m}(b_i)\frac{f_B(v_i|s)}{x'_B(v_i)}\right) - \right.$$
$$\left. - \quad \kappa M^s_{n,m}(b_i)\right] dH(s|v_i) + \varepsilon(v_i - b_i) = 0 \tag{8}$$

where:

$K^s_{n,m}(b_i) \equiv$ the probability that bid b_i lies between the nth and (n+1)th highest bid in a sample of $m-1$ buyers not including buyer i and $n-1$ sellers and conditional on the common signal s

$L^s_{n,m}(b_i) \equiv$ the probability that bid b_i lies between the nth and (n+1)th highest bid in a sample of $m-2$ buyers not including buyer i and n sellers and conditional on the common signal s

$M^s_{n,m}(b_i) \equiv$ the probability that bid b_i lies between the (n-1)th and nth highest bid in a sample of $m-1$ buyers not including buyer i and n sellers and conditional on the common signal s

These probabilities are defined for the special case of independent private values as formulae A.6 to A.8 in the appendix of Rustichini, Satterthwaithe, and Williams (1994). For correlated private values we simply replace $F_B(v_i)$ with $F_B(v_i|s)$ and

$F_S(v_S)$ with $F_S(v_S|s)$. Equation 8 consists of two parts: with probability $1-\varepsilon$ the price in the auction is determined according to the double auction rules. In this case a slight increase in the bid b_i gives the buyer a chance to win an object and gain $v - b_i$. There are n sellers he can surpass and m buyers and the density of a seller's/ buyer's offer at b_i is $\frac{f_S(v_S|s)}{x'_S(v_S)}$ and $\frac{f_B(v_i|s)}{x'_B(v_i)}$ respectively. The probabilities $K^s_{n,m}$ and $L^s_{n,m}$ are the respective probabilities that the other seller and buyers valuations are ordered accordingly. The cost to buyer i of increasing her bid b_i is increasing the price if she is just pivotal which happens with probability $M^s_{n,m}(b_i)$. The second part of equation 8 is the marginal benefit to buyer i of increasing her bid if the price is a uniform draw from the unit interval.

It follows that

$$\begin{aligned} v_i - b_i &\leq \frac{\kappa(1-\varepsilon)\int_s M^s_{n,m}(b_i)\,dH(s|v_i)}{(m-1)(1-\varepsilon)\int_s L^s_{n,m}(b_i)\,f_B(v_i|s)dH(s|v_i)+\epsilon} x'_B(v_i) \\ v - b_i &\leq \frac{\kappa\int_s M^s_{n,m}(b_i)\,dH(s|v_i)}{(m-1)\int_s L^s_{n,m}(b_i)\,f_B(v_i|s)dH(s|v_i)} x'_B(v_i) \qquad (9) \end{aligned}$$

Now define $F_{min}(v) = \min_{s,i} F_i(v|s)$ and $F_{max}(v) = \max_{s,i} F_i(v|s)$. Similarly, define $f_{max} = \max_{i,s,v_i} f_i(v_i|s)$ and $f_{min} = \min_{i,s,v_i} f_i(v_i|s)$. Note, that $0 < f_{min} < f_{max} < \infty$.

Lemma 6 *The ratio $M^v_{n,m}(b_i)/L^v_{n,m}(b_i)$ satisfies*

$$\frac{M^v_{n,m}(b_i)}{L^v_{n,m}(b_i)} \leq 2F_{max}(v_i) + \frac{2n}{m}\frac{(1-F_{min}(v_i))\,F_{max}(v_i)}{1-F_{max}(v_i)} < D \qquad (10)$$

for some constant D and $v_i \in [0,1]$.

Proof The proof for the independent private values case can be found in the appendix of Rustichini, Satterthwaithe, and Williams (1994) (proof of inequality 3.12). The proof for correlated private values is exactly the same - we simply integrate all expressions over s. $\diamondsuit$

We also know that $f_B(v_i|s) \geq f_{min} > 0$ by assumption. Hence we obtain:

$$v_i - b_i < \frac{\kappa D}{f_{min}(m-1)} x'_B(v_i) \qquad (11)$$

This implies that for large enough auctions, the truncated best response remains arbitrarily close to the diagonal at all points v_i where the truncation is not effective. Indeed, at points where $x'_B(v_i)$ is less then 1 this follows from the above inequality. At points where $x'_B(v_i)$ is greater than one, this follows because at those points the trajectory $x_B(v_i)$ is actually getting closer to the diagonal! The same argument extends to seller.

If the truncated best response stays closer than $C/2$ at all point where truncation is ineffective then truncation can never apply. At the no trade region for buyers with low values of v_i, the truncated best response is just bidding one's valuation as we have seen. Once we enter the trade region, the only way the truncated best response can start to be truncated is if it hits the truncation border $v_i - C$. But this never happens, as before hitting the border the above inequality applies, and it guarantees that the trajectory does not even get below the $v_i - C/2$ border. $\diamondsuit$

3.5 Taking ε to zero

The only remaining step is to take ε to zero. Consider a sequence ε^k going to zero, and let z^k be an equilibrium of the ε^k-perturbed game. As usual, we can select a convergent subsequence with a limit z. By Lemma 5 (b), at all points v that have positive probability of trade, z is increasing. By part (c) of Corollary 1, the best response property is preserved under the limit.

For the sake of completeness, here we show that if at value v_i the limiting bid $x_i(v_i)$ is such that no trade occurs, then no trade would occur for this buyer no matter what her bid would be (given her value is v_i). One might think that a bid higher than $x_i(v_i)$ (still not greater than v_i) may actually secure trade with positive probability. But in that case, she should have bid higher for k large enough, because bidding close to her current bid gives vanishingly small utility, whereas e.g., bidding v_i would have given her utility bounded away from zero for k large. This is a contradiction.

It follows that the limiting profile z is a symmetric, increasing equilibrium of the κ-double auction. We have just proved Theorem 1.

4 Conclusion

Independent private values. This case has been studied in Rustichini, Satterthwaithe, and Williams (1994) and Williams (1991) among others. For that auction the best responses are automatically increasing, so we do not need to truncate the best response functions to apply our fixed point argument, and we can show that equilibria exist for any n. Of course, the part of our argument that shows that equilibrium is approximately truthful does depend on n being large, as in small auctions each agent has some market power.

Relaxing symmetry. Our proof relies on all buyers respectively all sellers being identical, and on the symmetric nature of the equilibrium. This assumption can be relaxed: Our techniques apply to auctions with j classes of buyers and k classes of sellers, where all agents of a given type have identically distributed values and use the same strategies.

Affiliated values We would like to extend our analysis to the affiliated values case, where the analog of "bidding truthfully" is "bidding one's value conditional on being pivotal." We conjecture that this extension is feasible; the main difficulty that we see is in providing the appropriate extension of the Rustichini et al characterization.

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A Proof of Theorem 2

The proof of the theorem requires the following Lemma.

Lemma 7 *In the (ε, φ)-perturbed auction, with $\varepsilon, \varphi > 0$, if opponents' strategies are M' close to the diagonal, then the following inequality holds:*

$$\Pr(b_i \textit{ is the } n+1\textit{-st bid}) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} \cdot C \geq \Pr(b_i \textit{ is the } n\textit{-th bid}).$$

Proof: Fix the common part of all valuations, s, and a set of n players i_1,...,i_n. Suppose the state of the world (conditional on s) is such that exactly these players have bids above b_i. Denote the probability of this event by

$$P^{i_1,\ldots,i_n}.$$

Now pick one of these n players, say i_1, and consider the event when i_1 bids below b_i, while i_2,...,i_n are the players who bid above b_i. Call the corresponding new event the transformed event. This event has a probability of at most

$$P^{i_1,\ldots,i_n} \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'}.$$

Indeed, given s, individual bids are independent. If player i_1 has value greater than $b_i + 2M'$, then she certainly bids above b_i, because bids are M' close to valuations. Thus the conditional probability that i_1 bids above b_i is at least $f_{\min} \cdot (1 - b_i - 3M')$. Likewise, the conditional probability that bids i_1 bids below b_i is at most $f_{\max} \cdot (b_i + 3M')$.

Next pick instead player i_2, and let her bid below b_i, while the others continue to bid above. We can estimate the probability of this event the same way as above; adding up these bounds for all players gives

$$n \cdot P^{i_1,\ldots,i_n} \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} \geq Q(i_1, \ldots, i_n).$$

Here $Q(i_1, \ldots, i_n)$ denotes the probability of the event where exactly one of the players in the set i_1,...,i_n bids below b_i (all the others in the set bid above, and the rest of the players also bid below).

Summing this last inequality for all subsets i_1,...,i_n yields

$$n \cdot \Pr(b_i \text{ is the } n+1\text{-st bid}) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} \geq \sum Q(i_1, \ldots, i_n).$$

The sum on the right hand side covers all states of the world where there are exactly $n-1$ bids above b_i, and moreover that it counts each such state m times. It is clear that any such state of the world is covered, because there is at least one i_1,...,i_n set that can give rise to it. More importantly, a state appears in the summation when one of the bids currently below b_i is shifted there from above b_i. There are exactly m bids below b; thus there are m different $P^{i_1,\dots,i_n}$ cases that give rise to this particular state of the world through the event transformation. This proves that

$$n \cdot \Pr(b_i \text{ is the } n{+}1\text{-st bid} \mid s) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} \geq m \cdot \Pr(b_i \text{ is the } n{+}1\text{-st bid} \mid s)$$

and integrating over s gives

$$\Pr(b_i \text{ is the } n+1\text{-st bid}) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} \cdot \frac{n}{m} \geq \Pr(b_i \text{ is the } n\text{-th bid}).$$

Since n/m is bounded away from zero and infinity the claim follows. $\diamondsuit$

To prove the theorem, define $M = kM'$. We need to find a k such that the theorem holds. Suppose a buyer i has optimal bid b_i, and $b_i < v_i - M$ (otherwise we are done). We consider whether the buyer would prefer to bid instead $b_i + 4M'$. If yes, that would be a contradiction, showing that the optimal bid in fact has to be at least M close to the diagonal.

Note that for any opponent bidding function $x_j(.)$ satisfying the condition of lemma 7, we have the set inclusion

$$x_j^{-1}((b_i, b_i + 3M')) \supseteq (b_i + M', b_i + 2M').$$

This is because any value in the interval on the right hand side would induce a bid (both in the case of a buyer and a seller) that is contained in the interval $(b_i, b_i + 3M')$.

Assume that the buyer bids $b_i + 4M'$. Her gain will be an increased probability of winning. This gain is realized for example if her bid was the $n+1$-th highest previously, the φ-perturbation biased the final price upwards, and by increasing the bid, she overtook some opponent by sufficient distance so that even the φ-perturbation cannot make her lose. The size of the gain in this case is at least the distance between her current bid and her value v_i, minus possibly φ. Therefore the following formula is a lower bound for the expected gain from raising the bid

$$\frac{1}{2} \Pr(b_i \text{ is the } n+1\text{-st bid and } \exists \text{ opponent value in } (b_i + M', b_i + 2M')) \cdot (k-5)M'.$$

To see why, note that with probability 1/2 the φ perturbation is biased upwards. Now if there is an opponent value in $(b_i + M', b_i + 2M')$, that leads to a bid no greater than $b_i + 3M'$. If φ is small enough relative to M', then the realized price will still be lower than $b_i + 4M'$, thus our buyer wins. She wins at least $(k-4)M' - \varphi > (k-5)M'$. This proves the above inequality.

Next note that the probability in this formula will be arbitrarily close to Pr(b_i is the $n+1$-st bid) as the auction size increases in the sense that

$$\frac{\Pr(b_i \text{ is the } n+1\text{-st bid and } \exists \text{ opponent value in } (b_i + M', b_i + 2M'))}{\Pr(b_i \text{ is the } n+1\text{-st bid })} \to 1.$$

Therefore the gain is bounded from below by

$$\Pr(b_i \text{ is the } n+1\text{-st bid}) \cdot \frac{(k-5)M'}{4}$$

for a large enough auction.

Next consider the loss from increasing the bid. A loss will take place when b_i was exactly the n-th bid; the size of the loss is bounded from above by $4M'$. Thus the total expected loss is not more than

$$\Pr(b_i \text{ is the } n\text{-th bid}) \cdot 4M'.$$

We need to compare our bounds for the gain and the loss. This involves comparing the probability terms in the two formulas. Lemma 7 shows that

$$\Pr(b_i \text{ is the } n+1\text{-st bid}) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b - 3M'} \cdot C \geq \Pr(b_i \text{ is the } n\text{-th bid})$$

when C is an appropriate constant that does not vary with the size of the auction. Thus the gain will be greater than the loss if

$$\Pr(b_i \text{ is the } n{+}1\text{-st bid}) \cdot \frac{(k-5)M'}{4} \geq \Pr(b_i \text{ is the } n{+}1\text{-st bid}) \cdot \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'} 4M'$$

or

$$\frac{(k-5)}{4} \geq 4 \frac{f_{\max}}{f_{\min}} \cdot \frac{b_i + 3M'}{1 - b_i - 3M'}$$

which is implied by

$$\frac{(k-5)}{4} \geq 4 \frac{f_{\max}}{f_{\min}} \cdot \left(\frac{2}{kM'} - 1 \right)$$

for k large enough. For k large, this condition is satisfied when

$$k^2 \geq 64 \frac{f_{\max}}{f_{\min}} \cdot \frac{1}{M'}.$$

We can choose k large so that this final inequality holds. Moreover, we can choose k such that $kM' = M$ is still going to zero as M' is going to zero (because k is of order $M'^{-1/2}$).

For the appropriately chosen M the above argument shows that $b_i + 3M'$ is a better bid than b_i. This completes the proof. $\diamondsuit$

B Proof of Theorem 3

We show the claim by contradiction. Assume there is an agent i and two private values $v_i < v_i'$ with corresponding best responses b_i and b_i' such that $b_i > b_i'$.

Incentive compatibility implies

$$\begin{aligned} &P^{v_i}(p(b_i,\omega) < b_i)E^{v_i}\left[v_i - p(b_i,\omega)|p(b_i,\omega) < b_i\right] \geq \\ &P^{v_i}(p(b_i',\omega) < b_i')E^{v_i}\left[v_i - p(b_i',\omega)|p(b_i',\omega) < b_i'\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} &P^{v_i'}(p(b_i',\omega) < b_i')E^{v_i'}\left[v_i' - p(b_i',\omega)|p(b_i',\omega) < b_i'\right] \geq \\ &P^{v_i'}(p(b_i,\omega) < b_i)E^{v_i'}\left[v_i' - p(b_i,\omega)|p(b_i,\omega) < b_i\right]. \end{aligned} \tag{13}$$

We can rewrite these conditions as follows

$$\begin{aligned} 0 &\geq \left[\int_{p(b_i',\omega)<b_i'} (v_i - p(b_i',\omega))\ \mathrm{d}\mu^{v_i}(\omega) - \int_{p(b_i,\omega)<b_i} (v_i - p(b_i,\omega))\ \mathrm{d}\mu^{v_i}(\omega)\right] \\ 0 &\geq -\left[\int_{p(b_i',\omega)<b_i'} (v_i' - p(b_i',\omega))\ \mathrm{d}\mu^{v_i'}(\omega) - \int_{p(b_i,\omega)<b_i} (v_i' - p(b_i,\omega))\ \mathrm{d}\mu^{v_i'}(\omega)\right]. \end{aligned}$$

We introduce the notation $\Delta v_i = v_i - v_i'$ as well as the operators $\Delta_{v_i} A(\tilde{v}) = A(v_i) - A(v_i')$ and $\Delta_{b_i} A(\tilde{b}) = A(b_i) - A(b_i')$. We now add the two inequalities and organize terms to obtain

$$\begin{aligned} &-\Delta v_i\left[\int_{p(b_i',\omega)<b_i'} \mathrm{d}\mu^{v_i'}(\omega) - \int_{p(b_i,\omega)<b_i} \mathrm{d}\mu^{v_i'}(\omega)\right] \geq \\ &\Delta_{v_i}\left[\int_{p(b_i',\omega)<b_i'} (v_i - p(b_i',\omega))\ \mathrm{d}\mu^{\tilde{v}}(\omega) - \int_{p(b_i,\omega)<b_i} (v_i - p(b_i,\omega))\ \mathrm{d}\mu^{\tilde{v}}(\omega)\right]. \end{aligned} \tag{14}$$

Further reorganization yields

$$\begin{aligned} &\Delta v_i \Delta_{b_i} P^{v_i'}\left(p(\tilde{b},\omega) < \tilde{b}\right) \geq \\ &\Delta_{v_i}\left[\underbrace{\int_{p(b_i',\omega)<b_i'} (p(b_i,\omega) - p(b_i',\omega))\ \mathrm{d}\mu^{\tilde{v}}(\omega)}_{\text{Term I}} - \underbrace{\int_{\substack{p(b_i,\omega)<b_i \\ p(b_i'|\omega)\geq b_i'}} (v_i - p(b_i,\omega))\ \mathrm{d}\mu^{\tilde{v}}(\omega)}_{\text{Term II}}\right]. \end{aligned} \tag{15}$$

On the right-hand side we made use of the monotonicity of the disturbance φ which ensures that the ω-set $p(b_i,\omega) < b_i$ is larger than $p(b_i',\omega) < b_i'$ (note that $b_i > b_i'$ by assumption).

Term I is non-negative for all ω. By the argument in lemma 2 , we have

$$\left|\Delta_{v_i}\int_{p(b'_i,\omega)<b'_i}\left(p(b_i,\omega)-p(b'_i,\omega)\right)\ \mathrm{d}\mu^{\tilde{v}}(\omega)\right| \leq$$
$$K\cdot|\Delta v_i|\cdot\int_{p(b'_i,\omega)<b'_i}|p(b_i,\omega)-p(b'_i,\omega)|\ \mathrm{d}\mu^{v_i}(\omega) \tag{16}$$

for some constant K which is independent of v_i, v'_i, b_i, b'_i. Now (12) is easily seen to imply that

$$\int_{p(b'_i,\omega)<b'_i}\left(p(b_i,\omega)-p(b'_i,\omega)\right)\ \mathrm{d}\mu^{v_i}(\omega) \quad\leq\quad \int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\left(v_i-p(b_i,\omega)\right)\ \mathrm{d}\mu^{v_i}(\omega)$$

Furthermore, lemma 2 also allows us to bound term II

$$\left|\Delta_{v_i}\int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\left(v_i-p(b_i,\omega)\right)\ \mathrm{d}\mu^{\tilde{v}}(\omega)\right|\leq K\cdot|\Delta v_i|\cdot\int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\left(v_i-p(b_i,\omega)\right)\ \mathrm{d}\mu^{v_i}(\omega).$$

It follows from all of these above that the right hand side RHS of (15) can be estimated as

$$|RHS|\leq 2K\,|\Delta v_i|\int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\left(v_i-p(b_i,\omega)\right)\ \mathrm{d}\mu^{v_i}(\omega) \tag{17}$$

The integral is evaluated over an ω-set where the inequality $v_i > b_i > p(b_i,\omega) > p(b'_i,\omega) > b'_i$ holds. We therefore know that $v_i - p(b_i,\omega) < v_i - b'_i$. This allows us to simplify the above inequality further

$$|RHS| \quad\leq\quad 2K\,|\Delta v_i|\,(v_i-b'_i)\int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\mathrm{d}\mu^{v_i}(\omega) \tag{18}$$

By Lipschitz continuity we know that

$$|h(s|v_i)-h(s|v'_i)| \quad\leq\quad Kh(s|v'_i). \tag{19}$$

This allows us to simplify the inequality further to get

$$\begin{aligned}|RHS| \quad&\leq\quad 2K(K+1)\,|\Delta v_i|\,(v_i-b'_i)\int_{\substack{p(b_i,\omega)<b_i\\ p(b'_i,\omega)\geq b'_i}}\mathrm{d}\mu^{v'_i}(\omega)\\ &=\quad 2K(K+1)\,|\Delta v_i|\,(v_i-b'_i)\,\Delta_{b_i}P^{v'_i}\left(p(\tilde{b},\omega)<\tilde{b}\right).\end{aligned} \tag{20}$$

Now note that $|\Delta v_i| = -\Delta v_i$ such that we can plug it back into (15) and obtain

$$\Delta v_i \Delta_{b_i} P^{v_i'}\left(p(\tilde{b},\omega) < \tilde{b}\right) \geq 2K(K+1)\Delta v_i\left(v_i - b_i'\right)\Delta_{b_i} P^{v_i'}\left(p(\tilde{b},\omega) < \tilde{b}\right) \quad (21)$$

which yields

$$\Delta v_i \Delta_{b_i} P^{v_i'}\left(p(\tilde{b},\omega) < \tilde{b}\right)\left[1 - 2K(K+1)(\Delta v_i + v_i' - b_i')\right] \geq 0. \quad (22)$$

We can actually choose $\Delta v_i < \frac{1}{4K(K+1)}$ because if agent i's best response function is non-monotonic then we can find v and v' arbitrarily close such that $v_i < v_i'$ and $b_i > b_i'$. Therefore the final inequality implies that as long as

$$v_i' - b_i' < \frac{1}{4K(K+1)} \quad (23)$$

holds we have

$$\Delta_{b_i} P^{v_i'}\left(p(\tilde{b}|\omega) < \tilde{b}\right) \leq 0. \quad (24)$$

But this implies $b_i \leq b_i'$ which is a contradiction. $\diamondsuit$

C Proof of Lemma 4

The argument in the main text shows that the payoff to bidding b_i in the k-th game converges to that of bidding b_i in the limit game. That is, the payoff function is convergent pointwise. However, this is not quite enough to show that the maximum also converges. For this we need a stronger result, which is the following.

Let $b_i^k \to b_i$. Then

$$\Pi^k(v_i, b_i^k) \to \Pi(v_i, b_i),$$

that is, the payoff from bidding b_i^k in game k converges to that of bidding b_i in the limiting game. To see why, rewrite the payoff as an integral on the probability space:

$$\Pi^k(v_i, b_i^k) = \int_{\{p^k(b_i^k) < b_i\}} v_i - p^k(b_i^k,\omega)\mathrm{d}\mu^{v_i}(\omega)$$

where $p^k(b_i^k,\omega)$ is the price (a random variable) in game k. Denote the domain of integration in the above formula by $Z^k = \{\omega | p^k(b_i^k,\omega) < b_i^k\}$, and fix δ a small positive number. Then

$$\begin{aligned}\Pi^k(v_i, b_i^k) = &\int_{\{p(b_i) < b_i - \delta\}\cap Z^k} v_i - p^k(b_i^k,\omega)\mathrm{d}\mu^{v_i}(\omega) + \\ &+ \int_{\{b_i - \delta \leq p(b_i) \leq b_i + \delta\}\cap Z^k} v_i - p^k(b_i^k,\omega)\mathrm{d}\mu^{v_i}(\omega) + \\ &+ \int_{\{p(b_i) > b_i + \delta\}\cap Z^k} v_i - p^k(b_i^k,\omega)\mathrm{d}\mu^{v_i}(\omega).\end{aligned}$$

Because the distribution of $p(b_i)$ is atomless, the middle term can be made arbitrarily small by an appropriate choice of δ. Fix a δ. We also have that $p^k(b_i^k, \omega)$ converges to $p(b_i, \omega)$ in probability; therefore for k large enough (given δ), outside of a small probability event we will have that $\omega \in \{p(b_i) < b_i - \delta\}$ implies $\omega \in Z_k$. Therefore, controlling for the small approximation error, the first term can be considered to be integrated over $\{p(b_i) < b_i - \delta\}$. Likewise, for k large, the domain of integration of the final term will have arbitrarily small measure. Therefore we can write that

$$\Pi^k(v_i, b_i^k) = \text{small}(\delta) + \text{small}(k, \text{given } \delta) + \int_{\{p(b_i)<b_i-\delta\}} v - p^k(b_i^k, \omega) \mathrm{d}\mu^{v_i}(\omega) .$$

Furthermore, because the price distribution is atomless, the domain of integration in this formula can be replaced by $\{p(b_i) < b_i\}$; that introduces approximation errors smaller than what we currently have. Finally, because $p^k(b_i^k, \omega)$ converges in probability to $p(b, \omega)$, we can write that

$$\Pi^k(v_i, b_i^k) = \text{small}(\delta) + \text{small}(k, \text{given } \delta) + \int_{\{p(b_i)<b_i\}} v - p(b_i, \omega) \mathrm{d}\mu^{v_i}(\omega) =$$
$$= \text{small}(\delta) + \text{small}(k, \text{given } \delta) + \Pi(v_i, b_i).$$

This is what we wanted to prove. By choosing δ small enough, and then accordingly k large enough, we can show that $\Pi^k(v_i, b_i^k)$ gets arbitrarily close to $\Pi(v_i, b_i)$.

To get the statement of the lemma, assume that b_i^k is the best response in game k, but b_i, the limit, is not a best response in the limiting game. Then there is a b_i' that does better then b_i, so that $\Pi(v_i, b_i) < \Pi(v_i, b_i')$. However, in game k it has to be the case that $\Pi^k(v_i, b_i^k) \geq \Pi^k(v_i, b_i')$. Taking k to infinity, the left hand side converges to $\Pi(v_i, b_i)$, and the right hand side to $\Pi(v_i, b_i')$, thus giving $\Pi(v_i, b_i) \geq \Pi(v_i, b_i')$. This is a contradiction. $\diamondsuit$