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Working Paper no. 13/2003
June 2003

APPLIED MATHEMATICS
WORKING PAPER SERIES

Ultramodular Functions

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June 2003

Abstract

We study the properties of ultramodular functions, a class of functions that generalizes scalar convexity and that naturally arises in some economic and statistical applications.

1 Introduction

1.1 Outline

In this paper we study in detail the properties of ultramodular functions, a class of functions that generalizes scalar convexity. They arise naturally in some economic and statistical applications and it seems to us that ultramodular functions provide, at least for some uses, the appropriate extension of one-dimensional convexity.

A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be ultramodular if its increments are increasing, namely

$$f(x+h) - f(x) \leq f(y+h) - f(y) \tag{1}$$

for all $x, y \in A$ and $h \geq 0$ with $x \leq y$ and $x + h, y + h \in A$. Scalar convex functions satisfy (1), but this is no longer the case for convex functions of

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several variables. As a matter of fact, for such functions ultramodularity and convexity are quite unrelated properties, as it will be seen later.

Ultramodular functions are supermodular, that is, for them it holds

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

for all suitable x and y. The converse, however, is in general false and this observation motivates our choice of the terminology "ultramodular." Specifically, the set of bounded ultramodular functions is the intersection of the class of supermodular functions and the class of the functions that are separately convex in each variable. To exemplify further, functions having second partial derivatives are supermodular if and only if it holds $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all $i \neq j$, while they are ultramodular if $\partial^2 f/\partial x_i \partial x_j \geq 0$ for all i, j. From an economic viewpoint ultramodular functions then reflect a stronger form of complementary than supermodular ones (cf. Topkis [24]). In this vein, ultramodular functions appeared in papers dealing with cost games (see Sharkey and Telser [21], Sharkey [22], and Moulin [14] and [15]).

Ultramodular functions are much better behaved than supermodular ones. Our main purpose in the paper is to show that they feature nice regularity properties, comparable to those of multidimensional convex functions, which make them analytically tractable. This will be seen in Sections 4 and 5. In particular, Section 4 gives the basic properties of ultramodular functions, while Section 5, the heart of paper, is devoted to their main regularity properties. We will show that under mild assumptions ultramodular functions feature neat continuity, Lipschitzianity, and differentiability properties. In Section 6 we provide an application of such properties to the representation of cores of convex measure games, thus extending previous results we established in [12].

1.2 Related Literature

Before moving to study these properties, we devote the rest of the introduction to some historical remarks on the different research areas, both pure and applied, where ultramodular functions came up.

We have already mentioned how ultramodular functions have been used in economics to model complementarities. In game theory they pop up in dealing with transferable utility measure games, that is, cooperative games of the form $f \circ P$, where P is a vector measure and f a function defined over the range of P. This class of games plays an important role in mathematical economics, and standard examples include exchange economies with transferable utilities and models of production technology (see, e.g., [2] and [7]). For our purposes the key observation, elaborated in Example 5 of Subsection 3.3, is that a measure game is convex provided f is ultramodular. Therefore, ultramodular functions come up in game theory when dealing with the important class of convex measure games.

In statistics, ultramodular functions now play a central role in modelling stochastic orders and positive dependence among random vectors, as discussed at length in Muller and Scarsini [16] (see also the references therein contained). In this field they have been called "directionally convex functions," a term we prefer not to use as it might give the wrong impression that they are defined in terms of properties of directional derivatives.

In mathematics, to the best of our knowledge ultramodular functions first appeared in Wright [26], who defined them on the real line. He just called them "convex functions," and some authors use the term Wright convexity for them (see, e.g., Roberts and Varberg [18]). *Inter alia*, he observed that scalar functions satisfying (1) are mid-convex, and so they fall into a class of functions known since Jensen [8]. In two related papers, Kenyon [9] and Klee [10] showed that Wright convex functions are a proper subclass of the mid-convex functions. Brunk [3] later proved some Jensen-type inequalities for ultramodular functions defined on intervals of $\mathbb{R}^n$ (see Proposition 19 below). He called them "functions having nondecreasing increments."

From a rather different angle, Choquet [4, p. 172] defines a similar class of functions, even though he requires also the first difference to be non-negative, and so the function itself to be non-decreasing. He realized that definition (1) can be extended to functions defined on abstract domains with some algebraic structure. This important observation permits to deal with set functions – the so-called convex (supermodular) capacities – that enjoy properties similar to (1) and that play an important role in mathematical economics.

Finally, a classic area of mathematics where ultramodular functions arise is the Bernstein-Hausdorff theory of absolutely and completely monotonic functions on the real line (see Widder [25]). They are analytic functions representable by Laplace-Stieltjes integrals and having derivatives such that either $f^{(k)}(x) \geq 0$ for all k or $(-1)^k f^{(k)}(x) \geq 0$ for all k. Interestingly, Bernstein proposed a definition of these functions by means of finite differences, which is somewhat related to (1).

2 Preliminaries

2.1 Sets

Given any $x, y \in \mathbb{R}^n$, $x \leq y$ means $x_i \leq y_i$ for each i, while we write $x \ll y$ when $x_i < y_i$ for each i. Given any $x, y \in \mathbb{R}^n$, the bounded *order interval* $[x, y]$ is the set $\{z \in \mathbb{R}^n : x \leq z \leq y\}$. In a similar way, it is possible to introduce unbounded order intervals, which include $\mathbb{R}^n$. For instance, $[a, +\infty] = \{x \geq a\}$ and $[-\infty, a] = \{x \leq a\}$. An order interval $[x, y]$ is called *solid* if $x \ll y$. For convenience, throughout the paper I denotes a generic order interval in $\mathbb{R}^n$, bounded or unbounded.

Special importance will have the unit order interval $[0, 1]^n$, for which we keep this notation in place of $[\underline{0}, \underline{1}]$. It plays a special role as we can transform any solid interval $[a, b]$ into $[0, 1]^n$ via the order-preserving isomorphism $x \to A(x - a)$, where A is the diagonal matrix with entries $\lambda_i = (b_i - a_i)^{-1}$.

As usual, e_i denotes the i-th unit vector of the canonical base of $\mathbb{R}^n$, while $int\,(A)$, $cl\,(A)$, and $ext\,(A)$ denote the interior, the closure, and the set of extreme points of A, respectively. Finally, we denote by $|x|_1$ the l_1 norm $\sum_{i=1}^n |x_i|$ of $x \in \mathbb{R}^n$, by $|x|$ the Euclidean norm $\sum_{i=1}^n x_i^2$, and by $x \cdot y$ the scalar product in $\mathbb{R}^n$.

2.2 Functions

A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is:

(i) *increasing* (*decreasing*, resp.) if $f(x) \leq f(y)$ ($f(x) \geq f(y)$, resp.) whenever $x \leq y$ in $\mathbb{R}^n$.

(ii) *calm from below* at $a \in A$ if there are $K > 0$ and $\varepsilon > 0$ such that $f(x) \geq f(a) - K\|x - a\|$ for all $x \in A$ with $\|x - a\| \leq \varepsilon$.

(iii) *upper Lipschitz* at $a \in A$ if there is $K > 0$ such that $f(x) - f(a) \leq K\|x - a\|$ for each $x \in A$;

(iv) *(globally) Lipschitz* if there is $K > 0$ such that $|f(x) - f(y)| \leq K\|x - y\|$ for all $x, y \in A$;

(v) *locally Lipschitz* if for all $a \in A$ there are $K > 0$ and $\varepsilon > 0$ such that $|f(x_1) - f(x_2)| \leq K\|x_1 - x_2\|$ for all $x_1, x_2 \in A$ with $\|x_1 - a\| < \varepsilon$ and $\|x_2 - a\| < \varepsilon$.

Given a bounded function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, with A convex, the *upper (concave) envelope* $\overline{f}$ of f is defined as

$$\overline{f}(x) = \inf\{h(x) : h \geq f,\ \ h \text{ is affine}\}.$$

The function $\overline{f}$ is concave and upper semicontinuous.

For a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, the standard directional derivative at x in the direction h is given by

$$f'(x;h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t};$$

while the Clarke directional derivative at x in the direction h is given by

$$f^0(x;h) = \lim \sup_{y \to x,\ t \downarrow 0} \frac{f(y+th) - f(y)}{t}.$$

The Clarke differential $\partial^0 f(x)$ at x is given by

$$\partial^0 f(x) = \left\{p \in \mathbb{R}^n : f^0(x;h) \geq p \cdot h \quad \forall h \in \mathbb{R}^n\right\}.$$

When f is convex, $\partial^0 f(x)$ reduces to the standard subdifferential

$$\{m \in \mathbb{R}^n : f(y) \geq f(x) + m(y-x) \quad \forall y \in A\}.$$

2.3 Set Functions

Ultramodular functions are closely connected with a class of set functions called *games.* Given a σ-algebra Σ of subsets of a space Ω, a game $\nu : \Sigma \to \mathbb{R}$ is a set function such that $\nu(\varnothing) = 0$. In game theory, Ω is the set of players, Σ is the collection of admissible coalitions that players can form, and $\nu(E)$ is the worth of coalition E (see [2]).

A game ν is *monotone* if $\nu(E) \leq \nu(F)$ whenever $E \subseteq F$, it is *convex* (supermodular) if $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \in \Sigma$, and it is *additive* if $\nu(E \cup F) = \nu(E) + \nu(F)$ for all pairwise disjoint sets $E, F \in \Sigma$. Additive games are often called *charges*; *ba*(Σ) is the set of all bounded charges.

A fundamental set associated with a game ν is the *core*, defined by:

$$\textit{core}(\nu) = \{m \in \textit{ba}(\Sigma) : m(\Omega) = \nu(\Omega) \text{ and } m(E) \geq \nu(E)\ \forall E \in \Sigma\}. \quad (2)$$

The core is therefore the set of suitably normalized charges that setwise dominate the game ν.

Finally, let $B(\Sigma)$ be the set of all bounded Σ-measurable functions $X : \Omega \to \mathbb{R}$. For all games ν of bounded variation (i.e., games that can be written as differences of two monotone games) we can define the Choquet integral

$$\int X d\nu = \int_0^\infty \nu(X \geq t)\, dt + \int_{-\infty}^0 [\nu(X \geq t) - \nu(\Omega)]\, dt,$$

for all $X \in B(\Sigma)$, where on the right we have two Riemann integrals (since ν is of bounded variation, these integrals exist for all $X \in B(\Sigma)$ as $\nu(X \geq t)$ is of bounded variation in t).

3 Ultramodular Sets and Functions

3.1 Ultramodular Sets

A collection $\{x, y, z, w\}$ of vectors in $\mathbb{R}^n$ is a *test quadruple* if $x \leq y \leq w$ and $x + w = y + z$. The elements of a test quadruple can be viewed as the vertices of a quadrilateral centered at $(x + w)/2$.

We now introduce a class of sets that are closed under the formation of test quadruples.

Definition 1 *A set $A \subseteq \mathbb{R}^n$ is said to be ultramodular if given any triple $x, y, w \in A$ with $x \leq y \leq w$, we have $z \in A$ whenever $x + w = y + z$.*

For example, order intervals are ultramodular sets. More generally, any set $\prod_{i=1}^n A_i$, with each $A_i \subseteq \mathbb{R}$ ultramodular, is ultramodular. Notice that ultramodular sets can be finite; for example, a test quadruple itself is an ultramodular set.

The next result shows that a rather good description of ultramodular set is possible, provided the set is sufficiently "solid."

Proposition 1 *A set A in $\mathbb{R}^n$ is ultramodular whenever the following condition holds:*

$$x, y \in A \text{ and } x \leq y \Longrightarrow [x, y] \in A. \tag{3}$$

The converse is true provided at least one of the following properties holds:

(i) A *is open,*

(ii) $A = cl\,(int\,(A))\,,$

(iii) $int\,(A) \neq \varnothing$ *and* A *has a smallest and largest element.*

Note that (3) implies that the intersection of A with any straight line with non-negative slope is necessarily convex.

By Proposition 1, we have the following characterization of ultramodular sets in the real line: a set A in $\mathbb{R}$ is ultramodular if it is convex; the converse is false unless A has nonempty interior (finite ultramodular sets in $\mathbb{R}$ are simple examples of ultramodular sets in $\mathbb{R}$ with empty interiors that are not convex).

The betweenness property (3) suggests that order intervals are the most relevant examples of ultramodular sets in $\mathbb{R}^n$ and for this reason we will often consider ultramodular functions defined on order intervals.

3.2 Ultramodular Functions

We now present ultramodular functions. Though ultramodular sets are the natural domain for this class of functions, in the definition we do not require the domain to be ultramodular.

Definition 2 *A function* $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ *is said to be ultramodular if*

$$f\,(z) - f\,(x) \leq f\,(w) - f\,(y) \tag{4}$$

for all test quadruples $\{x, y, z, w\}$ *in* A*.*

If we set $h = z - x = w - y$, (4) can be written as

$$f\,(x + h) - f\,(x) \leq f\,(y + h) - f\,(y)\,, \tag{5}$$

where by construction $x \leq y$ and $h \geq 0$. Though the equivalent form (5) is more intuitive because it uses increasing increments, formulation (4) is superior. In fact, even when the domain A is ultramodular, in (5) we must always specify that it has to hold for all $x \leq y$ and $h \geq 0$ such that $x + h$, $y + h \in A$.

Note that ultramodularity can be also defined by saying that the second difference

$$\triangle_{h,k}^{2} f(x) = f(x+h+k) - f(x+h) - f(x+k) + f(x)$$

is non-negative for all admissible $h, k \geq 0$.

Ultramodular functions are supermodular provided the domain A is a lattice. In fact, the collection $\{x \wedge y, x, y, x \vee y\}$ is a test quadruple for all $x, y \in A$, and so (4) reduces to $f(x) + f(y) \leq f(x \vee y) + f(x \wedge y)$, which is the definition of supermodularity.

Ultramodular functions f defined on an order interval $[0, a]$, with $f(0) = 0$, are superadditive as well. In fact, $\{0, x, y, x+y\}$ is a test quadruple for all $x, y \in [0, a]$ with $x + y \in [0, a]$, and so, by (4), for all such x, y we have $f(x+y) - f(x) \geq f(y) - f(0) = f(y)$. In turn, this implies that non-negative ultramodular functions are increasing.

Besides supermodular and superadditive functions, convex functions are the other classic class of functions related with ultramodular functions. Wright [26] showed that in the scalar case ultramodularity is equivalent to convexity, provided f is continuous. This equivalence fails without continuity. For example, consider any solution f of the Cauchy functional equation $\psi(x+y) = \psi(x) + \psi(y)$ on $[0, 1]$. The function f is clearly ultramodular, but it might well be non convex unless it is assumed to be continuous. In this case the ultramodular function f is unbounded on $[0, 1]$ and it is a quite "wild" function.

When $n > 1$, ultramodularity and convexity are altogether independent notions. There are convex functions that are not ultramodular (e.g., $f(x) = \|x\|$) and, vice versa, ultramodular functions that are not convex (e.g., $f(x) = \prod_{i=1}^{n} x_i$). In the sequel we will see other examples.

3.3 Examples

Next we present some examples that illustrate the previous definitions of ultramodular sets and functions.

1. Discrete ultramodular sets A in $\mathbb{R}$ are simple to classify. If we assume that A contains a first element a_0, then A is a countable (finite or infinite) collection $A = \{a_0, a_1, ..., a_n, ...\}$, with $a_n = a_0 + nh$ for some $h > 0$. In particular, a function $f : A \to \mathbb{R}$ is ultramodular if and only if

$$f(a_n) - f(a_{n-1}) \leq f(a_{n+1}) - f(a_n)$$

for all $n \geq 1$.

2. Let H be an Hamel basis of $\mathbb{R}$ on the field of rationals, including the number 1. Consider the set $A = \left\{x \in \mathbb{R} : x = \sum_{h \in H} r_h h\right\}$ where $h \to r_h$ are functions with finite support and taking values on the set of the integers $\mathbb{Z}$. Actually, A is a subgroup of $(\mathbb{R}, +)$ invariant for the translation, namely, $A + n = A$ for all $n \in \mathbb{Z}$. Clearly, the set $A \cap [0,1]$ is an uncountable ultramodular set. Indeed, the set A is not measurable. Suppose A were measurable with Lebesgue measure $\lambda(A) > 0$. Given that $A + A = A$, by a classical result of Steinhaus, A should contain an open set. Hence $A = \mathbb{R}$, a contradiction. We infer that $\lambda(A) = 0$, but this leads to a new contradiction. Clearly, even the groups $(1/p)\,A$, with $p \in \mathbb{N}$ have zero measure. On the other hand, $(1/1!)\,A \subseteq (1/2!)\,A \subseteq ... \subseteq (1/n!)\,A \subseteq ...$, and $\cup_{n=1}^{\infty} (1/n!)\,A = \mathbb{R}$. We deduce that $\lambda(\mathbb{R}) = 0$. Therefore, A and $A \cap [0,1]$ are not measurable.

3. An interesting finite ultramodular set is $ext\,[0,1]^n = \{0,1\}^n$. This set, which is both ultramodular and a lattice, will play an important role in the sequel. The properties of ultramodularity and supermodularity agree for functions defined on $\{0,1\}^n$. Moreover, all such functions have a natural extension $\widetilde{f}$ to the whole order interval $[0,1]^n$ given by

$$\widetilde{f}(x) = \sum_{k=(k_1,\ldots,k_n)\in\{0,1\}^n} f(k)\, x_1^{k_1} \cdots x_n^{k_n} (1-x_1)^{1-k_1} \cdots (1-x_n)^{1-k_n}.$$

This extension preserves ultramodularity and it is the multivariate Bernstein polynomial of degree at most one in each variable. We will see more on this in Section 7.

4. Though it is outside the scope of this paper to extend the notion of ultramodularity to abstract algebraic structures, an important example has to be mentioned. Let Σ be an algebra of subset of a space Ω. The algebra Σ is a lattice under the natural partial order $\subseteq$ and it can be viewed as an ultramodular set. In fact, using indicator functions, given any chain $A \subseteq B \subseteq C$ there always exists an element $Z \in \Sigma$ such that $1_A + 1_C = 1_Z + 1_B$. The unique solution is $Z = C \setminus B$. Therefore, $\{A, B, C \setminus B, C\}$ is a test quadruple. This argument leads easily to the following two conditions on games $\nu : \Sigma \to \mathbb{R}$,

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B), \tag{6}$$
$$\nu(A \cup H) - \nu(A) \leq \nu(B \cup H) - \nu(B), \tag{7}$$

where the latter inequality must hold for all $A, B, H \in \Sigma$ such that $A \subseteq B$ and $B \cap H = \varnothing$. By Choquet [4], the supermodularity condition (6) is equivalent to the ultramodularity condition (7). As already mentioned, in economics literature games ν satisfying either (6) or (7) are known as convex games (see Shapley [20]).

5. A *measure game* is a game of the form $\nu = f \circ P$, where $P = (P_1, ..., P_n) : \Sigma \to \mathbb{R}^n$ is a vector measure with each P_i non-atomic, finite, and positive, and where $f : R(P) \to \mathbb{R}$ is a function defined on the range $R(P)$ of the vector measure, with $f(0) = 0$. The next result, whose main part is due to Choquet [4, pp. 193-194], relates convex measure games and ultramodular functions. We refer to [11] for a proof.

Proposition 2 *A measure game $f \circ P : \Sigma \to \mathbb{R}$ is convex whenever $f : R(P) \to \mathbb{R}$ is ultramodular. The converse holds when $R(P) = [0,1]^n$.*

Proposition 2 extends to non-continuous f the well-known fact that, when f is continuous, a scalar measure game $f \circ P$ is supermodular if and only if f is convex. As a matter of fact, in the next subsection it will shown that scalar convexity and ultramodularity are no longer equivalent without continuity, and so the standard result fails, while Proposition 2 still holds.

4 Basic Properties

We begin with a simple characterization of ultramodular functions, related to similar results for supermodular functions (see [24, Thms 2.61 and 2.62]).

Proposition 3 *Let $f : A = \prod_{i=1}^n A_i \to \mathbb{R}$ be a function defined on the direct product of ultramodular sets A_i. Then, f is ultramodular if and only if it is separately ultramodular and has increasing differences on $\prod_{i=1}^n A_i$.*

For a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, define the difference operator $\Delta_i^\varepsilon f(x)$ by

$$f(x) = f(x + \varepsilon e_i) - f(x),$$

where $\varepsilon > 0$, and $x + \varepsilon e_i \in A$. The next result is a simple consequence of Proposition 3.

Corollary 4 *For a function $f : I \subseteq \mathbb{R}^n \to \mathbb{R}$, the following conditions are equivalent:*

(i) f is ultramodular.

(ii) f is supermodular and it is separately ultramodular.

(iii) For all $x \in \mathbb{R}^n$, $1 \leq i, j \leq n$, and all $\varepsilon, \delta > 0$,

$$\Delta_i^{\varepsilon} \Delta_j^{\delta} f(x) \geq 0.$$

Next we consider the closure properties of ultramodular functions.

Proposition 5 *Let $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be two ultramodular functions and let $h : B \subseteq \mathbb{R} \to \mathbb{R}$ be an ultramodular function defined on an ultramodular set B containing the range of f. Then:*

(i) the sum $\alpha f + \beta g$ is ultramodular if α and β are non-negative scalars;

(ii) the product fg is ultramodular provided f and g are both non-negative and monotone;

(iii) the composition $h \circ f$ is ultramodular provided h is increasing.

The closure property (ii) of Proposition 5 is noteworthy. For instance, consider the function $f : \mathbb{R}_{++}^n \to \mathbb{R}$ defined by $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, with $\alpha_i \geq 1$ for each $1 \leq i \leq n$. It is ultramodular since it is the product of the powers x^α with $\alpha \geq 1$, which, being scalar positive convex functions, are ultramodular. This function $f(x)$ is not convex: in fact, for convex functions property (ii) does not hold, that is, in general they are not closed under products, unless they are scalar. Therefore, (ii) is an important closure property satisfied by ultramodular functions, but not by convex functions.

The next Lemma provides a useful minorization for supermodular functions. For any function $f : I \subseteq \mathbb{R}^n \to \mathbb{R}$, with $n > 1$, and any point $\overline{x}$ in I, define the scalar function

$$f_i(t; \overline{x}) = f(\overline{x}_1, ..., \overline{x}_{i-1}, t, \overline{x}_{i+1}, ..., \overline{x}_n)$$

for $i = 1, 2, ..., n$.

Lemma 6 *Let $f : A = \prod_{i=1}^{n} A_i \to \mathbb{R}$ be a function defined on the direct product of lattices A_i. For all $\overline{x} \in A$, we have*

$$f(x) \geq \sum_{i=1}^{n} f_i(x_i; \overline{x}) - (n-1) f(\overline{x}) \tag{8}$$

if either $x \in A \cap [\overline{x}, +\infty]$ or $x \in A \cap [-\infty, \overline{x}]$. Moreover, the remainder

$$R(x) = f(x) - \sum_{i=1}^{n} f_i(x_i; \overline{x}) - (n-1) f(\overline{x}), \tag{9}$$

is supermodular, increasing on $A \cap [\overline{x}, +\infty]$ and decreasing on $A \cap [-\infty, \overline{x}]$.

Inspection of the proof shows that (8) holds also for functions satisfying the single crossing property, though property (9) no longer holds in this case.[1]

Ultramodularity per se does not preclude very irregular behavior, as shown by the discontinuous solutions of the Cauchy functional equation mentioned in Section 3.2. Fortunately, this is prevented by a mild local boundedness assumption, as it will be seen in the next section. The next result provides a first instance where such condition has interesting implications.

Proposition 7 *Let $f : [a, b] \subseteq \mathbb{R}^n \to \mathbb{R}$ be ultramodular. Then, the following conditions are equivalent:*

(i) f is locally bounded from below at a;

(ii) f is bounded;

(iii) f is separately convex in $[a, b]$.

In this case, f is convex along all the line segments in $[a, b]$ with non-negative slope.

In view of Proposition 7, for the sake of brevity from now on we replace the condition "locally bounded from below" with "bounded."

Proposition 7 has two consequences. The first one completes, in a sense, Corollary 4.

[1] This class of functions, which extend supermodular functions, is studied in depth in [13].

Corollary 8 *Let $f : [a, b] \subseteq \mathbb{R}^n \to \mathbb{R}$ be a bounded function. Then, f is ultramodular if and only if it is supermodular and separately convex.*

Corollary 9 *Let $f : [a, b] \subseteq \mathbb{R}^n \to \mathbb{R}$ be a bounded function. Then, both f and $-f$ are ultramodular if and only if f is affine.*

To see why Corollary 9 holds, notice that f, being both supermodular and submodular, is separable (see [23]). Therefore, f is affine by Proposition 7.

5 Main Properties

In this section we present the main properties of ultramodular functions that we discovered. In establishing them, we make use of the basic results obtained in the previous section, as well as of the auxiliary results on Bernstein polynomials and linear cores that will be presented in the next section.

5.1 Homogeneity

We begin by showing that ultramodular functions are never positively homogeneous, unless they are linear.

Theorem 10 *Let $f : [0, 1]^n \to \mathbb{R}$ be an ultramodular function, with $f(0) = 0$. Then, f is positively homogeneous if and only if it is linear.*

Besides its intrinsic interest, this result also shows another important difference between convex and ultramodular functions. In fact, Theorem 10 is altogether false for convex functions: there are important convex functions that are positively homogeneous, such as norm functions, gauge functions, and support functions $f(x) = \max_{a \in K} a \cdot x$. By Theorem 10, none of them is ultramodular. The same is true for the Choquet functionals $f(x) = \int x d\nu$, where $v : 2^N \to \mathbb{R}$ is any finite game, and for the functions $f(x) = \max_{a \in K} a \cdot x - \min_{b \in T} b \cdot x$.

We can also establish a further related property. Given a function $f : [0, 1]^n \to \mathbb{R}$, say that a point $x \in [0, 1]^n$ is *linear* if $f(x) + f(1 - x) = f(1)$. Clearly, all points $x \in [0, 1]^n$ are linear whenever f is linear. The converse is false, even for ultramodular functions; in fact, all points of a non continuous solution of the Cauchy functional equation are linear. The next result shows that things go much better under continuity: in this case the existence of even a single interior linear point forces an ultramodular function to be linear.

Theorem 11 *Let* $f : [0,1]^n \to \mathbb{R}$ *be a continuous ultramodular function, with* $f(0) = 0$*. Then,* f *admits a linear point* $x \in (0,1)^n$ *if and only if it is linear.*

For continuous positively homogenous functions, Theorem 11 implies Theorem 10. In fact, all points $\alpha 1$, with $\alpha \in (0,1)$, are interior linear points of a positively homogeneous function. Without continuity, however, Theorem 11 is false, while Theorem 10 still holds.

5.2 Continuity

Despite their differences, ultramodular functions share some of the remarkable continuity and Lipschitzianity properties of convex functions defined on polytopes (see Gale et al. [6]).

Theorem 12 *Let* $f : I = [a,b] \subseteq \mathbb{R}^n \to \mathbb{R}$ *be ultramodular and bounded. The following properties hold:*

(i) f *is Lipschitz continuous on any compact set* $C \subseteq int(I)$*;*

(ii) f *is locally Lipschitz (and so continuous) on* $int(I)$*;*

(iii) f *is upper Lipschitz (and so upper semicontinuous) at each point* $x \in I$*.*

Next we show that by adding mild conditions at the endpoints a and b, we get full-fledged continuity and Lipschitzianity on the whole I.

Theorem 13 *Let* $f : I = [a,b] \subseteq \mathbb{R}^n \to \mathbb{R}$ *be ultramodular and bounded. Then,*

(i) f *is continuous on* I *if and only if it is lower semicontinuous at both* a *and* b*,*

(ii) f *is globally Lipschitz on* I *if and only if it is calm from below at both* a *and* b*.*

Remark. Theorem 12 has a weaker version if $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is ultramodular and bounded on a convex set C, which is not necessarily an order interval. In this case, we can still prove that f is locally Lipschitz on $Int(C)$.

5.3 Differentiability

Turn now to the differentiability properties of ultramodular functions. The two main ingredients have already been singled out: a bounded ultramodular function is convex along the positive directions and it is locally Lipschitz. Therefore, two differential notions will play a role, the subdifferential and the Clarke differential, both introduced in the Preliminaries.

In order to state the results, we need some notions. For a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, its "positive" subdifferential $\partial_{\pm} f(x)$ at x is given by the set

$$\{m \in \mathbb{R}^n : f(y) \geq f(x) + m(y - x) \;\; \forall y \in A \cap ([x, \infty] \cup [-\infty, x])\}.$$

In the scalar case, $\partial_{\pm} f(x)$ agrees with the standard subdifferential. We denote by $f_i^+(x)$ and $f_i^-(x)$ the one-sided partial derivatives

$$f_i^+(x) = \lim_{t \downarrow 0} \frac{f(x + te^i) - f(x)}{t} \quad \text{and} \quad f_i^-(x) = \lim_{t \uparrow 0} \frac{f(x + te^i) - f(x)}{t}.$$

The one-sided gradients are denoted by $\nabla^+ f(x) = \left(f_1^+(x), ..., f_n^+(x)\right)$ and $\nabla^- f(x) = \left(f_1^-(x), ..., f_n^-(x)\right)$. Clearly, $\nabla^+ f(x) = \nabla^- f(x)$ if and only if f has partial derivatives at x. In this case, the one-sided gradients reduce to the standard gradient $\nabla f(x)$.

Finally, $\partial_i f(x)$ denotes the "partial" subdifferential of f, namely, the subdifferential of the scalar function $f_i(t; x)$.

We begin with two lemmas of independent interest, which describe subdifferentials and directional derivatives of ultramodular functions.

Lemma 14 *Let $f : (a, b) \subseteq \mathbb{R}^n \to \mathbb{R}$ be ultramodular and bounded. For any $x \in (a, b)$ the set $\partial_{\pm} f(x)$ is not empty and compact, and*

$$\partial_{\pm} f(x) = \left[\nabla^- f(x), \nabla^+ f(x)\right] = \prod_{i=1}^{n} \partial_i f(x). \tag{10}$$

Moreover, it holds

$$\nabla^- f(x), \nabla^+ f(x) \in \partial^0 f(x) \subseteq \partial_{\pm} f(x). \tag{11}$$

A noteworthy immediate implication of (10) is that $\partial_{\pm} f(x)$ is a singleton, consisting of the gradient $\nabla f(x)$, if and only if the function f has partial derivatives at x.

Lemma 15 *Let $f : (a,b) \subseteq \mathbb{R}^n \to \mathbb{R}$ be ultramodular and bounded. Then, $f'(x;.)$ is linear over both $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. Moreover,*

$$f'(\overline{x};h) = f^0(\overline{x};h) = \begin{cases} \nabla^+ f(x) \cdot h & \text{if } h \geq 0 \\ \nabla^- f(x) \cdot h & \text{if } h \leq 0. \end{cases}.$$

By Theorem 12(ii), a bounded ultramodular function $f : [a,b] \subseteq \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz on (a,b). Hence, by the Rademacher Theorem, there is a full measure set $A \subseteq [a,b]$ on which f is strictly differentiable. The next theorem, which rests on Lemmas 14 and 15, shows that ultramodular functions have further nice differentiability properties.

Theorem 16 *Let $f : [a,b] \subseteq \mathbb{R}^n \to \mathbb{R}$ be ultramodular and bounded. Then,*

(i) f is strictly differentiable at $x \in (a,b)$ if and only if it has partial derivatives at x;

(ii) if f has partial derivatives on an open subset of $[a,b]$, then it is of class C^1 there.

(iii) if $x \in (a,b)$, then $f'(x;h)$ exists for all $h \in \mathbb{R}^n_+ \cup \mathbb{R}^n_-$, it is increasing in x, and $f'(x;h) = f^0(x;h)$ for all $h \in \mathbb{R}^n_+ \cup \mathbb{R}^n_-$.

Remark. These properties still hold for bounded ultramodular functions defined over an open convex set.

By point (iii), for bounded ultramodular functions the directional derivatives $f'(x;h)$ exist at each x in the positive directions, and they are increasing in x. The next theorem characterizes continuous ultramodular function via the monotonicity of directional derivatives in the canonical directions e_i.

Theorem 17 *Let $f : (a,b) \to \mathbb{R}$ be separately continuous. Then, f is ultramodular if and only if the directional derivative $f'(x;e_i)$ exists for each $i = 1,...,n$ and it is increasing in x.*

As an immediate corollary of this result we have the neat differential characterizations of ultramodularity mentioned in the introduction.

Corollary 18 *Suppose $f : (a,b) \to \mathbb{R}$ has first order partial derivatives. Then, f is ultramodular if and only if its gradient $\nabla f(x)$ is increasing. If, in addition, f has the second order partial derivatives, then f is ultramodular if and only if $\partial^2 f / \partial x_i \partial x_j \geq 0$ for all $i,j = 1,...,n$.*

We close the section by considering some Jensen-type inequalities of ultramodular functions. We have already seen in Proposition 7 that ultramodular functions are convex along the positively oriented directions. By elaborating on this result we can prove the next result, essentially due to Brunk [3].

Proposition 19 *Let $f : [0,1]^n \to \mathbb{R}$ be an ultramodular function. If f is continuous, then*

$$\int (f \circ \gamma)\, d\mu \geq f\left(\int \gamma d\mu\right) \tag{12}$$

for each continuous and non-decreasing curve $\gamma : [0,1] \to [0,1]^n$ and each Borel probability measure μ on $[0,1]$.

An important special case of Proposition 19 is when the measure μ has finite support. In this case, the result says that for each sequence $\{x_i\}_{i=1}^n \subseteq [0,1]^n$ with $x_1 \leq x_2 \leq \cdots \leq x_n$, it holds $\sum_{i=1}^n t_i f(x_i) \geq f\left(\sum_{i=1}^n t_i x_i\right)$ whenever $\{t_i\}_{i=1}^n$ is a sequence of positive numbers such that $\sum_{i=1}^n t_i = 1$.

6 An Application to Convex Games

The differentiability properties of ultramodular functions can be used to establish some useful properties of convex measure games, which we have introduced in Example 5 of Subsection 3.3.

Let $\nu : B(\Sigma) \to \mathbb{R}$ be the Choquet functional associated with a convex measure game $\nu = g \circ P$, namely, $\nu(X) = \int X d(f \circ P)$ for all $X \in B(\Sigma)$. Using the differentiability properties of ultramodular functions, we can establish the following result about the Gateaux differentiability of the Choquet functional ν. It significantly sharpens condition (i) of Theorem 10 in [12] by disposing of all differentiability assumptions on f.

In the statement, $BI(\Sigma)$ denotes the space of all bounded Σ-measurable injective functions, while $G_X = (G_X^1, ..., G_X^n)$ is the vector of cumulative distribution functions G_X^i defined by $G_X^i(q) = P_i(X \geq q)$ for each $q \in \mathbb{R}$ and each $i = 1, ..., n$.

Theorem 20 *Let $\nu = f \circ P$ be a measure game over a Borel space (Ω, Σ), with f continuous and ultramodular on $[0,1]^n$. The associated Choquet functional $\nu : B(\Sigma) \to \mathbb{R}$ is Gateaux differentiable at each $X \in BI(\Sigma)$, and its* *differential is given by*

$$\langle D\nu(X), Y\rangle = \sum_{i=1}^{n} \int f_i^+ (G_X \circ X) Y dP_i. \tag{13}$$

In [12] we established a core representation of convex games based on Gateaux derivatives and we showed that this representation takes a particularly stark form for measure games. Using Theorem 20, we can improve such representation for cores of measure games by disposing of all differentiability assumptions on f.

In order to do so, we need some further notation. Observe that, by the continuity of f, $core(f \circ P)$ consists of countably additive measures that are absolutely continuous with respect to $\overline{P} = \sum_{i=1}^{n} P_i$. Hence, by the Radon-Nikodym Theorem, we can write $core(f \circ P) \subseteq L_1(\Omega, \Sigma, \overline{P})$; in particular, we denote by $\overline{co}$ the closed convex hull in the norm topology of $L_1(\Omega, \Sigma, \overline{P})$.

Given any two $X, Y \in B(\Sigma)$, we write $X \sim Y$ if they are comonotonic, that is, if $[X(\omega) - X(\omega')][Y(\omega) - Y(\omega')] \geq 0$ for any $\omega, \omega' \in \Sigma$. Once restricted to $BI(\Sigma)$, $\sim$ is an equivalence relation, and $BI(\Sigma)/\sim$ denotes the set of equivalence classes determined by $\sim$ on $BI(\Sigma)$. With a slight abuse of notation, $X \in BI(\Sigma)/\sim$ means that X is a representative of one of the equivalence classes determined by $\sim$.

We can now state the result. We omit its proof since we can use the same argument used to prove Theorem 12 in [12], with Theorem 20 now in place of Theorem 10 of [12].

Theorem 21 *Let $\nu = f \circ P$ be a measure game over a Borel space (Ω, Σ), with f continuous and ultramodular on $[0,1]^n$. We have*

$$core(\nu) = \overline{co}\left\{ \sum_{i=1}^{n} \int f_i^+ (G_X \circ X) \frac{dP_i}{d\overline{P}} : X \in BI(\Sigma)/\sim \right\}$$

where $dP_i/d\overline{P}$ is the Radon-Nikodym derivative of P_i with respect to $\overline{P}$.

7 Some Useful Tools

In this section we introduce Bernstein polynomials and linear cores, two objects naturally associated with ultramodular functions. They will play a central role in proving the main properties of ultramodular functions stated in Section 5.

7.1 Bernstein Polynomials

We begin by studying the Bernstein polynomials associated with an ultramodular function; in the Appendix we will use them to prove Theorems 10 and 11.

Given a function $f : [0,1]^n \to \mathbb{R}$ and an n-tuple $m = (m_1, m_2, ..., m_n)$ with non-negative integer components, the Bernstein polynomial $B^m f$ is

$$B^m f(x) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} .... \sum_{k_n=0}^{m_n} f\left(\frac{k_1}{m_1}, ..., \frac{k_n}{m_n}\right) \prod_{i=1}^{n} \binom{m_i}{k_i} x_i^{k_i} (1-x_i)^{m_i - k_i}.$$

If f is continuous, then $B^m f \to f$ uniformly over $[0,1]^n$ as $m_1 \to \infty, m_2 \to \infty, ..., m_n \to \infty$ (see, e.g., [19]). The next result, essentially due to Brunk [3], relates ultramodular functions and its Bernstein polynomials (we omit its proof).

Proposition 22 *If a function $f : [0,1]^n \to \mathbb{R}$ is ultramodular, then all its Bernstein polynomials $B^m f$ are ultramodular on $[0,1]^n$. The converse is true provided f is continuous.*

Of special interest is $B^{(1,\dots,1)} f$, the least-degree Bernstein polynomial associated with $f : [0,1]^n \to \mathbb{R}$. For convenience, set $Bf = B^{(1,\dots,1)} f$. We have

$$Bf(x) = \sum_{k=(k_1,...,k_n) \in ext[0,1]^n} f(k)\, x_1^{k_1} \cdots x_n^{k_n} (1-x_1)^{1-k_1} \cdots (1-x_n)^{1-k_n},$$

which is a polynomial of first degree in each variable (see Example 3 of Section 3). Clearly, all functions that agree on the extreme points of $[0,1]^n$ share the same polynomial Bf.

We have the following useful inequalities (the game ν_f will be introduced momentarily before Lemma 24).

Lemma 23 *Let $f : [0,1]^n \to \mathbb{R}$ be an ultramodular function, with $f(0) = 0$ and locally bounded from below at 0. Then,*

$$f(x) \le B^m f(x) \le Bf(x) \le \overline{Bf}(x) = \int x d\nu_f. \tag{14}$$

for all $x \in [0,1]^n$ and all $m = (m_1, m_2, ..., m_n)$.[2]

[2] Here $\int x d\nu_f$ is a Choquet integral with respect to the finite game ν_f, which will be introduced momentarily before Lemma 24.

In the scalar case, Bf represents the chord joining the points $(0, f(0))$ and $(1, f(1))$, which by convexity lies above the function f. The inequality $f \leq Bf$ can be viewed as the multidimensional version of this simple geometric property of scalar convex functions. The polynomial Bf is ultramodular and it is common to all ultramodular functions agreeing with f on $ext\,[0,1]^n$; in particular, it is Owen's multilinear extension of the game ν_f.

7.2 Linear Core

Definition 3 *Given a function $f : [a,b] \to \mathbb{R}$, its linear core $\mathcal{L}core\,(f)$ is the set:*

$$\{m \in \mathbb{R}^n : m \cdot (b-a) = f(b) - f(a) \ \ and\ m \cdot (x-a) \geq f(x) - f(a) \ \ \forall x \in [a,b]\}$$

Definition 3 is inspired by the standard definition of the core of a game, reported in (2). The linear core consists of all suitably normalized vectors whose associated linear functionals $\langle m, \cdot \rangle$ pointwise dominate the function f. There is an immediate but useful characterization of linear cores in terms of superdifferentials ∂f of f, given by (cf. [11]):

$$\mathcal{L}core\,(f) = \partial f(a) \cap \partial f(b)\,. \tag{15}$$

Before moving on, we give couple of examples.

Example. When $[a,b]$ is the unit order interval $[0,1]^n$ and $f : [0,1]^n \to \mathbb{R}$ is such that $f(0) = 0$, we have

$$\mathcal{L}core\,(f) = \{m \in \mathbb{R}^n : m \cdot x \geq f(x) \ \text{ for all } x \in \mathbb{R}^n \text{ and } m \cdot 1 = f(1)\}\,.$$

For example, consider the ultramodular function $f : [0,1]^n \to \mathbb{R}$ defined by $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $\alpha_i \geq 1$ for each $1 \leq i \leq n$. It is easy to check that

$$\mathcal{L}core\,(f) = \left\{ m \in \mathbb{R}^n_+ : \sum_{i=1}^n m_i = 1 \right\} \tag{16}$$

Example. Consider the convex (but not ultramodular) function $f(x) = \max\{x_1, x_2, ..., x_n\}$. It is easy to see that $\mathcal{L}core\,(f) = \varnothing$.

As shown by the last example, the linear core may be empty. Fortunately, ultramodular functions always have nonempty linear cores. To see why this is the case and, more generally, to study the properties of $\mathcal{L}core\,(f)$, it is important to introduce a class of finite games naturally associated with the functions $f : [0,1]^n \to \mathbb{R}$ having $f(0) = 0$.

Consider the restriction of f on the extreme points of $[0,1]^n$. These points can be identified with elements of the space $\{0,1\}^n$, as well as with the points of the range $\{\delta(E) : E \subseteq \{1,...,n\}\}$ of the vector Dirac measure $\delta = (\delta_1, ..., \delta_n) : 2^{\{1,...,n\}} \to \mathbb{R}^n$, where each δ_i is the Dirac measure concentrated on the singleton $\{i\}$. This makes it possible to associate to each function f the finite game $\nu_f : 2^{\{1,...,n\}} \to \mathbb{R}$ defined by

$$\nu_f(E) = f(\delta(E)),$$

for all subsets E of $\{1,...,n\}$. The game ν_f is the extremal game associated with f.

The next lemma shows two key properties of extremal games. In reading the lemma, recall that since each $m \in core\,(\nu_f)$ can be written as $m = \sum_{i=1}^n m_i\delta_i$, the measure m can be identified as a vector $m = (m_1, ..., m_n) \in \mathbb{R}^n$. Hence, up to this identification, $core\,(\nu_f)$ can be viewed as a subset of $\mathbb{R}^n$ and this is why we use below the symbol $\cong$.

Lemma 24 *Let $f : [0,1]^n \to \mathbb{R}$ be a function on $[0,1]^n$ with $f(0) = 0$. Then:*

(i) the game ν_f is convex if f is supermodular;

(ii) it holds $core\,(\nu_f) \cong \mathcal{L}core\,(f)$ if f is separately convex.

By Corollary 8, bounded ultramodular functions are characterized by being both supermodular and separately convex. Therefore, since cores of convex games are nonempty (see [20]), Lemma 24 implies that the linear cores of bounded ultramodular functions are nonempty. This and other properties will be established in the next result.

Theorem 25 *Let $f : [0,1]^n \to \mathbb{R}$, with $f(0) = 0$, be ultramodular and bounded. Then, $\mathcal{L}core\,(f)$ is nonempty and*

$$core\,(\nu_f) \cong \mathcal{L}core\,(f) = \mathcal{L}core\left(\overline{f}\right). \tag{17}$$

Moreover,

$$\overline{f}(x) = \min_{m \in \mathcal{L}core(f)} m \cdot x = \int x d\nu_f \tag{18}$$

for all $x \in [0,1]^n$, *and* $\nu_f = \nu_{\overline{f}}$.

Example. Consider again the function $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $\alpha_i \geq 1$ for each $1 \leq i \leq n$. Its upper concave envelope $\overline{f}$ is given by the function $\min\{x_1, x_2, ..., x_n\}$, which is concave and supermodular. In particular, $\overline{f}$ has a nonempty linear core given by (16), in accordance with (17).

As a corollary, we have the following useful characterization of the singleton case in terms of Bernstein polynomials.

Corollary 26 *Let* $f : [0,1]^n \to \mathbb{R}$, *with* $f(0) = 0$, *be ultramodular and bounded. Then,* $\mathcal{L}core(f)$ *is a singleton if and only if* Bf *is a linear polynomial.*

Notice that the relation $core(\nu_f) \cong \mathcal{L}core(f)$ makes it possible to describe $\mathcal{L}core(f)$ by means of known results on cores of finite convex games. For example, from some classic results of Shapley [20] on cores of convex games, it follow immediately that $\mathcal{L}core(f)$ is a compact and convex polytope.

We close by observing that, up to some obvious modifications, Theorem 25 holds for any ultramodular function defined on an order interval. Only for convenience we stated it for functions f defined on $[0,1]^n$ and such that $f(0) = 0$. In particular, if we interpret $c = -f$ as a firm's cost function, the nonemptiness of the linear core in the general case is related to some results on natural monopolies of Sharkey and Telser [21] (see also [24, Lemma 2.63]). For example, in their language the nonemptiness of the linear core takes the following form:

Proposition 27 *Let* $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ *be ultramodular and bounded from below, and suppose* A *satisfies property (3). Then,* f *is supportable on* A, *that is, for any* $x, y \in A$ *with* $x \leq y$ *there exists a price vector* $p \in \mathbb{R}^n$ *such that*

$$f(z) \leq f(x) + p \cdot (z - x) \tag{19}$$

for all $z \in [x, y]$ *and* $f(y) = f(x) + p \cdot (y - x)$.

Naturally, the set of supporting price vectors is no longer compact, unless $[x, y]$ is solid.

8 Proofs

Proposition 1. The sufficiency of (3) is obvious. For, given any chain $x \leq y \leq w$, consider the interval $[x, w]$. Let z be the point which makes $\{x, y, z, w\}$ a test quadruple. It is immediate to check that $z \in [x, w]$.

As to the converse, we first prove (iii). Let a and b be the first and last element of A, respectively. Then $A \subseteq [a, b]$. Moreover, as $int(A) \neq \varnothing$, we have $a \ll b$. By normalization, it is then without loss of generality to set $A \subseteq [0,1]^n$, with 0 and 1 in A. Fixed an integer p, we divide the cube $[0,1]^n$ into p^n small cubes of size $1/p$. More precisely, set $A_1 = [0, 1/p]$, $A_2 = [1/p, 2/p]$,....., $A_p = [(p-1)/p, 1]$, which are real intervals. We can decompose the cube $[0,1]^n$ as

$$[0,1]^n = \bigcup_i A^1_{i_1} \times A^2_{i_2} \times .... \times A^n_{i_n},$$

where $C_i = A^1_{i_1} \times A^2_{i_2} \times .... \times A^n_{i_n}$ is a generic cube of size $1/p$ and the multi-index $i = (i_1, i_2, .., i_n)$ runs over $\{1, 2, ..., p\}^n$. All these cubes are labelled by i.

It is easy to see that $C_i \leq C_j$ whenever $i \leq j$. As $int(A) \neq \varnothing$, there exists some integer p such that at least one cube of size $1/p$ such that $C_{\bar{i}} \subseteq A$. Hence, the cubes $(1, 1, ..., 1)$ and $(p, p, ..., p)$ are contained in A. In turn, this implies that all the cubes $(p-1, p, ..., p)$,....., $(p, p, ..., p-1)$ are in A. By iterating, we get that all cubes C_i are contained in A, so that $A = [0,1]^n$. This completes the proof of (iii), since property (3) clearly holds in $[0,1]^n$.

Next, consider (i). Let $x, y \in A$ with $x \leq y$. As A is open, there exist two points x_1 and y_1 in A such that $x_1 \leq x \leq y \leq y_1$, with $x_1 \ll y_1$. By what we just proved, $[x_1, y_1] \subseteq A$.

Finally, we prove (ii). Let $x, y \in A$ with $x \leq y$. Assume first that $x \ll y$. By assumption, there are two sequences $\{a_n\}_n$ and $\{b_n\}_n$ such that $a_n \to x$, $b_n \to y$, with $a_n, b_n \in int(A)$. Since $x \ll y$, we can assume that $a_n \ll b_n$ for each n. On the other hand, as A is ultramodular, the set $int(A)$ is ultramodular as well. We deduce that $[a_n, b_n] \subseteq int(A)$. Now, we select subsequences of a_n and b_n in such a way that they converge monotonically to x and y, coordinatewise (it is easy to see that this always possible). Denote them again by a_n and b_n. It is simple to prove that $[a_n, b_n] \subseteq int(A)$ implies $(x, y) \subseteq int(A)$. Hence, $[x, y] \subseteq cl(int(A)) = A$ and this proves the theorem, as long as $x \ll y$. The extension to the case $x \leq y$ is easy. ■

Proposition 3. The condition is clearly necessary. Let us prove that it suffices. Let $x = (x_i) \leq y = (y_i) \in A$, and $h = (h_i) \geq 0$. We can write

$$\begin{aligned} & f(x+h) - f(x) \\ = & \sum_{i=1}^{n} f(x_1 + h_1, ..., x_i + h_i, x_{i+1}, ..., x_n) - f(x_1 + h_1, ..., x_{i-1} + h_{i-1}, x_i, ..., x_n) \\ \leq & \sum_{i=1}^{n} f(y_1 + h_1, ..., x_i + h_i, y_{i+1}, ..., y_n) - f(y_1 + h_1, ..., y_{i-1} + h_{i-1}, x_i, y_i, ..., y_n) \\ \\ \leq & \sum_{i=1}^{n} f(y_1 + h_1, ..., y_i + h_i, y_{i+1}, ..., y_n) - f(y_1 + h_1, ..., y_{i-1} + h_{i-1}, y_i, ..., y_n) \\ = & f(y+h) - f(y), \end{aligned}$$

where the first inequality is based on the property on increasing differences, and the second one holds as the function is separately ultramodular. ■

Proposition 5. (i) is obvious. (ii) Set $\Delta_h f(x) = f(x+h) - f(x)$. We want to prove that

$$\Delta_h (fg)(y) - \Delta_h (fg)(x) \geq 0$$

if $x \leq y$ and $h \geq 0$. Some tedious algebra yields

$$\begin{aligned} & \Delta_h (fg)(y) - \Delta_h (fg)(x) \\ & = g(y+h) [\Delta_h f(y) - \Delta_h f(x)] + \Delta_h f(x) [g(y+h) - g(x+h)] \\ & + [f(y) - f(x)] \Delta_h g(x) + f(y) [\Delta_h g(y) - \Delta_h g(x)], \end{aligned}$$

which is non-negative under our assumptions.

(iii) Let $\{x, y, z, w\} \subseteq A$ be a test quadruple and let z' be such that $z' - g(x) = g(w) - g(y)$. Since B is ultramodular, $z' \in B$. Moreover, $g(z) \leq z'$ since g is ultramodular. By the ultramodularity and monotonicity of h, we then have

$$h(g(z)) - h(g(x)) \leq h(z') - h(g(x)) \leq h(g(w)) - h(g(y)),$$

as desired. ■

Lemma 6. Let us prove that the function

$$R(x) = f(x) - \sum_{i=1}^{n} f_i(x_i; \overline{x}) + (n-1) f(\overline{x})$$

is increasing in $[\overline{x}, +\infty]$ and decreasing on $[-\infty, \overline{x}]$, where $f_i(t; \overline{x}) = f(t, \overline{x}_{\widehat{i}})$.

Note that, by construction, $R(y) = 0$ for all points $y = (y_i, \overline{x}_{\widehat{i}})$ and for each index i. Further, R is necessarily supermodular, as the separable function $\sum_{i=1}^{n} f_i(x_i; \overline{x})$ does not affect the supermodularity of f. Consequently, R has the increasing increment property. Fix two points $\overline{x} \leq x \leq y$. As usual we can write

$$\begin{aligned} & R(y) - R(x) \\ = & \sum_{i=1}^{n} R(y_1, .., y_i, x_{i+1}, .., x_n) - R(y_1, .., y_{i-1}, x_i, .., x_n) \end{aligned}$$

By repeated applications of the property of increasing increments, we have

$$\begin{aligned} & R(y) - R(x) \\ \geq & \sum_{i=1}^{n} R(\overline{x}_1, .., y_i, \overline{x}_{i+1}, .., \overline{x}_n) - R(\overline{x}_1, .., \overline{x}_{i-1}, x_i, .., \overline{x}_n) = 0 \end{aligned}$$

Likewise, by using the points $y \leq x \leq \overline{x}$, the same argument leads to $R(y) \geq R(x)$ and our claim is proved.■

Proposition 7. (ii) trivially implies (i). To prove that (i) implies (iii), assume f to be ultramodular and bounded from below in a neighborhood of a. Thanks to the minorization stated in Lemma 6, we have

$$f(x) \geq \sum_{i=1}^{n} f_i(x_i) - (n-1) f(a) \tag{20}$$

where each $f_i(x_i)$, defined on the interval $[a_i, b_i]$, is ultramodular and locally bounded at a_i. Clearly, each $f_i(x_i)$ is mid-convex. Thus, by Bernstein and Doetsch's theorem (see [18, p. 219]), the scalar functions $f_i(x_i)$ are convex and continuous on (a_i, b_i). In particular, $f_i(x_i)$ are bounded from below on $[a_i, b_i]$, provided b_i is finite. If we now consider any scalar function $\varphi(t) =$

$f(m+tn)$, with $t \in \mathbb{R}$ and $m \in \mathbb{R}^n$, $n \in \mathbb{R}^n_+$, it is ultramodular and, consequently, mid-convex. From (20), we have

$$\varphi(t) \geq \sum_{i=1}^{n} f_i(m_i + tn_i) - (n-1) f(a)$$

Thus $\varphi(t)$ is bounded from below on some interval. The previous theorem can be again invoked, and thus $\varphi(t)$ is convex.

It remains to prove that (iii) implies (ii). If f is separately convex, $f_i(x_i)$ are locally bounded from below at a_i. Hence, minorization (20) implies that $f(x)$ is locally bounded from below at a. A simple application Lemma 23 delivers (ii). ■

Theorem 10. Assume f is positively homogeneous. The function f is bounded from below at 0. For, (8) with $\overline{x} = 0$ becomes

$$f(x) \geq \sum_{i=1}^{n} f_i(x_i),$$

where all f_i are homogeneous, i.e., linear. This implies that f is bounded from below on $[0,1]^n$ and, in turn, that f is bounded. By Theorem 12, f is upper semicontinuous on $[0,1]^n$. On the other hand, f is concave since it is superadditive and linearly homogeneous. By Lemma 23 and Theorem 25, it holds $f \leq Bf \leq \overline{f}$. Therefore, since $\overline{f}$ is the least upper semicontinuous concave function greater than f, we have $f \leq Bf \leq \overline{f} \leq f$. Then, $f = Bf$, and so, f being positively homogeneous, the polynomial Bf must be linear. ■

Theorem 11. Assume that f is not linear. We want to show that $f(x) + f(1-x) \neq f(1)$ for all $x \in (0,1)^n$. Consider the Bernstein polynomial $B^m f$, with $m = (m_1, m_2, ...m_n)$. We have

$$B^m f(x) = \sum_{k} f(k) A_k(x),$$

where $k = (k_1/m_1, k_2/m_2, ..., k_n/m_n)$ and

$$A_k(x) = \prod_{i=1}^{n} \binom{m_i}{k_i} x_i^{k_i} (1-x_i)^{m_i - k_i}.$$

Clearly,

$$B^m f(1-x) = \sum_k f(1-k) A_k(x),$$

and so

$$B^m f(x) + B^m f(1-x) = \sum_k [f(1-k) + f(k)] A_k(x). \tag{21}$$

Using (21), we first show that there exists some $k \in [0,1]^n$ with rational coordinates such that $f(k) + f(1-k) < f(1)$. In fact, suppose *per contra* that $f(k) + f(1-k) = f(1)$ for all such $k \in [0,1]^n$. By Eq. (21), $B^m f(x) + B^m f(1-x) = f(1)$, since $\sum_k A_k(x) = 1$. Since by Proposition 22 $B^m f$ is ultramodular, it follows that $B^m f$ is linear. Hence, by Proposition 9, both $B^m f$ and $-B^m f$ are ultramodular. On the other hand, since f is continuous, again by Proposition 22 we have that both f and $-f$ are ultramodular. In turn, this implies that f is linear, a contradiction. We conclude that there exists some $k \in [0,1]^n$ with rational coordinates such that $f(k) + f(1-k) < f(1)$. We denote it by $\overline{k}$.

Since f is ultramodular, we have $f(1-k) + f(k) \leq f(1)$ for all $k \in [0,1]^n$ with rational coordinates. As $f(\overline{k}) + f(1-\overline{k}) < f(1)$, Eq. (21) then implies

$$B^m f(x) + B^m f(1-x) < B^m f(1)$$

for all $x \in (0,1)^n$. Hence, by Eq. (14), we conclude that

$$f(x) + f(1-x) \leq B^m f(x) + B^m f(1-x) < B^m f(1) = f(1)$$

for all $x \in (0,1)^n$, as desired. ■

Theorem 12. (i) W.l.o.g., consider an order interval $[a_1, b_1] \subseteq int(I)$. Hence, there exists some $\varepsilon > 0$ such that $a_1 \geq a + \varepsilon 1$ and $b_1 \leq b - \varepsilon 1$. Take two elements $x, y \in [a_1, b_1]$, with $x \leq y$ and $x \neq y$. Consider the scalar convex function

$$\varphi(t) = f\left[x + |y-x|_1^{-1} t (y-x)\right]$$

The domain of this function is a real interval $J \supseteq [-\varepsilon, |y-x|_1 + \varepsilon]$. In fact, it suffices to observe that

$$x - \varepsilon |y-x|_1^{-1} (y-x) = x - \varepsilon u,$$

with $u \geq 0$ and $|u|_1 = 1$. Consequently, $x - \varepsilon u \geq a_1 - \varepsilon u \geq a$. The same argument applies to the point $t = |y - x|_1 + \varepsilon$. To conclude, the convex function $\varphi(t)$ is Lipschitz on the interval $[0, |y - x|_1]$ with Lipschitz constant $K = (M - m)/\varepsilon$ (see, e.g., [18, p. 4]), where M and m are upper and lower bounds of f on I. Hence,

$$|f(y) - f(x)| = |\varphi(|y - x|_1) - \varphi(0)| \leq K |y - x|_1 .$$

Now, consider any two points $x, y \in [a_1, b_1]$. We can write

$$\begin{aligned} |f(y) - f(x)| &= |f(x \vee y) - f(x) + f(y) - f(x \vee y)| \\ &\leq |f(x \vee y) - f(x)| + |f(y) - f(x \vee y)| \\ &\leq K[|x \vee y - x|_1 + |x \vee y - y|_1] = K |y - x|_1 , \end{aligned}$$

as desired.

(ii) It is an immediate consequence of (i).

(iii) Take a point $x_0 \in I$ and set $I^- = [a, x_0]$ and $I^+ = [x_0, b]$. In view of Proposition 27, there exists $m \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) \leq m \cdot (x - x_0)$$

for all $x \in I^-$. This implies in turn that

$$f(x) - f(x_0) \leq K |x - x_0|_1 \tag{22}$$

holds for all $x \in I^-$ and for some positive scalar K. The same argument can be used for the interval I^+. Summarizing, Eq. (22) holds for all $x \in I^- \cup I^+$ and for some positive constant K. Now, pick any $x \in I$; we have

$$\begin{aligned} f(x \vee x_0) - f(x_0) &\leq K |x \vee x_0 - x_0|_1 , \\ f(x \wedge x_0) - f(x_0) &\leq K |x \wedge x_0 - x_0|_1 . \end{aligned}$$

Summing up and recalling that $f(x) + f(x_0) \leq f(x \vee x_0) + f(x \wedge x_0)$, we then obtain

$$f(x) - f(x_0) \leq K[|x \vee x_0 - x_0|_1 + |x \wedge x_0 - x_0|_1] = K |x - x_0|_1 ,$$

which is the desired result. ■

Theorem 13. We begin by proving point (i). According to point (iv) of Theorem 12, it suffices to show that f is lower semicontinuous on I, as long as f is lower semicontinuous at a and b. Fix a point $\overline{x} \in I$. Given a sequence $y_n \to \overline{x}$, with $y_n \leq \overline{x}$, by ultramodularity,

$$f(b + y_n - \overline{x}) - f(b) \leq f(y_n) - f(\overline{x})$$

Thus $\liminf_n f(y_n) \geq f(\overline{x})$. Suppose now $x_n \to \overline{x}$. Consider the sequence $x_n^- = x_n \wedge \overline{x}$. Clearly, by ultramodularity

$$\begin{aligned} f(x_n) - f\left(x_n^-\right) &\geq f\left(a + x_n - x_n^-\right) - f(a) \\ f(x_n) &\geq f\left(x_n^-\right) + f\left(a + x_n - x_n^-\right) - f(a) \end{aligned}$$

Taking the liminf, we have easily $\liminf_n f(x_n) \geq f(\overline{x})$, given that $x_n^- \to \overline{x}$. This proves point (i). As to (ii), suppose

$$f(x) \geq f(a) - \gamma_1 |x - a|_1 \tag{23}$$

for x close to a, and

$$f(x) \geq f(b) - \gamma_2 |x - b|_1 \tag{24}$$

for x close to b. As a first step, we want to prove that

$$-\gamma_1 |x - y|_1 \leq f(y) - f(x) \leq \gamma_2 |x - y|_1 \tag{25}$$

holds for all $x, y \in I$, with $x \leq y$. If we consider the "partial" functions $f_i(x_i; a)$ that appear in (8) of Lemma 6, Eq. (23) implies that $f_i(x_i; a) \geq f(a) - \gamma_1 (x_i - a_i)$ near a_i. As $f_i(x_i; a)$ is convex, the previous one must hold for all $a_i \leq x_i \leq b_i$. By (8), in turn this implies that (23) holds for all $x \in I$. A similar argument shows that (24) holds for all $x \in I$. Consider the two functions $\rho_1 = f(x) + \gamma_1 |x - a|_1 - f(a)$ and $\rho_2 = f(x) + \gamma_2 |x - b|_1 - f(b)$. We have $\rho_1(a) = 0$ and $\rho_2(b) = 0$; moreover, both functions are ultramodular. Hence, ρ_1 is increasing and ρ_2 is decreasing on I. This leads to (25). To conclude, let $x, y \in I$ and set $h = x - y$. We have

$$\begin{aligned} f(y) - f(x) &= f(x + h) - f(x) = f\left(x + h^+ - h^-\right) - f\left(x + h^+\right) \\ +f\left(x + h^+\right) - f(x) &\leq \gamma_1 \left|h^-\right|_1 + \gamma_2 \left|h^+\right|_1 \leq \gamma |h|_1, \end{aligned}$$

with $\gamma = \max\{\gamma_1, \gamma_2\}$. ■

Lemma 14. Fix a point $\overline{x} \in (a, b)$. Each scalar convex function $f_i(x_i, \overline{x})$ is defined on a complete neighborhood of $\overline{x}_i$. Its subdifferential is thus non-empty and is given by

$$\partial f_i(x_i, \overline{x}) = \partial_i f(\overline{x}) = \left[f_i^-(\overline{x}), f_i^+(\overline{x})\right]$$

for all i. This implies that

$$\prod_{i=1}^{n}\left[f_i^{-}(\overline{x}),f_i^{+}(\overline{x})\right]\subseteq\partial\varphi(\overline{x}),$$

where φ is the convex function $\sum_{i=1}^{n} f_i(x_i;\overline{x})$. By Lemma 6, we have

$$\prod_{i=1}^{n}\left[f_i^{-}(\overline{x}),f_i^{+}(\overline{x})\right]\subseteq\partial_{\pm}f(\overline{x}). \tag{26}$$

Conversely, if $m\in\partial_{\pm}f(\overline{x})$, then $m_i\in\partial f_i(x_i,\overline{x})=\left[f_i^{-}(\overline{x}),f_i^{+}(\overline{x})\right]$. Therefore, the inclusion in (26) is an equality. The relation (11) will be a consequence of the next Lemma. ■

Lemma 15. The proof requires several steps.

Step 1. We first check that $f'(\overline{x};.)$ is ultramodular over the positive cone $\mathbb{R}^n_+$. Take $h,k,h_1\in\mathbb{R}^n_+$, with $h_1\geq h$. From

$$f(\overline{x}+t(h+k))-f(\overline{x}+th)\leq f(\overline{x}+t(h_1+k))-f(\overline{x}+th_1),$$

we have

$$\begin{aligned}&[f(\overline{x}+t(h+k))-f(\overline{x})]-[f(\overline{x}+th)-f(\overline{x})]\\ \leq\ &[f(\overline{x}+t(h_1+k))-f(\overline{x})]-[f(\overline{x}+th_1)-f(\overline{x})].\end{aligned}$$

Dividing by t and letting $t\downarrow 0$, we have

$$f'(\overline{x};h+k)-f'(\overline{x};h)\leq f'(\overline{x};h_1+k)-f'(\overline{x};h_1),$$

which shows that $f'(\overline{x};.)$ is ultramodular. As $f'(\overline{x};.)$ is homogeneous, by Theorem 10 we have that $f'(\overline{x};.)$ is linear. To prove the same result on the negative cone, it suffices to consider the ultramodular function $\varphi(x)=f(b-x)$. Clearly, $\varphi'(b-\overline{x};h)=f'(\overline{x};-h)$ if h is positive.

Step 2. Now we prove that $f'(\overline{x};.)$ agrees with $f^0(\overline{x};.)$ both on $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. By the convexity of f along positive directions, we have

$$\frac{f(x+th)-f(x)}{t}\leq f(x+h)-f(x)$$

for all x, all $h \geq 0$ and $0 < t < 1$. Taking the limsup as $x \to \overline{x}$ and $t \downarrow 0$,

$$f^0(\overline{x}; h) \leq \lim_{x \to \overline{x}} [f(x+h) - f(x)] = f(\overline{x}+h) - f(\overline{x}).$$

By homogeneity,

$$tf^0(\overline{x}; h) = f^0(\overline{x}; th) \leq f(\overline{x}+th) - f(\overline{x}).$$

Dividing by t and taking the limit, we get $f^0(\overline{x}; h) \leq f'(\overline{x}; h)$, which implies $f^0(\overline{x}; h) = f'(\overline{x}; h)$. The proof for $h \leq 0$ is similar.

Step 3. Steps 1 and 2 imply

$$f^0(\overline{x}; h) = \begin{cases} \sum_{i=1}^n f_i^+(\overline{x}) h_i = \nabla^+ f(\overline{x}) \cdot h & \text{if } h \geq 0 \\ \sum_{i=1}^n f_i^-(\overline{x}) h_i = \nabla^- f(\overline{x}) \cdot h & \text{if } h \leq 0 \end{cases}.$$

Let $m \in \partial^0 f(\overline{x})$. This means $m \cdot h \leq f^0(\overline{x}; h)$ for all $h \in \mathbb{R}^n$. Setting $h = \pm e_i$, we have $f_i^-(\overline{x}) \leq m_i \leq f_i^+(\overline{x})$ for all i. Hence $\partial^0 f(\overline{x}) \subseteq \partial^{\pm} f(\overline{x})$. By the well known duality relation, we have

$$f^0(\overline{x}; h) = \max_{m \in \partial^0 f(\overline{x})} m \cdot h.$$

Set $h = \sum_{i=1}^n e_i$. Then,

$$\sum_{i=1}^n f_i^+(\overline{x}) = \max_{m \in \partial^0 f(\overline{x})} \sum_{i=1}^n m_i.$$

As $m_i \leq f_i^+(\overline{x})$, the previous relation implies that $\nabla^+ f(\overline{x}) \in \partial^0 f(\overline{x})$. Likewise, by using $h = -\sum_{i=1}^n e_i$, we obtain that $\nabla^- f(\overline{x}) \in \partial^0 f(\overline{x})$. ■

Theorem 16. (i) Assume that at the point $\overline{x}$ all the partial derivatives exist. This implies that $\partial_{\pm} f(\overline{x}) = \{\nabla f(\overline{x})\}$. Hence, $\partial f(\overline{x}) = \{\nabla f(\overline{x})\}$. Therefore, f is Frechet differentiable at $\overline{x}$ and the derivative is strict. The converse is trivial.

(ii) As f is Lipschitz, by the Rademacher Theorem, it is differentiable almost everywhere. Thus the desired property follows.

(iii) It is enough to consider $int(I)$. By Theorem 12, f is locally Lipschitz and, by Lemma 14, $\partial f(x)$ is a singleton for each $x \in int(I)$. Hence, by [5, p. 33] f is continuously differentiable on $int(I)$. ■

Theorem 17. Given $x \in (a, b)$, set $f_i(t) = f(x_{-i}, t)$ for each $t \in (a_i, b_i)$. As

$$f_i^+(t) = \lim_{h \downarrow 0} \frac{f(x_{-i}, t+h) - f(x_{-i}, t)}{h} = f'((x_{-i}, t); e_i),$$

each f_i^+ is increasing. By hypothesis, each f_i is also continuous, and so it is convex. Therefore,

$$f_i(x_i + h) - f_i(x_i) = h \int_0^1 f_i^+(x_i + th)\, dt.$$

By using the decomposition of the proof of Proposition 3, with $x \leq y$ and $h \geq 0$, we get:

$$\begin{aligned}
& f(x+h) - f(x) \\
= & \sum_{i=1}^n f(x_1 + h_1, ..., x_i + h_i, x_{i+1}, ..., x_n) - f(x_1 + h_1, ..., x_{i-1} + h_{i-1}, x_i, ..., x_n) \\
= & \sum_{i=1}^n h_i \int_0^1 f_i^+(x_1 + h_1, ..., x_i + th_i, x_{i+1}, ..., x_n)\, dt \\
\leq & \sum_{i=1}^n h_i \int_0^1 f_i^+(y_1 + h_1, ..., y_i + th_i, y_{i+1}, ..., y_n)\, dt \\
= & \sum_{i=1}^n f(y_1 + h_1, ..., y_i + h_i, y_{i+1}, ..., y_n) - f(y_1 + h_1, ..., y_{i-1} + h_{i-1}, y_i, ..., y_n) \\
= & f(y+h) - f(y),
\end{aligned}$$

as desired. ■

Proposition 19. Assume first that μ has finite support. In this case, Eq. (12) holds provided $\sum_{i=1}^n t_i f(x_i) \geq f\left(\sum_{i=1}^n t_i x_i\right)$ for each sequence $\{x_i\}_{i=1}^n \subseteq [0,1]^n$ with $x_1 \leq x_2 \leq \cdots \leq x_n$, and whenever $\{t_i\}_{i=1}^n$ is a sequence of positive numbers such that $\sum_{i=1}^n t_i = 1$. This is easily proved by induction (for $n = 2$ it is obviously true). We conclude that Eq. (12) holds when μ has finite support. On the other hand, this class of probability measures form a dense subset of the set of all Borel probability measures on $[0,1]$ w.r.t. the standard vague topology. By Proposition 13, f is continuous. Hence, both $f \circ \gamma$ and

γ are continuous functions, and so it is easy to see that Eq. (12) holds for any Borel probability measures on $[0,1]$. ■

Theorem 20. The map G_X can be viewed as a curve in $R(P)$ with endpoints $\underline{0}$ and $\underline{1}$. We denote by $C_X \subseteq R(P)$ its range. The following Claim is proved in [12].

Claim 1. Let $\nu = f \circ P$ be a measure game with f ultramodular and lower semicontinuous at 0 and $P(\Omega)$. If $X \in BI(\Sigma)$ and there exists a locally integrable function $\rho_X : \mathbb{R}_+ \to \mathbb{R}$ such that

$$f(y) = \int_0^{|y|_1} \rho_X(t)\, dt \tag{27}$$

for all $y \in C_X$, then the Gateaux derivative is given by

$$\langle D\nu(X), Y\rangle = \int_\Omega \rho_X(|G_X|_1 \circ X)\, Y d\overline{P}, \tag{28}$$

where $|G_X|_1 = \sum_{i=1}^n G_X^i$.

The curve $q \to G_X(q)$ is continuous since X is injective and each P_i is non-atomic. Consider the arc-length parametrization $\gamma : [0,n] \to C_X$, with $|\gamma(t)|_1 = t$ for each $t \in [0,n]$; that is, γ is the inverse of the map $x \to |x|_1$ restricted to C_X. The arc-length parametrization $\gamma : [0,n] \to C_X$ is an isometry. Since f is Lipschitz, the function $f \circ \gamma$ is Lipschitz over $[0,n]$. To deal with $f \circ \gamma$ we need the following Claim, which exploits the differentiability properties of ultramodular functions.

Claim 2. Let $\gamma : [0,1] \to [0,1]^n$, with $\gamma(0) = \underline{0}$ and $\gamma(1) = \underline{1}$, be an increasing Lipschitz curve. If $f : [0,1]^n \to \mathbb{R}$ is continuous ultramodular, then $f \circ \gamma : [0,1] \to \mathbb{R}$ is absolutely continuous. Moreover,

$$f(\gamma(t)) = \int_0^t p(s) \cdot \gamma'(s)\, ds \tag{29}$$

where $p(s) \in \partial_\pm f(\gamma(s))$. For instance, we can set $p(s) = \nabla^+ f(\gamma(s))$.

Proof of the Claim. Let us prove the first statement. It is trivial if $\gamma(t) \in (0,1)^n$ for all t. Actually, f is locally Lipschitz on $(0,1)^n$. Therefore, $f(\gamma(t))$ is Lipschitz and, hence, absolutely continuous.

Suppose now that $\gamma(t)$ fails to lie in the interior of $[0,1]^n$. The crucial assumption is that $\gamma(t)$ must be non-decreasing. Consider the two functions $Z, U : [0,1]^n \to \{0,1,...,n\}$ defined as $Z(x) =$ number of zeros of x, $U(x) =$ number of ones of x. The function $Z \circ \gamma$ decreases, while the function $U \circ \gamma$ increases. Consequently, there is a finite sequence $0 = t_0 < t_1 < .... < t_{k-1} < t_k = 1$, such that on each open interval (t_r, t_{r+1}), $r = 0, ..., k-1$, the two functions $Z \circ \gamma$ and $U \circ \gamma$ are constant. That means that $\gamma(t)$ lies on a face of $[0,1]^n$ when $t \in (t_r, t_{r+1})$. Clearly, f is locally Lipschitz over any open face. To conclude, $f(\gamma(t))$ is locally Lipschitz on $[0,1] \setminus \{t_0, t_1, ..., t_k\}$ and continuous over $[0,1]$. This is enough to infer that $f(\gamma(t))$ is absolutely continuous over $[0,1]$.

We now prove the last statement. Assume that $\gamma(t)$ is differentiable at a point $\bar{t}$ in $[0,1]$. We have

$$\begin{aligned}\limsup_{h\downarrow 0} \frac{f\left(\gamma\left(\bar{t}+h\right)\right) - f\left(\gamma\left(\bar{t}\right)\right)}{h} &= \limsup_{h\downarrow 0} \frac{f\left(\gamma\left(\bar{t}\right) + h\gamma'\left(\bar{t}\right) + ho\left(1\right)\right) - f\left(\gamma\left(\bar{t}\right)\right)}{h} \\ &\leq f^0\left(\gamma\left(\bar{t}\right); \gamma'\left(\bar{t}\right)\right) = \nabla^+ f\left(\gamma\left(\bar{t}\right)\right) \cdot \gamma'\left(\bar{t}\right).\end{aligned}$$

On the other hand,

$$\frac{f\left(\gamma\left(\bar{t}+h\right)\right) - f\left(\gamma\left(\bar{t}\right)\right)}{h} \geq \nabla^+ f\left(\gamma\left(\bar{t}\right)\right) \cdot \left(\frac{\gamma\left(\bar{t}+h\right) - \gamma\left(\bar{t}\right)}{h}\right)$$

for $h \geq 0$. Hence,

$$\liminf_{h\downarrow 0} \frac{f\left(\gamma\left(\bar{t}+h\right)\right) - f\left(\gamma\left(\bar{t}\right)\right)}{h} \geq \nabla^+ f\left(\gamma\left(\bar{t}\right)\right) \cdot \gamma'\left(\bar{t}\right).$$

We infer that the right derivative of $f(\gamma(t))$ equals $\nabla^+ f(\gamma(t)) \cdot \gamma'(t)$, provided γ is differentiable at t. Now, as $f(\gamma(t))$ is absolutely continuous and γ is Lipschitz, there is a full measure subset Ω of $[0,1]$ such that

$$\frac{d}{dt} f\left(\gamma\left(t\right)\right) = \nabla^+ f\left(\gamma\left(t\right)\right) \cdot \gamma'\left(t\right).$$

The same argument, obtained by calculating the left derivative, leads to the formula

$$\frac{d}{dt} f\left(\gamma\left(t\right)\right) = \nabla_- f\left(\gamma\left(t\right)\right) \cdot \gamma'\left(t\right)$$

which is true over a set of full measure. We conclude that $\nabla^- f\left(\gamma\left(\overline{t}\right)\right) = \nabla^+ f\left(\gamma\left(\overline{t}\right)\right)$ almost everywhere and, in turn, Eq. (29). This proves the Claim.

By Claim 2, $f \circ \gamma$ is absolutely continuous and $f(\gamma(t)) = \int_0^t \nabla^+ f(\gamma(s)) \cdot \gamma'(s)\, ds$ for all $t \in [0, n]$.[3] If we set $\rho(t) = \nabla^+ f(\gamma(t)) \cdot \gamma'(t)$, we have $f(x) = \int_0^{|x|_1} \rho(u)\, du$ for all $x \in C_X$. Plugging it into (28), we get that the derivative is

$$\langle D\nu(X), Y\rangle = \int \left\langle \nabla^+ f(G_X \circ X), \gamma'(|G_X|_1 \circ X)\right\rangle Y d\overline{P}. \tag{30}$$

Eq. (30) holds for all continuous and ultramodular f. Fix now X and set $f_i(x) = x_i$, with $i = 1, ..., n$. The corresponding game is $\nu = f_i \circ P = P_i$. Eq. (30) becomes

$$\int Y dP_i = \int \gamma_i'(|G_X|_1 \circ X)\, Y d\overline{P}$$

where $\gamma_i'(t)$ is the i^{th} component of the vector $\gamma'(t)$. If we set $Y = 1_E$, where E is any element of Σ, we have

$$P_i(E) = \int_E \gamma_i'(|G_X|_1 \circ X)\, d\overline{P}. \tag{31}$$

As P_i is absolutely continuous with respect to $\overline{P}$, by (31), $\gamma_i'(|G_X| \circ X)$ is the Radon-Nikodym derivative $dP_i/d\overline{P}$. Consequently, getting back to (30), we can write

$$\begin{aligned}\langle D\nu(X), Y\rangle &= \sum_{i=1}^n f_i^+(G_X \circ X)\, \gamma_i'(|G_X|_1 \circ X)\, Y d\overline{P} \\ &= \sum_{i=1}^n \int f_i^+(G_X \circ X) \frac{dP_i}{d\overline{P}} Y d\overline{P} = \sum_{i=1}^n \int f_i^+(G_X \circ X)\, Y dP_i,\end{aligned}$$

which is the desired result. ■

Lemma 23. We start with a convex function $\varphi(t)$ of a single variable in $[0, 1]$. We claim that

$$\varphi \leq B^n\varphi \leq B^1\varphi$$

[3] Though we picked $\nabla^+ f(\gamma(s))$, the same is true for $\nabla^- f(\gamma(s))$.

on $[0,1]$. Actually, all functions $B^n\varphi$ are convex and agree with φ at 0 and 1. By convexity, the cord $B^1\varphi$ lies above all $B^n\varphi$. It remains to check that $B^n\varphi \geq \varphi$. This is a more or less known property of univariate Bernstein polynomials. [1] provided the following representation of the remainders of Bernstein approximation

$$\varphi(t) = (B^n\varphi)(t) - \frac{t(1-t)}{n}[\varphi; t_1, t_2, t_3]$$

where $[\varphi; t_1, t_2, t_3]$ represents the divided difference of φ evaluated at some points $t_i \in [0,1]$, $i = 1, 2, 3$. Consequently, $B^n\varphi \geq \varphi$ as the divided difference are non-negative, provided φ is convex.

To consider the multidimensional case, we need a piece of notation. Let L be a linear map $L : \mathcal{F}[0,1] \to \mathcal{C}[0,1]$, where $\mathcal{F}[0,1]$ is the space of all functions defined on $[0,1]$ and $\mathcal{C}[0,1]$ is the linear subspace of the continuous ones. Given $i \in \{1, 2, ..., n\}$, by $L_i : \mathcal{F}[0,1]^n \to \mathcal{C}[0,1]^n$ we mean L applied to f viewed as a function of the i-th variable x_i, with the other variables x_j, $j \neq i$, held fixed. It is easy to check (see [19]) that

$$B^{(m_1,m_2,...,m_n)} = B_n^{m_n}.....B_2^{m_2}B_1^{m_1}, \tag{32}$$

where B^{m_i} are the univariate Bernstein operators.

Let $f : [0,1]^n \to \mathbb{R}$ be ultramodular and bounded from below near 0. By Proposition 7, f is separately convex. We then have

$$f \leq B_i^{m_i} f \leq B_i^1 f$$

for each $i = 1, ..., n$ and all integers m_i. By the monotonicity property of Bernstein operators, $f \leq B_1^{m_1} f$ implies $f \leq B_2^{m_2} f \leq B_2^{m_2} B_1^{m_1} f$. By proceeding iteratively, we have at last

$$f \leq B_n^{m_n}.....B_2^{m_2}B_1^{m_1} f = B^{(m_1,m_2,...,m_n)} f.$$

In particular, $f \leq B^{(1,1,...,1)} f$. On the other hand, all Bernstein polynomials $B^{(m_1,m_2,...,m_n)} f$ agree with f at the extremal points of $[0,1]^n$. Hence, we have $B^{(m_1,m_2,...,m_n)} f \leq B^{(1,1,...,1)} f$ and this concludes the first part of our claim. The rest follows from Theorem 25. ■

Lemma 24. (i) If f is supermodular, then, for all $x, y \in [0,1]^n$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

Let $A, B \subseteq \{1, ...n\}$ and let $\delta : 2^{\{1,...,n\}} \rightarrow \mathbb{R}^n$ be the vector Dirac measure. Then, $\delta(A)$ and $\delta(B)$ can be viewed as extreme points of $[0,1]^n$. Hence,

$$\begin{aligned} \nu_f(A \cup B) + \nu_f(A \cap B) &= f(\delta(A) \vee \delta(B)) + f(\delta(A) \wedge \delta(B)) \\ &\geq f(\delta(A)) + f(\delta(B)) = \nu_f(A) + \nu_f(B), \end{aligned}$$

as desired.

(ii) It is immediate to see that $\mathcal{L}core(f) \subseteq core(\nu_f)$. To prove the converse inclusion, let $m \in core(\nu_f)$. The element m can be identified as a vector $m \in \mathbb{R}^n$. In this way, the function $f(x) - m \cdot x$ is non-positive at the extreme points of $[0,1]^n$ and $f(1) = m \cdot 1$. We want to prove that $m \in core(f)$, namely $f(x) - m \cdot x \leq 0$ for all x in $[0,1]^n$. Suppose *per contra* that this not true, that is, suppose there exists $\overline{x} \in [0,1]^n$, not extremal, such that $f(\overline{x}) - m \cdot \overline{x} > 0$. The point $\overline{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$ is such that $\overline{x}_i \notin \{0,1\}$ for some i. Consider the scalar function $t \rightarrow g(\overline{x}_1, \overline{x}_2, ..., \overline{x}_{i-1}, t, \overline{x}_{i+1}, ..., \overline{x}_n)$ over $[0,1]$. By hypothesis, g is convex on $[0,1]$, and so its maximum is attained at the boundary. Therefore, g is strictly positive at $(\overline{x}_1, \overline{x}_2, ., \delta_i, .., \overline{x}_n)$ for δ_i equal to either 0 or 1. By iterating this argument, we eventually get an extreme point $(\delta_1, \delta_2, ..., \delta_n)$ where g is strictly positive, a contradiction. Therefore, $m \in core(f)$, as desired. ■

Theorem 25. By point (i) of Lemma 24, ν_f is convex. Hence, $core(\nu_f) \neq \emptyset$. Since f is ultramodular and bounded from below at 0, it is separately convex (see Proposition 7). Hence, by point (ii) of Lemma 24, $core(\nu_f) \cong \mathcal{L}core(f)$, and so we infer that $\mathcal{L}core(f) \neq \emptyset$. To conclude the proof of the Theorem, we formulate a general statement for a game $\nu : \Sigma \rightarrow \mathbb{R}$ where Σ is a σ-algebra of a set Ω.

Let $\mathcal{B}_1(\Sigma)$ be the set of the Σ-measurable functions $X : \Omega \rightarrow \mathbb{R}$ such that $0 \leq X \leq 1_\Omega$. Given ν, we define the upper envelope of ν, to be the function $\overline{\nu} : \mathcal{B}_1(\Sigma) \rightarrow \mathbb{R}$ given by

$$\overline{\nu}(X) = \inf\{h(X) : h(A) \geq \nu(A) \text{ for all } A \in \Sigma\}$$

where h is an affine function, i.e., $h(X) = \langle m, X \rangle + k$, with $m \in bv(\Sigma)$ and $k \in \mathbb{R}$.

In reading the following Claim, notice that a convex and bounded game is of bounded variation, as proved in [11].

Claim. Suppose ν is of bounded variation and with nonempty core. Then,

$$\int X d\nu \leq \overline{\nu}(X) \leq \min_{m \in core(\nu)} \langle m, X \rangle \tag{33}$$

for all $X \in \mathcal{B}_1(\Sigma)$. If, in addition, ν is convex and bounded, then

$$\int X d\nu = \overline{\nu}(X) = \min_{m \in core(\nu)} \langle m, X \rangle \tag{34}$$

Proof of the Claim. The Choquet integral is well defined whenever ν is of bounded variation (see [12]). Let h be any affine function such that $h(E) \geq \nu(E)$ for all $E \in \Sigma$. Then

$$\nu(X \geq t) \leq h(X \geq t) = m(X \geq t) + k$$

with $X \in \mathcal{B}_1(\Sigma)$ and $t \in [0,1]$. By integrating we get,

$$\begin{aligned} \int_0^1 \nu(X \geq t)\, dt &\leq \int_0^1 m(X \geq t)\, dt + k \\ \int X d\nu &\leq \langle m, X \rangle + k = h(X) \end{aligned}$$

Taking the inf, we get the first inequality in Eq. (33). On the other hand, if $m \in core(\nu)$, $m(E) \geq \nu(E)$ for all $E \in \Sigma$. Thus any element of the core is an affine function greater than ν. It follows

$$\overline{\nu}(X) \leq \min_{m \in core(\nu)} \langle m, X \rangle$$

and the second inequality holds as well.

To conclude, suppose ν is convex and bounded. Then it is of bounded variation and $\int X d\nu = \min_{m \in core(\nu)} \langle m, X \rangle$. Consequently, Eq. (34) holds and this completes the proof of the Claim.

Having established the Claim, we now conclude the proof of Theorem 25. In view of the previous claim, it suffices to prove that $\overline{\nu}(x) = \overline{f}(x)$. Define the two functions

$$\begin{aligned} f_1^*(m) &= \inf_{x \in [0,1]^n} \{m \cdot x - f(x)\} \\ f_2^*(m) &= \min_{x \in ext[0,1]^n} \{m \cdot x - f(x)\} \end{aligned}$$

for all $x \in \mathbb{R}^n$. It is easy to see that the two upper envelope can written in the "Fenchel form"

$$\begin{aligned} \overline{f}(x) &= \inf_m \{\langle m, x \rangle - f_1^*(m)\} \\ \overline{\nu}(x) &= \inf_m \{\langle m, x \rangle - f_2^*(m)\} \end{aligned}$$

On the other hand, the two functions $f_1^*(m)$ and $f_2^*(m)$ are identical. Actually, it will be seen that an ultramodular function is upper semicontinuous. Thus the inf in $f_1^*(m)$ is attained. By the usual arguments, as $f(x)$ is separately convex, the maximum is attained at an extremal point. Hence, $f_1^* = f_2^*$ and $\overline{f} = \overline{\nu}$. Moreover, as $\overline{f}(x) = \int x d\nu_f$, we have that $\nu_f = \nu_{\overline{f}}$. Finally, as a consequence of the Choquet representation of $\overline{f}$, we have $\mathcal{L}core\left(\overline{f}\right) = \partial\overline{f}(0) \cap \partial\overline{f}(1) = core(\nu_f) = core(f)$ and this ends the proof. ■

Corollary 26. If Bf is linear, the extremal game ν_f is additive. Hence $core(\nu_f) = \mathcal{L}core(f)$ is a singleton. Conversely, suppose $\mathcal{L}core(f)$ is a singleton. Then $core(\nu_f)$ is a singleton as well. But convex games with singleton cores are necessarily additive. Hence Bf is linear. ■

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