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UNDER PARTIAL INFORMATION

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Exponential Utility Maximization under Partial Information

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Abstract

We consider the exponential utility maximization problem under partial information. The underlying asset price process follows a continuous semimartingale and strategies have to be constructed when only part of the information in the market is available. We show that this problem is equivalent to a new exponential optimization problem, which is formulated in terms of observable processes. We prove that the value process of the reduced problem is the unique solution of a backward stochastic differential equation (BSDE), which characterizes the optimal strategy. We examine two particular cases of diffusion market models, for which an explicit solution has been provided. Finally, we study the issue of sufficiency of partial information.

Key words: Backward stochastic differential equation; semimartingale market model; exponential utility maximization problem; partial information; sufficient filtration.

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1 Introduction

Investors acting in the market often have only limited access to the information flow. Besides, they may not be able or may not want to use all available information even if they have access to the full market flow. In such cases investors take their decisions using only part of the market information.

We study the problem of maximizing the expected exponential utility of terminal net wealth, when the asset price process is a continuous semimartingale and the flow of observable events does not necessarily contain all information on the underlying asset's prices.

We assume that the dynamics of the price process of the asset traded on the market is described by a continuous semimartingale $S = (S_t, t \in [0,T])$ defined on a filtered probability space $(\Omega, \mathscr{A}, \mathcal{A} = (\mathcal{A}_t, t \in [0,T]), P)$, satisfying the usual conditions, where $\mathscr{A} = \mathcal{A}_T$ and $T < \infty$ is a fixed time horizon. Suppose the interest rate to be equal to zero and the asset price process to satisfy the structure condition, i.e., the process S admits the decomposition

$$S_t = S_0 + N_t + \int_0^t \lambda_u d\langle N\rangle_u, \quad \langle \lambda \cdot N\rangle_T < \infty \quad a.s., \tag{1}$$

where N is a continuous $\mathcal{A}$-local martingale and λ is a $\mathcal{A}$-predictable process.
Let G be a filtration smaller than $\mathcal{A}$

$$G_t \subseteq \mathcal{A}_t, \quad \text{for every} \quad t \in [0,T].$$

The filtration G represents the information that the investor has at his disposal. Hence, hedging strategies have to be constructed using only information available in G.

Let $U(x) = -e^{-\alpha x}$ be an exponential utility function, where $\alpha > 0$ is a fixed constant.

We consider the exponential utility maximization problem with random payoff H at time T,

$$\text{to maximize} \quad E[-e^{-\alpha(x+\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(G),$$

which is equivalent to the problem (without loss of generality we can take $x = 0$)

$$\text{to minimize} \quad E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(G), \tag{2}$$

where $\Pi(G)$ is a certain class of G-predictable S-integrable processes (to be specified later) and $(\int_0^t \pi_u dS_u, t \in [0,T])$ represents the wealth process related to the self-financing strategy π.

The utility maximization problem with partial information has been considered in the literature under various setups. In most papers (see, e.g., [14, 11, 25, 31]) the problem was studied for market models where only stock prices are observed, while the drift can not be directly observed, i.e. under the hypothesis $F^S \subseteq G$.
We also consider the case when the flow of observable events G does not necessarily contain all information on prices of the underlying asset, i.e., when S is not a G-semimartingale in general. We show that the initial problem is equivalent to a certain problem of maximizing the filtered terminal net wealth and apply the dynamic programming method to the reduced problem. Such an approach, in the context of mean variance hedging, was considered in [18] (for the mean-variance hedging problem under partial information see also [27, 4, 24]).

Let us introduce an additional filtration $F = (F_t, t \in [0,T])$, which is the augmented filtration generated by F^S and G. The price process S is also a F-semimartingale and the canonical decomposition of S with respect to the filtration F is of the form (see, e.g., [13])

$$S_t = S_0 + \int_0^t \widehat{\lambda}_u^F d\langle M \rangle_u + M_t, \tag{3}$$

where $\widehat{\lambda}_u^F$ denotes the F-predictable projection of λ and

$$M_t = N_t + \int_0^t [\lambda_u - \widehat{\lambda}_u^F] d\langle N \rangle_u \tag{4}$$

is a F-local martingale. Besides $\langle M \rangle = \langle N \rangle$ and these brackets are F^S-predictable.

Let us consider the following assumptions:

A) $\langle M \rangle$ is G-predictable and $d\langle M \rangle_t dP$ a.e. $\widehat{\lambda}^F = \widehat{\lambda}^G$, hence for each t

$$E(\lambda_t | F_{t-}^S \vee G_t) = E(\lambda_t | G_t), \quad P-\text{a.s.} \tag{5}$$

B) any G-martingale is an F-local martingale,

C) the filtration G is continuous,

D) for any G-local martingale $m(g)$, $\langle M, m(g)\rangle$ is G-predictable,

E) H is an $\mathcal{A}_T$-measurable bounded random variable, such that P- a.s.

$$E[e^{\alpha H}|F_T] = E[e^{\alpha H}|G_T]. \tag{6}$$

Let us make some remarks on conditions A)-E). It is evident, that if $F^S \subseteq G$, then $G = F$ and conditions A), B), D) and equality (6) of condition E) are satisfied. Condition B) is satisfied if and only if the σ-algebras $F_t^S \vee G_t$ and G_T are conditionally independent given G_t for all $t \in [0,T]$ (see Theorem 9.29 in Jacod (1979)). Recall that Condition C) means that all G-local martingales are continuous. Under condition B) the continuity of F implies the continuity of G, but not vice-versa. So, the filtration F may be not continuous in general. Equality (6) is satisfied if, e.g., H is of the form $f(\eta, \xi)$, where η is a G_T-measurable random variable and ξ is $\mathcal{A}_T$-measurable random variable independent of F_T.

We shall use the notation $\widehat{Y}_t$ for the G-optional projection of Y_t (note that under the presence conditions for all processes we considered the optional projection coincides with the predictable projection and therefore we use for them the same notation). Under condition A) S admits the decomposition

$$S_t = S_0 + \int_0^t \widehat{\lambda}_u d\langle M\rangle_u + M_t, \tag{7}$$

where $\widehat{\lambda}_t = \widehat{\lambda}_t^G$. Moreover, condition A) implies that

$$\widehat{S}_t = E(S_t|G_t) = S_0 + \int_0^t \widehat{\lambda}_u d\langle M\rangle_u + \widehat{M}_t, \tag{8}$$

where $\widehat{M}_t$ is the G-local martingale $E(M_t|G_t)$.

Under these assumptions, we show in Proposition 3.1 that the initial optimization problem (2) is equivalent to the problem

$$\text{to minimize} \qquad E[e^{-\alpha(\int_0^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2}\int_0^T \pi_u^2 (1-\kappa_u^2) d\langle M\rangle_u}] \tag{9}$$

over all $\pi \in \Pi(G)$, where

$$\kappa_t^2 = \frac{d\langle \widehat{M}\rangle_t}{d\langle M\rangle_t} \quad \text{and} \quad \widetilde{H} = \frac{1}{\alpha}\ln E(e^{\alpha H}|G_T).$$

To prove the main result we will also make the following assumption:

F) $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \le C, \;\; P-$a.s..

We prove (Theorem 1) that under assumptions A)-F) the value process V of the problem (9) is the unique bounded strictly positive solution of the following BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u \kappa_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \quad Y_T = E[e^{\alpha H}|G_T], \tag{10}$$

where L is a G-local martingale orthogonal to $\widehat{M}$. To show the existence of a solution of this equation we use recent results of [30] and [21] on BSDEs with quadratic growth and driven by martingales.

Moreover the optimal strategy exists and is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t}). \tag{11}$$

We also examine two particular market diffusion models.

In section 4 we consider a market model with one risky asset, when the drift of the return process S changes value from 0 to $\mu \neq 0$ at some random time τ

$$dS_t = \mu I_{(t\geq\tau)}dt + \sigma dW_t.$$

The change-point τ admits a known prior distribution, although the variable τ itself is unknown and it cannot be directly observed. Agents in the market have only knowledge of the measurement process S and not of the Brownian motion and of the random variable τ. In this case we give an explicit solution of problem (2) in terms of the a posteriori probability process $p_t = P(\tau \leq t|F_t^S)$ which satisfies the stochastic differential equation (SDE)

$$p_t = p_0 + \frac{\mu}{\sigma}\int_0^t p_u(1-p_u)d\widetilde{W}_u + \gamma\int_0^t (1-p_u)du.$$

For this and for the formulation of the change-point problem, the reader is referred to [28].

In section 5, we consider a diffusion market model consisting of two risky assets with the following dynamics

$$\begin{aligned} dS_t =& \mu(t,\eta)dt + \sigma(t,\eta)dW_t^1, \\ d\eta_t =& b(t,\eta)dt + a(t,\eta)dW_t, \end{aligned}$$

where W^1 and W are standard Brownian motions with correlation ρ. η represents the price of a nontraded asset (e.g. an index) and S denotes the process of returns of the tradable one. We consider problem (2): an agent is hedging a contingent claim H trading with the liquid asset S and using only the information on η. Under suitable conditions on μ, σ, b, a and H, namely 1)-4) of section 5, we give an explicit expression of the optimal amount of money which should be invested in the liquid asset.

In section 6 we study the issue of sufficiency of partial information of the optimization problem (2). The filtration G is said to be sufficient (for $\mathcal{A}$) if

$$\inf_{\pi\in\Pi(\mathcal{A})} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] = \inf_{\pi\in\Pi(G)} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}]. \tag{12}$$

We give conditions (Proposition 5 and Theorem 2) which guarantee the sufficiency of filtration G.

2 Main definitions and auxiliary facts

Denote by $\mathcal{M}^e(F)$ the set of equivalent martingale measures for S, i.e., the set of probability measures Q equivalent to P such that S is a F-local martingale under Q. For any $Q \in \mathcal{M}^e(F)$, let $Z_t(Q)$ be

the density process (with respect to the filtration F) of Q relative to P.
It follows from (7) that the density process $Z_t(Q)$ of any element Q of $\mathcal{M}^e(F)$ is expressed as an exponential martingale of the form

$$\mathcal{E}_t(-\widehat{\lambda} \cdot M + L),$$

where L is a F-local martingale strongly orthogonal to M and $\mathcal{E}_t(X)$ is the Doléans-Dade exponential of X. If condition F) is satisfied, the local martingale $Z_t^{min} = \mathcal{E}_t(-\widehat{\lambda} \cdot M)$ is a true martingale, $dQ^{min}/dP = Z_T^{min}$ defines the minimal martingale measure for S on F.

Let us define the classes of admissible strategies

$$\Pi(F) = \{\pi : F - \text{predictable}, \pi \cdot M \in BMO(F)\},$$

$$\Pi(G) = \{\pi : G - \text{predictable}, \pi \cdot M \in BMO(F)\}.$$

It is evident that $\Pi(G) \subseteq \Pi(F)$.

Remark 1 Note that if $\pi \in \Pi(G)$, then $\pi \cdot \widehat{M} \in BMO(G)$. Indeed, since $\kappa^2 \leq 1$ (see, e.g., [18]) for any G-stopping time τ (which is also F-stopping time)

$$E(\langle \pi \cdot \widehat{M} \rangle_T - \langle \pi \cdot \widehat{M} \rangle_\tau | G_\tau) = E(\int_\tau^T \pi_u^2 \kappa_u^2 d\langle M \rangle_u | G_\tau) \leq$$

$$\leq E(E(\int_\tau^T \pi_u^2 d\langle M \rangle_u | F_\tau) | G_\tau) \leq ||\pi \cdot M||^2_{BMO(F)}.$$

Under conditions A)-C) the process $\widehat{M}_t = E(M_t|G_t)$ admits the representation (see, e.g., Proposition 2.2 of [18])

$$\widehat{M}_t = \int_0^t (\frac{d\widehat{\langle M, m(g) \rangle}_u}{d\langle m(g) \rangle_u}) dm_u(g) + L_t(g), \tag{13}$$

where $m(g)$ is any G-local martingale and $L(g)$ is a G-local martingale orthogonal to $m(g)$. In particular, if condition D) is also satisfied, then

$$\widehat{M}_t = \int_0^t \frac{d\langle M, m(g) \rangle_u}{d\langle m(g) \rangle_u} dm_u(g) + L_t(g), \tag{14}$$

and

$$\langle M, m(g) \rangle_t = \langle \widehat{M}, m(g) \rangle_t \tag{15}$$

for any G-local martingale $m(g)$.

Lemma 1 *Let conditions A)-D) be satisfied. Then*

$$E(\mathcal{E}_t(M)|G_t) = \mathcal{E}_t(\widehat{M}), \tag{16}$$

where $\widehat{M}_t = E(M_t|G_t)$.

Proof. The process $E(\mathcal{E}_t(M)|G_t)$ is a strictly positive G-local martingale and there exists a G-local martingale $\widetilde{M}$ such that

$$E(\mathcal{E}_t(M)|G_t) = \mathcal{E}_t(\widetilde{M}).$$

Therefore

$$\widetilde{M}_t = \int_0^t \frac{1}{E(\mathcal{E}_u(M)|G_u)} dE(\mathcal{E}_u(M)|G_u). \tag{17}$$

From (13), applied to the continuous F-local martingale

$$\mathcal{E}_t(M) = 1 + \int_0^t \mathcal{E}_u(M) dM_u$$

we have

$$E(\mathcal{E}_t(M)|G_t) = 1 + \int_0^t E\big(\mathcal{E}_u(M)\frac{d\langle M, m(g)\rangle_u}{d\langle m(g)\rangle_u}|G_u\big) dm_u(g) + \widetilde{L}_t,$$

where $\widetilde{L}$ is a G-local martingale orthogonal to $m(g)$. Therefore, by condition D)

$$E(\mathcal{E}_t(M)|G_t) = 1 + \int_0^t E\big(\mathcal{E}_u(M)|G_u\big)\frac{d\langle M, m(g)\rangle_u}{d\langle m(g)\rangle_u} dm_u(g) + \widetilde{L}_t$$

and from (17) we obtain that

$$\widetilde{M}_t = \int_0^t \frac{d\langle M, m(g)\rangle_u}{d\langle m(g)\rangle_u} dm_u(g) + \int_0^t \frac{1}{E(\mathcal{E}_u(M)|G_u)} d\widetilde{L}_t.$$

Taking the mutual characteristics with respect to $m(g)$ of the both sides of this equality we have

$$\langle \widetilde{M}, m(g)\rangle_t = \langle M, m(g)\rangle_t$$

and by (15)

$$\langle \widetilde{M}, m(g)\rangle_t = \langle \widehat{M}, m(g)\rangle_t$$

for any G-local martingale $m(g)$. By the hypothesis $\langle M\rangle$ is G-predictable, we know that M can be localized by G-stopping times and this implies that $\widehat{M}$ is a G-local martingale. Since also $\widetilde{M}$ is a G-local martingale, by arbitrariness of $m(g)$, we have that $\widehat{M}$ and $\widetilde{M}$ are indistinguishable. □

Corollary 1 *If conditions A)-D) are satisfied, then for any $\pi \in \Pi(G)$*

$$E(\mathcal{E}_t(\pi \cdot M)|G_t) = \mathcal{E}_t(\widehat{\pi \cdot M}) = \mathcal{E}_t(\pi \cdot \widehat{M}).$$

For all unexplained notations concerning the martingale theory used below we refer the reader to [3, 10, 15].

3 Utility Maximization Problem

Let us introduce the value process of problem (9):

$$V_t = \operatorname*{ess\,inf}_{\pi\in\Pi(G)} E[e^{-\alpha(\int_t^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}|G_t], \tag{18}$$

where, we recall that,

$$\widetilde{H} = \frac{1}{\alpha}\ln E[e^{\alpha H}|G_T], \quad \kappa_t^2 = \frac{d\langle\widehat{M}\rangle_t}{d\langle M\rangle_t}.$$

Let

$$V_t(G) = \operatorname*{ess\,inf}_{\pi\in\Pi(G)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|G_t] \tag{19}$$

be the value process of the problem (2).

Proposition 1 *Let conditions A)-E) be satisfied. Then $V_t = V_t(G)$ and the problems (2) and (9) are equivalent. Moreover, for any $\pi \in \Pi(G)$*

$$E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|G_t] = E[e^{-\alpha(\int_t^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}|G_t].$$

Proof. Taking the conditional expectation with respect to F_T and using condition E), we have that

$$\begin{aligned} E[e^{-\alpha\int_t^T \pi_u dS_u + \alpha H}|G_t] &= E\left(e^{-\alpha\int_t^T \pi_u dS_u} E[e^{\alpha H}|F_T]|G_t\right) \\ &= E\left(e^{-\alpha(\int_t^T \pi_u dS_u - \widetilde{H})}|G_t\right). \end{aligned} \tag{20}$$

Let $s \in [0,T], t \geq s$ and denote by $\mathcal{E}_{st}(M) = \frac{\mathcal{E}_t(M)}{\mathcal{E}_s(M)}$. Using successively decomposition (7), condition A), Lemma 1 for the strictly positive G-local martingale $\{E(\mathcal{E}_{st}(M)|G_t), t \geq s\}$ with $s \in [0,T]$, decomposition (8) of $\widehat{S}$ and equality $\kappa_t^2 = d\langle\widehat{M}\rangle_t / d\langle M\rangle_t$, we have

$$\begin{aligned} &E[e^{-\alpha(\int_t^T \pi_u dS_u - \widetilde{H})}|G_t] \\ &= E[e^{-\alpha\int_t^T \pi_u dM_u - \frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u + \frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u - \alpha\int_t^T \pi_u\widehat{\lambda}_u d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] \\ &= E[\mathcal{E}_{tT}(-\alpha\pi\cdot M)e^{\frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u - \alpha\int_t^T \pi_u\widehat{\lambda}_u d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] \\ &= E[E(\mathcal{E}_{tT}(-\alpha\pi\cdot M)|G_T)e^{\frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u - \alpha\int_t^T \pi_u\widehat{\lambda}_u d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] \\ &= E[\mathcal{E}_{tT}(-\alpha\pi\cdot \widehat{M})e^{\frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u - \alpha\int_t^T \pi_u\widehat{\lambda}_u d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] \\ &= E[e^{-\alpha\int_t^T \pi_u d\widehat{M}_u - \frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle\widehat{M}\rangle_u + \frac{\alpha^2}{2}\int_t^T \pi_u^2 d\langle M\rangle_u - \alpha\int_t^T \pi_u\widehat{\lambda}_u d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] \\ &= E[e^{-\alpha\int_t^T \pi_u d\widehat{S}_u + \frac{\alpha^2}{2}\int_t^T \pi_u^2(d\langle M\rangle_u - d\langle\widehat{M}\rangle_u) + \alpha\widetilde{H}}|G_t] \\ &= E[e^{-\alpha(\int_t^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}|G_t] \end{aligned}$$

which together with (20) implies the equivalence of (2) and (9). □

Remark 2 If conditions A)-D) are satisfied and H is bounded and G_T-measurable, then $\widetilde{H} = H$ and problem (2) is equivalent

$$\text{to minimize} \qquad E[e^{-\alpha(\int_0^T \pi_u d\widehat{S}_u - H) + \frac{\alpha^2}{2}\int_0^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}].$$

Remark 3 It is evident that since H is bounded, there will be a positive constant C, $V_t \le C$. Besides if condition F) is satisfied then we have that $V_t \ge c$ for a positive constant c directly by duality issues. Indeed, if $V_t(F) = \operatorname{ess\,inf}_{\pi\in\Pi(F)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|F_t]$, then it is well known that under condition F), $V_t(F) \ge c$, (see e.g. [2] or [17]). Therefore

$$V_t = \operatorname*{ess\,inf}_{\pi\in\Pi(G)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|G_t] = \tag{21}$$

$$\operatorname*{ess\,inf}_{\pi\in\Pi(G)} E\left[E\left(e^{-\alpha(\int_t^T \pi_u dS_u - H)}|F_t\right)|G_t\right] \ge E(V_t^F|G_t) \ge c. \tag{22}$$

Lemma 2 *The martingale part of any bounded strictly positive solution of (10) is in BMO.*

Proof. Let Y be a bounded strictly positive solution of (10).

By Ito's formula and the boundary condition we write

$$Y_T^2 - Y_\tau^2 = E^2(e^{\alpha H}|G_T) - Y_\tau^2 = \int_\tau^T 2Y_t(\psi_t d\widehat{M}_t + dL_t) + \int_\tau^T (\psi_t\kappa_t^2 + \widehat{\lambda}_t Y_t)^2 d\langle M\rangle_t$$
$$+ \langle\psi\cdot\widehat{M}\rangle_T - \langle\psi\cdot\widehat{M}\rangle_\tau + \langle L\rangle_T - \langle L\rangle_\tau,$$

where τ is a G-stopping time. Without loss of generality we may assume that $\psi\cdot\widehat{M}+L$ is a square integrable martingale, otherwise we can use localization arguments. After taking conditional expectation, we see that

$$E[\langle\psi\cdot\widehat{M}\rangle_T - \langle\psi\cdot\widehat{M}\rangle_\tau|G_\tau] + E[\langle L\rangle_T - \langle L\rangle_\tau|G_\tau] \le C, \tag{23}$$

which implies assertion of the Lemma. □

Theorem 1 *Let H be a bounded $\mathcal{A}_T$-measurable random variable, $\langle M\rangle$ be G-predictable and let conditions B), C) and F) be satisfied. Then the value process V_t is the unique solution of the BSDE*

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u\kappa_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \tag{24}$$
$$Y_T = E(e^{\alpha H}|G_T), \quad \langle\widehat{M}, L\rangle = 0$$

in the class of processes satisfying the two-sided inequality

$$c \le Y_t \le C, \tag{25}$$

where c and C are strictly positive constants. Besides the optimal strategy of the problem (9) exists in the class $\Pi(G)$ and is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\kappa_t^2\psi_t}{Y_t}). \tag{26}$$

If conditions A)-F) are satisfied, then $V_t = V_t(G)$ and π^* defined by (26) is the optimal strategy also for the problem (2).

Proof. Let Y_t be a strictly positive process satisfying Equation (24).
Writing Ito's formula for the process $Z_t = \ln Y_t$, we find that Z satisfies

$$dZ_t = \frac{1}{2}\left[(\tilde{\psi}_t\kappa_t^2 + \widehat{\lambda}_t)^2 - \tilde{\psi}_t^2\kappa_t^2\right] d\langle M\rangle_t - \frac{1}{2}d\langle\tilde{L}\rangle_t + \tilde{\psi}_t d\widehat{M}_t + d\tilde{L}_t, \tag{27}$$

with the boundary condition $Z_T = \ln E(e^{\alpha H}|G_T) = \alpha\widetilde{H}$. Recall that we denoted by $\widetilde{H} = \frac{1}{\alpha}\ln E(e^{\alpha H}|G_T)$. In the previous equation we set $\tilde{\psi}_t = \frac{1}{Y_t}\psi_t$ and $\tilde{L}_t = (\frac{1}{Y}\cdot L)_t$. The existence of a solution of the previous BSDE follows from [30] or [21], where new results on the existence and uniqueness of solutions are proved for BSDEs with quadratic growth driven by continuous martingales.

Now we should show that the solution is unique and coincides with the value process V and that the strategy π^* is optimal.
For any $\pi \in \Pi(G)$, let us denote the process $e^{-\alpha\int_0^t \pi_u d\widehat{S}_u + \frac{\alpha^2}{2}\int_0^t \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}$ by $J_t(\pi)$.
Let us consider Y_t, solution of (24) satisfying (25). By using Ito's formula for the product $Y_tJ_t(\pi)$,

$$\begin{aligned} d(Y_tJ_t(\pi)) = {} & Y_tJ_t(\pi)\left(-\alpha\pi_t d\widehat{S}_t + \frac{\alpha^2}{2}\pi_t^2(1-\kappa_t^2)d\langle M\rangle_t + \frac{\alpha^2}{2}\pi_t^2 d\langle\widehat{M}\rangle_t\right) \\ & + J_t(\pi)\left(\psi_t d\widehat{M}_t + dL_t + \frac{1}{2}\frac{(\widehat{\lambda}_tY_t + \psi_t\kappa_t^2)^2}{Y_t}d\langle M\rangle_t\right) - J_t(\pi)\psi_t\alpha\pi_t\kappa_t^2 d\langle M\rangle_t \end{aligned}$$

and

$$d(Y_tJ_t(\pi)) = Y_tJ_t(\pi)\left[(\frac{\psi_t}{Y_t} - \alpha\pi_t)d\widehat{M}_t + \frac{1}{Y_t}dL_t + \frac{1}{2}(\alpha\pi_t - (\widehat{\lambda}_t + \frac{\psi_t}{Y_t}\kappa_t^2))^2 d\langle M\rangle_t)\right].$$

Therefore

$$Y_tJ_t(\pi) = Y_0\mathcal{E}_t((\frac{\psi}{Y} - \alpha\pi)\cdot\widehat{M} + \frac{1}{Y}\cdot L)e^{\frac{1}{2}\int_0^t(\alpha\pi_u - (\widehat{\lambda}_u + \frac{\psi_u}{Y_u}\kappa_u^2))^2 d\langle M\rangle_u)}. \tag{28}$$

In Equation (28), $YJ(\pi)$ is written as the product of a strictly positive increasing process and a uniformly integrable martingale. In fact, by (23) and since $\pi \in \Pi(G)$ implies that $\pi\cdot\widehat{M} \in BMO(G)$ (see Remark 2.1), the process $(\frac{\psi}{Y} - \alpha\pi)\cdot\widehat{M} + \frac{1}{Y}\cdot L)$ is a BMO(G)-martingale, thus, the exponential martingale $\mathcal{E}_t((\frac{\psi}{Y} - \alpha\pi)\cdot\widehat{M} + \frac{1}{Y}\cdot L)$ is a uniformly integrable martingale by [12]. Therefore, $YJ(\pi)$ is a submartingale. Using the boundary condition $Y_T = E(e^{\alpha H}|G_T)$, we find

$$Y_tJ_t(\pi) \le E(J_T(\pi)e^{\alpha\widetilde{H}}|G_t), \quad \text{a.}s.$$

hence

$$Y_t \le E(e^{-\alpha\int_t^T \pi_u d\widehat{S}_u + \frac{\alpha^2}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u + \alpha\widetilde{H}}|G_t) \ \text{a.}s..$$

This is true for any $\pi \in \Pi(G)$, and, recalling the definition of V_t, we immediately see that

$$Y_t \le \operatorname*{ess\,inf}_{\pi\in\Pi(G)} E[e^{-\alpha\int_t^T \pi_u d\widehat{S}_u + \frac{\alpha^2}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u + \alpha\widetilde{H}}|G_t] = V_t \ \text{a.}s.. \tag{29}$$

Let us check the opposite inequality $Y_t \ge V_t$ a.s..

Let us take $\tilde{\pi}_u = \frac{\widehat{\lambda}_u Y_u + \psi_u \kappa_u^2}{\alpha Y_u}$. By Ito's formula for $Y_t J_t(\tilde{\pi})$, similarly to (28), we find that the process is a strictly positive local martingale hence a supermartingale. The supermartingale property and the boundary condition give

$$Y_t \geq E[e^{-\alpha \int_t^T \tilde{\pi}_u d\widehat{S}_u + \frac{\alpha^2}{2} \int_t^T \tilde{\pi}_u^2 (1-\kappa_u^2) d\langle M\rangle_u + \alpha \widetilde{H}} | G_t] \text{ a.}s.. \tag{30}$$

Let us show that $\tilde{\pi}$ belongs to the class $\Pi(G)$.
From condition F) follows that $\widehat{\lambda} \cdot M$ is in $BMO(F)$ and by (23) (since $Y \geq c$ and $\kappa_t^2 \leq 1$) we have that $\frac{\psi \kappa^2}{Y} \cdot M$ is also in $BMO(F)$. Thus

$$\frac{1}{\alpha}(\widehat{\lambda} + \frac{\psi \kappa^2}{Y}) \cdot \; M \in BMO(F)$$

which implies that $\tilde{\pi} \in \Pi(G)$. Therefore,

$$E[e^{-\alpha \int_t^T \tilde{\pi}_u d\widehat{S}_u + \frac{\alpha^2}{2} \int_t^T \tilde{\pi}_u^2 (1-\kappa_u^2) d\langle M\rangle_u + \alpha \widetilde{H}} | G_t] \geq V_t \text{ a.}s.. \tag{31}$$

Hence from (29), (30) and (31) we have that

$$Y_t = V_t, \text{ a.}s.. \tag{32}$$

Therefore, (30), (31) and (32) give the equality

$$E[e^{-\alpha \int_t^T \tilde{\pi}_u d\widehat{S}_u + \frac{\alpha^2}{2} \int_t^T \tilde{\pi}_u^2 (1-\kappa_u^2) d\langle M\rangle_u + \alpha \widetilde{H}} | G_t] = V_t \text{ a.}s..$$

which means that $\tilde{\pi} = \pi^*$ is optimal. It follows from Proposition 3.1 that if A)-F) are satisfied, then $V_t = V_t(G)$ and π^* is the optimal strategy of the problem (2). □

Remark 4 In terms of Equation (27) the optimal strategy is expressed as

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \widetilde{\psi}_t \kappa_t^2). \tag{33}$$

Remark 5 One can see from the proof of the unicity that condition F) can be replaced by the weaker condition $\widehat{\lambda} \cdot M$ is in $BMO(F)$.

If $F_t^S \subseteq G_t$, then $\langle M\rangle$ is G-predictable, since it is F^S-predictable. Besides, $G_t = F_t \equiv F_t^S \vee G_t$ and conditions A), B), D) and equality (6) of condition E) are satisfied. So, in this case, $\widehat{M_t} = M_t$ and $\kappa_t^2 = 1$ for all $t \in [0, T]$. Since $F^S \subseteq G$, S is a G-semimartingale with canonical G-decomposition

$$S_t = S_0 + \int_0^t \widehat{\lambda}_u d\langle M\rangle_u + M_t, \quad M \in \mathcal{M}_{\text{loc}}(G) \tag{34}$$

and for any $\pi \in \Pi(G)$

$$E\left(e^{-\alpha(\int_t^T \pi_u dS_u - H)} | G_t\right) = E\left(e^{-\alpha(\int_t^T \pi_u dS_u - \widetilde{H})} | G_t\right),$$

with $\widetilde{H} = \frac{1}{\alpha} \ln E(e^{\alpha H} | G_T)$, where $\widetilde{H} = H$ if H is G_T-measurable.
So, taking the G-decomposition (34) in mind, problem (2) is equivalent to the problem

$$\text{to minimize} \quad E(e^{-\alpha(\int_0^T \pi_u dS_u - \widetilde{H})}) \quad \text{over all} \quad \pi \in \Pi(G). \tag{35}$$

Thus, we have the following corollary of Theorem 1:

Corollary 2 *Let $F^S \subseteq G \subseteq \mathcal{A}$ and let conditions C), F) be satisfied and H be a bounded $\mathcal{A}_T$-measurable random variable. Then, the value process V is the unique solution of the BSDE*

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u dM_u + L_t, \quad Y_T = E(e^{\alpha H}|G_T), \tag{36}$$

satisfying $0 < c \le Y_t \le C$. Moreover, the optimal strategy is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\psi_t}{Y_t}). \tag{37}$$

Remark 6 Note that in the case of full information $G_t = \mathcal{A}_t$, we additionally have

$$\widehat{M}_t = M_t = N_t, \quad \widehat{\lambda}_t = \lambda_t, \quad Y_T = e^{\alpha H}$$

and Equation (36) takes the form

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u + \lambda_u Y_u)^2}{Y_u} d\langle N\rangle_u + \int_0^t \psi_u dN_u + L_t, \quad Y_T = e^{\alpha H}. \tag{38}$$

Equations of type (38) (or equivalent to (38)) were derived in [8, 9, 29] for BSDEs driven by Brownian motion and in [17] for BSDEs driven by martingales.

4 Application to the disorder problem

As an example we consider a market with one risky asset, where the drift of this asset changes value (from 0 to μ, $\mu \neq 0$) at a random time, which cannot be directly observed.

Let $(\Omega, \mathscr{A}, \mathcal{A} = (\mathcal{A}_t, t \in [0,T]), P)$, be a filtered probability space, where $\mathscr{A} = \mathcal{A}_T$, hosting a Brownian motion W and a random variable τ with distribution

$$P(\tau = 0) = p \quad \text{and} \quad P(\tau > t|\tau > 0) = e^{-\gamma t}, \quad \text{for all} \quad t \in [0,T]$$

for some known constants $p \in [0,1[$ and $\gamma > 0$. The Brownian motion W and the random variable τ are assumed to be independent.

The dynamics of the asset price $\widetilde{S}$ is determined by the SDE

$$d\widetilde{S}_t = \widetilde{S}_t(\mu I_{(t\ge\tau)}dt + \sigma dW_t),$$

where $\mu \neq 0$ and $\sigma^2 > 0$. By $dS_t = \frac{d\widetilde{S}_t}{\widetilde{S}_t}$, we introduce the return process which satisfies

$$dS_t = \mu I_{(t\ge\tau)}dt + \sigma dW_t. \tag{39}$$

So, agents in the market do not observe the Brownian motion and the random instant τ, but only the measurement process S (or $\widetilde{S}$). The filtration G is generated by the observations process $G_t = F_t^S = \sigma(S_u, 0 \le u \le t)$. So, we have

$$F_t^S = G_t \subseteq \mathcal{A}_t.$$

We consider the optimization problem (2), where the strategy π_t is interpreted as a dollar amount invested in the stock at time t. Assume that the contingent claim H is of the form $H = g(S_T, \zeta)$, where g is a positive bounded function of two variables and ζ is an $\mathcal{A}_T$-measurable random variable independent of F^S_T. ζ can be the terminal value of a non traded asset.
Let $p_t = P(\tau \le t|F^S_t)$ be the a posteriori probability process. With respect to the filtration $G = F^S$ the process S admits the decomposition

$$S_t = S_0 + \mu \int_0^t p_u du + \sigma \widetilde{W}_t, \tag{40}$$

where $\widetilde{W}_t = \frac{1}{\sigma}(S_t - S_0 - \mu \int_0^t p_u du)$ is a Brownian motion, an innovation process, with respect to the filtration F^S (see, e.g., [16]).
The measure Q defined by

$$dQ = \mathcal{E}_T(-\frac{\mu}{\sigma} p \cdot \widetilde{W}) dP$$

is a martingale measure for S on F^S. It is evident that F) is satisfied.
It follows from [28] that the process p_t satisfies the stochastic differential equation

$$p_t = p_0 + \frac{\mu}{\sigma} \int_0^t p_u(1-p_u) d\widetilde{W}_u + \gamma \int_0^t (1-p_u) du. \tag{41}$$

Moreover, (p_t, F^S_t) is a strong Markov process.

In this case

$$N_t = \sigma W_t, \quad M_t = \widehat{M}_t = \sigma \widetilde{W}_t, \quad \kappa_t^2 = 1 \quad \lambda_t = \frac{\mu}{\sigma^2} I_{(\tau \le t)}, \quad \widehat{\lambda}_t = \frac{\mu}{\sigma^2} p_t.$$

By Corollary 2, the value process of problem (35) satisfies the following BSDE

$$Y_t = Y_0 + \frac{1}{2} \int_0^t \frac{(\sigma \psi_u + \frac{\mu}{\sigma} p_u Y_u)^2}{Y_u} du + \int_0^t \sigma \psi_u d\widetilde{W}_u, \quad Y_T = E(e^{\alpha H}|F^S_T), \tag{42}$$

with $L_t = 0$, since any F^S-local martingale is representable as a stochastic integral with respect to $\widetilde{W}$ (see, e.g., Theorem 5.17 in [16]). Besides, the process $Z_t = \ln Y_t$ is the solution of the linear equation (here we set $\widetilde{\psi} = \frac{\psi}{Y}$)

$$Z_t = Z_0 + \int_0^t (\mu p_u \widetilde{\psi}_u + \frac{\mu^2}{2\sigma^2} p_u^2) du + \int_0^t \sigma \widetilde{\psi}_u d\widetilde{W}_u, \quad Z_T = \ln E(e^{\alpha H}|F^S_T). \tag{43}$$

Since $\mu p_t dt + \sigma d\widetilde{W}_t = dS_t$, (43) can be written in the following equivalent form

$$Z_t = Z_0 + \frac{1}{2} \frac{\mu^2}{\sigma^2} \int_0^t p_u^2 du + \int_0^t \widetilde{\psi}_u dS_u, \quad Z_T = \ln E(e^{\alpha H}|F^S_T). \tag{44}$$

The solution of this equation is expressed as

$$Z_t = E\big(\mathcal{E}_{tT}(-\frac{\mu}{\sigma} p \cdot \widetilde{W})(\ln E(e^{\alpha H}|F^S_T) - \frac{\mu^2}{2\sigma^2} \int_t^T p_u^2 du)|F^S_t\big). \tag{45}$$

Besides, taking $t = T$ in Equation (44), one can find the integrand $\widetilde{\psi}$ by the martingale representation property

$$\ln E(e^{\alpha H}|F_T^S) - \frac{\mu^2}{2\sigma^2}\int_0^T p_u^2 du = c + \int_0^T \widetilde{\psi}_u dS_u. \tag{46}$$

Since the left side of (46) is the difference of two F_T^S-measurable random variables, by the martingale representation theorem, there exist two predictable S-integrable processes $\widetilde{\psi}(1)$ and $\widetilde{\psi}(2)$, such that

$$\ln E(e^{\alpha H}|F_T^S) = c_1 + \int_0^T \widetilde{\psi}_u(1) dS_u \tag{47}$$

$$-\frac{\mu^2}{2\sigma^2}\int_0^T p_u^2 du = c_2 + \int_0^T \widetilde{\psi}_u(2) dS_u. \tag{48}$$

Thus, the logarithm of the value process is the sum of two parts $Z_t(1)$ and $Z_t(2)$

$$Z_t = E^Q\big(\ln E(e^{\alpha H}|F_T^S)|F_t^S\big) - E^Q\big(\frac{\mu^2}{2\sigma^2}\int_t^T p_u^2 du|F_t^S\big) = Z_t(1) + Z_t(2).$$

With regard to $Z_t(1)$, we first observe that

$$E(e^{\alpha H}|F_T^S) = E(e^{\alpha g(\zeta, S_T)}|F_T^S) = f(S_T),$$

where

$$f(S) = E(e^{\alpha g(\zeta, S_T)}|S_T = S) = \int_0^\infty e^{\alpha g(u,S)} dF_\zeta(u).$$

In the last equality we used the independence of ζ and S_T. Note that $f > 1$, since g is positive. Since $S_t = S_0 + \sigma\overline{W}_t$ and $\overline{W}_t = \widetilde{W}_t + \frac{\mu}{\sigma}\int_0^t p_u du$ is a Q-Brownian motion, by the Markov property of S under (Q, F^S) we easily see that

$$E^Q\big(\ln E(e^{\alpha H}|F_T^S)|F_t^S\big) = E^Q\big(\ln f(S_T)|S_t\big)$$

and

$$G(t,x) \equiv E^Q(\ln f(S_T)|S_t = x) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_R \ln f(y) e^{-\frac{(y-x)^2}{2\sigma^2(T-t)}} dy. \tag{49}$$

Since $G \in C^{1,2}([0,T[\times R)$ and $Z_t(1)$ is a Q-martingale, by Ito's formula

$$Z_t(1) = G(t, S_t) = G(0, S_0) + \int_0^t G_x(t, S_t) dS_t. \tag{50}$$

Comparing (50) and (47) we see that $\widetilde{\psi}_t(1) = G_x(t, S_t), \quad dPdt$ a.e..

While, with regard to $Z_t(2)$, let us observe that under the measure Q the process p_t satisfies the SDE

$$p_t = p_0 + \frac{\mu}{\sigma}\int_0^t p_u(1-p_u)\frac{1}{\sigma} dS_u + \int_0^t (1-p_u)(\gamma - \frac{\mu^2}{\sigma^2}p_u^2) du, \tag{51}$$

where $\frac{1}{\sigma}S_t$ is a Brownian motion under the measure Q.

Therefore $Z_t(2)$ can be represented as

$$Z_t(2) = -\frac{\mu^2}{2\sigma^2} U(t, p_t),$$

where

$$U(t, x) = E^Q(\int_t^T p_u^2 du | p_t = x). \tag{52}$$

Let $R(t, x)$ be a solution of the linear PDE

$$R_t(t, x) + \frac{\mu^2}{2\sigma^2} x^2(1-x)^2 R_{xx}(t, x) + \big(\gamma - \frac{\mu^2}{\sigma^2} x^2\big)(1-x) R_x(t, x) + x^2 = 0, \quad R(T, x) = 0, \tag{53}$$

where R_t, R_x and R_{xx} are partial derivatives of R. The existence of a solution of (53) from the class $\in C^{1,2}([0, T] \times (0, 1))$ follows from [23].

It follows from the Ito formula for $R(t, p_t)$, taking in mind that $R(t, x)$ solves equation (53), that the process

$$R(t, p_t) + \int_0^t p_u^2 du = R(0, p_0) + \frac{\mu}{\sigma^2} \int_0^t R_x(u, p_u) dS_u \tag{54}$$

is a martingale under Q. Using the martingale property and the boundary condition

$$R(t, p_t) = E^Q(\int_t^T p_u^2 du | F_t^p)$$

and from (52) we obtain that $R(t, p_t) = U(t, p_t)$. Comparing the Q martingale parts of (48)and (54) we obtain that $dPdt$ a.e..

$$\widetilde{\psi}_t(2) = -\frac{\mu^3}{2\sigma^4} p_t(1 - p_t) R_x(t, p_t).$$

Thus, we proved

Proposition 2 *The value process of problem (35) is equal to*

$$V_t = e^{G(t, S_t) - \frac{\mu^2}{2\sigma^2} R(t, p_t)}$$

and the optimal strategy π^* is equal to

$$\pi_t^* = \frac{\mu p_t}{\alpha\sigma^2}(1 - \frac{\mu^2}{2\sigma^2}(1 - p_t) R_x(t, p_t)) + \frac{1}{\alpha} G_x(t, S_t),$$

where $G(t, x)$ is defined by (49), $R(t, x)$ satisfies the linear PDE (53) and p_t is a solution of SDE (51).

Remark 7 The optimal wealth process is the sum of two components, a hedging fund

$$\frac{1}{\alpha} \int_0^t G_x(u, S_u) dS_u$$

(which is zero if $H = 0$) and an investment fund

$$\frac{\mu}{\alpha\sigma^2} \int_0^t p_u\big(1 - \frac{\mu^2}{2\sigma^2}(1 - p_u) R_x(u, p_u)\big) dS_u.$$

This fact is a consequence of wealth independent risk aversion of the exponential utility function, which makes the investors behavior not depending on his initial endowment.

Remark 8 If $H = 0$ and τ is deterministic, $\tau = t_0$, then $R_x = 0$ and the optimal strategy is $\frac{\mu}{\alpha\sigma^2} I_{(t \geq t_0)}$. If $t_0 = 0$, then we obtain Merton's optimal strategy $\pi^* = \frac{\mu}{\alpha\sigma^2}$, (see [19]).

5 Diffusion model with correlation.

In this section we deal with a model with two assets one of which has no liquid market. First we consider an agent who is finding the best strategy to hedge a contingent claim H, trading with the liquid asset but using just the information on the non tradable one. Then we specialize the result to the case when the model is Markovian and H depends only on the nontraded asset's price at time T. Finally, we solve the problem for full information when H is function only of the nontraded asset and compare the obtained result.
Let S and η denote respectively the return of the tradable and the price of the non tradable assets. We assume S and η have the following dynamics

$$dS_t = \mu(t,\eta)dt + \sigma(t,\eta)dW_t^1, \tag{55}$$

$$d\eta_t = b(t,\eta)dt + a(t,\eta)dW_t, \tag{56}$$

where W^1 and W are correlated Brownian motions with constant correlation $\rho \in (-1,1)$.
The coefficients μ, σ, a and b are non anticipative functionals such that

1) $\int_0^T \frac{\mu^2(t,\eta)}{\sigma^2(t,\eta)} dt$ is bounded,

2) $\sigma^2 > 0$, $a^2 > 0$,

3) equation (56) admits a unique strong solution,

4) H is bounded and measurable with respect to the σ-algebra $F_T^\eta \vee \sigma(\xi)$, where ξ is a random variable independent of S.

Note that under conditions 2) and 3), $F^{S,\eta} = F^{W^1,W}$, $F^\eta = F^W$. So, we will have

$$F_t = F_t^{S,\eta} \subseteq \mathcal{A}_t \quad \text{and} \quad G_t = F_t^\eta.$$

Here $F^{W^1,W}$ (resp. F^W) is an augmented filtration generated by W^1 and W (resp. W). So the filtration F_t^η is continuous and the condition C) is satisfied.
Since $F_T = F_T^{S,\eta}$, assumption 4) implies that condition E) is fulfilled.
Condition F) is here represented by 1).

We consider the optimization problem

$$\text{to minimize} \quad E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(F^\eta), \tag{57}$$

where π represents the dollar amount the agent invests in the stock and, constructing the optimal strategy, he uses only the information based on η.
Let us observe that

$$M_t = \int_0^t \sigma(u,\eta)dW_u^1, \quad \langle M \rangle_t = \int_0^t \sigma^2(u,\eta)du, \quad \lambda_t = \frac{\mu(t,\eta)}{\sigma^2(t,\eta)}.$$

We denote the market price of risk by

$$\theta_t = \frac{\mu(t,\eta)}{\sigma(t,\eta)}. \tag{58}$$

It is convenient to think of W as a linear combination of two independent Brownian motions W^0 and W^1, thus

$$W_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^0. \tag{59}$$

It is evident that W is a Brownian motion also with respect to the filtration F^{W^0,W^1} (which is equal to $F^{W^1,W}$) and condition B) is satisfied. Therefore, by Proposition 2.2 in [18],

$$\widehat{M}_t = \rho \int_0^t \sigma(u,\eta) dW_u.$$

Hence, $\langle \widehat{M} \rangle_t = \rho^2 \int_0^t \sigma^2(u,\eta)du = \rho^2 \langle M \rangle_t$ and $\kappa_t^2 = \rho^2$, therefore $\widehat{S}$ satisfies

$$d\widehat{S}_t = \mu(t,\eta)dt + \rho\sigma(t,\eta)dW_t.$$

Since any F^η local martingale is represented as a stochastic integral with respect to W and $\sigma(t,\eta)$ is F_t^η-adapted, condition D) is also satisfied.

Therefore, it follows from Proposition 1, that under conditions 1)-4) the optimization problem (57) is equivalent to the problem

$$\text{to minimize} \quad E[e^{-\alpha(\int_0^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2}(1-\rho^2)\int_0^T \pi_u^2 \sigma^2(u,\eta)du}] \quad \text{over all} \quad \pi \in \Pi(F^\eta), \tag{60}$$

where $\widetilde{H} = \frac{1}{\alpha} \ln E(e^{\alpha H} | F_T^\eta)$.

We shall show that in this case problem (60) admits an explicit solution.

Let $\widetilde{Q}$ be the measure defined by

$$\frac{d\widetilde{Q}}{dP} = \mathcal{E}_T(-\rho\, \theta \cdot W). \tag{61}$$

Note that $\widetilde{Q}$ is a martingale measure for $\widehat{S}$ (on F^η)and by Girsanov's theorem, under the measure $\widetilde{Q}$,

$$\widetilde{W}_t = W_t + \rho \int_0^t \theta_u du$$

is a Brownian motion.

Proposition 3 *Let us assume conditions 1)-4). Then, the value process related to problem (60) is equal to*

$$V_t = \left(E^{\widetilde{Q}}[e^{(1-\rho^2)(\alpha\widetilde{H} - \frac{1}{2}\int_t^T \theta_u^2 du)} | F_t^\eta] \right)^{\frac{1}{1-\rho^2}}. \tag{62}$$

Moreover, the optimal strategy π^* is identified by

$$\pi_t^* = \frac{1}{\alpha\sigma(t,\eta)} \big(\theta_t + \frac{\rho h_t}{(1-\rho^2)(c + \int_0^t h_u d\widetilde{W}_u)}\big), \tag{63}$$

where h_t is the integrand of the integral representation

$$e^{(1-\rho^2)(\alpha\widetilde{H} - \frac{1}{2}\int_0^T \theta_t^2 dt)} = c + \int_0^T h_t d\widetilde{W}_t. \tag{64}$$

Proof. Conditions 1)-4) imply that the assumptions A)-F) are satisfied. Thus, according to Theorem 1, the value process V_t related to the problem (60) is the unique bounded solution of Equation (24). In this case, after the change of variables $x \equiv \rho\sigma\psi$, the BSDE (24) can be written

$$Y_t = Y_0 + \frac{1}{2}\int_0^t [\frac{(\theta_u Y_u + x_u\rho)^2}{Y_u}]du + \int_0^t x_u dW_u, \quad Y_T = E(e^{\alpha H}|F_T^\eta). \tag{65}$$

Besides, the equation for the logarithm of the value process takes the following form

$$dZ_t = -\frac{1}{2}[\varphi_t^2(1-\rho^2) - 2\varphi_t\rho\theta_t - \theta_t^2]dt + \varphi_t dW_t, \quad Z_T = \alpha\widetilde{H}. \tag{66}$$

Note that $\varphi_t = \frac{x_t}{Y_t}$.
Under $\widetilde{Q}$, Equation (66) becomes

$$dZ_t = -\frac{1}{2}[\varphi_t^2(1-\rho^2) - \theta_t^2]dt + \varphi_t d\widetilde{W}_t, \quad Z_T = \alpha\widetilde{H}. \tag{67}$$

From (67), using the boundary condition $Z_T = \alpha\widetilde{H}$, we have

$$Z_0 + \int_0^T \varphi_t d\widetilde{W}_t - \frac{1-\rho^2}{2}\int_0^T \varphi_t^2 dt = -\frac{1}{2}\int_0^T \theta_t^2 dt + \alpha\widetilde{H}. \tag{68}$$

Multiplying both parts of this equation by $1-\rho^2$ and taking exponentials we obtain

$$c\,\mathcal{E}_T((1-\rho^2)\varphi\cdot\widetilde{W}) = e^{(1-\rho^2)(\alpha\widetilde{H} - \frac{1}{2}\int_0^T \theta_t^2 dt)}, \tag{69}$$

where $c = e^{(1-\rho^2)Z_0}$.
Since $\int_0^T \theta_t^2 dt$ and $\widetilde{H}$ are F_T^η-measurable and $\widetilde{H}$ is bounded, by the martingale representation theorem and Bayes rule, there exists a F^η-predictable function h, such that the martingale $E^{\widetilde{Q}}\big(e^{(1-\rho^2)(\alpha\widetilde{H} - \frac{1}{2}\int_0^T \theta_u^2 du)}|F_t^\eta\big)$ admits the representation

$$E^{\widetilde{Q}}[e^{(1-\rho^2)(\alpha\widetilde{H} - \frac{1}{2}\int_0^T \theta_u^2 du)}|F_t^\eta] = c + \int_0^t h_u d\widetilde{W}_u. \tag{70}$$

Therefore, taking conditional expectations in (69),

$$\mathcal{E}_t((1-\rho^2)\varphi\cdot\widetilde{W}) = 1 + c^{-1}\int_0^t h_u d\widetilde{W}_u. \tag{71}$$

Thus,

$$(\varphi\cdot\widetilde{W})_t = (1-\rho^2)^{-1}\int_0^t (c + \int_0^s h_u d\widetilde{W}_u)^{-1} h_s d\widetilde{W}_s$$

and $dPdt$ a.e.

$$\varphi_t = (1-\rho^2)^{-1}(c + \int_0^t h_u d\widetilde{W}_u)^{-1} h_t. \tag{72}$$

Note that, since we changed the variables ($x \equiv \psi\rho\sigma$) in (65), the processes $\widetilde{\psi}$ from (27) and φ are related by the equality $\widetilde{\psi} = \varphi/\rho\sigma$. Therefore, it follows from (33) that the optimal hedging strategy is (63).

Using the martingale property of $\varphi \cdot \widetilde{W}$ and the boundary condition, from (67) we obtain

$$Z_t = E^{\widetilde{Q}} \left(\alpha \widetilde{H} + \frac{1}{2} \int_t^T \left((1-\rho^2)\varphi_u^2 - \theta_u^2 \right) du | F_t^\eta \right). \tag{73}$$

Now let us express Z_t in a more explicit form.
From (64), we have

$$(1-\rho^2)(\alpha \widetilde{H} - \frac{1}{2}\int_0^T \theta_t^2 dt) = \ln(c + \int_0^T h_u d\widetilde{W}_u)$$

and, by Ito's formula,

$$d\ln(c + \int_0^t h_u d\widetilde{W}_u) = \frac{h_t}{c + \int_0^t h_u d\widetilde{W}_u} d\widetilde{W}_t - \frac{1}{2}\frac{h_t^2}{(c + \int_0^t h_u d\widetilde{W}_u)^2} dt.$$

From this equation, using (72), we have

$$(1-\rho^2)^{-1} \left(\ln(c + \int_0^T h_u d\widetilde{W}_u) - \ln(c + \int_0^t h_u d\widetilde{W}_u) \right) = \int_t^T \varphi_u d\widetilde{W}_u - \frac{1}{2}(1-\rho^2)\int_t^T \varphi_u^2 du.$$

Taking conditional expectation with respect to the measure $\widetilde{Q}$, we find that

$$\frac{(1-\rho^2)}{2} E^{\widetilde{Q}}(\int_t^T \varphi_u^2 du | F_t^\eta) = E^{\widetilde{Q}}(-\alpha \widetilde{H} + \frac{1}{2}\int_0^T \theta_u^2 du | F_t^\eta) + \ln(c + \int_0^t h_u d\widetilde{W}_u)^{(1-\rho^2)^{-1}} \tag{74}$$

If we substitute (74) in (73), we have that

$$Z_t = \frac{1}{2}\int_0^t \theta_u^2 du + \ln(c + \int_0^t h_u d\widetilde{W}_u)^{(1-\rho^2)^{-1}}$$

which implies that the value process of problem (60) is

$$V_t = e^{\frac{1}{2}\int_0^t \theta_u^2 du}(c + \int_0^t h_u d\widetilde{W}_u)^{(1-\rho^2)^{-1}},$$

which is equal to (62), by (70). □

Remark 9 If $H = 0$, like in the pure investment problem, and μ, σ are constants (or if the mean variance tradeoff $\int_0^T \theta_t^2 dt$ is deterministic), then $h = 0$ by (70),

$$V_t = e^{-\frac{1}{2}\theta^2(T-t)} \quad \text{and} \quad \pi_t^* = \frac{\mu}{\alpha\sigma^2}$$

is the optimal investment strategy for maximizing exponential utility (as in the Merton's model), which keeps a constant dollar amount invested in the stock.

Remark 10 If $H \neq 0$, μ, σ are constants, but $\rho = 0$, it follows from (63) that the same strategy is optimal. In this case $\widetilde{H} = f(\eta)$ is independent of the traded asset and the risk is "completely unhedgeable".

Remark 11 If $\rho^2 = 1$, then the solution of Equation (66) gives the value process is

$$V_t = e^{E^{\widetilde{Q}}(\alpha \widetilde{H} - \frac{1}{2}\int_t^T \theta_u^2 du | F_t^\eta)}.$$

The same result is obtained by taking the limit as $\rho \to \pm 1$ in (62), since for any t, we have

$$\lim_{\rho \to \pm 1} \left(E^{\widetilde{Q}}[e^{(1-\rho^2)(\alpha \widetilde{H} - \frac{1}{2}\int_t^T \theta_u^2 du)} | F_t^\eta] \right)^{\frac{1}{1-\rho^2}} = e^{E^{\widetilde{Q}}(\alpha \widetilde{H} - \frac{1}{2}\int_t^T \theta_u^2 du | F_t^\eta)}.$$

Note that if $\rho^2 = 1$, to avoid arbitrage opportunities, the coefficients of (55), (56) must be related by

$$\mu(t, \eta) = \pm \frac{b(t, \eta)}{a(t, \eta)} \sigma(t, \eta), \quad \text{if} \quad \rho = \pm 1.$$

We move on to considering a contingent claim $H = f(\eta_T)$ and we assume also the Markov structure of the coefficients. Introduce the function

$$R(t, x) = E^{\widetilde{Q}}[e^{(1-\rho^2)(\alpha f(\eta_T) - \frac{1}{2}\int_t^T \theta^2(u, \eta_u) du)} | \eta_t = x].$$

Then we can express $R(t, x)$ as a solution of a linear PDE and h in terms of the first derivative of R. Classical results yield that, under some regularity conditions on the coefficients, $R \in C^{1,2}([0, T] \times \mathbb{R})$ and satisfies the following PDE (see, e.g.,[5])

$$R_t(t, x) + \frac{1}{2} a^2(t, x) R_{xx}(t, x) + R_x(t, x)(b(t, x) - \rho \theta(t, x) a(t, x)) - R(t, x) \frac{1 - \rho^2}{2} \theta^2(t, x) = 0, \quad (75)$$

with terminal condition $R(T, x) = e^{\alpha(1-\rho^2) f(x)}$, where R_t, R_x and R_{xx} are partial derivatives of R. On the one hand, let us observe that

$$R(t, \eta_t) = c \, e^{\frac{1-\rho^2}{2}\int_0^t \theta^2(u, \eta_u) du} \mathcal{E}_t((1 - \rho^2)\varphi \cdot \widetilde{W}),$$

hence $R(t, \eta_t)$ is solution of the following backward equation

$$dR(t, \eta_t) = R(t, \eta_t)[(1 - \rho^2)\varphi_t d\widetilde{W}_t + \frac{(1 - \rho^2)}{2} \theta^2(t, \eta_t) dt], \quad R(T, \eta_T) = e^{\alpha(1-\rho^2) f(\eta_T)}. \quad (76)$$

Writing the Ito formula for $R(t, \eta_t)$ and comparing with (76) we obtain that

$$\varphi_t = \frac{R_x(t, x) a(t, x)}{(1 - \rho^2) R(t, x)}.$$

Using representation (64), in an analogous way, one can write

$$R(t, \eta_t) = e^{\frac{1-\rho^2}{2}\int_0^t \theta^2(u, \eta_u) du}(c + \int_0^t h_u d\widetilde{W}_u).$$

Thus,

$$h_t = e^{-\frac{1-\rho^2}{2}\int_0^t \theta^2(u, \eta_u) du} R_x(t, \eta_t) a(t, \eta_t).$$

The optimal strategy π^* in this case takes the following form

$$\pi_t^* = \frac{1}{\alpha\sigma(t,\eta_t)}\left(\theta(t,\eta_t) + \frac{R_x(t,\eta_t)}{(1-\rho^2)R(t,\eta_t)}\rho a(t,\eta_t)\right),$$

where $R(t,x)$ satisfies the PDE (75).

Now we deal with the case of full information. To this end, we consider the same market model (55)-(56) consisting of two risky assets, one of which is non traded, and denote by $\widetilde{S}_t$ the asset price of the traded one. Thus, the dynamics of the prices are respectively:

$$d\widetilde{S}_t = \widetilde{S}_t(\mu(t,\eta)dt + \sigma(t,\eta)dW_t^1) \tag{77}$$

$$d\eta_t = b(t,\eta)dt + a(t,\eta)dW_t. \tag{78}$$

Let the coefficients μ, σ, a and b satisfy conditions 1)-3) and let H be a bounded F_T^η-measurable random variable.

We consider the problem

$$\text{to minimize } E\left(e^{-\alpha\left(\int_0^T \pi_t d\widetilde{S}_t - H\right)}\right) \quad \text{over all} \quad \pi \in \Pi(F^{\widetilde{S},\eta}), \tag{79}$$

where π_t represents the number of stocks held at time t and is adapted to the filtration $F_t^{\widetilde{S},\eta}$.
We assume an agent is trading with a portfolio of stocks $\widetilde{S}$ in order to hedge a contingent claim $H = f(\eta)$, written on the non traded asset. Notice that the agent builds his strategy using all market information, and $G_t = F_t^{\widetilde{S},\eta} = \mathcal{A}_t$. The market is incomplete, since η is non tradable and the contingent claim H is not attainable by the wealth process $X_t^\pi = x + \int_0^t \pi_u d\widetilde{S}_u$, by means of a self financing strategy π.

This problem was earlier studied in [1, 6, 7, 20, 22], for the model (77)-(78) (which is sometimes also called the "basis risk model") with constant coefficients μ and σ. In the case of constant coefficients μ, σ, assuming the Markov structure of b and a, in [22] (see also [6]) an explicit expression for the value function related to the problem (79) is given, which is

$$v(t,y) = \left(E^Q[e^{(1-\rho^2)(\alpha f(\eta_T) - \frac{1}{2}(T-t)\frac{\mu^2}{\sigma^2}}|\eta_t = y]\right)^{\frac{1}{1-\rho^2}}, \tag{80}$$

where the measure Q is defined by

$$\frac{dQ}{dP} = \mathcal{E}_T(-\theta \cdot W^1). \tag{81}$$

Note that, although the measures Q and $\widetilde{Q}$, defined in (61), are different, the process η_t has the same law under both of them and the value process V_t of Equation (62) (in case $H = f(\eta)$) and the value function (80) are related by the equality

$$V_t = v(t,\eta_t).$$

Note that (75) coincides with Equation (15) of [22] for constant coefficients μ and σ.

Using Theorem 1, we shall show that if H is F_T^η-measurable, then V_t, defined by (62), coincides with the value process related to problem (79) also in a non Markovian setting and that the optimal amount of money depends only on the observation coming from the non traded asset.
Under assumptions 1)-3), straightforward calculations yield that

$M_t = \widehat{M}_t = \int_0^t \widetilde{S}_u \sigma(u,\eta) dW_u^1, \qquad \langle M \rangle_t = \langle \widehat{M} \rangle_t = \int_0^t \widetilde{S}_u^2 \sigma^2(u,\eta) du$ and $\lambda_t = \frac{\mu(t,\eta)}{\widetilde{S}_t \sigma^2(t,\eta)}$, besides the orthogonal part will be expressed as a stochastic integral with respect to W^0.
Thus, according to Theorem 1, the value process related to problem (79) is the unique bounded solution of the BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u + \theta_u Y_u)^2}{Y_u} du + \int_0^t \psi_u dW_u^1 + \int_0^t \psi_u^\perp dW_u^0, \;\; Y_T = e^{\alpha H} \tag{82}$$

and the optimal strategy is of the form

$$\pi_t^* = \frac{1}{\alpha\sigma(t,\eta)\widetilde{S}_t}\big(\theta_t + \frac{\psi_t}{Y_t}\big), \tag{83}$$

where θ_t is defined in (58).

Proposition 4 *The process*

$$Y_t = \Big(E^Q[e^{(1-\rho^2)(\alpha H - \frac{1}{2}\int_t^T \theta_u^2 du)}|F_t^\eta]\Big)^{\frac{1}{1-\rho^2}} \tag{84}$$

is the unique bounded strictly positive solution of Equation (82). Besides, the optimal strategy is

$$\widetilde{\pi}_t^* = \frac{1}{\alpha\sigma(t,\eta)\widetilde{S}_t}\left(\theta_t + \frac{\rho h_t}{(1-\rho^2)(c + \int_0^t h_u d\widetilde{W}_u)}\right), \tag{85}$$

where h is defined by (64).

Proof.
Using (64) and Ito's formula, we find that

$$\begin{aligned} dY_t &= d\left(e^{\frac{1}{2}\int_0^t \theta_u^2 du}(E^Q[e^{(1-\rho^2)(\alpha H - \frac{1}{2}\int_0^T \theta_u^2 du)}|F_t^\eta])^{\frac{1}{1-\rho^2}}\right) \\ &= d\left(e^{\frac{1}{2}\int_0^t \theta_u^2 du}(c + \int_0^t h_u d\widetilde{W}_u)^{\frac{1}{1-\rho^2}}\right) \\ &= e^{\frac{1}{2}\int_0^t \theta_u^2 du}(c + \int_0^t h_u d\widetilde{W}_u)^{\frac{1}{1-\rho^2}} \; \frac{1}{2}\theta_t^2 dt + e^{\frac{1}{2}\int_0^t \theta_u^2 du} \times \\ &\times \left[\frac{1}{1-\rho^2}(c + \int_0^t h_u d\widetilde{W}_u)^{\frac{\rho^2}{1-\rho^2}} \; h_t d\widetilde{W}_t + \frac{1}{2}\frac{\rho^2}{(1-\rho^2)^2}(c + \int_0^t h_u d\widetilde{W}_u)^{\frac{\rho^2}{1-\rho^2}-1} h_t^2 dt\right] \\ &= \frac{1}{2}[\frac{(\theta_t Y_t + \phi_t \rho)^2}{Y_t}]dt + \phi_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^0), \end{aligned} \tag{86}$$

where in the last equality, we introduced the symbol ϕ to denote

$$\phi_t = e^{\frac{1}{2}\int_0^t \theta_u^2 du} \frac{1}{1-\rho^2}(c + \int_0^t h_u d\widetilde{W}_u)^{\frac{\rho^2}{1-\rho^2}}\ h_t. \tag{87}$$

Comparing (86) with (82), we see that Y_t is a bounded solution of (82), where $\psi_t = \phi_t\rho$ and $\psi_t^\perp = \phi_t\sqrt{1-\rho^2}$. By the uniqueness of (82), we obtain (84).
To obtain the optimal strategy, we substitute the expressions of $\psi_t = \phi_t\rho$, where ϕ is specified by (87), and of Y_t, written in terms of the representation (64), in (83). □

Remark 12 Note that Y_t satisfies also the BSDE (65), related to problem (57), which is simpler than (82). This follows either from the proof of Proposition 3 or directly from (86), since $\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^0 = W_t$.

So, the optimization problems (57) and (79) are equivalent since the corresponding value processes coincide and the optimal strategies of these problems are related by the equality

$$\pi_t^* = \widetilde{\pi}_t^* \widetilde{S}_t.$$

This means that, if $H = f(\eta)$, the optimal dollar amount invested in the assets is the same in both problems and is based only on the information coming from the non traded asset η. We deduce that if the contingent claim does not depend on the stock but only on the non traded asset η, we need the same optimal amount of money to hedge with the stock S in both situations: if we consider just the information on η and if we use all the information of the market.

Remark 13 If we consider the optimization problem

$$\text{to minimize } E\left(e^{-\alpha(\int_0^T \pi_t d\widetilde{S}_t - H)}\right) \quad \text{over all} \quad \pi \in \Pi(F^\eta), \tag{88}$$

where π_t represents the number of stocks held at time t and is adapted to the filtration F_t^η, then this problem is not equivalent to (57) and (79). Indeed, the optimal quantity of assets in (79) is not F^η-predictable and then it is evident that

$$\pi_t^* \neq \widetilde{\pi}_t^* \widetilde{S}_t,$$

where π^* and $\widetilde{\pi}^*$ denote respectively the optimal strategies of (57) and (88).
Theorem 1 cannot be applied directly to problem (88). In fact, for the process $\widetilde{S}$, condition A) is not satisfied and the equivalent F^η-adapted problem for (88) is more complicated.

6 Sufficiency of filtrations

In this section we study the issue of sufficiency of partial information for the optimization problem (2).

Let $V_t(\mathcal{A})$ and $V_t(G)$ be the value processes of the problem corresponding respectively to the cases of full and partial information

$$V_t(\mathcal{A}) = \operatorname*{ess\,inf}_{\pi\in\Pi(\mathcal{A})} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|\mathcal{A}_t],$$

$$V_t(G) = \operatorname*{ess\,inf}_{\pi\in\Pi(G)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|G_t].$$

It is convenient to express the optimization problem as the set

$$O\Pi = \{(\Omega, \mathscr{A}, \mathcal{A}, P), S, H\}$$

formed by the filtered probability space, the asset price process S and the terminal reward H.

Definition 1 The filtration G is said to be sufficient for the optimization problem $O\Pi$ if $V_0(G) = V_0(\mathcal{A})$, i.e., if

$$\inf_{\pi\in\Pi(\mathcal{A})} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] = \inf_{\pi\in\Pi(G)} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}], \tag{89}$$

where $\Pi(\mathcal{A})$ (respectively $\Pi(G)$) is the class of $\mathcal{A}$- predictable (respectively G- predictable) processes π such that $\pi\cdot N \in BMO(\mathcal{A})$.

Note that for the model (55)-(56) satisfying conditions 1)-3) the filtration $G_t = F_t^\eta$ is sufficient, if $\mathcal{A}_t$ coincides with the filtration $F_t^{S,\eta}$ and if H is F_T^η-measurable. Our aim is to give sufficient conditions for (89) for a more general model.

In this section we shall use the following assumptions:

$A')$ $\langle N\rangle$ is G-predictable,

$B')$ any G-martingale is an $\mathcal{A}$-local martingale,

$C')$ the filtration G is continuous,

$D')$ λ and $\langle N, m(g)\rangle$ (for any G-local martingale $m(g)$) are G-predictable,

$E')$ H is a G_T-measurable bounded random variable,

$F')$ $\int_0^T \lambda_t^2 d\langle N\rangle_t \le C, \quad P-$a.s..

Remark 14 These assumptions are similar to A)-F), but there are several differences. Here we don't use the auxiliary filtration F and conditions $A') - F')$ are formulated in terms of filtration $\mathcal{A}$, e.g., $D')$ and $E')$ are conditions on $\mathcal{A}$-decomposition terms of S and in assumptions $B')$ and $E')$ the filtration F is replaced by $\mathcal{A}$) (so, they are stronger). On the other hand, condition $A')$ is the first part of A) and $C')$ is the same as C).

Throughout this section we shall assume that conditions $A'), B'), C')$ and $F')$ are satisfied.
Under conditions $A'), B'), C')$

$$\widehat{S}_t = E(S_t|G_t) = S_0 + \int_0^t \widehat{\lambda}_u d\langle N\rangle_u + \overline{N}_t, \tag{90}$$

where $\overline{N}_t$ admits the representation (the proof is similar to [15] (Theorem 1 of Ch.4.10))

$$\overline{N}_t = \int_0^t (\frac{d\widehat{\langle N, m(g)\rangle}_u}{d\langle m(g)\rangle_u}) dm_u(g) + L_t(g), \tag{91}$$

for any G-local martingale $m(g)$. Here $L(g)$ is a G-local martingale orthogonal to $m(g)$. Note that $\overline{N}_t$ is equal to $E(N_t|G_t)$ if λ is G-predictable.
It follows from (91) that

$$\langle N, m(g)\rangle^G = \langle \overline{N}, m(g)\rangle \tag{92}$$

for any G-local martingale $m(g)$, where by A^G we denote the dual G-predictable projection of the process A.

Note that $k_t^2 = d\langle\overline{N}\rangle_t / d\langle N\rangle_t \le 1$ and, like in section 2, one can show that $\pi \cdot \overline{N} \in BMO(G)$ for any $\pi \in \Pi(G)$.

Lemma 3 *Let $V_0(\mathcal{A}) = V_0(G)$. Then $V_t(G) = E(V_t(\mathcal{A})|G_t)$. If in addition $F^S \subseteq G$, and conditions $B^{'}), E^{'})$ are satisfied, then $V_t(\mathcal{A}) = V_t(G)$ for all $t \in [0, T]$.*

Proof. If $V_0(\mathcal{A}) = V_0(G)$ then the optimal strategy π^* for $V_0(\mathcal{A})$ is G-predictable. Therefore

$$V_t(\mathcal{A}) = E\left(e^{-\alpha(\int_t^T \pi_u^* dS_u - H)} |\mathcal{A}_t\right) \tag{93}$$

and taking conditional expectations with respect to G_t we obtain the equality $V_t(G) = E(V_t(\mathcal{A})|G_t)$. If $F^S \subseteq G$ and H is G_T-measurable then

$$E\left(e^{-\alpha(\int_t^T \pi_u^* dS_u - H)} |\mathcal{A}_t\right) = E\left(e^{-\alpha(\int_t^T \pi_u^* dS_u - H)} |G_t\right) = V_t(G), \tag{94}$$

since by condition $B^{'})$ the σ-algebras $\mathcal{A}_t$ and G_T are conditionally independent with respect to G_t. Thus, (93) and (94) imply the equality $V_t(\mathcal{A}) = V_t(G)$. □

Proposition 5 *Let conditions $A^{'}), B^{'}), C^{'})$ and $F^{'})$ be satisfied. Then $V_t(\mathcal{A}) = V_t(G)$ if and only if H is G_T-measurable and the process $V_t(\mathcal{A})$ satisfies the BSDE*

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u k_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle N\rangle_u + \int_0^t \psi_u d\overline{N}_u + \widetilde{N}_t, \quad Y_T = e^{\alpha H}, \tag{95}$$

where $\widetilde{N}$ is a G-local martingale orthogonal to $\overline{N}$ and $k_t^2 = d\langle\overline{N}\rangle_t / d\langle N\rangle_t$.

Proof. Let $V_t(\mathcal{A}) = V_t(G)$. This implies that $e^{\alpha H} = E(e^{\alpha H}|G_T)$ and H is G_T-measurable. According to Corollary 2 (more exactly, by (36)) $V_t(\mathcal{A})$ is the unique (bounded strictly positive) solution of the BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t Y_u\big(\lambda_u + \frac{\varphi_u}{Y_u}\big)^2 d\langle N\rangle_u + \int_0^t \varphi_u dN_u + L_t, \quad Y_T = e^{\alpha H} \tag{96}$$

and the optimal strategy is of the form

$$\pi_t^* = \frac{1}{\alpha}(\lambda_t + \frac{\varphi_t}{Y_t}). \tag{97}$$

Since $V_t(\mathcal{A}) = V_t(G)$, the optimal strategy π^* is G-predictable, hence $\pi^* = \widehat{\pi}^*$, i.e.,

$$\lambda_t + \frac{\varphi_t}{V_t(\mathcal{A})} = \widehat{\lambda}_t + \frac{\widehat{\varphi}_t}{V_t(\mathcal{A})}. \tag{98}$$

Therefore (96), (98) and condition $A^{'}$) imply that the martingale part $m_t = \int_0^t \varphi_u dN_u + L_t$ is a G-local martingale. By the Galtchouk-Kunita-Watanabe (G-K-W) decomposition of m with respect to $\overline{N}$

$$\int_0^t \varphi_u dN_u + L_t = \int_0^t \psi_u d\overline{N}_u + \widetilde{N}_t, \tag{99}$$

where ψ is a G-predictable $\overline{N}$-integrable process and $\widetilde{N}$ is a G-local martingale orthogonal to $\overline{N}$ (since $m_t = \int_0^t \varphi_u dN_u + L_t$ is G-adapted). It is evident that

$$\psi_t = \frac{d\langle m, \overline{N}\rangle_t}{\langle \overline{N}\rangle_t} \quad \text{and} \quad \psi_t k_t^2 = \frac{d\langle m, \overline{N}\rangle_t}{\langle N\rangle_t}. \tag{100}$$

Taking first the mutual characteristics (with respect to N) and then G-dual predictable projections of both parts in (99)

$$\int_0^t \widehat{\varphi}_u d\langle N\rangle_u = \int_0^t \frac{d\langle m, \overline{N}\rangle_u}{d\langle \overline{N}\rangle_u} d\langle \overline{N}_u, N\rangle_u^G + \langle N, \widetilde{N}\rangle_t^G. \tag{101}$$

It follows from (92) that $\langle \overline{N}, N\rangle^G = \langle \overline{N}\rangle$ and $\langle N, \widetilde{N}\rangle^G = \langle \overline{N}, \widetilde{N}\rangle = 0$. Therefore

$$\int_0^t \widehat{\varphi}_u d\langle N\rangle_u = \langle \overline{N}, m\rangle_t$$

and

$$\widehat{\varphi}_t = \frac{d\langle m, \overline{N}\rangle_t}{d\langle N\rangle_t} = \psi_t \kappa_t^2. \tag{102}$$

Substituting (98), (99) and (102) in (96) we obtain that $V_t(\mathcal{A})$ satisfies (95).

Let us assume now, that $V_t(\mathcal{A})$ satisfies equation (95). It follows from the proof of Theorem 1 that the unique bounded strictly positive solution of equation (24) is the process

$$V_t = \operatorname*{ess\,inf}_{\pi \in \Pi(G)} E[e^{-\alpha(\int_t^T \pi_u d\widehat{S}_u - H) + \frac{\alpha^2}{2}\int_t^T \pi_u^2 (1-k_u^2) d\langle N\rangle_u} | G_t]. \tag{103}$$

Note that here we don't need condition $D^{'}$), which is needed just to show the equivalence $V_t(G) = V_t$ in Proposition 1.

Thus $V_t(\mathcal{A}) = V_t$. This implies that $V_t(\mathcal{A})$ is G-adapted and the optimal strategy π^* for $V_t(\mathcal{A})$ is G-predictable. Therefore

$$V_t(\mathcal{A}) = \operatorname*{ess\,inf}_{\pi \in \Pi(G)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)} | \mathcal{A}_t] = E[e^{-\alpha(\int_t^T \pi_u^* dS_u - H)} | \mathcal{A}_t]$$

and taking conditional expectations (with respect to G_t) in this equality we have

$$V_t(\mathcal{A}) = E[e^{-\alpha(\int_t^T \pi_u^* dS_u - H)} | G_t] \geq \operatorname*{ess\,inf}_{\pi \in \Pi(G)} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)} | G_t] = V_t(G). \tag{104}$$

On the other hand $V_t(G) \geq E(V_t(\mathcal{A})|G_t)$ (this is proved similarly to (21)) which together with inequality (104) implies the equality $V_t(\mathcal{A}) = V_t(G)$. □

Corollary 3 *Let conditions $C^{'}$) and $F^{'}$) be satisfied and let $F^S \subseteq G$. Then*

a) $V_t(\mathcal{A}) = V_t(G)$ if and only if H is G_T-measurable and λ is G-predictable.

b) If in addition condition $B^{'}$) is satisfied and H is G_T-measurable, then $V_0(\mathcal{A}) = V_0(G)$ if and only if λ is G-predictable.

Proof. Let H be a G_T-measurable and λ is G-predictable. Since $F^S \subseteq G$ the square bracket $\langle N\rangle$ is G-predictable and from (1) follows that N is G-adapted. Therefore, $N = \overline{N}$ and $k^2 = 1$. This implies that equations (95) and (96) coincide (note that $\lambda = \widehat{\lambda}$), hence $V_t(\mathcal{A})$ satisfies (95) and $V_t(\mathcal{A}) = V_t(G)$ by Proposition 5.

Let $V_t(\mathcal{A}) = V_t(G)$. Since $\langle m, N\rangle = \langle V(G), S\rangle$ be G-predictable, $\varphi = \frac{d\langle m,N\rangle}{d\langle N\rangle}$ is G-predictable and it follows from equality (98) that λ is also G-predictable.

The proof of the part b) follows from part a) and Lemma 3. □

Theorem 2 *Let conditions $A^{'}$)–$F^{'}$) be satisfied. Then $V_t(\mathcal{A}) = V_t(G)$ and the filtration G is sufficient.*

Proof. It follows from the proof of Theorem 1 (and Proposition (1)) that under conditions $A^{'}) - F^{'}$) the value process $V(G)$ coincides with V (defined by (103)) and satisfies equation (95). It follows from (92) and condition $D^{'}$) that $\langle N, \overline{N}\rangle = \langle \overline{N}\rangle$, which implies that

$$\overline{N}_t = \int_0^t k_t^2 dN_t + \widetilde{L}_t, \tag{105}$$

where $\widetilde{L}$ is orthogonal to N. On the other hand under conditions $A^{'}) - F^{'}$) and equality (92) also imply that $\langle \widetilde{N}, N\rangle = \langle \widetilde{N}, \overline{N}\rangle = 0$ ($\widetilde{N}$ is defined by (95)), i.e., a local martingale orthogonal to $\overline{N}$ is also orthogonal to N. Therefore plugging (105) into (95), by a change of variables $\varphi = \psi k^2$, we obtain that equations (95) and (96) are equivalent (note that $\lambda = \widehat{\lambda}$ by $D^{'}$)). This implies that $V_t(G) = V_t(\mathcal{A})$. □

Remark 15 Note that under conditions A)–F) the filtration G_t is sufficient for the auxiliary filtration $F_t = F_t^S \vee G_t$.

Remark 16 Of course, condition $E^{'}$) is not necessary for the filtration G to be sufficient, i.e. the filtration G can be sufficient for some claims H which are not G_T-measurable. One can show that Theorem 2 remains true if we replace condition $E^{'}$) by assuming that in the G-K-W decomposition $H = EH + \int_0^T \varphi_u^H dN_u + L_T^H$ the processes φ^H, $\langle L^H\rangle$ and $\langle L^H, m(g)\rangle$ (for any G-local martingale $m(g)$) are G-predictable.

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