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Some Counterexamples in Positive Dependence*

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Abstract

We provide some counterexamples showing that some concepts of positive dependence are *strictly* stronger than others. In particular we will settle two questions posed by Pemantle (2000) and Pellerey (2002) concerning respectively association versus weak association, weak association versus supermodular dependence, and supermodular dependence versus positive orthant dependence.

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1 Introduction

Several concepts of stochastic dependence have been introduced in the past forty years. Some of them can be derived from positive dependence orderings by comparing a random vector with a vector having the same marginals, but independent components. For instance supermodular dependence and positive orthant dependence are of this type.

As all dependence concepts can be used to derive descriptive statistics for multivariate data sets, it is of great importance to know the relations between them. Many implications among different dependence concepts are well known. The reader is referred to Joe (1997) or Müller and Stoyan (2002) for an extensive treatment of the topic.

In this note we will provide some counterexamples and show that some concepts of dependence are *strictly* stronger than others. In particular we will show that association and weak association are not equivalent, and we will give an example of a random vector which is positive orthant dependent but not positive supermodular dependent. This settles two questions posed by Pemantle (2000) and Pellerey (2002), respectively.

We will also prove that supermodular dependence does not imply weak association.

2 Main results

In the following the terms increasing and decreasing will be used in the weak sense. The space $\mathbb{R}^d$ will be equipped with the componentwise order, i.e. $\mathbf{x} \leq \mathbf{y}$ will mean $x_i \leq y_i$ for all $i \in \{1, \ldots, d\}$.

A random vector $\mathbf{X}$ is stochastically increasing in the random vector $\mathbf{Y}$ if $E[\phi(\mathbf{X}) \big| \mathbf{Y} = \mathbf{y}]$ is an increasing function of $\mathbf{y}$ for all increasing functions ϕ for which the expectation is defined.

Given a random vector $\mathbf{X}$ we indicate by $\mathbf{X}^{\perp}$ the random vector whose univariate marginal distributions coincide with the marginals of $\mathbf{X}$, and whose components are independent.

Given $(X_1, \ldots, X_d)$ and $I \subset \{1, \ldots, d\}$, we denote by $\mathbf{X}_I$ the vector $(X_i : i \in I)$.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is *supermodular*, if

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}) \quad \text{for all } \mathbf{x} \text{ and } \mathbf{y}, \tag{1}$$

where the lattice operators $\wedge$ and $\vee$ are defined as

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \dots, \min\{x_d, y_d\})$$

and

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \dots, \max\{x_d, y_d\}).$$

We call $P_{\mathbf{X}}$ the distribution of $\mathbf{X}$. Given two probability measures $P_{\mathbf{X}}, P_{\mathbf{Y}}$ on $\mathbb{R}^d$ we say that $P_{\mathbf{X}} \leq_{\mathrm{sm}} P_{\mathbf{Y}}$ if

$$\int \phi \, \mathrm{d}P_{\mathbf{X}} \leq \int \phi \, \mathrm{d}P_{\mathbf{Y}} \tag{2}$$

for all supermodular functions ϕ on $\mathbb{R}^d$; see Müller and Stoyan (2002), Chapter 3.9 for a detailed treatment of that order relation.

If (2) holds for all functions ϕ that are indicators of upper (lower) orthants, then we say that $P_{\mathbf{X}} \leq_{\mathrm{uo}} P_{\mathbf{Y}}$ ($P_{\mathbf{X}} \leq_{\mathrm{lo}} P_{\mathbf{Y}}$).

If $P_{\mathbf{X}} \leq_{\mathrm{uo}} P_{\mathbf{Y}}$ and $P_{\mathbf{X}} \leq_{\mathrm{lo}} P_{\mathbf{Y}}$ hold simultaneously, then we say that they are comparable in *concordance order*, written as $P_{\mathbf{X}} \leq_{\mathrm{c}} P_{\mathbf{Y}}$. This definition is due to Joe (1990).

Definition. A random vector $\mathbf{X} = (X_1, \dots, X_d)$ is

- *conditionally increasing (CI)* if $\mathbf{X}_I$ is stochastically increasing in X_J for all $I, J \subset \{1, \dots, d\}$ with $I \cap J = \varnothing$,
- *associated* if $\mathrm{Cov}[f(\mathbf{X}), g(\mathbf{X})] \geq 0$, for all increasing functions f, g,
- *weakly associated* if $\mathrm{Cov}[f(\mathbf{X}_I), g(\mathbf{X}_J)] \geq 0$, for all $I, J \subset \{1, \dots, d\}$ with $I \cap J = \varnothing$, for all increasing functions f, g,
- *positive supermodular dependent (PSMD)* if $P_{\mathbf{X}^\perp} \leq_{\mathrm{sm}} P_{\mathbf{X}}$,
- *positive upper orthant dependent (PUOD)* if $P_{\mathbf{X}^\perp} \leq_{\mathrm{uo}} P_{\mathbf{X}}$,
- *positive lower orthant dependent (PLOD)* if $P_{\mathbf{X}^\perp} \leq_{\mathrm{lo}} P_{\mathbf{X}}$,

- *positive orthant dependent (POD)* if $P_{\mathbf{X}^\perp} \leq_c P_{\mathbf{X}}$.

The following implications are well known.

(a) If $\mathbf{X}$ is CI, then it is associated (Müller and Scarsini (2001)).

(b) If $\mathbf{X}$ is associated, then it is weakly associated (obvious).

(c) If $\mathbf{X}$ is weakly associated, then it is PSMD (Christofides and Vaggelatou (2003)).

(d) If $\mathbf{X}$ is PSMD then it is POD (Müller and Stoyan (2002)).

When $d = 2$ the situation is far simpler: For instance PUOD, PLOD, POD, PSMD, and weak association are equivalent.

The question whether in general weak association implies association was given as an open problem in Pemantle (2000). The following counterexample settles the question in the negative.

Example 1. It is well known that there are bivariate random vectors, which are POD but not associated. Take for example (X_1, X_2) such that

$$\begin{aligned} &P(X_1 = 0, X_2 = 0) = P(X_1 = 2, X_2 = 2) = 2/9, \\ &P(X_1 = 0, X_2 = 2) = P(X_1 = 2, X_2 = 0) = 1/9, \\ &P(X_1 = 1, X_2 = 1) = 3/9. \end{aligned}$$

Let $X_3, \ldots, X_d$ be independent and independent of (X_1, X_2).

By choosing the increasing indicator functions

$$f(x_1, x_2) = I_{[2,\infty)}(\max(x_1, x_2)) \quad \text{and} \quad g(x_1, x_2) = I_{[1,\infty)\times[1,\infty)}(x_1, x_2)$$

we see that $\mathrm{Cov}(f(X_1, X_2), g(X_1, X_2)) = -2/81 < 0$ and hence (X_1, X_2) is not associated, but it is POD, and therefore weakly associated (since the two latter concepts coincide for $d = 2$).

Concatenations of weakly associated random vectors that are independent among each other are weakly associated again, therefore $(X_1, \ldots, X_d)$ is weakly associated, but not associated.

Christofides and Vaggelatou (2003) proved that weak association implies supermodular dependence. The following example shows that the converse implication fails to hold.

Example 2. Take $\mathbf{X}$ as in Example 5.1 of Block et al. (1985). This means that it has a distribution on $\{0,1\}^4$ with $P(\mathbf{X}=(0,0,0,0))=4/24$,

$$\begin{aligned}P(\mathbf{X}=(0,1,1,1)) = P(\mathbf{X}=(1,0,1,1)) = P(\mathbf{X}=(1,1,0,1)) = P(\mathbf{X}=(1,1,1,0))\\ = P(\mathbf{X}=(1,1,1,1)) = 2/24,\end{aligned}$$

and the probabilities of the ten other possible outcomes are all 1/24.

To prove that $\mathbf{X}$ is PSMD it suffices to check that $\mathbb{E}[\phi(\mathbf{X})]-\mathbb{E}[\phi(\mathbf{X}^{\perp})]\geq 0$ for all symmetric supermodular functions ϕ, since the distribution of $\mathbf{X}$ is symmetric (for the interaction between integral stochastic orderings of symmetric distributions and integral stochastic orderings defined through symmetric functions, see Scarsini and Shaked (1990)). Without loss of generality assume that $\phi(0,0,0,0)=0$. Then if we define $a=1/24$, we have

$$\mathbb{E}[\phi(\mathbf{X})]-\mathbb{E}[\phi(\mathbf{X}^{\perp})] = \frac{a}{2}[\phi(1,1,1,1)+4\phi(1,1,1,0)-6\phi(1,1,0,0)-4\phi(1,0,0,0)] \geq 0 \quad (3)$$

for all symmetric supermodular ϕ. To see this notice that by symmetry

$$\begin{aligned}&\phi(1,1,1,1)+4\phi(1,1,1,0)-6\phi(1,1,0,0)-4\phi(1,0,0,0)\\ &\quad = [4\phi(1,1,1,0)-4\phi(1,1,0,0)-4\phi(0,0,1,0)+4\phi(0,0,0,0)]\\ &\qquad + [\phi(1,1,1,1)-\phi(1,1,0,0)-\phi(0,0,1,1)+\phi(0,0,0,0)]\end{aligned}$$

and both terms in square brackets are nonnegative by the assumption of supermodularity of ϕ.

The inequality in (3) implies that $\mathbf{X}$ is PSMD. However $\mathbf{X}$ is not weakly associated, since

$$P(X_1=X_2=X_3=X_4=1) = \frac{1}{16}+\frac{a}{2} < \left(\frac{1}{4}+a\right)^2 = P(X_1=X_2=1)P(X_3=X_4=1).$$

The POD concept is strictly weaker than all other notions of positive dependence mentioned in this paper. Our final counterexample in dimension $d=3$ shows that POD does not imply PSMD (the question was posed to us by Pellerey (2002)).

Example 3. For $d \geq 4$ Joe (1990) and for $d = 3$ Müller and Scarsini (2000) proved that $P_{\mathbf{X}} \leq_{\mathrm{c}} P_{\mathbf{Y}}$ does not imply $P_{\mathbf{X}} \leq_{\mathrm{sm}} P_{\mathbf{Y}}$. The counterexamples involve discrete distributions.

Any counterexample for the dependence orderings, involving discrete distributions on a finite support, can be transformed into a counterexample for the corresponding dependence concept.

Indeed, let $P_{\mathbf{X}}, P_{\mathbf{Y}}$ be such that $P_{\mathbf{X}} \leq_{\mathrm{c}} P_{\mathbf{Y}}$ but not $P_{\mathbf{X}} \leq_{\mathrm{sm}} P_{\mathbf{Y}}$, and let the support of $P_{\mathbf{X}}, P_{\mathbf{Y}}$ be a subset of a finite lattice. Take $\alpha > 0$ small enough such that for some measure Q, the measure $\alpha P_{\mathbf{X}} + (1-\alpha)Q$ is a uniform distribution on the finite lattice, and therefore a product measure. Since the above orders are preserved under mixtures, we have

$$\alpha P_{\mathbf{X}} + (1-\alpha)Q \leq_{\mathrm{c}} \alpha P_{\mathbf{Y}} + (1-\alpha)Q,$$

but not

$$\alpha P_{\mathbf{X}} + (1-\alpha)Q \leq_{\mathrm{sm}} \alpha P_{\mathbf{Y}} + (1-\alpha)Q.$$

Hence if $\mathbf{Z}$ is distributed according to $\alpha P_{\mathbf{Y}} + (1-\alpha)Q$, then $\mathbf{Z}$ is POD, but not PSMD.

Notice that $P_{\mathbf{X}} \leq_{\mathrm{sm}} P_{\mathbf{Y}}$ implies that they have the same marginals, therefore the marginals of $\alpha P_{\mathbf{X}} + (1-\alpha)Q$ and $\alpha P_{\mathbf{Y}} + (1-\alpha)Q$ are equal, too. From the construction we then have that $\mathbf{Z}^{\perp}$ has distribution $\alpha P_{\mathbf{X}} + (1-\alpha)Q$.

As an explicit counterexample consider the following case, with $d = 3$, taken from Müller and Scarsini (2000). There $\mathbf{X}$ is uniformly distributed on the set

$$A = \{(2,2,1),(2,1,2),(1,2,2),(1,1,1),(0,0,2),(2,0,0)\}$$

and $\mathbf{Y}$ is uniformly distributed on the set

$$B = \{(2,2,2),(2,1,1),(1,2,1),(1,1,2),(2,0,2),(0,0,0)\}.$$

These vectors are such that $P_{\mathbf{X}} \leq_{\mathrm{c}} P_{\mathbf{Y}}$, but not $P_{\mathbf{X}} \leq_{\mathrm{sm}} P_{\mathbf{Y}}$.

Choose $\alpha = 6/27$, and Q uniformly distributed on the 21 points of the set $\{0,1,2\}^3 \setminus A$. Then the probability measure $\alpha P_{\mathbf{Y}} + (1-\alpha)Q$ assigns probability $2/27$ to the points in B and probability $1/27$ to the points in $\{0,1,2\}^3 \setminus (A \cup B)$, and the probability measure $\alpha P_{\mathbf{X}} + (1-\alpha)Q$ is uniform on the lattice $\{0,1,2\}^3$. Therefore $\alpha P_{\mathbf{Y}} + (1-\alpha)Q$ is POD, but not PSMD.

Remark. Notice that of the various definitions of positive dependence some are in principle directly checkable once the distribution functions are given (e.g. PUOD, PLOD, CI). However association, weak association and PSMD are not directly checkable, because they would require checking an inequality for all possible multivariate increasing functions f, g, or for all supermodular functions ϕ.

It is known that for association it is sufficient to consider the indicator functions of upper sets. In the case of distributions supported on a finite lattice some enumeration criteria for upper sets have been studied by Sampson and Whitaker (1988, 1989). Even in this case the complexity of the problem is very high, which makes it often difficult to find counterexamples.

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