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Sign Tests for Dependent Observations

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Rustam Ibragimov
and
Donald J. Brown

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Rustam Ibragimov

Department of Economics, Harvard University

Donald J. Brown

Department of Economics, Yale University

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Rustam Ibragimov
Department of Economics
Harvard University
Littauer Center
1875 Cambridge St.
Cambridge, MA 02138

Email: ribragim@fas.harvard.edu
Phone: +1-203-887-1631
Fax: +1-617-495-7730

Donald Brown
Department of Mathematics
PO Box 208283
Yale University
New Haven, CT 06520-8283

Email: donald.brown@yale.edu
Phone: +1-203-432-6323
Fax: +1-203-432-2128

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ABSTRACT

The present paper introduces new sign tests for testing equality of conditional distributions of two (arbitrary) adapted processes as well as for testing conditionally symmetric martingale-difference assumptions. Our analysis is based on results that demonstrate randomization over ties in sign tests for equality of conditional distributions of two adapted sequences produces a stream of i.i.d. symmetric Bernoulli random variables. This reduces the problem of estimating the critical values of the tests to computing the quantiles or moments of Binomial or normal distributions. A similar proposition holds for randomization over zero values of three-valued random variables in a conditionally symmetric martyingale-difference sequence.

Key words and phrases: Sign tests, dependence, adapted processes, martingale-difference sequences, Bernoulli random variables, conservative tests, exact tests

JEL Classification: C12, C14, G12, G14

1 Introduction and discussion

1.1 Objectives and key results

This paper introduces new sign tests for testing equality of conditional distributions of two (arbitrary) adapted sequences of random variables (r.v.'s) as well as for testing conditionally symmetric martingale-difference assumptions. Our analysis is based on results that demonstrate randomization over ties in sign tests for equality of conditional distributions of two adapted processes produces a stream of i.i.d. symmetric Bernoulli r.v.'s. This reduces the problem of estimating the critical values of the tests to computing the quantiles or moments of Binomial or normal distributions (see Theorem 2.4 and Corollary 2.4). A similar proposition holds for randomization over zero values of three-valued r.v.'s in a conditionally symmetric martingale-difference sequence (Theorem 2.1 and Corollary 2.1). We wish to point out that the results obtained in the present paper can also be used to test the hypothesis that the conditional median of a sequence of r.v.'s X_t adapted to a filtration $(\Im_t)$ equals some constant $\mu \neq 0$. More precisely, the results can be used to test the null hypothesis that the conditional distributions $\mathcal{L}(X_t|\Im_{t-1})$ are symmetric about μ using tests based on $sign(X_t - \mu)$. These results allow us to obtain general estimates for the tail probabilities of sums of signs of random variables forming a conditionally symmetric martingale-difference sequence or signs of differences of the components of two adapted sequences. Such estimates give sharp (i.e., attainable either in finite samples or in the limit) bounds for the tail probabilities in terms of (generalized) moments of sums of i.i.d. Bernoulli r.v.'s (or corresponding moments of Binomial distributions) and standard normal r.v.'s (see Corollaries 2.2, 2.3, 2.6 and 2.7). Similar estimates hold as well for expectations of arbitrary functions of the signs that are convex in each of their arguments (Theorem 2.2 and Corollary 2.5).

The analysis in this paper is based, in large part, on general characterization results for two-valued martingale difference sequences and multiplicative forms obtained recently in Sharakhmetov and Ibragimov (2002). Their results allow one to reduce the study of many problems for three-valued martingales to the case of i.i.d. symmetric Bernoulli r.v.'s and provide the key to the development of sign tests for dependent observations.

1.2 Sign tests

There are many studies focusing on procedures for dealing with ties in sign tests if observations are independent (see Coakley and Heise, 1996, for a review and comparisons of sign tests in the presence of ties). Using the conclusions derived from a size and power study, Coakley and Heise (1996) recommended using the asymptotic uniformly most powerful nonrandomized (ANU) test due to Putter (1955) if ties occur in the sign test. The results obtained by Putter (1955) show that randomization over ties reduces the exact power of the sign test and the asymptotic efficiency of the sign test. It is known, however, that the exact version of the ANU test is conservative for small samples compared to both its randomized conditional version as well as to ANU (see Coakley and Heise, 1996; Wittkowski, 1998). The estimates obtained in the present paper shed new light on sign tests comparisons and suggest that randomization over ties leads, in general, to more conservative unconditional sign tests since it provides bounds for the tail probabilities of signs in terms of generalized moments of i.i.d. Bernoulli r.v.'s. An advantage of randomization over ties or zero observations is that it allows one to use sign tests in the presence of dependence while nonrandomized sign tests can only be used in the case of independent data. In this regard, our results demonstrate that, in addition to their other appealing properties, sign tests also have the important property of robustness to dependence.

An appealing property of sign tests is that a simple linear transformation of a test statistic based on signs leads to a Binomial distribution, and, thus, its distribution can be computed exactly. This is in contrast to other commonly used test statistics for which the exact distributions are frequently unknown. Even if known, the exact distributions of such test statistics are usually difficult to compute and have to be obtained by relying on computationally intensive algorithms or Monte-Carlo techniques.

Another important property of sign tests is that they can be applied in the case of a small number of observations. This is very important since large sample approximations, e.g., those based on central limit theorems, require special regularity assumptions on the distribution of the observations such as existence of the second or higher moments or identical distribution.

1.3 Applications to experimental game theory

Finite-sample tests for equality of conditional distributions of adapted processes are especially important in experimental game theory and experimental economics. In particular, reliable tests that perform well with a relatively small number of dependent observations are necessary in analyzing experimental data in these fields. The high costs of implementing experiments prohibits large sample size and the unavoidable presence of dependence in observations is caused by subjects' intertemporal learning.

Bracha (2005) proposed a new paradigm for decision making under uncertainty. In her model, there is an interaction between cognitive and affective neural processes described as an intrapersonal potential game where observed behavior is a Nash equilibrium of the game resulting from simultaneous play of cognitive and affective processes. This interaction is termed affective decision making.

Bracha, Gray, Ibragimov, Nadler, Shapiro, Ames and Brown (2005) consider the implications of Bracha (2005)'s model for a hypothetical experiment in discrete choice under risk. The hypothetical experiment considered by Bracha et al. (2005) produces a finite sequence of dependent observations on one group of subjects choosing between decks of cards with random monetary payoffs paired with another group of subjects choosing between decks of cards with the same random monetary payoffs, but containing affective payoffs, i.e., images. Bracha's model prediction is that the sequences of the players payoffs in the two groups have different conditional distributions. Thus, Bracha et al. (2005) test the null hypothesis that the conditional distributions of choices are the same in both groups against the alternative hypothesis that the conditional distributions of choices differ as predicted by Bracha's model using the tests proposed in this paper. In order to be able to conduct our tests of the above hypotheses, it is important that the players' choices are adapted to the same filtration generated by the monetary outcomes that both groups observe. In other words, if seeing the images has no effect on a subject's decisions, then the players' decisions are determined only by the monetary outcomes in the previous rounds observed by the subjects in both groups.

1.4 Organization of the paper

The paper is organized as follows. In Section 2, we obtain the main results of the paper on the distributional properties of sign tests for dependent r.v.'s that provide the key to the development of statistical procedures based on signs of dependent observations and to obtaining sharp bounds in the trinomial contingent claim pricing model in subsequent sections. Section 3 describes the new sign tests based on the results obtained in Section 2. The sign tests provide the statistical procedures for testing for conditionally symmetric martingale-difference assumptions as well as for testing that conditional distributions of two (arbitrary) adapted sequences are the same. Section

4 is an appendix that recalls the relevant result from Sharakhmetov and Ibragimov (2002) that is the basis for the analysis in this paper, Proposition 4.1, and a technical lemma, Proposition 4.2, that is a corollary of Proposition 4.1.

2 Distributions of sign test statistics for dependent observations

The present section of the paper establishes the results on the distributional properties of the sign tests for adapted processes that provide the basis for the development of statistical procedures based on signs of dependent observations in the Section 3.

Let $(\Omega, \Im, P)$ be a probability space equipped with a filtration $\Im_0 = (\Omega, \emptyset) \subseteq \Im_1 \subseteq ...\Im_t \subseteq ... \subseteq \Im$.

Let Z_t, $t = 1, 2, ...$, be an $(\Im_t)-$conditionally symmetric martingale-difference sequence (so that $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $t = 1, 2, ...$, for all $x > 0$) consisting of r.v.'s each of which takes three values $\{-a_t, 0, a_t\}$. Further, let, for $z \in \mathbf{R}$, $sign(z)$ denote the sign of z defined by $sign(z) = 1$, if $z > 0$, $sign(z) = -1$, if $z < 0$, and $sign(0) = 0$.

Throughout Sections 2 and 3, ϵ_t, $t = 1, 2, ...$, stand for a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of Z_t, $t = 1, 2, ...$; in addition to that, in what follows, we denote by $\mathcal{N}$ the standard normal r.v. if not stated otherwise.

Theorem 2.1 *The r.v.'s $\eta_t = sign(Z_t) + \epsilon_t I(Z_t = 0)$ are i.i.d. symmetric Bernoulli r.v.'s.*

Proof. The theorem follows from Proposition 4.1 since, as it is easy to see, the r.v.'s (η_t) form an $(\Im_t)-$martingale-difference sequence and each of them takes two values -1 and 1. ■

Corollary 2.1 *The statistic $S_n = (\sum_{t=1}^{n} sign(Z_t) + \epsilon_t I(Z_t = 0) + n)/2$ has Binomial distribution $Bin(n, 1/2)$ with parameters n and $p = 1/2$.*

Proof. The corollary is an immediate consequence of Theorem 2.1. ■

Theorem 2.2 *For any function $f : \mathbf{R}^n \to \mathbf{R}$ convex in each of its arguments,*

$$Ef(sign(Z_1), sign(Z_2), ..., sign(Z_n)) \leq Ef(\epsilon_1, \epsilon_2, ..., \epsilon_n).$$

Proof. The theorem follows from Proposition 4.2 applied to the martingale-difference sequence $\eta_t = sign(Z_t)$, $t = 1, 2, ...$, consisting of r.v.'s each of which takes three values $\{-1, 0, 1\}$. ■

Corollary 2.2 *For any* $x > 0$,

$$P\Big(\sum_{t=1}^{n} sign(Z_t) > x\Big) \leq \inf_{0<c<x} \frac{E \max\Big(\sum_{t=1}^{n} \epsilon_t - c, 0\Big)}{(x-c)}. \tag{2.1}$$

Proof. The corollary is an immediate consequence of Markov's inequality and Theorem 2.2 applied to the functions $f_c(x_1, x_2, ..., x_n) = max\Big(\sum_{t=1}^{n} x_t - c, 0\Big)$, $0 < c < x$. ■

Remark 2.1 *For a fixed* $x > 0$, *consider the class of functions* ϕ *satisfying* $\phi(y) = \int_0^y max(y - u, 0)dF(u)$, $y \geq 0$, $\phi(y) = 0$, $y < 0$, *and* $\phi(x) = \int_0^x max(x - u, 0)dF(u) = 1$, *for a nonnegative bounded nondecreasing function* $F(x)$ *on* $[0, +\infty)$ *with* $F(0) = 0$. *Similar to the proof of Corollary 2.2 we obtain*

$$P\Big(\sum_{t=1}^{n} sign(Z_t) > x\Big) \leq E\phi\Big(\sum_{t=1}^{n} \epsilon_t\Big) \tag{2.2}$$

for all ϕ. *It is not difficult to show, similar to Proposition 4 in Eaton (1974) (see also the discussion following Theorem 5 in de la Peña, Ibragimov and Jordan, 2004, for related optimality results for bounds on the expected payoffs of contingent claims in the binomial model) that bound (2.1) is the best among all estimates (2.2), that is,*

$$\inf_{\phi} E\phi\Big(\sum_{t=1}^{n} \epsilon_t\Big) = \inf_{0<c<x} \frac{E \max\Big(\sum_{t=1}^{n} \epsilon_t - c, 0\Big)}{(x-c)}.$$

The following result gives sharp bounds for the tail probabilities of the normalized sum of signs of the r.v.'s Z_t in terms of (generalized) moments of the standard normal r.v.

Corollary 2.3 *For any* $x > 0$,

$$P\Big(\frac{\sum_{t=1}^{n} sign(Z_t)}{\sqrt{n}} > x\Big) \leq \inf_{0<c<x} \frac{E \max\Big(\frac{\sum_{t=1}^{n} \epsilon_t}{\sqrt{n}} - c, 0\Big)}{(x-c)} \leq \inf_{0<c<x} \frac{(E[max(\mathcal{N} - c, 0)]^3)^{1/3}}{x-c}. \tag{2.3}$$

Proof. Using Markov's inequality and Theorem 2.2 applied to the functions

$$f_c(x_1, x_2, ..., x_n) = max\Big(\frac{\sum_{t=1}^{n} x_t}{\sqrt{n}} - c, 0\Big),$$

$0 < c < x$, we get the first estimate in (2.3). From Jensen's inequality we obtain

$$E \max\Big(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\Big) \le \Big\{E\Big[\max\Big(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\Big)\Big]^3\Big\}^{1/3}, \tag{2.4}$$

$0 < c < x$. The second bound in (2.3) is a consequence of estimate (2.4) and the inequality

$$E\Big[\max\Big(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\Big)\Big]^3 \le E[max(\mathcal{N} - c, 0)]^3 \tag{2.5}$$

for all $c > 0$ implied by the results in Eaton (1974). ■

Let (X_t), $t = 1, 2, ...,$ and (Y_t), $t = 1, 2, ...,$ be two $(\Im_t)-$adapted sequences of r.v.'s.

The following results provide analogues of Theorem 2.1 and Corollaries 2.1-2.3 that concern the distributional properties of sign tests for equality of conditional distributions of (X_t) and (Y_t). They follow from Theorem 2.1 and Corollaries 2.1-2.3 applied to the r.v.'s $Z_t = X_t - Y_t$ that form a conditionally symmetric martingale-difference sequence under the assumption that the conditional distributions of (X_t) and (Y_t) are the same.

Theorem 2.3 *If the conditional (on $\Im_{t-1}$) distributions of (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$, then the r.v.'s $\tilde{\eta}_t = sign(X_t - Y_t) + \epsilon_t I(X_t = Y_t)$ are i.i.d. symmetric Bernoulli r.v.'s*

Corollary 2.4 *If the conditional (on $\Im_{t-1}$) distributions of (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$, then the statistic $\tilde{S}_n = (\sum_{t=1}^n sign(X_t - Y_t) + \epsilon_t I(X_t = Y_t) + n)/2$ has Binomial distribution $Bin(n, 1/2)$ with parameters n and $p = 1/2$.*

Corollary 2.5 *If the conditional (on $\Im_{t-1}$) distributions of (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$, then, for any function $f : \mathbf{R}^n \to \mathbf{R}$ convex in each of its arguments,*

$$Ef(sign(X_1 - Y_1), sign(X_2 - Y_2), ..., sign(X_n - Y_n)) \le Ef(\epsilon_1, \epsilon_2, ..., \epsilon_n).$$

Corollary 2.6 *If the conditional (on $\Im_{t-1}$) distributions of (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$, then, for any $x > 0$,*

$$P\Big(\sum_{t=1}^n sign(X_t - Y_t) > x\Big) \le \inf_{0<c<x} \frac{E \max\Big(\sum_{t=1}^n \epsilon_t - c, 0\Big)}{(x - c)}.$$

Corollary 2.7 *If the conditional (on $\Im_{t-1}$) distributions of (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$, then, for any $x > 0$,*

$$P\Big(\frac{\sum_{t=1}^{n} sign(X_t - Y_t)}{\sqrt{n}} > x\Big) \le \inf_{0<c<x} \frac{E\max\left(\frac{\sum_{t=1}^{n}\epsilon_t}{\sqrt{n}} - c, 0\right)}{(x-c)} \le \inf_{0<c<x} \frac{(E[max(Z-c,0)]^3)^{1/3}}{x-c}.$$

Remark 2.2 *Bounds for the tail probabilities of sums of bounded r.v.'s forming a conditionally symmetric martingale-difference sequence implied by the results in the present section provide better estimates than many inequalities implied, in the trinomial setting, by well-known estimates in martingale theory. In particular, from Markov's inequality and Theorem 2.2 applied to the function $f(x_1, x_2, ..., x_n) = \exp(h\sum_{i=1}^{n} u_t x_t)$, $h > 0$, it follows that the tail probability $P\Big(\sum_{t=1}^{n} X_t > x\Big)$, $x > 0$, of the sum of r.v.'s X_t that take three values $\{-u_t, 0, u_t\}$ is bounded from above by $\exp(-hx)E\exp\left(h\sum_{t=1}^{n} u_t\epsilon_t\right)$, $h > 0$:*

$$P\Big(\sum_{t=1}^{n} X_t > x\Big) \le \inf_{h>0} \exp(-hx)E\exp\Big(h\sum_{t=1}^{n} u_t\epsilon_t\Big). \tag{2.6}$$

From estimate (2.6) it follows that Hoeffding-Azuma inequality for martingale-differences in the above setting

$$P\Big(\sum_{t=1}^{n} X_t > x\Big) \le \exp\Big(-\frac{x^2}{2\sum_{t=1}^{n} u_t^2}\Big) \tag{2.7}$$

is implied by the corresponding bounds on the expectation of exponents of weighted i.i.d. Bernoulli r.v.'s $E\exp\left(h\sum_{t=1}^{n} u_t\epsilon_t\right)$ (see Hoeffding, 1963; Azuma, 1967). More generally, Markov's inequality and Theorem 2.2 imply the following bound for the tail probabilities of three-valued r.v.'s forming a conditionally symmetric martingale-difference sequence with the support on $\{-u_t, 0, u_t\}$:

$$P\Big(\sum_{t=1}^{n} X_t > x\Big) \le \inf_{\phi} \frac{\phi\Big(\sum_{t=1}^{n} u_t\epsilon_t\Big)}{\phi(x)}, \tag{2.8}$$

where the infimum is taken over convex increasing functions $\phi : \mathbf{R} \to \mathbf{R}_+$. It is easy to see that estimate (2.8) is better than Hoeffding-Azuma inequality (2.7) since the latter follows from choosing a particular (close to optimal) h in estimates for the right-hand side of (2.6) which is a particular case of (2.8) (see Hoeffding, 1963, and also Remark 2.1 on the optimality of bound (2.2))

3 Sign tests under dependence

As follows from the results in the previous section, sign tests for testing the null hypothesis that the conditional distributions of two adapted processes (X_t) and (Y_t) are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$ for all t or that (Z_t) is an $(\Im_t)-$conditionally symmetric martingale-difference sequence with $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, can be based on the procedures described below. As most of the testing procedures in statistics and econometrics, they can be classified as falling into one of the following classes: exact tests, conservative tests and testing procedures based on asymptotic approximations. The exact tests are based on the fact that, according to Corollaries 2.1 and 2.4, the distributions of the transformation of signs in the model is known precisely to be Binomial and thus the statistical inference can be based on critical values for the sum of i.i.d. Bernoulli r.v.'s (the case of exact randomized ER tests below). The asymptotic tests use approximations for the quantiles of the Binomial distribution in terms of the limiting normal distribution (the case of asymptotic randomized AR tests). The conservative testing procedures in the present section are based on sharp estimates for the tail probabilities of sums of dependent signs in the model in terms of sums of i.i.d. Bernoulli or normal r.v.'s implied by Corollaries 2.2, 2.3, 2.6 and 2.7 and corresponding estimates for the critical values of the sign tests for dependent observations in terms of quantiles of the Binomial or Gaussian distributions (Binomial conservative non-randomized BCN and normal conservative non-randomized NCN testing procedures). The classification of the sign tests in the present section as non-randomized or randomized refers, respectively, to whether the inference is based on the original (three-valued) signs $sign(X_t - Y_t)$ (resp., $sign(X_t)$) in the model with dependent observations or the r.v.'s $sign(X_t - Y_t) + \epsilon_t I(X_t = Y_t)$ (resp., $sign(X_t) + \epsilon_t I(X_t = 0)$) that form, according to the results in the previous section, a sequence of symmetric i.i.d. Bernoulli r.v.'s.

The following are statistical procedures for testing the null hypothesis that conditional distributions of components of two adapted sequences of r.v.'s are the same: $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$.[5]

1. The exact randomized (ER) sign test with the test statistic $\tilde{S}_n^{(1)} = (\sum_{t=1}^n sign(X_t - Y_t) + \epsilon_t I(X_t = Y_t) + n)/2$ rejects the null hypothesis $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$ for all t in favor of the (two-sided) hypothesis $\mathcal{L}(X_t|\Im_{t-1}) \neq \mathcal{L}(Y_t|\Im_{t-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $\tilde{S}_n^{(1)} < B_{\alpha/2}^{(1)}$ or $\tilde{S}_n^{(1)} > B_{\alpha/2}^{(2)}$ where $B_{\alpha/2}^{(1)}$ and $B_{\alpha/2}^{(2)}$ are, respectively, the $(\alpha/2)-$ and $(1-\alpha/2)-$quantiles of the Binomial distribution $Bin(n, 1/2)$.

[5]We describe the tests for the two-sided alternative since this is usually the case of interest in most of the applications.

2. The asymptotic randomized (AR) sign test with the test statistic $\tilde{S}_n^{(2)} = (\sum_{t=1}^n sign(X_t - Y_t) + \epsilon_t I(X_t = Y_t))/\sqrt{n}$ rejects the null hypothesis the null hypothesis $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$ for all t in favor of $\mathcal{L}(X_t|\Im_{t-1}) \neq \mathcal{L}(Y_t|\Im_{t-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $|\tilde{S}_n^{(2)}| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the $(1-\alpha/2)-$quantile of the standard normal distribution $\mathcal{N}(0,1)$.

3. The binomial conservative non-randomized (BCN) sign test with the test statistic $\tilde{S}_n^{(3)} = \sum_{t=1}^n sign(X_t - Y_t)$ rejects the null hypothesis $\mathcal{L}(X_n|\Im_{n-1}) = \mathcal{L}(Y_n|\Im_{n-1})$ for all n in favor of $\mathcal{L}(X_n|\Im_{n-1}) \neq \mathcal{L}(Y_n|\Im_{n-1})$ for all n at the significance level $\alpha \in (0, 1/2)$, if $|\tilde{S}_n^{(3)}| > B_{\alpha/2}$, where $B_{\alpha/2}$ is such that

$$\inf_{0<c<B_{\alpha/2}} \frac{E \max\Big(\sum_{t=1}^n \epsilon_t - c, 0\Big)}{(B_{\alpha/2} - c)} < \alpha/2.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic $\tilde{S}_n^{(4)} = \sum_{t=1}^n sign(X_t - Y_t)$ rejects the null hypothesis $\mathcal{L}(X_t|\Im_{t-1}) = \mathcal{L}(Y_t|\Im_{t-1})$ for all t in favor of $\mathcal{L}(X_t|\Im_{t-1}) \neq \mathcal{L}(Y_t|\Im_{t-1})$ for all t at the significance level $\alpha \in (0, 1/2)$, if $|\tilde{S}_n^{(4)}| > z_{\alpha/2}$, where $z_{\alpha/2}$ is such that

$$\inf_{0<c<z_{\alpha/2}} \frac{(E[max(Z-c,0)]^3)^{1/3}}{z_{\alpha/2} - c} < \alpha/2.$$

The following are the analogues of the above procedures for testing the null hypothesis that (Z_t) is an $(\Im_t)-$conditionally symmetric martingale-difference sequence with $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$.

1. The exact randomized (ER) sign test with the test statistic $S_n^{(1)} = (\sum_{t=1}^n sign(X_t) + \epsilon_t I(X_t = 0) + n)/2$ rejects the null hypothesis $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t in favor of $P(Z_t > x|\Im_{t-1}) > P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all n at the significance level $\alpha \in (0, 1/2)$, if $S_n > B_\alpha$, where B_α is the $(1-\alpha)-$quantile of the Binomial distribution $Bin(n, 1/2)$.

Using the central limit theorem for the statistic $(\sum_{t=1}^n sign(X_t) + \epsilon_t I(X_t = 0))/\sqrt{n}$, in the case of large sample sizes n one can also use the following asymptotic version of the previous testing procedure.

2. The asymptotic randomized (AR) sign test with the test statistic $S_n^{(2)} = (\sum_{t=1}^n sign(X_t) + \epsilon_t I(X_t = 0))/\sqrt{n}$ rejects the null hypothesis $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t in favor of $P(Z_t > x|\Im_{t-1}) > P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t at the significance level $\alpha \in (0, 1/2)$, if $S_n > z_\alpha$, where z_α is the $(1-\alpha)-$quantile of the standard normal distribution $\mathcal{N}(0,1)$.

3. The binomial conservative non-randomized (BCN) sign test with the test statistic $S_n^{(3)} =$

$\sum_{t=1}^{n} sign(X_t)$ rejects the null hypothesis $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t in favor of $P(Z_t > x|\Im_{t-1}) > P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t at the significance level $\alpha \in (0, 1/2)$, if $S_n > B_\alpha$, where B_α is such that

$$\inf_{0<c<B_\alpha} \frac{E \max\left(\sum_{t=1}^{n} \epsilon_t - c, 0\right)}{(B_\alpha - c)} < \alpha.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic $S_n^{(4)} = \sum_{t=1}^{n} sign(X_t)$ rejects the null hypothesis $P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t in favor of $P(Z_t > x|\Im_{t-1}) > P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t at the significance level $\alpha \in (0, 1/2)$, if $S_n > z_\alpha$, where z_α is such that

$$\inf_{0<c<z_\alpha} \frac{(E[max(\mathcal{N} - c, 0)]^3)^{1/3}}{z_\alpha - c} < \alpha.$$

The analogues of the above tests in the case of the two-sided alternative $P(Z_t > x|\Im_{t-1}) \neq P(Z_t < -x|\Im_{t-1})$ are completely similar.

For illustration, in Table 1 in the Appendix, we provide the results on calculations of the power of the AR sign test for testing the null hypothesis $H_0 : P(Z_t > x|\Im_{t-1}) = P(Z_t < -x|\Im_{t-1})$, $x > 0$, for all t against a particular case of the alternative hypothesis, namely, against the assumption that $P(Z_t > x|\Im_{t-1}) = p > 1 - p = P(Z_t < -x|\Im_{t-1})$, $x > 0$, where $p \in (1/2, 1]$ (the power of other tests discussed in the present section against this particular alternative may be calculated in complete similarity). One should note that, as it is not difficult to see, the power calculations are the same for the AR test for testing H_0 against the alternative $P(Z_t > x|\Im_{t-1}) = p_1 > q_1 = P(Z_t < -x|\Im_{t-1})$, $x > 0$, $P(Z_t = 0|\Im_{t-1}) = 1 - p_1 - q_1$, where $p_1, q_1 \in [0, 1]$ are such that $1/2 + (p_1 - q_1)/2 = p$. They are also the same for the AR sign test for testing the null hypothesis of equality of conditional distributions of two $(\Im_t)-$adapted processes X_t and Y_t against the alternative that $P(X_t > Y_t|\Im_{t-1}) = p_2 > 1/2 > q_2 = P(Y_t > X_t|\Im_{t-1})$, where $p_2, q_2 \in [0, 1]$ are such that $1/2 + (p_2 - q_2)/6 = p$. According to the table, the test has very good power properties, even in the case of small samples.

4 Appendix A1. Probabilistic foundations for the analysis

Let $(a_t)_{t=1}^{\infty}$ and $(b_t)_{t=1}^{\infty}$ be arbitrary sequences of real numbers such that $a_t \neq b_t$ for all t.

The key to the analysis in this paper is provided by Propostion 4.1. This proposition is a consequence of more general results obtained in Sharakhmetov and Ibragimov (2002) that show that r.v.'s taking $k+1$ values form a multiplicative system of order k if and only if they are jointly independent (see also de la Peña and Ibragimov, 2003; de la Peña, Ibragimov and Sharakhmetov, 2003).

Proposition 4.1 *If r.v.'s X_t, $t = 1, 2, ...,$ form a martingale-difference sequence with respect to a filtration $(\Im_t)_t$ and each of them takes two (not necessarily the same for all t) values $\{a_t, b_t\}$, then they are jointly independent.*

Let X_t, $t = 1, 2, ...,$ be an $(\Im_t)-$martingale-difference sequence consisting of r.v.'s each of which takes three values $\{-a_t, 0, a_t\}$. Denote by ϵ_t, $t = 1, 2, ...,$ a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of $(X_t)_{t=1}^{\infty}$. The following proposition provides an upper bound for the expectation of arbitrary convex function of X_t in terms of the expectation of the same function of the r.v.'s ϵ_t.

Proposition 4.2 *If $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a function convex in each of its arguments, then the following inequality holds:*

$$Ef(X_1, ..., X_n) \leq Ef(a_1\epsilon_1, ..., a_n\epsilon_n). \tag{4.9}$$

Proof. Let $\tilde{\Im}_0 = \Im_n$. For $t = 1, 2, ..., n$, denote by $\tilde{\Im}_t$ the $\sigma-$algebra spanned by the r.v.'s $X_1, X_2, ..., X_n, \epsilon_1, ..., \epsilon_t$. Further, let, for $t = 0, 1, ..., n$, E_t stand for the conditional expectation operator $E(\cdot|\tilde{\Im}_t)$ and let η_t, $t = 1, ..., n$, denote the r.v.'s $\eta_t = X_t + \epsilon_t I(X_t = 0)$.

Using conditional Jensen's inequality, we have

$$\begin{aligned} Ef(X_1, X_2, ..., X_n) = Ef(X_1 + E_0[\epsilon_1 I(X_1 = 0)], X_2, ..., X_n) \leq \\ E[E_0 f(X_1 + \epsilon_1 I(X_1 = 0), X_2, ..., X_n)] = Ef(\eta_1, X_2, ..., X_n). \end{aligned} \tag{4.10}$$

Similarly, for $t = 2, ..., n$,

$$\begin{aligned} Ef(\eta_1, \eta_2, ..., \eta_{t-1}, X_t, X_{t+1}, ..., X_n) = \\ Ef(\eta_1, \eta_2, ..., \eta_{t-1}, X_t + E_{t-1}[\epsilon_t I(X_t = 0)], X_{t+1}, ..., X_n) \leq \\ E[E_{t-1} f(\eta_1, \eta_2, ..., \eta_{t-1}, X_t + \epsilon_t I(X_t = 0), X_{t+1}, ..., X_n)] = \\ Ef(\eta_1, \eta_2, ..., \eta_{t-1}, \eta_t, X_{t+1}, ..., X_n). \end{aligned} \tag{4.11}$$

From equations (4.10) and (4.11) by induction it follows that

$$Ef(X_1, X_2, ..., X_n) \leq Ef(\eta_1, \eta_2, ..., \eta_n). \tag{4.12}$$

It is easy to see that the r.v.'s η_t, $t = 1, 2, ..., n$, form a martingale-difference sequence with respect to the sequence of $\sigma-$algebras $\tilde{\Im}_0 \subseteq \tilde{\Im}_1 \subseteq ... \subseteq \tilde{\Im}_t \subseteq ...$, and each of them takes two values $\{-a_t, a_t\}$. Therefore, from Proposition 4.1 we get that η_t, $t = 1, 2, ..., n$, are jointly independent and, therefore, the random vector $(\eta_1, \eta_2, ..., \eta_n)$ has the same distribution as $(a_1\epsilon_1, a_2\epsilon_2, ..., a_n\epsilon_n)$. This and (4.12) implies estimate (4.9). ■

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