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The convexity-cones approach to comparative risk and downside risk*

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Abstract

Based on Jewitt (1986) we try to find a characterization of comparative downside risk aversion and love. The desired characterizations involve the decomposition of the dual of the intersection of two convexity cones. The decomposition holds in the case of downside risk love, but not in the case of downside risk aversion. A counterexample is provided.

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1 Introduction

Jewitt (1986) proposed a general framework to compare attitudes towards different forms of risk. In particular given a relation $\mathcal{R}$ on a space of distribution functions he characterized the dual relation $\mathcal{R}^\circ$ on a space of utility functions such that

$$\int u \, dF \geq \int u \, dG \quad \text{implies} \quad \int v \, dF \geq \int v \, dG, \tag{1.1}$$

whenever $(F, G) \in \mathcal{R}$ and $(u, v) \in \mathcal{R}^\circ$.

In particular, if $(F, G) \in \mathcal{R}$ holds when G is a degenerate distribution function and F is any distribution function, (1.1) provides the usual Arrow-Pratt characterization of comparative risk aversion (see Pratt (1964)): $(u, v) \in \mathcal{R}^\circ$ iff $u(\cdot) = k \circ v(\cdot)$, with k increasing concave.

If $(F, G) \in \mathcal{R}$ means that F is preferred to G by all agents with an increasing convex utility, we obtain the characterization due to Ross (1981), according to which $(u, v) \in \mathcal{R}^\circ$ iff $v(\cdot) = \alpha u(\cdot) + w(\cdot)$ with $\alpha \geq 0$ and w increasing convex.

The concept of risk used by Ross (1981), is based on the idea of mean preserving spread, namely of a shift of probability from the center to the tail of a distribution that leaves the expectation unaltered (see Rothschild and Stiglitz (1970, 1972)). The opposite shift will be called a mean-preserving contraction.

Menezes et al. (1980) examined a concept of risk, called *downside risk*, that involves a mean preserving spread and a mean preserving contraction, where the contraction happens on the right of the spread. Formally, given $a < b < c$, we will consider a probability transfer such that every subinterval of $(-\infty, a) \cup (b, c)$ will have more mass, and every subinterval of $(a, b) \cup (c, \infty)$ will have less mass, and the first two moments μ_1, μ_2 do not change. We will call this transfer a (μ_1, μ_2)-preserving downside spread. We will call the opposite transfer a (μ_1, μ_2)-preserving downside contraction. A sequence of (μ_1, μ_2)-preserving downside spreads will be called an increase in downside risk. The above definition of (μ_1, μ_2)-preserving downside spread is quite general and does not imply the existence of a density.

The following simple example of (μ_1, μ_2)-preserving downside spread is taken from Menezes et al. (1980). On the set $\{0, 1, 2, 3\}$ consider the two lotteries given by the probability vectors $p = (0, 3/4, 0, 1/4)$ and $p' = (1/4, 0, 3/4, 0)$; they have same mean and variance, and most people report preference for the former. In fact $p' = p + s + c$ where $s = (1/4, -1/2, 1/4, 0)$ is a spread and $c = (0, -1/4, 1/2, -1/4)$ is a contraction *occurring on the right of the spread*; thus p' is obtained from p by shifting dispersion from right to left, and the change from p to p' is the prototype increase in downside risk.

As shown by Menezes et al. (1980), every decision maker, whose utility function u has convex derivative, will dislike a increase in downside risk. For smooth utility functions, convex derivative is equivalent to nonnegative third derivative.

The tools that Jewitt (1986) employed are drawn from the theory of convexity cones, as developed by Karlin and Novikoff (1963), based on ideas of Hopf and

Popoviciu (for the relevant references see Karlin and Studden (1966)).

A central result in Jewitt's paper is a theorem due to Amir and Ziegler (1968) that allows to decompose the dual of the intersection of two cones of (utility) functions.

In this note we will show how a characterization of comparative risk love à la Ross can be easily obtained from Jewitt's result, and we will try to apply similar ideas to downside risk. We will see that a useful decomposition holds for downside risk love, but not for downside risk aversion. We will provide a counterexample and give a heuristics for the non-decomposability. Section 2 introduces some notation, Section 3 deals with risk, and Section 4 treats the case of downside risk. Proofs are contained in Section 5.

2 Notation

We introduce some concepts and notations that are needed for the construction. We work with utilities in $C = C[0,1]$, the Banach space of continuous real functions on $[0,1]$ with the supremum norm. Its dual space is $M = M[0,1]$, the space of Radon measures on $[0,1]$ (representable as functions of bounded variation on $[0,1]$, and including all distribution functions on $[0,1]$ as well as their differences). The duality is $\mu(f) = \int f \, d\mu$ (see Edwards (1995)).

For $n \geq 0$, let $\mathcal{C}_n \subseteq C$ be the cone of functions ϕ that are convex with respect to the extended Tchebycheff system $(1, t, t^2, \ldots, t^{n-1})$, that is functions such that

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ t_0 & t_1 & \ldots & t_{n+1} \\ t_0^2 & t_1^2 & \ldots & t_{n+1}^2 \\ \vdots & & & \\ t_0^{n-1} & t_1^{n-1} & \ldots & t_{n+1}^{n-1} \\ \phi(t_0) & \phi(t_1) & \ldots & \phi(t_{n+1}) \end{vmatrix} \geq 0,$$

for all choices of $\{t_i\}_0^{n+1}$ satisfying $0 < t_0 < t_1 < \cdots < t_{n+1} < 1$. These cones are called convexity cones.

The cone $\mathcal{C}_n$ is the closure of the set of smooth functions with nonnegative n-th derivative. The cone $\mathcal{C}_1$ is the set of nondecreasing functions. The cone $\mathcal{C}_2$ is the set of convex functions. The cone $\mathcal{C}_3$ is the set of functions with convex derivative.

Given a set of functions $K \subseteq C$, the set $K^* = \{\mu \in M : \mu(f) \geq 0 \ \forall f \in K\}$ is the dual cone of K. Notice that all convexity cones contain the constant functions. Therefore for any $\mu \in \mathcal{C}_n^*$ we have $\mu(1) = \int d\mu = 0$. Hence, modulo re-normalization, μ can be interpreted as as a difference of probability measures. The bi-dual of K is $K^{**} = \{f \in C : \mu(f) \geq 0 \ \forall \mu \in K^*\}$, which is the closed convex hull of K.

3 Risk

A probability change in the dual of a convexity cone is one which is favored by all utility functions in the cone. For instance a first-order stochastic dominance shift is in $\mathcal{C}_1^*$ since it is favored by all non-decreasing functions. An increase in risk is in $\mathcal{C}_2^*$ since it is favored by all convex functions.

An element of $\mathcal{C}_1^* + \mathcal{C}_2^*$ is the sum of two shifts of the above types, namely, it corresponds to an increase in return and in risk. As a particular case of a general result due to Amir and Ziegler (1968) we have the following proposition

Proposition 3.1. *$\mathcal{C}_1^* + \mathcal{C}_2^* = [\mathcal{C}_1 \cap \mathcal{C}_2]^*$.*

Recall that u more risk averse (à la Ross) than v if

$$\left(\mu \in [\mathcal{C}_1^* + \mathcal{C}_2^*] \quad \text{and} \quad \int u \, d\mu \geq 0\right) \quad \text{imply} \quad \int v \, d\mu \geq 0. \tag{3.1}$$

As Jewitt (1986) noticed, this corresponds to

$$v \in ([\mathcal{C}_1 \cap \mathcal{C}_2] \cup \{u\})^{**},$$

that is, $v(x) = \alpha\, u(x) + w(x)$ for some $\alpha \geq 0$ and $w \in [\mathcal{C}_1 \cap \mathcal{C}_2]$. This way he obtained the characterization of Ross. For increasing utility functions, it is also equivalent, as is easily seen, to

$$\exists \lambda > 0 \quad \text{such that} \quad \forall x, y \in \mathbb{R} \quad \frac{u''(x)}{v''(x)} > \lambda > \frac{u'(y)}{v'(y)}. \tag{3.2}$$

To compare intensities of risk *love* in the Ross-Jewitt vein one clearly needs another trade-off, namely, increase in return versus *decrease* in risk, i.e. we will assume $\mu \in \mathcal{C}_1^* - \mathcal{C}_2^*$ (a decrease in risk is what the concave functions, i.e. those in $-\mathcal{C}_2$, favor). The following proposition is an immediate consequence of Proposition 3.1.

Proposition 3.2. *$\mathcal{C}_1^* - \mathcal{C}_2^* = [\mathcal{C}_1 \cap -\mathcal{C}_2]^*$.*

Proof. For each $\mu \in M$ define a $\nu \in M$ as follows:

$$\mu\left(\mathbf{1}_{[0,x]}\right) = -\nu\left(\mathbf{1}_{[1-x,1]}\right), \quad \forall x \in [0,1],$$

where $\mathbf{1}_A$ is the indicator of the set A.

We can associate to each $\phi \in C$ a $\psi \in C$ such that

$$\phi(x) = -\psi(1-x), \quad \forall x \in [0,1].$$

As a consequence of the above definitions we have $\mu(\phi) = \nu(\psi)$.

Since $\psi \in \mathcal{C}_1$ iff $\phi \in \mathcal{C}_1$ and $\psi \in -\mathcal{C}_2$ iff $\phi \in \mathcal{C}_2$, we have

$$\begin{aligned} \mu \in \mathcal{C}_1^* \quad &\text{iff} \quad \nu \in \mathcal{C}_1^*, \\ \mu \in \mathcal{C}_2^* \quad &\text{iff} \quad \nu \in -\mathcal{C}_2^*. \end{aligned}$$

Furthermore $\mu \in \mathcal{C}_1^* + \mathcal{C}_2^*$ (i.e. $\mu = \mu_1 + \mu_2$, with $\mu_1 \in \mathcal{C}_1^*$ and $\mu_2 \in \mathcal{C}_2^*$) iff $\nu \in \mathcal{C}_1^* - \mathcal{C}_2^*$ (i.e. $\nu = \nu_1 + \nu_2$, with $\nu_1 \in \mathcal{C}_1^*$ and $\nu_2 \in -\mathcal{C}_2^*$).

In order to see this it is enough to choose, for $i = 1, 2$, ν_i such that

$$\mu_i \left(\mathbf{1}_{[0,x]}\right) = -\nu_i \left(\mathbf{1}_{[1-x,1]}\right),$$

As a consequence of Proposition 3.1 we obtain $\mathcal{C}_1^* - \mathcal{C}_2^* = [\mathcal{C}_1 \cap -\mathcal{C}_2]^*$. □

One then defines v to be more risk attracted than u if

$$\left(\mu \in \mathcal{C}_1^* - \mathcal{C}_2^* \quad \text{and} \quad \int v \, d\mu \geq 0\right) \quad \text{imply} \quad \int u \, d\mu \geq 0. \tag{3.3}$$

The above proposition gives the desired characterization: v is more risk loving than u iff $v(x) = \alpha u(x) + w(x)$ for some $\alpha \geq 0$ and w *decreasing* convex. This is also equivalent to the following separation of derivative-ratios

$$\exists \lambda > 0 \quad \text{such that} \quad \forall x, y \in \mathbb{R} \quad \frac{u''(x)}{v''(x)} < \lambda < \frac{u'(y)}{v'(y)}. \tag{3.4}$$

It is interesting to see that the fact that u is more risk averse than v does not imply that v is more risk loving than u. This is different from the comparative characterization of risk aversion à la Arrow-Pratt, based on the idea of total insurance, namely of substitution of a random variable with a constant. In the Ross-Jewitt approach, partial insurance is allowed and a random variable is replaced by a less risky one, not necessarily a constant.

4 Downside risk

Menezes et al. (1980) studied the attitude of an agent who experiences a mean-and-variance-preserving combination of an increase in risk on the left tail and a decrease in risk on the right tail of a distribution. They called this *downside risk*, and showed that downside risk aversion corresponds to a utility function with convex derivative. In our terminology a decision maker is downside risk averse if her utility function $u \in -\mathcal{C}_3$. Therefore a signed measure μ is an increase in downside risk if $\mu \in [-\mathcal{C}_3]^*$. Of course the love counterpart is: μ is a decrease in downside risk if $\mu \in \mathcal{C}_3^*$.

Analogous to what we have seen in the previous section, the convexity-cones approach to comparative downside risk aversion, based on the risk-return tradeoff, is to define u to be more downside risk averse than v if

$$\left(\mu \in \mathcal{C}_1^* - \mathcal{C}_3^* \quad \text{and} \quad \int u \, d\mu \geq 0\right) \quad \text{imply} \quad \int v \, d\mu \geq 0. \tag{4.1}$$

The love counterpart is: v is more downside risk loving than u if

$$\left(\mu \in \mathcal{C}_1^* + \mathcal{C}_3^* \quad \text{and} \quad \int v \, d\mu \geq 0\right) \quad \text{imply} \quad \int u \, d\mu \geq 0. \tag{4.2}$$

In both cases, one looks for a representation $v = \alpha u + w$ where w has concave derivative (and is increasing or decreasing in each to the two cases), and, as before, in each case the representation hinges on the decomposability of the relevant cone.

Unlike the case of risk, only one of the desired decompositions holds.

Proposition 4.1. $\mathcal{C}_1^* + \mathcal{C}_3^* = [\mathcal{C}_1 \cap \mathcal{C}_3]^*$.

Proposition 4.2. $\mathcal{C}_1^* - \mathcal{C}_3^* \subsetneqq [\mathcal{C}_1 \cap -\mathcal{C}_3]^*$.

The content of the last proposition is that there are u more downside risk averse than v according to (3.1) which are not of the type $u = \alpha v + w$ where w is (decreasing) with convex derivative. In other words, the convex-derivative property of the (affine) transformation is stronger than u being more downside risk averse than v. On the other hand, for smooth utility functions existence of the above transformation can be proved directly (see Modica and Scarsini (2002)).

The decomposability result of Proposition 4.1 and the parallel non-decomposability stated in Proposition 4.2 appear quite puzzling, at first sight. Already Amir and Ziegler (1968) proved the surprising result that a decomposition is available for the dual cone of the intersection of two or three consecutive convexity cones, but not for the dual cone of the intersection of four convexity cones. In our case the convexity cones whose intersection we are considering are not consecutive: We intersect either $\mathcal{C}_1$ and $\mathcal{C}_3$ or $\mathcal{C}_1$ and $-\mathcal{C}_3$. In the fist case, even if $\mathcal{C}_2$ is missing, the decomposition of the dual of the intersection is available. In the second case it is not. It is well known that good results can be obtained for the sequence of cones $\{\mathcal{C}_n\}_{n \in \mathbb{N}}$ or, equivalently, for the sequence $\{(-1)^{n+1}\mathcal{C}_n\}_{n \in \mathbb{N}}$. Proposition 4.1 refers to a pair of cones taken from either one of these sequences, and this justifies the availability of the decomposition, even if the cones are not consecutive.

On the other hand there is no well-behaved sequence of convexity cones where the pair of Proposition 4.2 can be embedded and that's why the decomposition is not available here.

5 Proofs

In all cases, the inclusion of the sum of duals in the dual of the intersection is a direct consequence of definitions; proofs are needed to show that a given μ in the dual of the intersection is decomposable in a sum. Most arguments in the sequel are applications of ideas of Amir and Ziegler (1968) and Karlin and Studden (1966). Some complications arise from dealing with non-consecutive cones.

In each case the starting point is a useful characterization of the dual in terms of the extreme rays of the corresponding cone. And in each case, it can be checked by applying the approximation methods of Karlin and Studden (1966, ch. XI) that the smooth functions are dense in the relevant cone.

Basic notation: for $\mu \in M$, and $t \in [0,1]$ let

$$P\mu(t) = \int_t^1 d\mu(x), \quad Q\mu(t) = \int_0^t d\mu(x).$$

We will write

$$P^2\mu(t) = \int_t^1 P\mu(x)\, dx, \quad Q^2\mu(t) = \int_0^t Q(x)\, dx,$$

and for $n > 2$

$$P^n\mu(t) = \int_t^1 P^{n-1}\mu(x)\, dx, \quad Q^n\mu(t) = \int_0^t Q^{n-1}\mu(x)\, dx.$$

Lemma 5.1. $\mu \in [\mathcal{C}_1 \cap \mathcal{C}_3]^*$ *iff*

$$\int_0^1 d\mu = 0, \quad P^3\mu(t) \geq 0 \quad \textit{and} \quad Q^3\mu(t) \geq 0 \quad \forall t \in [0,1]. \tag{5.1}$$

Proof. First observe that

$$P^3\mu(t) = \int_0^1 \tau_2(x;t)\, d\mu(x), \quad \text{and} \quad Q^3\mu(t) = \int_0^1 \psi_2(x;t)\, d\mu(x), \tag{5.2}$$

where

$$\tau_2(x;t) = \mathbf{1}_{[t,1]}(x)(x-t)^2/2, \quad \psi_2(x;t) = -\mathbf{1}_{[0,t]}(x)(x-t)^2/2,\ x \in [0,1].$$

Since the constant function, and the functions $\tau_2(\cdot;t), \psi_2(\cdot;t) \in \mathcal{C}_1 \cap \mathcal{C}_3$ for all $t \in [0,1]$, we get the necessity part of the proposition.

For sufficiency, suppose that μ satisfies conditions (5.1) and integrate a smooth

$\phi \in \mathcal{C}_1 \cap \mathcal{C}_3$. Then obtain (using $P\mu(0) = 0$)

$$
\begin{aligned}
\int_0^1 \phi(x)\ d\mu(x) &= -\int_0^1 \phi(x)\ dP\mu(x) \\
&= \int_0^1 \phi'(x)P\mu(x)\ dx \\
&= -\int_0^1 \phi'(x)\ dP^2\mu(x) \\
&= \phi'(0)P^2\mu(0) + \int_0^1 \phi''(x)P^2\mu(x)\ dx \\
&= \phi'(0)P^2\mu(0) + \int_0^t \phi''(x)[P^2\mu(0) - Q^2\mu(x)]\ dx - \int_t^1 \phi''(x)\ dP^3\mu(x) \\
&= \phi'(t)P^2\mu(0) - \int_0^t \phi''(x)\ dQ^3\mu(x) - \int_t^1 \phi''(x)\ dP^3\mu(x) \\
&= \phi'(t)P^2\mu(0) - \phi''(t)\left(Q^3\mu(t) - P^3\mu(t)\right) \\
&\quad + \int_0^t \phi^{(3)}(x)Q^3\mu(x)\ dx + \int_t^1 \phi^{(3)}(x)P^3\mu(x)\ dx.
\end{aligned}
$$

By positivity and the endpoint conditions at $t = 0, 1$ of $P^3\mu$ and $Q^3\mu$, there exists t such that $Q^3\mu(t) - P^3\mu(t) = 0$; take t equal to this value in the last expression above, and you are left with only positive terms (recall that $P^2\mu(0) = \int_0^1 x\ d\mu(x) \geq 0$). $\square$

Proof of Proposition 4.1. Given $\mu \in [\mathcal{C}_1 \cap \mathcal{C}_3]^*$, we want a decomposition $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathcal{C}_1^*$ and $\mu_2 \in \mathcal{C}_3^*$. Again from Karlin and Novikoff (1963), or Amir and Ziegler (1968), the conditions on μ_1 and μ_2 are

$$P\mu_1(0) = 0, \quad P\mu_1(t) \geq 0, \quad \forall t \in [0, 1], \tag{5.3}$$

$$P\mu_2(0) = 0, \quad P^2\mu_2(0) = 0, \quad P^3\mu_2(0) = 0, \quad P^3\mu_2(t) \geq 0,\ t \in (0, 1). \tag{5.4}$$

Recall that$P^2\mu_2(0) = \int_0^1 x\ d\mu_2(x)$ and $P^3\mu_2(t) = 2^{-1} \int_t^1 (x - t)^2\ d\mu_2(x)$.

Call $A = P^2\mu(0)$. Given $P\mu(0) = 0$, it easily checked that

$$Q^3\mu(t) = P^3\mu(t) + At - P^3\mu(0). \tag{5.5}$$

From Lemma 5.1, $A \geq 0$. Suppose $A = 0$. Since the last two conditions on μ in the lemma assert non-negativity of $P^3\mu$ and $Q^3\mu$ on $[0, 1]$, and equation (5.5) (with $t = 1$) and $A = 0$ then implies $P^3\mu(0) \leq 0$, we may conclude that if $A = 0$ then also $P^3\mu(0) = 0$. But in this case $\mu \in \mathcal{C}_3^*$ (apply (5.4) to μ), and decomposition obtains with $\mu_1 \equiv 0$.

So assume $A > 0$. Then, since

$$\frac{d}{dt}P^3\mu(t)\Big|_{t=0} = -P^2\mu(0) < 0,$$

and $P^3\mu(t) \geq 0$ all t, it must be $P^3\mu(0) > 0$. Letting $B = P^3\mu(0)$, and using (5.5), the conditions on μ in Lemma 5.1 can then be written as

$$\begin{aligned} &P\mu(0) = 0, \quad P^2\mu(0) = A > 0, \quad P^3\mu(0) = B > 0, \\ &P^3\mu(t) \geq 0, \quad P^3\mu(t) \geq B - At, \quad \forall t \in [0,1]. \end{aligned} \tag{5.6}$$

On the other hand, using $\mu_2 = \mu - \mu_1$ and writing conditions (5.4) in terms of μ and μ_1 we obtain conditions on μ_1 equivalent to (5.3) and (5.4), which in the present case are

$$\begin{aligned} P\mu_1(0) = 0, \quad P\mu_1(t) \geq 0 \quad \forall t, \ P^2\mu_1(0) = A, \\ P^3\mu_1(0) = B, \quad P^2\mu_1(t) \leq P^3\mu(t) \quad \forall t \in [0,1]. \end{aligned} \tag{5.7}$$

Starting with (5.6), we look for a μ_1 satisfying (5.7). Take a function F on $[0,1]$ such that $F(1) = DF(1) = D^2F(1) = 0$, and define μ_1 as

$$P^3\mu_1 = F. \tag{5.8}$$

Then μ_1 satisfies (5.7) iff F satisfies

$$\begin{aligned} &F(0) = B, \quad F(t) \leq P^3\mu(t) \quad \forall t \in [0,1], \\ &DF(0) = -A, \quad D^2F(0) = 0, \quad D^2F(t) \geq 0 \quad \forall t \in [0,1]. \end{aligned} \tag{5.9}$$

Notice that at $t = 0$, F is required to be equal to $P^3\mu$ with its first two derivatives; and at $t = 1$ it is required to be equal to $P^3\mu$ with its first derivative. For the rest, F has to be convex and dominated by $P^3\mu$. If an F constant on a left neighborhood of $t = 1$ and satisfying (5.9) exists, the proposition is proved (with μ_1 defined by (5.8)).

We already observed that $Q^2\mu(t) = P^2\mu(0) - P^2\mu(t)$; then $Q^2\mu(1) = A > 0$ which, together with $Q^3\mu(t) \geq 0$ all t, implies $Q^3\mu(1) > 0$; so from (5.5) letting $t = 1$ we get $A > B$. Hence there is $t_0 \in (0,1)$ such that $B - At_0 = 0$. Define

$$F_1(t) = \mathbf{1}_{[0,t_0]}(t)(B - At).$$

This F_1 satisfies (5.9) (with convexity replacing $D^2F \geq 0$) except smoothness at t_0. To smooth it around t_0 (and end up with a function still below $P^3\mu$) we need to exclude that $P^3\mu(t_0) = 0$. But if this were the case, by smoothness of $P^3\mu$ we would have $P^3\mu(t) < B - At$ on a left neighborhood of t_0, contradicting the last requirement of (5.6). Therefore we can smooth F_1 around t_0 (as done in Amir and Ziegler (1968) for example) to get the wanted F. □

To characterize the dual of $\mathcal{C}_1 \cap -\mathcal{C}_3$ define the following family of extreme rays for $t \in (0,1]$:

$$\begin{aligned} \pi_2(x;t) &= \mathbf{1}_{[0,t)}(x)\,\frac{(1-t)x^2}{2} + \mathbf{1}_{[t,1]}(x)\,\frac{-t^2 + tx(2-x)}{2} \\ &= \int_{x_1=0}^{x}\int_{x_2=0}^{x_1} \left[(1-t)\mathbf{1}_{[0,t)}(x_2) - t\mathbf{1}_{[t,1]}(x_2)\right]\, dx_2\, dx_1. \end{aligned}$$

The last equality is elementarily checked, and it easily gives

$$\int_0^1 \pi_2(x;t)d\mu(x) = \big[(1-t)B - P^3\mu(t)\big] \equiv H(t), \tag{5.10}$$

where as before $B = P^3\mu(0)$. Incidentally, it will be again $A = P^2\mu(0)$.

Lemma 5.2. *$\mu \in [\mathcal{C}_1 \cap -\mathcal{C}_3]^*$ iff*

$$\int_0^1 d\mu(x) = 0, \ \int_0^1 x^2 \ d\mu(x) \geq 0, \ \int_0^1 -(x-1)^2 \ d\mu(x) \geq 0, \ \textit{and}$$
$$\int_0^1 \pi_2(x;t) \ d\mu(x) \geq 0, \ \forall t \in (0,1).$$

Proof. Necessity is obvious since the constant function, and the functions x^2, $-(x-1)^2$, and $\pi_2(x;t)$ are in the cone $\mathcal{C}_1 \cap -\mathcal{C}_3$ for all $t \in [0,1]$.

For sufficiency take a smooth $\phi \in \mathcal{C}_1 \cap -\mathcal{C}_3$ and integrate. The first equality below is obtained by using as usual $d\mu(x) = -dP\mu(x)$, $P\mu(x)dx = -dP^2\mu(x)$ and $P\mu(0) = 0$; then we use the fact that for the H defined in (5.10) we have $dH(t) = [P^2\mu(t) - B]\ dt$, and $H(0) = H(1) = 0$.

$$\begin{aligned}
\int_0^1 \phi(x)\ d\mu(x) &= -\int_0^1 \phi'(x)\ dP^2\mu(x) \\
&= -\phi'(1)P^2\mu(1) + \phi'(0)P^2\mu(0) \int_0^1 \phi''(x)\ [dH(x) + B\ dx] \\
&= \phi'(0)P^2\mu(0) + B\int_0^1 \phi''(x)\ dx + \phi''(1)H(1) \\
&\quad - \phi''(0)H(0) - \int_0^1 \phi^{(3)}(x)H(x)\ dx \\
&= \phi'(0)A + B[\phi'(1) - \phi'(0)] - \int_0^1 \phi^{(3)}H(x)\ dx \\
&= \phi'(1)B + \phi'(0)[A-B] - \int_0^1 \phi^{(3)}(x)H(x)\ dx.
\end{aligned}$$

We have

$$B = P^3\mu(0) = \int_0^1 \frac{x^2}{2}\ d\mu(x) \geq 0$$

by hypothesis, and

$$A = P^2\mu(0) = \int_0^1 x\ d\mu(x),$$

so using $\int_0^1 d\mu(x) = 0$ we get

$$A - B = \int_0^1 \frac{x(2-x)}{2}\ d\mu(x) = \int_0^1 \left(\frac{x(2-x)}{2} - \frac{1}{2}\right)\ d\mu(x) = \int_0^1 -\frac{(x-1)^2}{2}\ d\mu(x) \geq 0$$

by hypothesis. Again by assumption, H, ϕ' and $\phi^{(3)}$ are also non-negative. The desired result then follows. $\square$

Notice that although the condition $\int_0^1 x\ d\mu(x) \geq 0$ does not appear in the lemma, we have seen that it does hold: $\int_0^1 x\ d\mu(x) = A \geq B \geq 0$.

Proof of Proposition 4.2. We show that there exists $\mu \in [\mathcal{C}_1 \cap -\mathcal{C}_3]^*$ which admits no representation of the form $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathcal{C}_1^*$ and $\mu_2 \in -\mathcal{C}_3^*$. The conditions for $\mu_1 \in \mathcal{C}_1^*$ are still those in (5.3); those on μ_2 are found from (5.4) by observing that $\mu_2 \in -\mathcal{C}_3^*$ iff $-\mu_2 \in \mathcal{C}_3^*$, hence they are

$$P\mu_2(0) = P^2\mu_2(0) = P^3\mu_2(0) = 0, \text{ and } P^3\mu_2(t) \leq 0,\ t \in (0,1).$$

As before using $\mu_2 = \mu - \mu_1$ and rewriting the above conditions, we find that μ admits representation of the wanted type iff there exists μ_1 satisfying

$$\begin{aligned} &P\mu_1(0) = 0,\ P\mu_1(t) \geq 0\ \forall t \in (0,1), \\ &P^2\mu_1(0) = A,\ P^3\mu_1(0) = B, \text{ and } P^3\mu_1(t) \geq P^3\mu(t)\ \forall t \in (0,1). \end{aligned} \tag{5.11}$$

We now rewrite the conditions on μ in Lemma 5.2. Given $\int_0^1\ d\mu(x) = 0$, the condition $\int_0^1 -(x-1)^2\ d\mu(x) \geq 0$ amounts to $A \geq B$; also, $\int_0^1 x^2\ d\mu(x) = 2B$. Finally, using (5.10) it is seen that the last condition in the Lemma 5.2 is $P^3\mu(t) \leq B(1-t)$ for all $t \in (0,1)$, which obviously holds also for $t = 0, 1$. Hence we conclude that $\mu \in [\mathcal{C}_1 \cap -\mathcal{C}_3]^*$ iff

$$\int_0^1 d\mu(x) = 0, \quad A \geq B \geq 0, \quad \text{and} \quad P^3\mu(t) \leq B(1-t) \quad \forall t \in [0,1], \tag{5.12}$$

where $A = P^2\mu(0) = \int_0^1 x\ d\mu(x)$ and $B = P^3\mu(0) = \frac{1}{2}\int_0^1 x^2\ d\mu(x)$.

If $A = 0$ then also $B = 0$, and the decomposition is trivially obtained, with $\mu_1 \equiv 0$. We now give an example of μ satisfying (5.12), with $A > 0$ and $B = 0$, which cannot be decomposed. Recalling that

$$P^3\mu(t) = \int_0^1 \tau_2(x;t)\ d\mu(x) = \int_t^1 \frac{(x-t)^2}{2}\ d\mu(x),$$

conditions (5.12) in the present case read

$$\begin{aligned} &\int_0^1 d\mu(x) = 0, \quad \int_0^1 x\ d\mu(x) > 0, \quad \int_0^1 x^2\ d\mu(x) = 0, \quad \text{and} \\ &\int_t^1 (x-t)^2\ d\mu(x) \leq 0 \quad \forall t \in [0,1]. \end{aligned} \tag{5.13}$$

Take a function S on $[0,1]$ satisfying the following conditions.

$$\begin{aligned} & S(0) = S(1) = 0, \quad \int_0^1 S(x)\ dx > 0, \quad \int_0^1 xS(x)\ dx = 0, \\ & \int_t^1 (x-t)S(x)\ dx \leq 0 \quad \forall t \in [0,1]. \end{aligned} \tag{5.14}$$

If we define μ through

$$P\mu(x) = S(x),$$

we see that μ satisfies (5.13) (and viceversa).

Furthermore if we define $R(t) = \int_t^1 (x-t)S(x)\ dx$, then

$$R'(t) = -\int_t^1 S(x)\ dx = \int_0^t S(x)\ dx - A, \tag{5.15}$$

and if $\int_0^1 xS(x)\ dx = 0$ we also have $R(0) = 0$.

If we choose

$$S(t) = \mathbf{1}_{[0,\frac{1}{8}]}(t)\, t + \mathbf{1}_{(\frac{1}{8},\frac{1}{4}]}(t) \left(-t + \tfrac{1}{4}\right) + \mathbf{1}_{[\frac{3}{4},\frac{7}{8}]}(t)\, \frac{1}{7} \left(\tfrac{3}{4} - t\right) + \mathbf{1}_{(\frac{7}{8},1]}(t)\, \frac{1}{7}\, (x-1)\,,$$

then conditions (5.14) are satisfied. The first three conditions are easily verified. For the last one, which is $R(t) \leq 0$ for all $t \in [0,1]$, consider that $R'(t)$, increases on $[0,1/4]$ from $-A$ to a positive value, then remains constant up to $t = 3/4$, then decreases to zero, which it reaches at $t = 1$. Therefore $R(t)$ decreases on $[0,1/8]$, then increases; since $R(0) = R(1) = 0$, it must be $R(t) \leq 0$ for all $t \in [0,1]$.

To finish the proof we shall show that μ is not decomposable. Suppose it were, with $\mu = \mu_1 + \mu_2$, $\mu_1 \in \mathcal{C}_1^*$ and $\mu_2 \in -\mathcal{C}_3^*$. If we define, for $i = 1,2$,

$$S_i = \int_t^1 d\mu_i,$$

and ν_i as $d\nu_i(x) = S_i(x)\ dx$, then we have

$$\nu_1 \in (C^+)^*,\ \nu_2 \in -\mathcal{C}_2^*,$$

C^+ being the cone of positive functions. Therefore their densities S_1 and S_2 should in particular satisfy

$$S_1(t) \geq 0\ \forall t, \quad \int_0^1 S_2(x)\ dx = 0, \quad \int_0^1 xS_2(x)\ dx = 0. \tag{5.16}$$

This and $S = S_1 + S_2$ give $\int_0^1 xS(x)\ dx = \int_0^1 xS_1(x)\ dx$; and the integral on the left is zero by (5.14), thus by non-negativity and right continuity of S_1 (5.16) implies that the latter is identically zero. On the other hand $\int_0^1 S_2(x)\ dx = 0$ implies $\int_0^1 S_1(x)\ dx = \int_0^1 S(x)\ dx$, which is strictly positive by (5.14). A contradiction has been reached. $\square$

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