SOME EXTREMAL PROBLEMS FOR GAUSSIAN AND EMPIRICAL RANDOM FIELDS

by

Alexander Nazarov
St.Petersburg University, Russia,

and

Yakov Nikitin
St.Petersburg University, Russia and ICER

December 2000

Abstract

Some important problems of Probability and Statistics can be reduced to the evaluation of supremum of some homogeneous functionals defined on the Strassen ball in the space of smooth functions on the square. We give the solution of this extremal problem when the functional is linear and continuous and when it is a superposition of two seminorms. As a result we obtain the large deviation asymptotics for L_p-norms of Brownian fields on the square, some Strassen type laws of iterated logarithm for functionals of Brownian fields, and describe the conditions of local Bahadur optimality for some nonparametric independence tests like rank correlation coefficients.

Keywords: Brownian field, Strassen ball, large deviations, tail asymptotics, rank correlations.

Acknowledgement: The authors are indebted to Professors I.Ibragimov and M. Lifshits for the discussion of results and valuable consultations. Parts of this paper were written when the second author was visiting ICER in Turin. ICER is gratefully acknowledged for providing financial support and excellent working conditions. The second author was also supported by grants of RFBR No.99-01-0724 and 00-15-96019.

1. Introduction

Some important problems of Probability and Statistics related to random fields on the unit square may be reduced to the same extremal problem.

Let U be the space of real functions $u(x,y)$ on the unit square I^2 having generalized mixed derivative $u_{xy}(x,y)$ and satisfying a set of boundary conditions including the condition

$$u(x,0) = u(0,y) = 0 \quad \forall x, y \in [0,1]. \tag{1.1}$$

Consider in U the so-called Strassen ball S determined by inequality

$$\int_0^1 \int_0^1 u_{xy}^2(x,y)dxdy \le 1. \tag{1.2}$$

Let $b(u)$ be a homogeneous functional of degree 2 defined on the space U. It is required to find the value

$$\lambda = \sup\{b(u) \,|\, u \in S\} \tag{1.3}$$

and whenever possible to specify the extremal u on which this extremum is attained.

We describe now three typical problems leading to such formulation.

A. *Large deviations for functionals of Gaussian fields.* Let $W(s,t)$ be the Brownian sheet or the Wiener-Chentsov field that is centered Gaussian field with covariance function

$$K(\bar{s},\bar{t}) = \min(s_1,t_1)\min(s_2,t_2),\ s_i \ge 0, t_i \ge 0, i = 1,2,$$

where $\bar{s} = (s_1,s_2)$, $\bar{t} = (t_1,t_2)$, and let $T(W)$ be a continuous and homogeneous functional of the field W. It follows from the large deviation principle for Gaussian fields (see [1], §§10, 12) and Varadhan's contraction principle [2] that

$$\lim_{r\to+\infty} r^{-2}\ln P(T(W)\ge r) = -\frac{1}{2}J_T,$$

where

$$J_T = \inf\{\int_0^1\int_0^1 u_{xy}^2(x,y)dxdy \,|\, u \in U,\, T(u)\ge 1\}.$$

However the arising extremal problem can be written also in the dual form, namely

$$J_T^{-1} = \sup\{T(u) \,|\, u \in S\},$$

so that the problem of evaluation of the constant J_T belongs to the class of problems (1.3).

It is possible to consider on I^2 similar Gaussian fields, for example, the Brownian pillow or the tucked Brownian sheet $W_1(s,t)$ with the covariance

$$K_1(\bar{s},\bar{t}) = (\min(s_1,t_1) - s_1t_1)(\min(s_2,t_2) - s_2t_2)\,,$$

Brownian pillow-slips, or Kiefer fields $W_2(s,t)$ and $W_3(s,t)$ with covariances

$$K_2(\bar{s},\bar{t}) = \min(s_1,t_1)(\min(s_2,t_2) - s_2t_2)\,,$$

$$K_3(\bar{s},\bar{t}) = (\min(s_1,t_1) - s_1t_1)\min(s_2,t_2)\,,$$

as well as the pinned Brownian sheet or Brownian bed-sheet $W_4(s,t)$ with the covariance

$$K_4(\bar{s},\bar{t}) = \min(s_1,t_1)\min(s_2,t_2) - s_1t_1s_2t_2\,.$$

In these cases the calculation of the constant J_T is again reduced to the problem (1.3) but in the definition of the space U, besides (1.1), there appear new boundary conditions on other sides of the unit square. The constant J_T is interesting for the description of large deviation asymptotics of the functional T. It is also important in some problems of Statistics, in particular when calculating the limiting Pitman and approximate Bahadur efficiencies of distribution-free tests based on functionals of empirical fields (see [3]).

One more problem important for Statistics which leads in the two-dimensional case to the calculation of constants J_T is a problem of *moderately large* deviations of empirical measures (see Borovkov and Mogulski [4] and Ermakov [5]). Such deviations are used when evaluating the so-called *intermediate* or Kallenberg efficiency (see [6] and [7]).

B. *Functional law of the iterated logarithm* .

Let $W(\bar{s},\bar{t})$ be again the Brownian sheet. We consider the random broken lines

$$f_{mn}(s,t) = [4mn\ln\ln mn]^{-\frac{1}{2}}W(ms,nt),\ m,n \geq 3\,.$$

It was proved in the paper of Park [8] (see also [9]), that for almost all trajectories the set of limiting points for the sequence $\{f_{mn}(s,t)\ :\ m,n \geq 3\}$ coincides with the Strassen ball S in U.

If T is the continuous mapping of the space $C(I^2)$ in R^1 it follows from the result of Park (see [1], §17), that almost surely

$$\limsup_{m,n\to\infty} T(f_{mn}) = \sup\{T(u)\,|\,u \in S\}. \tag{1.4}$$

Hence we obtain again the problem (1.3). The value of the constant in the right-hand side of (1.4) is known actually only for few functionals T. The technique developed in this paper permits us to enlarge considerably this set of functionals.

C. *Local asymptotic optimality of independence tests.*

Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a sample from bivariate continuous distribution function (d.f.) $F(x, y)$.

To test the independence of components X and Y the statistics based on the empirical distribution functions (e.d.f) are frequently used. Let $F_n(x, y)$ be the e.d.f. based on the initial sample, and $G_n(x)$ and $H_n(y)$ are the e.d.f. constructed on the basis of $X_1, ..., X_n$ and $Y_1, ..., Y_n$ correspondingly.

We can consider as statistics for testing of independence the functionals $T(\xi_n)$ of random fields

$$\xi_n(x, y) = \sqrt{n}(F_n(x, y - G_n(x)H_n(y)), (x, y) \in R^2. \tag{1.5}$$

If the functional $T(\xi_n)$ corresponds to the distribution-free statistic, we can assume that the distributions of X_i, $1 \le i \le n$, and Y_j, $1 \le j \le n$, are uniform on [0,1] and then the field $\xi_n(x, y)$ converges weakly in the Skorokhod space $D(I^2)$ to the Brownian pillow $W_1(x, y)$.

If one of the distributions of components is known it is natural to consider the empirical fields $\sqrt{n}(F_n(x, y) - xH_n(y))$ and $\sqrt{n}(F_n(x, y) - G_n(x)y)$ converging weakly to Brownian pillow-slips $W_2(x, y)$ and $W_3(x, y)$. In case when distributions of both components are known, we obtain the field $\sqrt{n}(F_n(x, y) - xy)$ converging weakly to the pinned Brownian sheet $W_4(x, y)$.

As the alternative to the independence hypothesis we consider the widespread model (see, for example, [11] - [14] or [3], Ch. V), when the joint d.f. of observations is

$$F(x, y, \theta) = G(x)H(y) + \theta\Omega(G(x), H(y)), \ \theta \ge 0, \tag{1.6}$$

where G and H are marginal d.f.'s and where the non-negative function Ω is called dependence function and is such that

$$\Omega \ge 0, \quad \Omega \in C^2(I^2), \quad \Omega(\partial I^2) = 0.$$

The alternative (1.6) is an important particular case of the so-called alternative of strict positive quadrant dependence when it is supposed that

$$F(x, y) \ge G(x)H(y)$$

for all x and y, and the sign of a strict inequality takes place at least for one point (x, y).

The measure of local Bahadur efficiency for statistics of the type $T_n = T(\xi_n)$ is the local index $b_T(\Omega)$ calculated according to certain rule and taking into account the large deviation asymptotics of T_n under the null hypothesis and its almost sure behavior under the alternative (see [10] or [3]) . Due to the local variant of Bahadur-Raghavachari inequality [10] it is true that

$$b_T(\Omega) \leq \int_0^1 \int_0^1 \Omega_{xy}^2(x,y)dxdy. \tag{1.7}$$

To describe the domain of local optimality for the sequence T_n we must find for which functions Ω the sign of equality is attained in (1.7) . It results again in problem (1.3). The technique developed below permits to solve a lot of such problems.

2. The space H and its properties .

Consider on the space

$$X = \{u \in C^2(I^2) \,|\, u(x,0) = u(0,y) = 0\}$$

a bilinear form

$$[u,v] = \int_{I^2} u_{xy}(x,y)\, v_{xy}(x,y)dxdy. \tag{2.1}$$

Proposition 1. *The form (2.1) defines on X a scalar product.*

Proof. All properties of scalar product are obvious unless (2.1). The latter follows from the identity

$$u(x,y) = \int_0^x \int_0^y u_{xy}(x,y)dxdy, \tag{2.2}$$

which is true for all $u \in X$ due to boundary conditions.

In fact, if $[u,u] = 0$ then $u_{xy} \equiv 0$, whence by virtue of (2.2) $u \equiv 0.\Box$

Denote by H the closure of the space X under the norm $\|\cdot\|$ induced by the scalar product (2.1). Clearly H is the Hilbert space.

Proposition 2. *The embedding of the space H into the space $C(I^2)$ is compact.*

Proof. Let us rewrite the formula (2.2) in the form

$$u = K\, D_x\, D_y\, u, \tag{2.3}$$

where D_x and D_y are the operators of differentiation and K is the Volterra integral operator. As the operator $D_x D_y$ acting from H into $L^2[0,1]$ is bounded and K is the compact operator acting from $L^2[0,1]$ into $C(I^2)$ we get immediately the required assertion. □

Proposition 3. *The embedding of the space H into the space $W_2^1(I^2)$ is compact.*

Proof. Similarly to (2.3) we have two formulas

$$u_x = K_y\, D_x\, D_y\, u \quad , \quad u_y = K_x\, D_x\, D_y\, u,$$

where K_x and K_y are the operators of taking the primitives with respect to corresponding variables. The compactness of the embedding of H into $W_2^1(I^2)$ follows from the compactness of operators K_x and K_y in $L^2(I^2)$. □

It follows from Proposition 2 that the functions from H satisfy the boundary conditions (1.1). For such functions in the space $W_2^1(I^2)$ it is possible to introduce the equivalent norm $\| \nabla u \|_{L^2(I^2)}$.

Remark 1. It follows from Fubini theorem that for any function $u \in H$ it is true that for almost all x $u_{xy}(x,\cdot) \in L^2[0,1]$. Hence $u_x(x,\cdot) \in W_2^1(I)$ for almost all $x \in I$ and similarly $u_y(\cdot, y \in W_2^1(I)$ for almost all $y \in I$.

Remark 2. The attempt to remove a part of boundary conditions (1.1) would result in that the form (2.1) becomes degenerate. But even if we correct it by the addition of lower term, for example $< u, v >_{L^2(I^2)}$, where the symbol $< \cdot, \cdot >$ means the scalar product, the obtained space cannot be embedded into $C(I^2)$ as well as into $W_2^1(I^2)$ because the existence of the mixed Sobolev derivative does not imply, generally speaking, the existence of the first derivatives.

Let us introduce the following subspaces of the space H

$$H_1 = \{u \in H \,|\, u(1,y) = u(x,1) = 0\},$$

$$H_2 = \{u \in H \,|\, u(1,y) = 0\},$$

$$H_3 = \{u \in H \,|\, u(x,1) = 0\},$$

as well as

$$H_4 = \{u \in H \,|\, u(1,1) = 0\}.$$

It is clear now that the problems of the type (1.3) described above in $\mathbf{A} - \mathbf{C}$ can be formalized as follows *find the maximum of the homogeneous functional $T(u)$ on the unit ball of the space H or one of its subspaces $H_1 - H_4$ and find (in the problem* **C***) those elements u, on which this maximum is attained.*

The space H should be used in the problem **A** for the analysis of large deviation asymptotics and in the problem **B** for the formulation of the law of iterated logarithm for functionals of Brownian sheet. The space H_1 should be used in the problem **A** for large deviations of functionals of the Brownian pillow W_1 and in the problem **C** when studying the local optimality of tests based on the random field (1.5). The spaces H_2 and H_3 occur at the analysis of Kiefer fields W_2 and W_3 and empirical fields when one of marginal distribution of bivariate observations is known. The space H_4 is suitable for the study of pinned Brownian sheet W_4 and empirical fields in the problems when both marginal distributions of initial observations are known.

One of sufficient conditions for the existence of the maximum which is fulfilled in almost all important cases is the continuity of the functional T on the space where H can be embedded compactly, for example, on $C(I^2)$.

Note that due to homogeneity of the functional T the maximum can be taken over the unit sphere S_1 instead of the unit ball S.

3. Problems with linear forms .

We begin with the easiest case when

$$T(u) = [b(u)]^2,$$

where b is the linear functional on H and we must find

$$\sup\{[b(u)]^2 \mid u \in S_1\},$$

or equivalently

$$\inf\{\| u \|^2 \mid b(u) = 1\}. \tag{3.1}$$

Let us assume in addition that the functional b is continuous on the space $C(I^2)$. Then by Riesz–Markov theorem

$$b(u) = \int_{I^2} u \, d\mu,$$

where μ is a finite charge on I^2.

The necessary condition of the minimum in (3.1) is thus reduced to the Euler-Lagrange equation

$$\lambda\, u_{xxyy} = \mu \tag{3.2}$$

under boundary conditions on ∂I^2 including natural ones. The numerical Lagrange multiplier λ is found from the norming condition $b(u) = 1$.

The solution of the boundary problem for the equation (3.2) is given by the formula

$$u(x,y) = \lambda^{-1} \int_{I^2} K(x,y;s,t)\,d\mu(s,t), \tag{3.3}$$

where K is the Green function of the operator $D_x^2 D_y^2$ for the appropriate set of boundary conditions. It follows from the norming condition and (3.3) that

$$\lambda = \int_{I^2}\int_{I^2} K(x,y;s,t)\,d\mu(x,y)\,d\mu(s,t). \tag{3.4}$$

The Green functions for the operator $D_x^2 D_y^2$ under the boundary conditions included in the definition of the space H and its subspaces $H_1 - H_4$ are well-known in Mathematical Physics. In fact they coincide with the covariances of Gaussian fields listed in the formulation of the problem **A**. For convenience we give them below.

1. If $u \in H$ and the boundary conditions are $u(x,0) = u(0,y) = 0$ together with the natural boundary conditions $u_x(1,y) = u_y(x,1) = 0$, the Green function is

$$K(x,y;s,t) = \min(x,s)\min(y,t)\,.$$

2. If $u \in H_1$ and the boundary conditions are $u(x,0) = u(0,y) = u(x,1) = u(1,y) = 0$, the Green function is

$$K_1(x,y;s,t) = (\min(x,s) - xs)\,(\min(y,t) - yt)\,.$$

3. If $u \in H_2$ with the boundary conditions $u(x,0) = u(0,y) = u(1,y) = u_y(x,1) = 0$, the Green function is

$$K_2(x,y;s,t) = (\min(x,s) - xs)\min(y,t)\,.$$

Similarly, for the space H_3 the Green function is

$$K_3(x,y;s,t) = \min(x,s)(\min(y,t) - yt)\,.$$

4. Finally, if $u \in H_4$ with the boundary conditions $u(x,0) = u(0,y) = u_{xxy}(x,1) = u_{xyy}(1,y) = u(1,1) = 0$, the Green function is equal to

$$K_4(x,y;s,t) = \min(x,s)\min(y,t) - xyst\,.$$

Consider three examples of applications of the obtained result to the problem **C**. The linear case is usually well-studied and the examples given below do not pretend to novelty but illustrate well the general approach described above.

First we consider the so-called first component of the omega-square statistic for independence testing that was introduced in [31]. In the notations of the problem **C** this statistic has an expression

$$V_n = \int_{R^2} [F_n(x,y) - G_n(x)H_n(y)] \sin \pi G_n(x) \sin \pi H_n(y) dG_n(x) dH_n(y)$$

From the arguments given in [3], Ch. 5,6 it follows that the inequality (1.7) takes the form

$$4\pi^4 \left(\int_{I^2} \Omega(x,y) \sin \pi x \sin \pi y dx dy\right)^2 \leq \int_{I^2} \Omega_{xy}^2(x,y) dx dy,$$

where the function Ω belongs to the subspace H_1. Therefore due to (3.3) the local Bahadur optimality of the statistic V_n in model (1.6) is attained whenever

$$\Omega(x,y) = C \sin \pi x \sin \pi y, C \geq 0.$$

(cf. with Corollary 2 of Theorem 6.6.3 in [3]).

Other example is related to the Gini rank association coefficient introduced in [16]. Long-term efforts of Italian statisticians were devoted to its study , see, for example, [17].

Denote by R_i the rank of X_i among X's and by S_i the rank of Y_i among Y's. The Gini rank association coefficient r_G is defined by the formula

$$r_G = \frac{2}{D_n} \sum_{i=1}^{n} (|n+1-R_i-S_i| - |R_i - S_i|),$$

where $D_n = n^2$ for even n and $D_n = n^2 - 1$ for odd n.

From [18] it follows that whenever the mixed derivative $\Omega_{xy}(x,y)$ in model (1.6) is bounded, the local Bahadur index of r_G is equal to

$$24\left(\int_0^1 [\Omega(x,x) + \Omega(1-x,x)] dx\right)^2,$$

Thus the inequality (1.7) takes the form

$$24\left(\int_0^1 [\Omega(x,x) + \Omega(1-x,x)] dx\right)^2 \leq \int_0^1 \int_0^1 \Omega_{xy}^2(x,y) dx dy.$$

In this case

$$\mu(x,y) = \delta(x-y) + \delta(1-x-y), \tag{3.4}$$

where $\delta(x)$ means a delta-measure in the point x. Substituting in the formula (3.3) the Green function $K_1(x, y; s, t) = (\min(x, s) - xs)\,(\min(y, t) - yt)$ and the measure (3.4), we obtain after some calculations

$$\Omega_G(x, y) = C(|x - y|^3 + |x + y - 1|^3 - 3(x^2 + y^2) + 3(x + y - 1)), C > 0.$$

It is curious that it is a natural example of dependence function and hence of a copula with *cubic* sections. The interest to such copulas has arisen after the publication of the recent paper [19].

Similar arguments may be used for the analysis of the Spearman's footrule which is based on the statistic

$$r_f = \sum_{i=1}^{n} |R_i - S_i|,$$

It was proposed by Spearman in [20], and the interest to it has increased after the papers [21] - [23] and a number of others.

In this case the local index has a form

$$b_f(\Omega) = 90(\int_{R^2} \Omega(x, x)dx)^2,$$

and the "optimum" dependence function has again cubic sections and looks as follows:

$$\Omega_f(x, y) = C[|x - y|^3 - (x + y)^3 + 2xy(x^2 + y^2 + 2)],\ C > 0.$$

The third example is connected with the generalization of Kendall rank correlation coefficient τ. This coefficient is the *nonlinear* rank statistic and in the notations of the previous example has a form

$$\tau_n = (n(n-1))^{-1} \sum_{1 \le i \neq j \le n} sign(R_i - R_j) sign(S_i - S_j)$$

Its properties are well-studied (see, for example, [24], [25].) It is known, in particular, that the statistic τ_n is locally Bahadur optimal in the problem of independence testing against alternatives (1.6) only for Farlie-Gumbel-Morgenstern dependence function

$$\Omega_{FGM}(x, y) = Cx(1 - x)y(1 - y), C > 0.$$

Kochar and Gupta proposed in [26] an interesting generalization of Kendall statistic of the order $k \ge 1$ which coincides with Kendall statistic at $k = 1$. It was shown in [27] that the local Bahadur optimality of Kochar-Gupta statistic results in the inequality

$$(k+1)^4(2k+1)^2(\int_{I^2}(xy)^{k-1}\Omega(x,y)dxdy)^2 \leq \int_{I^2}\Omega^2_{xy}(x,y)dxdy, \tag{3.5}$$

where the function Ω is taken again from the subspace H_1. From (3.3) it follows that the local Bahadur optimality is possible only for Woodworth dependence function

$$\Omega_W(x,y) = Cx(1-x^k)y(1-y^k), C > 0.$$

4. Problems with separated variables .

In this section we solve the problem (1.3) in the case when

$$b(u) = [p_x(p_y(u(x,y))]^2.$$

Here p_x and p_y are two seminorms defined on functions of one variable, and p_x has a property

$$p_x(|v|) \quad \geq \quad p_x(v). \tag{4.1}$$

The arguments presented below are valid for any space of H, H_1, H_2, H_3, but not for H_4.

Let consider two pairs of auxiliary one-dimensional extremal problems

$$\lambda_k^{(x)} = \sup_{A_k}[p_x(v)]^2 \ k = 1,2, \tag{4.2}$$

$$\lambda_j^{(y)} = \sup_{A_j}[p_y(v)]^2, j = 1,2, \tag{4.3}$$

where

$$A_1 = \{v \in W_2^1(I)|\, v(0) = 0, \textstyle\int_I v'^2(x)dx \leq 1\},$$

$$A_2 = \{v \in A_1|\, v(1) = 0\}.$$

The following theorem is our most essential result.

THEOREM 1 . *Suppose the upper bounds in problems (4.2) and (4.3) are attained. Then the maximum in (1.3) is also attained and equals:*
a). $\lambda = \lambda_1^{(x)}\lambda_1^{(y)}$ for the space H;

b). $\lambda = \lambda_2^{(x)}\lambda_2^{(y)}$ for the space H_1;
c). $\lambda = \lambda_2^{(x)}\lambda_1^{(y)}$ and $\lambda = \lambda_1^{(x)}\lambda_2^{(y)}$ for the spaces H_2 and H_3 correspondingly.

If the extremals on which the upper bounds in problems (4.2) and (4.3) are attained are unique (up to the factor ± 1) , the extremal in (1.3) is also unique up to the factor ± 1 and equals the product of appropriate univariate extremals.

Proof. All cases in the theorem are analyzed identically and we consider for definiteness case a). Let obtain first the estimate from above for the right-hand side in (1.3) and let prove that it is attained on the extremal mentioned above.

From the triangle inequality for p_y it follows that for any $x_1 \neq x_2$ one has

$$|p_y(u(x_1,y) - p_y(u(x_2,y)| \leq p_y(u(x_1,y) - u(x_2,y).$$

Hence for all $x \in I, h > 0$

$$h^{-1}|p_y(u(x,\cdot)) - p_y(u(x-h,\cdot))| \leq p_y\left(h^{-1}(u(x,\cdot) - u(x-h,\cdot))\right) \tag{4.4}$$

(for $x - h < 0$ we assume that $u(x-h,\cdot) \equiv 0$). Now we estimate the right part of (4.4) in the norm $L_2(I)$:

$$\int_0^1 p_y^2(h^{-1}(u(x,\cdot) - u(x-h,\cdot)))dx \leq$$

$$\lambda_1^{(y)} \int_0^1 \int_0^1 [h^{-1}(u_y(x,\cdot) - u_y(x-h,\cdot))]^2 dydx \leq \lambda_1^{(y)} \int_0^1 \int_0^1 u_{xy}^2(x,y)dxdy$$

Therefore the ratios

$$h^{-1}[p_y(u(x,\cdot)) - p_y(u(x-h,\cdot))]$$

are bounded in the norm $L_2(I)$ uniformly in $h > 0$. From Lemma 2.1 of Ch.6 in [28] it follows that the function $x \to p_y(u(x,\cdot))$ has generalized derivative belonging to $L_2(I)$. Passing to the limit in (4.4) as $h \to 0$, we obtain for almost all $x \in I$ the inequality

$$|\frac{d}{dx}p_y(u(x,\cdot)| \leq p_y(u_x(x,\cdot)). \tag{4.5}$$

(The right part of this inequality is defined for almost all $x \in I$ due to Remark 1 of Section 2).

Therefore for all $u \in H$

$$b(u) = [p_x(p_y(u(x,y))]^2 \leq$$

$$\leq \lambda_1^{(x)} \int_0^1 [\tfrac{d}{dx}(p_y(u(x,y))]^2 dx$$

$$\leq \lambda_1^{(x)} \int_0^1 [(p_y(u_x(x,y)]^2 dx$$

$$\leq \lambda_1^{(x)}\lambda_1^{(y)} \int_0^1 \int_0^1 u_{xy}^2(x,y)dxdy. \tag{4.6}$$

Thus

$$\sup\{b(u)\,|u\in S_1\}\leq\lambda_1^{(x)}\lambda_1^{(y)}.$$

Now let v_1 be the extremal in the problem (4.2) for $k=1$ and v_2 be the extremal in the problem (4.3) for $j=1$. Due to (4.1) we can assume that v_1 is non-negative. Put $u_0(x,y)=v_1(x)v_2(y)$. Then

$$\| u_0 \|_H^2=\int_{I^2} u_{xy}^2(x,y)dxdy=\int_0^1 v_1'^2(x)dx\int_0^1 v_2'^2(y)dy\leq 1,$$

so that $u_0\in S$. Let calculate now $b(u_0)$. Taking in account the non-negativity of v_1 we get:

$$\begin{aligned}b(u_0)&=[p_x(p_y(v_1(x)v_2(y)))]^2=[p_x(v_1(x)p_y(v_2(y))]^2\\&=[p_x(v_1(x))p_y(v_2(y))]^2=\lambda_1^{(x)}\lambda_1^{(y)}.\end{aligned}$$

Thus the theoretical maximum of $b(u)$ on S_1 is reached on u_0 and equals exactly to $\lambda_1^{(x)}\lambda_1^{(y)}$.

Suppose now that the extremals v_1 and v_2 are unique up to a constant. Then the last inequality in (4.6) becomes equality only if for almost all $x\in I$

$$u_x(x,\cdot)=\alpha(x)v_2(y).$$

But then

$$u(x,y)=\int_0^x\alpha(t)dt\cdot v_2(y)=\bar{\alpha}(x)v_2(y).$$

If the first inequality in (4.6) also becomes equality then $\bar{\alpha}(x)=C\cdot v_1(x)$. From the norming condition we obtain $C=\pm 1$ and hence the extremal function is necessary equal to $\pm u_0$. $\square$

5. Applications to problems A, B,C.

Very natural and important examples of seminorms p_x and p_y are norms in spaces $L_p, 1 \le p \le +\infty$. Let $1 \le p_1, p_2 \le +\infty$ and let

$$p_x(v) = \left[\int_0^1 |v(t)|^{p_1} dt\right]^{1/p_1}, \quad p_y \left[\int_0^1 |v(t)|^{p_2} dt\right]^{1/p_2},$$

then

$$p_x(p_y(v(x,y)) = \left[\int_0^1 [\int_0^1 |v(s,t|^{p_2} dt]^{p_1/p_2} ds\right]^{1/p_1}.$$

The constants λ_k^x and $\lambda_k^y, k = 1,2$, appearing in the theorem of the previous section are well-known and are equal to

$$\lambda_1^x = \sigma_1(p_1), \quad \lambda_1^y = \sigma_1(p_2),$$

$$\lambda_2^x = \sigma_2(p_1), \quad \lambda_2^y = \sigma_2(p_2),$$

where

$$\sigma_1(p) = \frac{2}{p\pi}(1+\frac{p}{2})^{(p-2)/p}\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{p})}{\Gamma(1+\frac{1}{p})}\right)^2, \quad \sigma_2(p) = \sigma_1(p)/4. \qquad (5.1)$$

For the first time the constants $\sigma_m(p), m = 1,2$ were calculated by Strassen [29], later his result was reproduced, supplemented or partially rediscovered in [30] - [34]. We write out the values of these constants for $p = 1,2$ and $p = +\infty$:

$$\sigma_1(1) = 1/3, \quad \sigma_1(2) = 4/\pi^2, \quad \sigma_1(+\infty) = 1;$$

$$\sigma_2(1) = 1/12, \quad \sigma_2(2) = 1/\pi^2, \quad \sigma_2(+\infty) = 1/4.$$

In [34] the tables of $\sigma_2(p)$ for small values of $p > 1$ are given.

We return now to problems **A**, **B** and **C** described above. Consider first the problem **A** on the rough large deviation asymptotics for functionals of Gaussian fields. We put for any field ξ on I^2

$$\| \xi \|_{p_1,p_2}^2 = \left[\int_0^1 [\int_0^1 |\xi(s,t)|^{p_2} dt]^{p_1/p_2} ds\right]^{2/p_1}.$$

This functional (*the square* of the anisotropic norm) is suitable for the application of Theorem 1. Substituting instead of ξ Gaussian fields W, W_1, W_2 and W_3, we obtain the following theorem.

THEOREM 2 . *For $1 \leq p_1, p_2 \leq \infty$ the following asymptotics are true:*

$$\lim_{r\to+\infty} r^{-2} \ln P\{\| W \|_{p_1,p_2}^2 \geq r\} = -(2\sigma_1(p_1)\sigma_1(p_2))^{-1},$$

$$\lim_{r\to+\infty} r^{-2} \ln P\{\| W_1 \|_{p_1,p_2}^2 \geq r\} = -(2\sigma_2(p_1)\sigma_2(p_2))^{-1},$$

$$\lim_{r\to+\infty} r^{-2} \ln P\{\| W_2 \|_{p_1,p_2}^2 \geq r\} = -(2\sigma_1(p_2)\sigma_2(p_1))^{-1},$$

$$\lim_{r\to+\infty} r^{-2} \ln P\{\| W_3 \|_{p_1,p_2}^2 \geq r\} = -(2\sigma_1(p_1)\sigma_2(p_2))^{-1}.$$

where the constants $\sigma_1(p), \sigma_2(p)$ are described in (5.1).

The asymptotics presented above are new except for two cases. From the book of Lifshits [1], §14, where the exact tail asymptotics for the suprema of considered fields are described, it is possible to extract our results for $p_1 = p_2 = +\infty$.

Another case when it is not difficult to obtain such asymptotics is the case $p_1 = p_2 = 2$. As the eigenfunctions and the eigenvalues of integral operators with symmetric kernels K and $K_1 - K_3$ are known in the explicit form, it is possible to expand the random fields W and $W_1 - W_3$ in orthogonal random series in two variables (the Karhunen-Loeve series, see, for example, [35], p. 75 - 76), converging in $L_2(I^2)$. The square of the norm in $L_2(I^2)$ of such fields is represented as an infinite weighted sum of squares of independent standard Gaussian variables with the weights equal to eigenvalues of corresponding kernels. It remains to apply the well-known result of Zolotarev [36] by virtue of which the rough large deviation asymptotics is defined by the first eigenvalue of an appropriate kernel. Close arguments are contained in [37], where the problem of small deviations is studied.

However the results of a type

$$\lim_{r\to+\infty} r^{-2} \ln P\{\sup_y \int_0^1 |W_1(x,y|dx \geq r\} = -24,$$

that corresponds to the case $p_1 = 1, p_2 = +\infty$ or

$$\lim_{r\to+\infty} r^{-2} \ln P\{[\int_0^1 [\int_0^1 |W_2(x,y|^3 dy]^{\frac{5}{3}} dx \geq r^5\} = -\frac{3 \cdot 5^{\frac{2}{3}} \cdot \Gamma^2(\frac{4}{3})\Gamma^2(\frac{6}{5})}{2^{\frac{1}{15}} \cdot 7^{\frac{3}{5}} \cdot \Gamma^2(\frac{5}{6})\Gamma^2(\frac{7}{10})}\pi^2,$$

that answers the case $p_1 = 5, p_2 = 3$, not speaking on arbitrary p_1 and p_2, are new.

In principle the norm L_p can be considered with non-negative weights χ, however the set of weights for which we can explicitly write out the values λ_1 and λ_2

of univariate extremal problems is not large.We know, for example, from [38] the classical weight of Anderson-Darling for $L^2(I)$

$$\chi^*(t) = [t(1-t)]^{-1}, 0 \leq t \leq 1,$$

for which

$$\sup\{\int_0^1 v^2(t)\chi^*(t)dt | v \in A_2\} = 1/2,$$

and the extremal is $\sqrt{3}t(1-t)$. Another example, which can be extracted from table 5 in the book of Collatz [39], is the weight in the space $L^2(I)$ equal to $\bar{\chi}(t) = (1+t^2)^2$. In this case

$$\sup\{\int_0^1 v^2(t)\bar{\chi}(t)dt | v \in A_2\} = 1/15,$$

and the extremal is $\sqrt{\frac{8}{15\pi}} \cdot \sqrt{1+t^2}\sin(4\arctan t)$. Combining these results for $p_1 = p_2 = 2$, we get

$$\lim_{r\to+\infty} r^{-2} \ln P\{\int_0^1 [\int_0^1 W_2^2(x,y/[x(1-x)(1+y^2)^2]dxdy \geq r^2\} = -15,$$

that is again a new result.

Applying the obtained results to the field $W(x,y)$ in the problem **B**, we can write, saving the notations, that with probability 1

$$\limsup_{m,n\to\infty} [4mn \ln\ln mn]^{-1} \sup_s \int_0^1 |W(ms,nt)|^2 dt \ = 4/\pi^2,$$

and a number of similar relations can be established analogously.

We give here one more application to the problem **C**. The functional

$$T^*(W_2) = \sup_x |\int_0^1 W_2(x,y)dy| \tag{5.2}$$

is of interest because it leads to the known distribution-free test of independence proposed by Durbin [40]. It can be obtained substituting in $T^*(\cdot)$ instead of the fields W_2 the empirical field (1.5). Similar but more complicated functionals were considered in the preprint [41].

The distribution of the functional (5.2) up to the scale factor coincides with the classical distribution of Kolmogorov statistic that increases the interest to this functional and corresponding independence criterion. The condition of Bahadur local optimality for alternatives (1.6) takes the form (see [7], Ch. 6):

$$48(\sup_x |\int_0^1 \Omega(x,y)dy|)^2 \leq \int_0^1 \int_0^1 \Omega_{xy}^2(x,y)dxdy.$$

The extremals for univariate problems of finding maxima on the set A_2 for functionals $\sup_t |v(t)|$ and $|\int_0^1 v(t)dt|$ are well known (see [7], Ch.6). They are unique up to the factors ± 1 and are equal correspondingly to $\min(t, 1-t)$ and $\sqrt{3}t(1-t)$. Taking their product as prescribed by Theorem 1, we get as a result that the Durbin test is locally optimal only for dependence function $\Omega(x, y) = C\min(x, 1-x)y(1-y)$, $C > 0$, that coincides with the result obtained in [7], §6.6 by means of much longer proof.

It is worth to emphasize that the discovery of *new* results in the problem **C** relies on the evaluation of rough large deviation asymptotics for functionals of *prelimit* empirical fields. However if this asymptotics is already found our theorem gives the simple and effective way for the description of local optimality conditions.

In conclusion we note that all results can be carried over multivariate case. The formulations are similar but more cumbersome because of sharp extension of the set of boundary conditions. This made us to be limited by the bivariate case.

REFERENCES

1. Lifshits, M.A. *Gaussian random functions.* Kluwer, 1995.

2. Varadhan, S.R.S. Large deviations and applications. SIAM, Philadelphia, 1984.

3. Nikitin, Ya.Yu. *Asymptotic efficiency of nonparametric tests.* Cambridge University Press, 1995.

4. Borovkov, A.A., Mogulski, A.A. On large deviations in topological spaces. I, II. *Siberian J. of Math.* , 1978, v.19, N 5, p. 988 - 1004; 1980, v.21, N 5, p.12 - 26.

5. Ermakov, M.S. Large deviation of empirical measures and statistical tests. *Notes of Scientific seminars of POMI* , 1993, v. 207, p.37 -59.

6. Inglot T. Generalized intermediate efficiency of goodness-of-fit tests. *Math. Meth. of Stat.*, 1999, v.8, N 4, p. 487 – 509.

7. Kallenberg W.C.M. Intermediate efficiency, theory and examples. *Ann.Statist.*, 1983, v.11, N 1, p.170 – 182.

8. Park, W.J. On Strassen's version of the law of the iterated logarithm for the two-parameter Gaussian process. *J.Multiv. Anal.* , 1974, v.4, N 3, p.479 - 485.

9. Park, W.J. The law of the iterated logarithm for Brownian sheets. *J. Appl. Prob.* , 1975, v.12, p.840 - 844.

10. Bahadur, R.R. *Some limit theorems in statistics* . SIAM: Philadelphia, 1971.

11. Farlie, D.J. The asymptotic efficiency of Daniels's generalized correlation coefficients. *J. Roy. Statist. Soc.* , 1961, v. B28, N 1, p.128 - 142.

12. Johnson, N.L., Kotz, S. On some generalized Farlie-GumbelMorgenstern distributions. *Commun. Statist. Theory and Meth.* , 1975, v. 4, N 3, p.415 - 427.

13. Ledwina, T. On the limiting Pitman efficiency of some rank tests of independence. *J.Multiv. Anal.* , 1986, v. 20, N 2, p. 265 - 271.

14. Bajorski, P. Local Bahadur optimality of some rank tests of independence. *Statist. and Prob. Lett.* , 1987, v. 5, N 6, p.255 - 262.

15. Koziol, J.A., Nemec, A.F. On a Cramér - von Mises type statistic for testing bivariate independence. *Canadian J.of Stat.* , 1979, v. 7, N 1, p.43-52.

16. Gini C. *L'ammontare e la composizione della ricchezza delle nazioni.* F.lli Bocca: Torino, 1914.

17. Cifarelli D.M., Regazzini E. On a distribution-free test of independence based on Gini's rank association coefficient. In: *Recent Developments in Statistics* (J.R. Barra et al., eds.), North Holland, Amsterdam, 1977, p.375-385

18. Conti, P.L., Nikitin, Ya. Yu. Asymptotic efficiency of independence tests based on Gini's rank association coefficient, Spearman's footrule and their generalizations. *Commun. Statist. Theory and Methods* , 1999, v.28, N 2, p.453 - 465.

19. Nelsen R., Quesada - Molina J.J., Rodriguez - Lallena J.A. Bivariate copulas with cubic sections. *J. of Nonparametric Statistics* , 1997, v. 7, N 2, p.205-220.

20. Spearman C. Footrule for measuring correlation. *Brit. J. Psych.* , 1906, v. 2, p.89 - 108.

21. Diaconis P., Graham R.L. Spearman's footrule as a measure of disarray. *J. Roy. Stat. Soc.* , 1977, v. B 39, N 2, p.262 - 268.

22. Tyurin, Yu.N., Schmerling, D.S. On some statistical methods for the analysis of rankings: a review. *Zavodskaya labor.* , 1985, v.51, N 7, p. 42 - 48.

23. Salama I.A., Quade D. A note on Spearman's footrule. *Commun. Statist. Simul. Comput.* , 1990, v. 19, p.591 - 601.

24. Kendall, M. *Rank correlation methods.* London, Griffin, 1970.

25. Hettmansperger, T.P. *Statistical methods based on ranks* . N.Y.: Wiley, 1984.

26. Kochar, S.C., Gupta, R.P. Competitors of the Kendall's tau test for testing independence against positive quadrant dependence. *Biometrika* , 1987, v. 74, N 3, p. 664 - 666.

27. Nikitin, Ya.Yu., Stepanova N.A. One generalization of Kendall's tau and the asymptotic efficiency of corresponding independence test. *Notes of Scientific*

seminars of POMI , 1997, v.244, p. 227-237.

28. Ladyzhenskaya, O.A. *Boundary problems of mathematical physics.* Moscow, Nauka, 1973.

29. Strassen, V. An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. Verw. Geb.* , 1964, v. 3, N 2, p. 211-226.

30. Kallianpur, G., Oodaira H. Freidlin-Wentzell type estimates for abstract Wiener spaces. *Sankhyā* , 1978, v. A40, N 1, p. 116 - 137.

31. Nikitin, Ya.Yu. Large deviations and asymptotic efficiency of statistics of integral type . II. *J.of Soviet Math.*, 1984,v.84, p.585 - 603.

32. Piterbarg, V.I., Fatalov V.R. The Laplace method for probability measures in Banach spaces. *Uspekhi Mat.nauk* , 1995, v. 50, N 6, p.57 - 150.

33. Fatalov, V.R. Large deviations of Gaussian measures in the spaces l^p and L^p. *Theory Probab.Appl.*, 1996, v. 41, p.682 - 689.

34. Gajek, L., Kaluszka, M., and Lenic, A. The law of the iterated logarithm for L_p - norms of empirical processes. *Statist. and Prob. Lett.*, 1996, v. 28, N 1, p.107 - 110.

35. Adler, R.J. *An introduction to continuity, extrema, and related topics for general Gaussian processes* . IMS Lecture Notes - Monograph Ser., 12, Hayward, California, 1990.

36. Zolotarev, V.M. On one problem of probability. *Theory of Probab. Appl.* , 1961, v.6, N 2, p. 219 - 222.

37. Csaki, E. On small values of the square integral of a multiparameter Wiener process. In:*Statistics and Probability, Proc. of the 3rd Pannonian Symposium on Math. Stat.*, 1982, p.19 - 26.

38. Anderson, T.W., Darling D.A. Asymptotic theory of certain "goodness-of-fit" criteria based on stochastic processes. *Ann. Math. Stat.* , 1952, v. 23, N 1, p.193 - 212.

39. Collatz, L. *Eigenwertaufgaben mit technischen Anwendungen* , Leipzig: Akademische Verlag, 1963.

40. Durbin, J. Asymptotic distributions of some statistics based on the bivariate sample distribution functions. In: *Nonparametric techniques in statistical inference* , (M.L.Puri, ed.), Cambridge University Press, 1970, p.435 - 451.

41. Horvath, L., Shao, Q. *Liminf for weighted empirical processes* . Preprint, 1999.