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A Characterization of the Core of Convex Games through Gateaux Derivatives

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Abstract

We establish a calculus characterization of the core of supermodular games, which reduces the description of the core to the computation of suitable Gateaux derivatives of the Choquet integrals associated with the game. Our result generalizes to infinite games a classic result of Shapley (1971). As a secondary contribution, we provide a fairly complete analysis of the Gateaux and Frechet differentiability of the Choquet integrals of supermodular measure games.

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1 Introduction

Even though the core of a transferable utility (TU) game is a fundamental solution concept, widely used in mathematical economics, fairly little is known about its structure in infinite games. In order to shed light on this issue, Epstein and Marinacci (2001) and Marinacci and Montrucchio (2001)

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have introduced a calculus and a subcalculus for set functions and used them to establish several characterizations of cores of infinite TU games. Besides providing a useful conceptual framework, this approach turned out to be especially fruitful in the study of infinite games having finite dimensional cores, as discussed at length in Marinacci and Montrucchio (2001).

In this paper we follow a different route and show that cores of infinite games can be also characterized by using the Gateaux derivatives of the associated Choquet integrals. Specifically, let $\nu : \Sigma \to \mathbb{R}$ be a bounded supermodular game defined on a σ-algebra Σ and let $B(\Sigma)$ be the set of all bounded Σ-measurable real valued functions. If we denote by $\nu(X) = \int X d\nu$ the Choquet integral of $X \in B(\Sigma)$ with respect to ν, we can view ν as a functional on $B(\Sigma)$. This makes it possible to talk of its Gateaux derivative $D\nu(X)$ at X, which is a finitely additive measure on Σ, whose associated linear functional $\langle D\nu(X), Y\rangle$ is the Gateaux differential at the direction $Y \in B(\Sigma)$. Given a supermodular game $\nu : \Sigma \to \mathbb{R}$, our main result, Theorem 7, shows that under some standard topological conditions, we have

$$core(\nu) = \overline{co}^{w^*}(\{D\nu(X) : \ X \text{ injective and in } B(\Sigma)\}), \tag{1}$$

namely, the core of ν is the weak*-closed convex hull of the Gateaux derivatives of ν computed at all injective functions belonging to $B(\Sigma)$.

This result provides a calculus characterization of the core and reduces its description to the computation of suitable Gateaux derivatives. Moreover, it generalizes the classic result of Shapley (1971) about the set of extreme points of a finite supermodular game. As a matter of fact, in the Concluding Remarks we show that Shapley's result implicitly rests on the Gateaux derivatives of injective functions, and that our characterization reduces to his result in the finite case.

To illustrate the usefulness of our calculus representation, in Section 4 we consider measure games, a widely used class of games of the form $f \circ P$, where $P = (P_1, ..., P_n) : \Sigma \to \mathbb{R}^n$ is a non-atomic vector probability measure and $f : R(P) \to \mathbb{R}$ is real-valued function defined on the range $R(P) = \{P(E) : E \in \Sigma\} \subseteq \mathbb{R}^n$ of P. For this class of games, Theorem 16 shows that the general representation given by Eq. (1) takes the following stark closed form:

$$core(\nu) = \overline{co}^{w^*}\left\{\sum_{i=1}^{n}\int_E \frac{\partial f}{\partial x_i}(G_X \circ X)\, dP_i : X \text{ injective and in } B(\Sigma)\right\}, \tag{2}$$

where $G_X : \mathbb{R} \to \mathbb{R}^n$ is the vector distribution function defined by $G_X(q) = (P_1(X \geq q), ..., P_n(X \geq q))$ for $q \in \mathbb{R}$. Several different representations of the core will be also proved.

A secondary contribution of this paper is a fairly detailed study of the Gateaux differentiability of Choquet integrals, which is a key issue for our representation. In particular, given a function $X \in B(\Sigma)$ with a continuous vector distribution function G_X, Theorem 11 shows that for a suitably differentiable function f the Gateaux derivative of the Choquet functional $\nu : B(\Sigma) \to \mathbb{R}$ associated with a supermodular measure game $\nu = f \circ P$ exists and it has the following neat closed form:

$$\langle D\nu(X), Y\rangle = \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X) Y dP_i.$$

Actually, more is true: the Choquet functional is Frechet differentiable provided f is continuously differentiable.

To the best of our knowledge, the recent paper by Carlier and Dana (2001) is the only other work that has investigated similar issues. They consider the special case of positive scalar measure games and use different "rearrangements" techniques, which do not seem to be easy to extend beyond the scalar case. Nevertheless, their paper contains several insightful results on both the structure of *core* $(f \circ P)$ and the Gateaux differentiability properties of the Choquet functional $\nu = f \circ P$, which we discuss in some detail in the Concluding Remarks.

The paper is organized as follows. After some preliminaries in Section 2, we prove in Section 3 our representation for general supermodular games, while Section 4 contains its version for measure games. In the Concluding Remarks we discuss in detail the relations of our paper with Shapley (1971) and Carlier and Dana (2001). The Appendix contains the proofs. Of special importance are the proofs of Theorem 5 and Lemmas 24 and 25, which provide our main technical tools. Theorem 5, which shows that Choquet functionals are Gateaux differentiable at injective functions, is based on a result of Mackey (1957), which says that in standard Borel spaces a countable and separating collection of sets generates the Borel σ-algebra. This important property of standard Borel spaces is key for our results. Lemma 24 provides the representation of the Gateaux derivative $D\nu(X)$ when X is injective and ν is a supermodular measure game; in particular, Eq. (2) is a simple specification of this lemma. Finally, Lemma 25 provides the key result needed to establish the Frechet differentiability of the Choquet functional. This lemma shows that, given any function X, it is possible to construct a function Y comonotonic with X and having a continuous distribution.

2 Preliminaries

Throughout the paper, Σ is a σ-algebra of sets of a space Ω. Subsets of Ω are understood to be in Σ even where not stated explicitly.

2.1 Set Functions

A set function $\nu : \Sigma \to \mathbb{R}$ is a *game* if $\nu(\varnothing) = 0$. A game ν is

positive if $\nu(A) \geq 0$ for all A,

bounded if $\sup_{A \in \Sigma} |\nu(A)| < \infty$.

monotone (or a capacity) if $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$,

continuous if $\lim_{n\to\infty} \nu(A_n) = \nu(\Omega)$ whenever $A_n \uparrow \Omega$ and $\lim_{n\to\infty} \nu(A_n) = 0$ whenever $A_n \downarrow \varnothing$.

supermodular (or convex) if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all sets A and B,

additive (or a charge) if $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint sets A and B,

countably additive (or a measure) if $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$ for all countable collections of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty}$.

The set of all charges (measures, resp.) that are bounded with respect to the variation norm $\|\cdot\|$ is denoted by $ba(\Sigma)$ ($ca(\Sigma)$, resp.). Generic elements of $ba(\Sigma)$ are denoted by m.

The set of all games that can be expressed as differences of two capacities is denoted by $bv(\Sigma)$. Aumann and Shapley (1974) show that a game ν belongs to $bv(\Sigma)$ if and only if the norm $\|\nu\| \equiv \sup \sum_{i=1}^{n} |\nu(E_i) - \nu(E_{i-1})|$, where the sup is taken over all finite chains $\{E_i\}_{i=0}^{n}$, is finite. Moreover, they prove that $(bv(\Sigma), \|\cdot\|)$ is a Banach space and that $\|\cdot\|$ reduces to the variation norm on games belonging to $ba(\Sigma)$. Finally, Maccheroni and Ruckle (2001) have shown that $(bv(\Sigma), \|\cdot\|)$ can be viewed as a dual space.

Let $B(\Sigma)$ be the set of all bounded Σ-measurable functions defined on Ω. The standard duality between $(ba(\Sigma), \|\cdot\|)$ and $(B(\Sigma), \|\cdot\|)$ will be denoted by $\langle X, \mu\rangle = \int X d\mu$, with $X \in B(\Sigma)$ and $\mu \in ba(\Sigma)$.

2.2 Cores and a Theorem of Shapley

The core of a game ν is the set of all charges in $ba(\Sigma)$ that setwise dominate ν, that is,

$$core(\nu) = \{m \in ba(\Sigma) : m(A) \geq \nu(A) \text{ for all } A \in \Sigma \text{ and } m(\Omega) = \nu(\Omega)\}.$$

The core is a (possibly empty) convex set and it is weak*-compact. The following result, which plays an important role in this paper, is a simple generalization to real-valued supermodular games of well-known properties of positive supermodular games, essentially due to Choquet (1953).

Lemma 1 *Let $\nu : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. Then $core(\nu) \neq \varnothing$ and*

(i) Given any chain $\{E_i\}_{i\in I}$ in Σ, there is $m \in core(\nu)$ such that $m(E_i) = \nu(E_i)$ for all $i \in I$.

(ii) ν is continuous if and only if $core(\nu) \subseteq ca(\Sigma)$.

(iii) ν belongs to $bv(\Sigma)$.

In a classic result, Shapley (1971) has characterized the extreme points of $core(\nu)$ when Ω is finite and Σ is its power set. Recall that a maximal chain $\mathcal{C}$ of a finite set $\Omega = \{\omega_1, ..., \omega_N\}$ is a collection of sets

$$\left\{\omega_{\sigma(1)}\right\}, \left\{\omega_{\sigma(1)}, \omega_{\sigma(2)}\right\}, ..., \left\{\omega_{\sigma(1)}, ..., \omega_{\sigma(N)}\right\},$$

where σ is a permutation over $\{1, ..., N\}$.

Theorem 2 ***(Shapley)***. *Let ν be a supermodular game defined on the power set of a finite set Ω. Then, a charge m is an extreme point of $core(\nu)$ if and only if there is a maximal chain $\mathcal{C}$ such that $\nu(A) = m(A)$ for all $A \in \mathcal{C}$.*

2.3 Choquet Integrals and Derivatives

Given a game ν and a function $X \in B(\Sigma)$, the Choquet integral $\nu : B(\Sigma) \to \mathbb{R}$ is defined as follows:

$$\nu(X) = \int_0^\infty \nu(X \geq t)\, dt + \int_{-\infty}^0 [\nu(X \geq t) - \nu(\Omega)]\, dt, \tag{3}$$

where in the r.h.s. we have two Riemann integrals. The Choquet integral exists for all $X \in B(\Sigma)$ whenever $\nu \in bv(\Sigma)$. For, in this case $\nu(X \geq t)$ is of bounded variation in t and the Riemann integrals in Eq. (3) are well-defined.

The Choquet integral is positive homogeneous and Lipschitz continuous (see Lemma 22 in the Appendix). It is also monotone when ν is a capacity, while it is superadditive (and so concave) when ν is supermodular. Finally, it is additive on any pair of comonotonic functions, that is, on any pair $X, Y \in B(\Sigma)$ such that $[X(\omega) - X(\omega')][Y(\omega) - Y(\omega')] \geq 0$ for any $\omega, \omega' \in \Sigma$ (see Schmeidler, 1986).

Another remarkable property of the Choquet integral is that, by point (i) of Lemma 1, for each $X \in B(\Sigma)$ it holds

$$\nu(X) = \min_{m \in core(\nu)} \langle X, m \rangle$$

whenever ν is supermodular. Consequently, in the supermodular case the Choquet integral can be viewed as a support function (though the converse is clearly false, as in general support functions are not comonotonic additive).

Given $\nu : B(\Sigma) \to \mathbb{R}$ and $X \in B(\Sigma)$, if there exists an element $D\nu(X) \in ba(\Sigma)$ such that

$$\langle Y, D\nu(X) \rangle = \lim_{t \downarrow 0} \frac{\nu(X + tY) - \nu(X)}{t} \tag{4}$$

for all $Y \in B(\Sigma)$, then we say that ν is Gateaux differentiable at X, and $D\nu(X)$ is the Gateaux derivative of ν at X.

If ν is Gateaux differentiable at $X \in B(\Sigma)$ and the limit (4) is uniform over $Y \in B(\Sigma)$ with $\|Y\| = 1$, then ν is said to be Frechet differentiable at X, and $D\nu(X)$ is the Frechet derivative of ν at X.

Finally, when ν is concave we denote by $\partial\nu(X)$ the standard superdifferential

$$\partial\nu(X) = \{m \in ba(\Sigma) : \langle Y - X, m \rangle \geq \nu(Y) - \nu(X) \text{ for all } Y \in B(\Sigma)\}.$$

3 General Results

We begin with a lemma that establishes some properties of $\partial\nu(X)$ in the supermodular case.

Lemma 3 *Let $\nu : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. Then, for all $X \in B(\Sigma)$, $\partial\nu(X)$ is non-empty and*

$$\partial\nu(X) = \{m \in core(\nu) : \langle X, m\rangle = \nu(X)\}. \tag{5}$$

Moreover, given any $X_1, X_2 \in B(\Sigma)$, the three following conditions are equivalent:

(i) $\partial\nu(X_1) \cap \partial\nu(X_2) \neq \varnothing$,

(ii) $\nu(X_1 + X_2) = \nu(X_1) + \nu(X_2)$,

(iii) $\partial\nu(X_1) \cap \partial\nu(X_2) = \partial\nu(X_1 + X_2)$.

If ν is supermodular, by a classic result in Convex Analysis the set $\partial\nu(X)$ is a singleton if and only if the Choquet integral $\nu : B(\Sigma) \to \mathbb{R}$, which is Lipschitz continuous, is Gateaux differentiable at X (see, e.g., Phelps (1993) p. 5). The following result is thus an immediate but interesting consequence of the second part of Lemma 3. It says that the derivative is invariant under comonotonicity.

Corollary 4 *Let $\nu : \Sigma \to \mathbb{R}$ be a bounded and supermodular game. If the Choquet integral $\nu : B(\Sigma) \to \mathbb{R}$ is Gateaux differentiable at X_1 and at X_2, then $D\nu(X_2) = D\nu(X_1)$ whenever X_1 and X_2 are comonotonic.*

Having established Lemma 3, we can now turn to our first result, in which we prove that the Choquet functional is Gateaux differentiable at all injective functions in $B(\Sigma)$. This is a significant class of functions in $B(\Sigma)$, as proved by Lemma 23 in the Appendix, which shows that the class of all injective functions is dense in $B(\Sigma)$.

To prove this result we need some more structure. In particular, we have to assume that ν is continuous and that (Ω, Σ) is a (standard) Borel space, that is, (Ω, Σ) is isomorphic to a pair (Ω', Σ'), where Ω' is a Borel subset of some Polish space and Σ' is its Borel σ-algebra (see, e.g., Srivastava, 1998).

Theorem 5 *Let* (Ω, Σ) *be a standard Borel space and let* $\nu : \Sigma \to \mathbb{R}$ *be a bounded, continuous and supermodular game. If* $X \in B(\Sigma)$ *is injective, then* $\partial\nu(X)$ *is a singleton consisting of the Gateaux derivative* $D\nu(X)$*.*

The next example shows that continuity is needed in Theorem 5.

Example. Set $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$. Consider the filter game $\nu : \Sigma \to \{0, 1\}$ defined as follows: $\nu(A) = 1$ if and only if $A \subseteq \mathbb{N}$ is cofinite (see Marinacci, 1996). This two-valued game ν is supermodular and discontinuous at Ω. It is easy to verify that $\int X d\nu = \liminf_{n\to\infty} X(n)$ for each $X \in B(\Sigma)$. This functional is nowhere Gateaux differentiable (see, e.g., Phelps (1993) example 1.21).

It is easy to see that the Gateaux derivatives $D\nu(X)$ of Theorem 5 are extreme points of *core* (ν). This suggests the possibility of representing *core* (ν) as a weak* closed convex hull of these derivatives. Theorem 7, our second main result, will establish such a representation. In order to state it, we need a final lemma, which refines the representation of $\partial\nu(X)$ given by Eq. (5). It is based on Lemma 23, Theorem 5, and on a "D-representation" result of Jofre and Thibault (1990). We denote by $BI(\Sigma)$ the class of all injective functions belonging to $B(\Sigma)$.

Lemma 6 *Let* (Ω, Σ) *be a standard Borel space and* $\nu : \Sigma \to \mathbb{R}$ *be a bounded, continuous and supermodular game. Then*

$$\partial\nu(X) = \overline{co}^{w^*}\left\{w^* - \lim D\nu(X_n) : \ \|X_n - X\| \to 0, \ \ X_n \in BI(\Sigma)\right\} \tag{6}$$

for all $X \in B(\Sigma)$*.*

We can now state our main result, which generalizes to infinite convex games the classic result of Shapley (1971) reported in Theorem 2 and, more importantly, provides a calculus characterization of the core. Let $\sim$ be the comonotonic relation on $B(\Sigma)$, that is, $X_1 \sim X_2$ if X_1 and X_2 are comonotonic. It is easy to check that $\sim$ is an equivalence relation when restricted to the collection $BI(\Sigma)$. As usual, $BI(\Sigma)/\sim$ denotes the set of equivalence classes determined by $\sim$, while, with a slight abuse of notation, $X \in BI(\Sigma)/\sim$ means that X is a representative of one of the equivalence classes determined by $\sim$.

Theorem 7 *Let* (Ω, Σ) *be a standard Borel space and* $\nu : \Sigma \to \mathbb{R}$ *be a bounded, continuous and supermodular game. Then*

$$core\,(\nu) = \overline{co}^{w^*} \{D\nu\,(X) : \; X \in BI\,(\Sigma) \,/ \sim\} \tag{7}$$

and, for all $Y \in B\,(\Sigma)$,

$$\nu\,(Y) = \inf \{\langle D\nu\,(X)\,, Y\rangle : \; X \in BI\,(\Sigma) \,/ \sim\}\,, \tag{8}$$

where the inf is a min if and only if there is an injective $X \in B\,(\Sigma)$ *such that* $\nu\,(X + Y) = \nu\,(X) + \nu\,(Y)$.

Theorem 7 is the announced calculus characterization of the core. In the Concluding Remarks we will discuss its relations with Shapley's Theorem 2, while in the next subsection we provide an illustration of the usefulness of this calculus characterization of the core by studying the important class of measure games.

4 Application: Measure Games

A game $\nu : \Sigma \to \mathbb{R}$ is a *(non-atomic) measure game* if there exists a vector measure $P = (P_1, ..., P_N) : \Sigma \to \mathbb{R}^N_+$, where each P_i is a non-atomic probability measure on Σ, and a function $f : R\,(P) \to \mathbb{R}$ defined over the range $R\,(P)$ of P and with $f\,(0) = 0$, such that

$$\nu\,(E) = f\,(P\,(E)) \quad \text{for all } E \in \Sigma.$$

When $n = 1$, $\nu = f \circ P$ is called a *scalar measure game.* Notice that, by the Lyapunov Theorem, $R\,(P)$ is a compact and convex subset of $\mathbb{R}^n$.

Measure games play in important role in mathematical economics, where they are widely used and studied. One of the reasons of the importance of measure games lies in their remarkable analytical tractability, due to the added structure guaranteed by the special form $f \circ P$. Despite their superior tractability, even for measure games little is known about the structures of their cores, except in the special case when its elements are linear combinations of the underlying vector measure P, that is, when $core\,(f \circ P) \subseteq span\,\{P_1, ..., P_n\}$. As far as we know, outside this special "linear" case – treated at length in Marinacci and Montrucchio (2001) and in the references therein contained – the only serious attempt to characterize the core of a measure game can be found in the recent work of Carlier and Dana (2001), who consider scalar measure games.

The purpose of this section is to show that, because of their special form, for supermodular measure games it is possible to establish an especially neat version of Theorem 7, which provides a sharp description of the structure of the cores of these games. This is achieved through a careful study of the differentiability properties of the associated Choquet integrals $\int X d\left(f \circ P\right)$, which is a key step in order to obtain the calculus characterization of Theorem 7. As a result, a secondary contribution of this subsection is a fairly complete analysis of the differentiability properties of the Choquet integrals $\int X d\left(f \circ P\right)$.

For the scalar case, our results sharpen the ones that Carlier and Dana (2001) have obtained in the scalar case with their different techniques, as we discuss below.

4.1 Functions Monotone of Order 2

In view of the importance in what follows of supermodular measure games, it is important to provide a characterization of this class of measure games in terms of the underlying function $f : R\left(P\right) \to \mathbb{R}$. Consider the following class of functions:

Definition 8 *A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ defined on a convex set A is monotone of order 2 if*

$$f\left(x+h\right) - f\left(x\right) \leq f\left(y+h\right) - f\left(y\right)$$

for all $x, y \in A$ with $x \leq y$ and for all $h \geq 0$ such that $x+h$ and $y+h$ belong to A.

In other words, a function is monotone of order 2 if its second difference

$$f\left(x+h+k\right) - f\left(x+h\right) - f\left(x+k\right) + f\left(x\right)$$

is non-negative for $h, k \geq 0$. Choquet (1953) defines on p. 172 a similar class of functions, even though he requires also the first difference to be non-negative, and so the function itself to be non-decreasing.

The next result, whose main part is due to Choquet (1953) pp. 193-194, shows the importance for our purposes of functions monotone of order 2.

Proposition 9 *A measure game $f \circ P : \Sigma \to \mathbb{R}$ is supermodular whenever $f : R(P) \to R$ is monotone of order 2. The converse holds when $R(P) = [0,1]^n$.*

Functions monotone of order 2 have several interesting characterizations, at least under mild assumptions of regularity (see, e.g., Marinacci, Montrucchio, and Scarsini (2002) and the references therein contained). The simplest one is the following: If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable on A, then f is monotone of order 2 if and only if $\nabla f(x) \leq \nabla f(y)$ whenever $x \leq y$. In turn, this implies that when f is twice differentiable, then f is monotone of order 2 if and only if $\partial^2 f / \partial x_i \partial x_j \geq 0$ for all i, j.

When $n = 1$, monotonicity of order 2 is equivalent to convexity, provided f is continuous. Without continuity, this equivalence fails. For example, any solution f of the Cauchy functional equation $\psi(x+y) = \psi(x) + \psi(y)$ is clearly monotone of order 2, but it might well be non convex if it is not assumed to be continuous. Notice that this example also shows that Proposition 9 extends to non continuous functions f the well-known fact that, when f is continuous, a scalar measure game $f \circ P$ is supermodular if and only if f is convex. As a matter of fact, without continuity, convexity and monotonicity of order 2 are no longer equivalent, and so the standard result fails, while Proposition 9 still holds.

Interestingly, when $n > 1$ monotonicity of order 2 and convexity are quite independent notions. There are convex functions that are not monotone of order 2 (e.g., $F(x) = \|x\|$) and, vice versa, functions monotone of order 2 that are not convex (e.g., $F(x) = \prod_{i=1}^n x_i$).

We close with some further interesting properties of regularity of monotone functions of order 2, proved in Marinacci, Montrucchio, and Scarsini (2002). Recall that a function $f : A \subseteq \mathbb{R}^N \to \mathbb{R}$ is calm from below at $x_0 \in A$ if there is a neighborhood $B(x_0, \varepsilon)$ and a constant $\gamma > 0$ such that, for all $x \in B(x_0, \varepsilon) \cap A$,

$$f(x) \geq f(x_0) - \gamma \|x - x_0\|.$$

For example, if f is differentiable at x_0 or it is locally Lipschitz at x_0, then it is calm from below at x_0 (see, e.g., Rockafellar and Wets (1998) p. 320).

Proposition 10 *Let $f : [0,1]^n \to \mathbb{R}$ be a function monotone of order 2. If f is lower semicontinuous at $\underline{0}$ and $\underline{1}$, then:*

(i) f *is continuous and of bounded variation on* $[0,1]^n$.

(ii) *given a continuous and non-decreasing curve* $\gamma : [0,1] \to [0,1]^n$, *it holds*

$$\int (f \circ \gamma)\, d\mu \geq f\left(\int \gamma d\mu\right) \tag{9}$$

for any Borel probability measure μ *on* $[0,1]$. *Moreover, the function* $f \circ \gamma : [0,1] \to \mathbb{R}$ *is absolutely continuous provided* γ *is Lipschitz.*

(iii) *there is a set* D *whose complement has Lebesgue measure zero, on which* f *is differentiable and the gradient mapping* ∇f *is continuous.*

If, in addition, f *is calm from below at* $\underline{0}$ *and* $\underline{1}$, *then* f *is Lipschitz on* $[0,1]^n$.

The first part of point (ii) is due to Brunk (1964), which also proves other versions of Eq. (9). An important special case of point (ii) is when the measure μ has finite support. In this case, point (ii) says that for each sequence $\{x_i\}_{i=1}^n \subseteq [0,1]^n$ with $x_1 \leq x_2 \leq \cdots \leq x_n$, it holds $\sum_{i=1}^n t_i f(x_i) \geq f\left(\sum_{i=1}^n t_i x_i\right)$ whenever $\{t_i\}_{i=1}^n$ is a sequence of positive numbers such that $\sum_{i=1}^n t_i = 1$. In particular, this means that all continuous functions monotone of order 2 are separately convex in each component and, more generally, are convex the restrictions over the straight-lines with non-negative slope.

4.2 Gateaux Differentiability

In view of Theorem 7, to characterize $core\,(f \circ P)$ we have to study the Gateaux differentiability of the Choquet functional $\int X d\,(f \circ P)$. This issue has been studied by Carlier and Dana (2001) in the case of scalar monotone games. Here we provide general results for measure games, which also sharpen their results in the scalar case (cf. Remark (i) below).

Some notation is in order. For a given $X \in B(\Sigma)$, $G_X^i(q)$ denotes the distribution function $G_X^i(q) = P_i(X \geq q)$ for $i = 1, 2, ..., n$, while $G_X : \mathbb{R} \to R(P)$ is the mapping defined by

$$G_X(q) = \left(G_X^1(q), ..., G_X^n(q)\right).$$

The symbol $\overline{P}$ denotes the measure $\overline{P} = P_1 + ... + P_n$. Finally, for a vector $x \in \mathbb{R}^n$, $|x|_1 = \sum_{i=1}^n |x_i|$ is the l^1-norm of $\mathbb{R}^n$.

Theorem 5 in Section 3 showed that under very general conditions the Choquet functional is Gateaux differentiable at all injective functions in $B(\Sigma)$. We now show that a stronger result can be proved for measure games, thanks to their special structure. In particular, the next result, Theorem 11, improves Theorem 5 in several ways. First, the derivative takes the stark closed form given by Eq. (10). Second, Theorem 11 shows that the Choquet functional $\int X d(f \circ P)$ is Gateaux differentiable at all functions $X \in B(\Sigma)$ that have a continuous distribution function $G_X(q)$; this a broad class of functions – a dense G_δ subset of $B(\Sigma)$, as shown in the remark (ii) below – which clearly includes all injective functions. Finally, Theorem 11 provides a simple condition under which the Choquet functional is actually Frechet differentiable.

We can state the result. As usual, a function $f : R(P) \to \mathbb{R}$ is differentiable on $R(P)$ whenever it can be extended to a differentiable function on some open set containing $R(P)$.

Theorem 11 *Let $\nu = f \circ P$ be a measure game over a standard Borel space (Ω, Σ), with f monotone of order 2. Given $X \in B(\Sigma)$, suppose one of the following holds:*

(i) X is injective and f is differentiable and Lipschitz;

(ii) X has a continuous distribution function G_X and f is continuously differentiable.

Then, the game ν is Gateaux differentiable at X and its differential is

$$\langle D\nu(X), Y\rangle = \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i} (G_X \circ X) Y dP_i. \tag{10}$$

Finally, if f is continuously differentiable, then ν is actually Frechet differentiable at X.

There is a "trade-off" between conditions (i) and (ii): condition (i) requires more on X (injectivity rather than just a continuous distribution function), while condition (ii) requires more on f (continuous differentiability rather than "plain" differentiability).

In the special case $R(P) = [0,1]^n$, Theorem 11 takes a simpler form as it becomes superfluous to require in point (i) that f be Lipschitz. In fact, differentiability implies calmness, and so by Proposition 10 a function f monotone of order 2 is Lipschitz whenever it is differentiable.

More is true when $n = 1$: in this case, the differentiability assumptions as well can be weakened. For convenience, we consider this case separately.

Proposition 12 *Let $\nu = f \circ P$ be a scalar measure game over a standard Borel space (Ω, Σ), with f convex and continuous. Given $X \in B(\Sigma)$, suppose one of the following holds:*

(i) X is injective;

(ii) X has a continuous distribution function G_X and f is Lipschitz.

Then, the game ν is Gateaux differentiable at X and its differential is

$$\langle D\nu(X), Y\rangle = \int f'_{+}(G_X \circ X) Y dP.$$

Finally, if f is differentiable, then ν is actually Frechet differentiable at X.

Condition (i) in the two previous results is all we need to establish the calculus characterization of $core(f \circ P)$ we are looking for. Condition (ii), however, provides an interesting sufficient condition for the Gateaux differentiability of $\int X d(f \circ P)$. The next result completes our analysis by showing that this condition is "almost" necessary, provided f is continuously differentiable. To state the result we need the following definition.

Definition 13 *A differentiable and monotone of order 2 function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly monotone of order 2 if, for all $x \geq y$,*

$$\frac{\partial f}{\partial x_i}(x) > \frac{\partial f}{\partial x_i}(y)$$

whenever $x_i > y_i$.

For example, in the scalar case a continuous function is strictly monotone of order 2 if and only if it is strictly convex. We can now state the announced result.

Theorem 14 *Let $\nu = f \circ P$ be a measure game over a standard Borel space (Ω, Σ), and suppose f is strictly monotone of order 2 and continuously differentiable on $R(P)$. Then, ν is not Gateaux differentiable at any $X \in B(\Sigma)$ such that the distribution function G_X is not continuous.*

Remark. Again, in the case $n = 1$ a weaker differentiability assumption is needed: it is enough to define strict monotonicity using the right derivative f'_+ and to assume in Theorem 14 that f'_+ is bounded.

Summing up, we can conclude that the continuity of the distribution function G_X is tightly connected with the Gateaux differentiability of the Choquet functional $\int X d(f \circ P)$. As a matter of fact, by Theorem 11 it is a sufficient condition, while by Theorem 14 it is “almost” necessary as well.

We close the subsection with few remarks:

(i) Carlier and Dana (2001) have proved the Gateaux part of Proposition 12 and Theorem 14 in the scalar case $n = 1$ and with f strictly increasing and differentiable (cf. the Concluding Remarks).

(ii) By Lemma 23, $BI(\Sigma)$ is dense in $B(\Sigma)$. This immediately implies that the set of all functions $X \in B(\Sigma)$ whose G_X is continuous is dense in $B(\Sigma)$. Interestingly, more is true: such a set is a G_δ subset of $B(\Sigma)$. For, let $\alpha_n \downarrow 0$ and set $V_n = \left\{X \in B(\Sigma) : \overline{P}(X = q) < \alpha_n \text{ for all } q \in \mathbb{R}\right\}$. It is easy to see that each set V_n is open, and so the set $\bigcap_n V_n$ is the desired G_δ dense subset. Since $B(\Sigma)$ is not a weak Asplund space, the fact that the domain of Gateaux differentiability of a concave Choquet integral contains a dense G_δ subset of $B(\Sigma)$ is noteworthy.

(iii) Unless $R(P) = [0,1]^n$, there might be measure games $f \circ P$ that are supermodular even though f is not monotone of order 2 (cf. Proposition 9). On the other hand, it can be shown that Theorems 11 and 14 hold if we just assume that the game $f \circ P$ is supermodular rather than the function f be monotone of order 2, provided we assume that f is continuously differentiable on $R(P)$.

4.3 Core Representation

We are now ready to state the version of Theorem 7 for measure games. One of the features of this version is the use of densities, made possible by the existence of an underlying vector probability P. As a matter of fact, we have the following simple result (cf. Marinacci and Montrucchio, 2001).

Lemma 15 *Let $\nu = f \circ P$ be a measure game, not necessarily supermodular, and suppose f is lower semicontinuous at $\underline{0}$ and at $\underline{1}$. Then, $core(\nu) \subseteq ca(\Sigma)$ and each $m \in core(\nu)$ is absolutely continuous with respect to $\overline{P}$.*

In view of Lemma 15 we can consider $core\,(\nu) \subseteq L_1\left(\Omega, \Sigma, \overline{P}\right)$, by identifying $m \in core\,(\nu)$ with its density $dm/d\overline{P} \in L_1\left(\Omega, \Sigma, \overline{P}\right)$. In particular, it is easy to check that $core\,(\nu)$ can be viewed as a $\sigma\left(L_1, L_\infty\right)$-compact subset of $L_1\left(\Omega, \Sigma, \overline{P}\right)$.

We can now state our calculus characterization of $core\,(f \circ P)$. Two pieces of notation:

(i) $\overline{co}$ denotes the closed convex hull in the norm topology of $L_1\left(\Omega, \Sigma, \overline{P}\right)$;

(ii) $\int_E \left(\partial f/\partial x_i\right)\left(G_X \circ X\right) dP_i$ denotes the measure naturally associated with the linear functional $Y \to \int \left(\partial f/\partial x_i\right)\left(G_X \circ X\right) Y dP_i$.

Theorem 16 *Let $\nu = f \circ P$ be a measure game over a standard Borel space (Ω, Σ), and suppose f is differentiable and monotone of order 2. Then,*

$$\begin{aligned} core\,(\nu) &= \overline{co}\left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(G_X \circ X\right) \frac{dP_i}{d\overline{P}} : X \in BI\,(\Sigma) / \sim\right\} \qquad (11) \\ &= \overline{co}^{w^*}\left\{\sum_{i=1}^{n} \int_E \frac{\partial f}{\partial x_i}\left(G_X \circ X\right) dP_i : X \in BI\,(\Sigma) / \sim\right\}, \end{aligned}$$

where $dP_i/d\overline{P}$ is the Radon-Nikodym derivative of P_i with respect to $\overline{P}$.

Remark. For brevity we have omitted in the statement the counterpart of Eq. (8).

The representation of $core\,(\nu)$ given in Theorem 16 can be further sharpened thanks to the $\sigma\left(L_1, L_\infty\right)$-compactness of $core\,(\nu)$, which makes it possible to apply the classic Vitali Convergence Theorem (see, e.g., Dunford and Schwartz (1957) p. 325). We thus get the following "concrete" non topological representation, based on almost sure convergence. In the statement, $\overline{P} - \lim f_n$ denotes a $\overline{P}$-a.e. limit of the sequence $\left\{f_n\right\}_n$.

Corollary 17 *Let $\nu = f \circ P$ be a measure game and suppose the hypotheses of Theorem 16 hold. If we set*

$$\mathcal{D} \equiv \left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(G_X \circ X\right) \frac{dP_i}{d\overline{P}} : X \in BI\,(\Sigma) / \sim\right\},$$

then,

$$core\,(\nu) = \left\{\overline{P} - \lim f_n : f_n \in co\,(\mathcal{D})\right\}.$$

The scalar case $n = 1$ deserves to be treated separately because in this case the differentiability of f becomes superfluous. We have the following result, which combines Theorem 16 and Corollary 17 in the scalar case (we omit the simple proof).

Proposition 18 *Let $\nu = f \circ P$ be a scalar measure game over a standard Borel space (Ω, Σ), and suppose f is continuous and convex. If we set*

$$\mathcal{D} \equiv \left\{ f'_+ (G_X \circ X) : X \in BI(\Sigma) / \sim \right\}$$

then,

$$\begin{aligned} core(\nu) &= \overline{co}\{\mathcal{D}\} = \{P - \lim f_n : f_n \in co(\mathcal{D})\} \\ &= \overline{co}^{w^*} \left\{ \int_E f'_+ (G_X \circ X)\, dP : X \in BI(\Sigma) / \sim \right\}. \end{aligned}$$

By Theorem 11, the measure game $\nu = f \circ P$ is Frechet differentiable at all $X \in BI(\Sigma)$ whenever f is continuously differentiable and monotone of order 2. By elaborating on Proposition 5.11 of Phelps (1993), it is easy to check that this implies that $D\nu(X)$ is a weak* strongly exposed point of $core(\nu)$. Hence, while under Gateaux differentiability the measure $D\nu(X)$ is an extreme point of $core(\nu)$, under Frechet differentiability it becomes a weak* strongly exposed point. As a result, we have the following corollary of Theorem 16, which is worth noting since $B(\Sigma)$ is not an Asplund space (cf. Proposition 5.12 of Phelps, 1993).

Corollary 19 *Let $\nu = f \circ P$ be a measure game over a standard Borel space (Ω, Σ), and suppose f is continuously differentiable and monotone of order 2. Then, $core(\nu)$ is the weak* closed convex hull of its weak* strongly exposed points.*

5 Concluding Remarks

We close the paper with some concluding remarks, the first two discuss some related papers, while the last one provides some issues for future research.

1. Theorem 7 extends to infinite supermodular games Theorem 2, due to Shapley (1971). In Shapley's result the maximal chains of the finite set Ω

play a key role. On the other hand, when Ω is finite each maximal chain can be viewed as the collection of the upper sets of a suitable injective function on Ω. In particular, up to comonotonicity, such injective function is unique. In fact, it is easy to see that two injective functions X_1 and X_2 share the same collection of upper sets if and only if they are comonotonic. Therefore, modulo comonotonicity, there is a one-to-one natural correspondence between injective functions and maximal chains.

This is the first key observation that leads from Shapley's result to Theorem 7. The second important observation comes from Theorem 3. As we already observed, $\partial\nu(X)$ is a singleton if and only if the Choquet integral $\nu : B(\Sigma) \to \mathbb{R}$ is Gateaux differentiable at X. By Theorem 3, $\partial\nu(X) = \left\{m \in core(\nu) : \int X dm = \int X d\nu\right\}$. Now, let $X_{\mathcal{C}}$ be the injective function associated to a given a maximal chain $\mathcal{C}$ by the natural correspondence discussed before. Since Ω is finite, it is very easy to check that the set $\left\{m \in core(\nu) : \langle X, m\rangle = \int X d\nu\right\}$ is a singleton consisting of the unique measure m such that $m_{|\mathcal{C}} = \nu_{|\mathcal{C}}$. We conclude that $\partial\nu(X)$ is a singleton consisting of the Gateaux derivative $D\nu(X_{\mathcal{C}})$. Finally, by Corollary 4 the Gateaux derivative is invariant under comonotonicity and, therefore, for the above argument it is immaterial which injective function $X_{\mathcal{C}}$ we choose among the ones whose collection of upper sets is the maximal chain $\mathcal{C}$.

Summing up these observations, we have the following equivalent form of Shapley's Theorem:

Proposition 20 *Let ν be a supermodular game defined on the power set of a finite set Ω. Then, a charge m is an extreme point of $core(\nu)$ if and only if it belongs to the set*

$$\{D\nu(X) : X \text{ is an injective function on } \Omega\}. \tag{12}$$

Since Eq. (7) reduces to Eq. (12) when Ω is finite, this completes our discussion.

El Kaabouchi (1994) provides a related generalization of Shapley's result for supermodular Choquet capacities defined on a compact metric space. In his richer setting, he shows that the core of a supermodular Choquet capacity ν is the weak*-closed convex hull of all the measures $m \in core(\nu)$ such that $\int X dm = \int X d\nu$ for some injective Borel function. Relative to his article, a key advance of our work is the observation that such a set is, for a general function $X \in B(\Sigma)$, the superdifferential of the Choquet integral at X. By elaborating on this observation, we can obtain our differential characterization of cores of general supermodular games and the applications to measure games.

2. Carlier and Dana (2001) investigate the core of a distortion $\nu = f \circ P$, where the function $f : [0,1] \to [0,1]$ is assumed to be strictly convex, increasing and differentiable. A function $s : \Omega \to [0,1]$ is measure preserving (m.p) with respect to P if $\lambda(B) = P(s^{-1}(B))$ for all Borel sets $B \subseteq [0,1]$. Using "rearrangements" techniques (see, e.g., Ryff 1970), Carlier and Dana (2001) prove that

$$core\,(f \circ P) = \overline{co}^{s}\left\{f'(s) : s \text{ is a m.p. function}\right\}.$$

Unlike the representation in Proposition 18, this representation does not have a closed form. However, we can prove the next result, which is key to understand the relations between their result and ours.

Proposition 21 *Let P be a nonatomic probability measure defined on a σ-algebra Σ of subsets of a space Ω. Then, a function $s : \Omega \to [0,1]$ is measure preserving if and only if there exists $X \in B(\Sigma)$ with a continuous distribution function and such that $G_X \circ X = s$.*

Hence, in our standard Borel space setting Proposition 18 improves their result since, under weaker hypotheses, it establishes a more economical representation of the core of a supermodular scalar measure game, which only uses the collection of injective functions rather than the collection of all functions having continuous distribution functions. To see that the latter collection is substantially larger than the former, consider $\Omega = [0,1]$ equipped with the Lebesgue measure λ, and the tent map $T : [0,1] \to [0,1]$ defined by $T(x) = 1 - |2x-1|$ for all $x \in [0,1]$. The tent map and all its iterates T^n are m.p. functions with respect to λ, and so by Proposition 21 they have a continuous distribution function G_{T^n}. But, they are not injective, and so while they are included in the Carlier and Dana representation, they are absent in ours.

In sum, Proposition 18 improves in two ways their result: (i) it provides a sharper and closed form characterization; (ii) it shows that the characterization of $core\,(f \circ P)$ is a special case of the general calculus characterization provided by Theorems 7 and 16, which apply to any supermodular game and any supermodular measure game, respectively.

The cost of this improvement is that, unlike Carlier and Dana (2001), we have to assume that the space (Ω, Σ) is a standard Borel space. As a matter of fact, while their investigation is based on properties of rearrangements, we use a very different approach based on some nice properties of standard Borel spaces, a class of spaces widely used in Probability Theory.

3. This work has provided a fairly complete analysis of the calculus representation of cores of supermodular games and of the Gateaux and Frechet differentiability of the associated Choquet functionals. The next natural step is to extend the analysis to games that are not necessarily supermodular, a step which is also motivated by recent applications of Choquet functionals in Decision Theory.

A first class of games to consider are those that can be represented as differences of supermodular games. This is a fairly large class of games; for example, it can be shown that it includes all measure games $f \circ P$ with f belonging to $C^{1,1}$, the class of differentiable functions that have a Lipschitz continuous gradient. Naturally, all our differentiability results immediately extend to this class of games. In contrast, the calculus representation of the core of these games turned out to be more complicated, and it is the subject of our current investigation, along with extensions to more general games.

6 Proofs

6.1 Proof of Lemma 1

Point (i) is proved in Marinacci and Montrucchio (2001). As to (ii), in view of (i), the "if" part is trivial. Actually, if $E_n \uparrow \Omega$, there exists some $m \in \mathit{core}(\nu)$, such that $m(E_n) = \nu(E_n)$. Consequently, $\nu(E_n) \to \nu(\Omega)$, analogously for $E_n \downarrow \varnothing$. As to the "only if" part, let $E_n \uparrow \Omega$ and assume that ν is continuous. Let $m \in \mathit{core}(\nu)$. Then $\liminf_n m(E_n) \geq \lim_n \nu(E_n) = m(\Omega)$ and $\liminf_n m(E_n^c) \geq \lim_n \nu(E_n^c) = 0$. In turn, the latter inequality implies that $\limsup_n m(E_n) = m(\Omega) - \liminf_n m(E_n^c) \leq m(\Omega)$, so that $\liminf_n m(E_n) \geq m(\Omega) \geq \limsup_n m(E_n)$. We conclude that $m \in ca(\Sigma)$.

We conclude with (iii). Since $\mathit{core}(\nu)$ is weak*-compact, it is norm bounded and so there is $M > 0$ such that $\|m\| \leq M$ for all $m \in \mathit{core}(\nu)$. Hence, for each finite chain $\{E_i\}_{i=0}^n$, we have $\sum_{i=1}^n |\nu(E_i) - \nu(E_{i-1})| \leq \sum_{i=1}^n |m(E_i) - m(E_{i-1})| \leq M$. This implies that $\|\nu\| < +\infty$. ■

6.2 Proof of Lemma 3

A routine separation argument proves the first part. In particular, as to the non-emptiness of $\partial\nu(X)$, by Lemma 1 there exists $m \in core(\nu)$ such that $m(X \geq t) = \nu(X \geq t)$ for all $t \in \mathbb{R}$.

Let us prove the equivalence among the three statements.
(i) $\Longrightarrow$ (ii). Let $m \in \partial\nu(X_1) \cap \partial\nu(X_2)$. Hence, $\nu(X_1) = \langle X_1, m\rangle$ and $\nu(X_2) = \langle X_2, m\rangle$, and so

$$\nu(X_1) + \nu(X_2) = \langle X_1 + X_2, m\rangle \geq \nu(X_1 + X_2).$$

As ν is superadditive, we conclude that $\nu(X_1) + \nu(X_2) = \nu(X_1 + X_2)$.
(ii) $\Longrightarrow$ (iii). We first prove that $\partial\nu(X_1) \cap \partial\nu(X_2) \subseteq \partial\nu(X_1 + X_2)$. Suppose that $\partial\nu(X_1) \cap \partial\nu(X_2) \neq \varnothing$, the inclusion being trivially true otherwise. Let $m \in \partial\nu(X_1) \cap \partial\nu(X_2)$, so that

$$\nu(X_1) = \langle X_1, m\rangle \text{ and } \nu(X_2) = \langle X_2, m\rangle.$$

Hence, $\nu(X_1 + X_2) = \nu(X_1) + \nu(X_2) = \langle X_1 + X_2, m\rangle$, which implies $m \in \partial\nu(X_1 + X_2)$. Conversely, let us prove that $\partial\nu(X_1 + X_2) \subseteq \partial\nu(X_1) \cap \partial\nu(X_2)$. Pick any $m \in \partial\nu(X_1 + X_2)$ and suppose, *per contra*, that $m \notin \partial\nu(X_1) \cap \partial\nu(X_2)$, say $m \notin \partial\nu(X_1)$. We have

$$\nu(X_1 + X_2) = \langle X_1 + X_2, m\rangle = \langle X_1, m\rangle + \langle X_2, m\rangle > \nu(X_1) + \nu(X_2),$$

a contradiction.
(iii) $\Longrightarrow$ (i). Since $\partial\nu(X_1 + X_2) \neq \varnothing$, we have $\partial\nu(X_1) \cap \partial\nu(X_2) \neq \varnothing$. $\blacksquare$

6.3 Proof of Theorem 5

It suffices to prove that $\partial\nu(X)$ is a singleton. Note that the continuity of ν at Ω implies that $core(\nu) \subseteq ca(\Sigma)$ and, consequently, $\partial\nu(X) \subseteq ca(\Sigma)$ (see Lemma 1). Take any element $m \in \partial\nu(X)$ and, w.l.o.g., assume that $X \geq 0$. It satisfies

$$\int_0^\infty \nu(X \geq t)\, dt = \int_0^\infty m(X \geq t)\, dt. \tag{13}$$

The function $m(X \geq t) - \nu(X \geq t)$ is of bounded variation and so it is continuous for all $t \in \mathbb{R} \setminus C$, where C contains at most countably many elements. As $m(X \geq t) - \nu(X \geq t) \geq 0$, in view of Eq. 13 we infer that $m(X \geq t) = \nu(X \geq t)$ for all $t \in \mathbb{R} \setminus C$. Therefore, given two supergradients $m_1, m_2 \in \partial\nu(X)$, we have

$$m_1(X \geq t) = m_2(X \geq t) \text{ for all } t \in [a,b] \setminus C$$

where $a < \inf_{\omega\in\Omega} X(\omega)$ and $b > \sup_{\omega\in\Omega} X(\omega)$ and C is as above. Let us consider the chain

$$\mathcal{C} = \{X \geq t\}_{t\in[a,b]\setminus C}$$

Our objective is that of choosing a countable subchain of $\mathcal{C}$. As $\lambda\{[a,b] \setminus C\} = b - a$, the set $[a,b] \setminus C$ is dense in $[a,b]$. Therefore, it is not difficult to construct a sequence $q_n \in [a,b] \setminus C$ which is dense in $[a,b] \setminus C$. Consider now the family

$$\mathcal{C}^* = \{q_m > X \geq q_n\}_{n,m}$$

with $q_m > q_n$. Clearly, m_1 and m_2 agree over the sets $\{q_m > X \geq q_n\}$. Since $\mathcal{C}^*$ is a π system, m_1 and m_2 agree over $\sigma(\mathcal{C}^*)$ as well. On the other hand, it is obvious that $\mathcal{C}^*$ is a separating system, provided X is injective. Actually, for any pair ω, $\omega' \in \Omega$, with $\omega \neq \omega'$, there exists some q_n such that $X(\omega) < q_n < X(\omega')$. Therefore, $\omega \in \{q_n > X \geq q_s\}$ for some q_s, while $\omega' \notin \{q_n > X \geq q_s\}$. If $X(\omega) > X(\omega')$ the argument is analogous. Hence, by the Mackey Theorem, $\sigma(\mathcal{C}^*)$ is the Borel σ-algebra Σ. We deduce $m_1 = m_2$ and this ends the proof. ■

6.4 Proof of Theorem 7

Lemma 22 *Let $\nu : \Sigma \to \mathbb{R}$ be a game in $bv(\Sigma)$. The Choquet integral $\nu : B(\Sigma) \to \mathbb{R}$ is Lipschitz continuous, with*

$$|\nu(X) - \nu(Y)| \leq \|\nu\| \|X - Y\| \tag{14}$$

for all $X, Y \in B(\Sigma)$.

Proof. Let us first prove the result when ν itself is a capacity. Suppose $\nu(X) \geq \nu(Y)$. As $X \leq Y + \|X - Y\|$, by monotonicity $\nu(X) \leq \nu(Y) + \|X - Y\| \nu(\Omega)$, which implies

$$|\nu(X) - \nu(Y)| \leq \nu(\Omega) \|X - Y\| \tag{15}$$

when ν is monotonic. Pick now any $\nu \in bv(\Sigma)$. We know that ν is representable as $\nu = \nu^+ - \nu^-$ where ν^+ and ν^- are the positive and negative semivariations of ν respectively. ν^+ and ν^- are monotonic and $\|\nu\| = \nu^+(\Omega) + \nu^-(\Omega)$. Thanks to (15), it follows straightforwardly

$$|\nu(X) - \nu(Y)| \leq \left[\nu^+(\Omega) + \nu^-(\Omega)\right] \|X - Y\|$$

which is (14). ■

Lemma 23 *The set $BI(\Sigma)$ is dense in $B(\Sigma)$ whenever it is non-empty (e.g., when Ω is a standard Borel space).*

Proof. Since the simple functions are dense in $B(\Sigma)$, it will suffice to prove that, given any simple function X, there are injective functions ε close to X, for all $\varepsilon > 0$. W.l.o.g., set $\|X_0\| = 1$, where X_0 is the existing injective function. Let us prove that, for $\lambda > 0$ small enough, the elements $X + \lambda X_0$ are injective. Let $R(X)$ be the (finite) range of X. Define

$$\sigma = \min\{|x_i - x_j| : \ x_i, x_j \in R(X), \, x_i \neq x_j\}.$$

The function $Y = X + \lambda X_0$ is injective when $0 < \lambda < \sigma/2$. For, take any two elements $\omega_1 \neq \omega_2$. If $X(\omega_1) = X(\omega_2)$, then $Y(\omega_1) \neq Y(\omega_2)$ for all $\lambda > 0$. Suppose, in contrast, that $X(\omega_1) \neq X(\omega_2)$, say $X(\omega_1) > X(\omega_2)$. Then,

$$\begin{aligned} Y(\omega_1) - Y(\omega_2) &= [X(\omega_1) - X(\omega_2)] + \lambda[X_0(\omega_1) - X_0(\omega_2)] \\ &\geq \sigma - 2\lambda > 0, \end{aligned}$$

as desired. ■

Proof of Theorem 7. By Eq. 5 of Lemma 3, *core* $(\nu) = \partial\nu(0)$ and, for all $\alpha > 0$ and $\beta \in \mathbb{R}$, it holds

$$\partial\nu(\alpha X + \beta) = \partial\nu(X)$$

whenever $X \in B(\Sigma)$. Hence, since $\alpha X \xrightarrow{\|\cdot\|} 0$ as $\alpha \downarrow 0$, by Eq. (6) we have $\{D\nu(X) : X \in BI(\Sigma)\} \subseteq \partial\nu(0)$, and so $\overline{co}^{w^*}(\{D\nu(X) : X \in BI(\Sigma)\}) \subseteq \partial\nu(0)$. Again by Eq. (6), also the converse inclusion holds, and we conclude that *core* $(\nu) = \overline{co}^{w^*}(\{D\nu(X) : X \in BI(\Sigma)\})$. The restriction to $BI(\Sigma)/\sim$ follows from Corollary 4.

The last part of the theorem is a direct consequence of Lemma 3. For, suppose there exists some $X \in BI(\Sigma)$ such that $\nu(Y+X) = \nu(Y) + \nu(X)$. By Lemma 3, $\partial\nu(Y) \cap \partial\nu(X) \neq \varnothing$, i.e., $D\nu(X) \in \partial\nu(Y)$. Hence, $\nu(Y) = \langle Y, D\nu(X)\rangle$ and the infimum is thus attained. Conversely, suppose the infimum is a minimum, i.e., $\nu(Y) = \langle Y, D\nu(X)\rangle$ for some $X \in BI(\Sigma)$. This implies $D\nu(X) \in \partial\nu(Y)$ and, in turn, $\partial\nu(Y) \cap \partial\nu(X) \neq \varnothing$. By Lemma 3 we conclude that $\nu(Y+X) = \nu(Y) + \nu(X)$. ■

6.5 Proof of Proposition 9

We only prove the converse, as the other direction is due to Choquet (1953). Since $R(P) = [0,1]^n$, there is a collection $\{E_i\}_{i=1}^n$ such that, for each i, $P_i(E_i) = 1$ and $P_i(E_j) = 0$ for all $i \neq j$. For each i, set $E_i^* = E_i \cap \left(\bigcup_{j\neq i} E_j\right)^c$. The collection $\{E_i^*\}_{i=1}^n$ is pairwise disjoint and is such that $P_i(E_i^*) = 1$ and $P_i\left(E_j^*\right) = 0$ for all $i \neq j$. Given $y \in R(P)$, since each component y_i belongs to $[0,1]$, by non-atomicity for each i there exists $E_i^y \subseteq E_i^*$ such that $P_i(E_i^y) = y_i$. Set $E^y = \bigcup_{i=1}^n E_i^y$. The sets $\{E_i^y\}_{i=1}^n$ are pairwise disjoint, and so $P_i(E^y) = \sum_{j=1}^n P_i\left(E_j^y\right) = y_i$. Hence, $P(E^y) = y$.

Now, let $R(P) \ni x \leq y$. Since each x_i belongs to $[0, y_i]$, again by non-atomicity for each i there exists $E_i^x \subseteq E_i^y$ such that $P_i(E_i^x) = x_i$. If we set $E^x = \bigcup_{i=1}^n E_i^x$, by proceeding as before it is easy to see that $P(E^x) = x$. Let $h \geq 0$ be such that $y + h \in R(P)$. Again, by proceeding as before, there exists $E^{y+h} \supseteq E^y$ such that $P\left(E^{y+h}\right) = y + h$.

Set $A = E^y$ and $B = E^x \cup \left(E^{y+h} \backslash E^y\right)$. Then, $A \cup B = E^{y+h}$ and $A \cap B = E^x$. By the supermodularity of $f \circ P$,

$$
\begin{aligned}
f(y) + f(x+h) &= f(P(E^y)) + f\left(P\left(E^x \cup \left(E^{y+h} - E^y\right)\right)\right) \\
&= f(P(A)) + f(P(B)) \leq f(P(A \cup B)) + f(P(A \cap B)) \\
&= f\left(P\left(E^{y+h}\right)\right) + f(P(E^x)) = f(y+h) + f(x),
\end{aligned}
$$

which shows that f is monotone of order 2. ■

6.6 Proof of Theorem 11

We begin with couple of lemmas of independent interest. The first one provides a simple condition, Eq. (16), under which the Choquet functional $\int X d(f \circ P)$ is Gateaux differentiable at an injective function $X \in B(\Sigma)$.

Notice that the mapping G_X can be viewed as a curve in $R(P)$ with endpoints $\underline{0}$ and $\underline{1}$. We denote by C_X its range, that is,

$$C_X = \{x \in \mathbb{R}^n : \; x = G_X(q) \; \text{ for some } q \in \mathbb{R}\}.$$

Clearly, $C_X \subseteq R(P)$.

Lemma 24 *Let $\nu = f \circ P$ be a supermodular measure game, with f lower semicontinuous at 0 and $P(\Omega)$. If $X \in B(\Sigma)$ is injective and if there exists a locally integrable function $\rho_X(t) : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$f(y) = \int_0^{|y|_1} \rho_X(t)\, dt \tag{16}$$

for all $y \in C_X$, then the Gateaux derivative is given by

$$\langle D\nu(X), Y\rangle = \int_\Omega \rho_X\left(|G_X|_1 \circ X\right) Y d\overline{P}, \tag{17}$$

where $|G_X|_1 = \sum_{i=1}^n G_X^i$.

Remark. It is important to note that, by point (ii) of Proposition 10, when $R(P) = [0,1]^n$ we can set $\rho_X(t) = (f \circ \gamma)'(t)$ whenever f is monotone of order 2. For, in this case the function $f \circ \gamma$ is absolutely continuous.

Proof. The curve $q \to G_X(q)$ is continuous since X is injective and each P_i is non-atomic. Consider the arc-length parametrization $\gamma : [0,n] \to C_X$, with $|\gamma(t)|_1 = t$ for each $t \in [0,n]$; that is, γ is the inverse of the map $x \to |x|_1$ restricted to C_X.

We begin by proving the map $s = \gamma^{-1} \circ G_X \circ X : \Omega \to [0,n]$ is measure preserving, namely,

$$\lambda(A) = \overline{P}\left(s^{-1}(A)\right) \tag{18}$$

for all Borel set $A \subseteq [0,n]$, where λ is the Lebesgue measure on $\mathbb{R}$. For, take an interval $[t_1,t_2] \subseteq [0,n]$, with $t_1 \leq t_2$. Let $x_1, x_2 \in C_X$ the unique points on the curve such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. Hence, $|x_1|_1 = t_1$ and $|x_2|_1 = t_2$. Since G_X is monotone, we have

$$\gamma([t_1,t_2]) = \{x \in C_X : \; x_1 \leq x \leq x_2\}$$

Clearly,

$$G_X^{-1}\left(\gamma\left([t_1, t_2]\right)\right) = [q_1, q_2]$$

where q_1 is the minimal element for which $G_X(q_1) = x_2$ and q_2 is the maximal element such that $G_X(q_2) = x_1$ (such elements exists because G_X is continuous). Now

$$\begin{aligned}
\overline{P}\left(s^{-1}\left([t_1, t_2]\right)\right) &= \overline{P}\left(\{q_1 \leq X \leq q_2\}\right) = \sum_{i=1}^{n} P_i\{q_1 \leq X \leq q_2\} \\
&= \sum_{i=1}^{n}\left[G_X^i(q_1) - G_X^i(q_2)\right] = \left|G_X(q_1) - G_X(q_2)\right|_1 \\
&= \left|x_2 - x_1\right|_1 = \left|t_2 - t_1\right| = \lambda\left([t_1, t_2]\right),
\end{aligned}$$

which proves that s is measure preserving.

Since the linear functional $Y \to \int_\Omega \rho\left(\left|G_X \circ X\right|_1\right) Y \, d\overline{P}$ is continuous over $B(\Sigma)$, it can be viewed as signed measure $\overline{m}$ over $B(\Sigma)$. Thanks to (16), fixed any scalar $q^* \in \mathbb{R}$, we have

$$f\left(G_X(q^*)\right) = \int_0^{\left|G_X(q^*)\right|_1} \rho(t)\, dt$$

In view of (18), we can change variable by means of $t = \gamma \circ G_X \circ X = \left|G_X \circ X\right|_1$. This implies:

$$f\left(G_X(q^*)\right) = \int_{X \geq q'} \rho\left(\left|G_X \circ X\right|_1\right) d\overline{P},$$

where q' is the minimal $q \in \mathbb{R}$ such that $G_X(q') = G_X(q^*)$. As $\overline{P}\{q' \leq X < q^*\} = 0$, we have

$$f\left(G_X(q^*)\right) = \int_{X \geq q^*} \rho\left(\left|G_X \circ X\right|_1\right) d\overline{P} = \overline{m}\{X \geq q^*\}$$

Consequently, as q^* was arbitrary, we conclude that for each $q \in \mathbb{R}$ it holds:

$$\overline{m}\{X \geq q\} = f\left(G_X(q)\right) = f\left(P(X \geq q)\right) = \nu(X \geq q). \tag{19}$$

By Theorem 5, the Gateaux derivative $D\nu(X)$ exists at X. On the other hand, the same uniqueness argument used to prove Theorem 5 shows that the equality $\overline{m}\{X \geq q\} = \nu(X \geq q)$ of Eq. (19) implies $\overline{m} = D\nu(X)$, as desired. ■

To state and prove the second lemma (which will be used in Step 3), we need some notation. Given an element $X \in B(\Sigma)$, $\overline{F}_X(q) = \overline{P}\{X \leq q\}$ denotes the distribution function. In the sequel, Z denotes a fixed function in $B(\Sigma)$ which is injective and satisfies $\|Z\| \leq 1$. Moreover, for any scalar q, define the function $Y_q : \Omega \to \mathbb{R}$ by

$$Y_q = I_{\{X>q\}} - I_{\{X<q\}} + \frac{Z}{2} I_{\{X=q\}} \tag{20}$$

Lemma 25 *Let $\nu = f \circ P$ be supermodular. For any $X \in B(\Sigma)$, ν is differentiable at*

$$Y(\omega) = X(\omega) + \int_{\mathbb{R}} Y_q(\omega)\, d\overline{F}_X(q) \tag{21}$$

and $D\nu(Y) \in \partial\nu(X)$, provided that:

(i) $|f'_+| \leq M$, if $n = 1$,

(ii) f is C^1, if $n > 1$.

Proof. It is easy to check that each Y_q is comonotonic with X. In turn, this implies that Y is comonotonic to X. Actually, the following stronger property holds

$$[Y(\omega) - Y(\omega')][X(\omega) - X(\omega')] \geq [X(\omega) - X(\omega')]^2 \tag{22}$$

Partition the space Ω in the following two sets:

$$\begin{aligned} D &= \left\{\omega : \overline{P}(X = X(\omega)) > 0\right\}, \\ D^c &= \left\{\omega : \overline{P}(X = X(\omega)) = 0\right\}. \end{aligned}$$

Clearly, each G^i_X is continuous at the points $q = X(\omega)$ with $\omega \in D^c$; in contrast, if $\omega \in D$, then some G^i_X is not continuous at $q = X(\omega)$. Next we list some important properties.

(i) $X(\omega) \neq X(\omega')$ if and only if $Y(\omega) \neq Y(\omega')$ over D^c;

(ii) Y is injective over D;

(iii) if $Y(\omega) = Y(\omega')$, then $\omega, \omega' \in D^c$.

Let us prove (i). Suppose $X(\omega) = X(\omega')$ with $\omega, \omega' \in D$. It is easy to see that (21) can be written down explicitly as

$$Y(\omega) = X(\omega) + n - 2|G_X|_1(X(\omega)) + \left(1 + \frac{Z(\omega)}{2}\right)\overline{P}(X = X(\omega)) \quad (23)$$

Therefore, $Y(\omega) = Y(\omega')$. On the other hand, if $Y(\omega) = Y(\omega')$ the (22) implies $X(\omega) = X(\omega')$.

As to (ii), let $\omega \neq \omega'$. If $X(\omega) = X(\omega')$, then Eq. (23) entails $X(\omega) \neq X(\omega')$ as Z is injective. If $X(\omega) \neq X(\omega')$, say $X(\omega) < X(\omega')$, then $Y(\omega) < Y(\omega')$ because of Eq. (22).

Finally, let us prove (iii). If $Y(\omega) = Y(\omega')$, Eq. (22) implies $X(\omega) = X(\omega')$. Since we just proved in (ii) that Y is injective on D, it then follows that $\omega, \omega' \in D^c$.

Using (i)-(iii), it is easy to see that, for all $q \in \mathbb{R}$

$$\overline{P}\{Y = q\} = 0,$$

namely, $P_i(Y = q) = 0$ for all i and all q. Hence, the G_Y^i are all continuous and therefore ν is differentiable at Y, as proved below in Step 2. Moreover, as Y is comonotone to X, by Lemma 3 we have $D\nu(Y) \in \partial\nu(X)$. ■

We can now prove Theorem 11. We divide the argument in several steps.

Step 1. Under the hypotheses of Theorem 11, Eq. (10) holds when $X \in B(\Sigma)$ is injective.

Proof. Suppose first that $n = 1$. In this case $C_X = [0, 1]$. Since f is convex and continuous, we can write

$$f(y) = \int_0^y \rho(t)\,dt$$

with $\rho = f'_+$, which is (16). Next, suppose $n > 1$. Since f is Lipschitz and γ is an isometry, the function $f \circ \gamma$ is Lipschitz over $[0, n]$. Consequently, $f \circ \gamma$ is absolutely continuous, and so, γ being differentiable a.e., we have $f(\gamma(t)) =$

$\int_0^t d/du\,[f(\gamma(u))]\,du$ for all $t \in [0,n]$. Hence, if we set $\rho(t) = d/dt\,[f(\gamma(t))]$, we have $f(x) = \int_0^{|x|_1} \rho(u)\,du$ for all $x \in C_X$. On the other hand, if $\nabla f(x)$ denotes the gradient of f, by the chain rule we have:

$$d/dt\,[f(\gamma(t))] = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle.$$

Hence, $\rho(t) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle$. Plugging it into Eq. (17), we get that the Gateaux derivative is

$$\langle D\nu(X), Y\rangle = \int \langle \nabla f(G_X \circ X), \gamma'(|G_X|_1 \circ X)\rangle\, Y\, d\overline{P} \tag{24}$$

Eq. (24) holds for all f which are differentiable and monotone of order 2. Fix now X and take $f(x) = x_i$, with $i = 1, ..., n$. The corresponding game is $\nu = f \circ P = P_i$. Eq. (24) becomes

$$\int Y\, dP_i = \int \gamma_i'(|G_X|_1 \circ X)\, Y\, d\overline{P}$$

where $\gamma_i'(t)$ is the i^{th} component of the vector $\gamma'(t)$. If we set $Y = 1_E$, where E is any element of Σ, we have

$$P_i(E) = \int_E \gamma_i'(|G_X|_1 \circ X)\, d\overline{P}. \tag{25}$$

Clearly, P_i is absolutely continuous with respect to $\overline{P}$ and, by Eq. (25), $\gamma_i'(|G_X| \circ X)$ is the Radon-Nikodym derivative $dP_i/d\overline{P}$. Consequently, getting back to Eq. (24), we can write

$$\begin{aligned}\langle D\nu(X), Y\rangle &= \sum_{i=1}^n \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, \gamma_i'(|G_X|_1 \circ X)\, Y\, d\overline{P} \\ &= \sum_{i=1}^n \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, \frac{dP_i}{d\overline{P}} Y\, d\overline{P} = \sum_{i=1}^n \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, Y\, dP_i,\end{aligned}$$

as desired. ■

Step 2. Under the hypotheses of Theorem 11, Eq. (10) holds when $X \in B(\Sigma)$ is such that G_X is continuous.

Proof. Suppose first that $n > 1$. It follows from our D-representation of the superdifferential given in Proposition 6. We know that

$$\partial\nu(X) = \overline{co}^{w^*}\{w^* - \lim D\nu(X_n)\}$$

where X_n are injective and approaching uniformly to X. Let then X_n be any sequence of injective functions with $\|X_n - X\| \to 0$. By Step 1,

$$\langle D\nu(X_n), Y\rangle = \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_{X_n} \circ X_n)\, Y dP_i.$$

If G_X is continuous, then $G_{X_n} \to G_X$ uniformly. Consequently, $G_{X_n} \circ X_n \to G_X \circ X$, and so $\frac{\partial f}{\partial x_i}(G_{X_n} \circ X_n) \to \frac{\partial f}{\partial x_i}(G_X \circ X)$. As $\left|\frac{\partial f}{\partial x_i}(G_{X_n} \circ X_n)\right| \leq M$, by the Lebesgue dominated convergence theorem, we have

$$\langle D\nu(X_n), Y\rangle \to \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, Y dP_i.$$

Therefore, all the w^*-limits are identical and, consequently, $\partial\nu(X)$ is a singleton. ■

In the last step we consider the Frechet differentiability.

Step 3. Under the hypotheses of Theorem 11, the Choquet integral $\int X d(f \circ P)$ is Frechet differentiable at all $X \in B(\Sigma)$ with continuous distribution function provided f is continuously differentiable on $R(P)$.

Proof. By Proposition 2.8 in Phelps (1993), it suffices to demonstrate that, given an X with continuous distribution, there exists a selection from the superdifferential map $\partial\nu : B(\Sigma) \to 2^{L_1(\Omega,\Sigma,\overline{P})}$, which is norm-to-norm continuous at X. The selection we use is given in Lemma 25. More precisely, given a sequence X_n with $\|X_n - X\| \to 0$, we take the sequence of supergradients $D\nu(Y_n) \in \partial\nu(X_n)$, where

$$\begin{aligned} Y_n(\omega) &= X_n(\omega) + \int_{\mathbb{R}} Y_q^n(\omega)\, d\overline{F}_X(q), \\ Y(\omega) &= X(\omega) + \int_{\mathbb{R}} Y_q(\omega)\, d\overline{F}_X(q), \end{aligned}$$

and the element Z is fixed. By Corollary 4, $D\nu(Y) = D\nu(X)$ Hence, it suffices to prove that $D\nu(Y_n) \to D\nu(Y) = D\nu(X)$ in the L_1 norm.

For convenience, denote $\overline{G}_X = |G_X|_1 = \sum_{i=1}^n G_X^i$. Assume X has a continuous distribution function. Then $\overline{G}_X(q)$ is uniformly continuous over $\mathbb{R}$, since it is continuous and constant outside a compact interval. Denote by $\delta(\varepsilon) > 0$ the function such that $|q_1 - q_2| \le \delta(\varepsilon)$ implies $\left|\overline{G}_X(q_1) - \overline{G}_X(q_1)\right| \le \varepsilon$. The function $\delta(\varepsilon)$ can be chosen strictly decreasing. From the inclusions

$$\{X \ge q + \|X - X_n\|\} \subset \{X_n \ge q\} \subset \{X \ge q - \|X - X_n\|\},$$

we deduce

$$\overline{G}_X(q + \|X - X_n\|) \le \overline{G}_{X_n}(q) \le \overline{G}_X(q - \|X - X_n\|). \tag{26}$$

By Eq. (23),

$$\begin{aligned} |Y_n(\omega) - Y(\omega)| \le \ & \|X_n - X\| + 2\left|\overline{G}_X(X(\omega)) - \overline{G}_X(X_n(\omega))\right| + \\ & 2\left|\overline{G}_X(X_n(\omega)) - \overline{G}_{X_n}(X_n(\omega))\right| + \frac{3}{2}\overline{P}(X_n = X_n(\omega)). \end{aligned}$$

We have $\left|\overline{G}_X(X(\omega)) - \overline{G}_X(X_n(\omega))\right| < \delta^{-1}(\|X - X_n\|)$. Moreover, by Eq. (26),

$$\left|\overline{G}_X(X_n(\omega)) - \overline{G}_{X_n}(X_n(\omega))\right| < \delta^{-1}(\|X - X_n\|),$$

and so

$$|Y_n(\omega) - Y(\omega)| \le \|X_n - X\| + 4\delta^{-1}(\|X - X_n\|) + \frac{3}{2}\overline{P}(X_n = X_n(\omega)).$$

As to the last addendum, we have:

$$\begin{aligned} \overline{P}(X_n = q) \le \ & \overline{P}(|X - q| \le \|X - X_n\|) \\ = \ & \overline{G}_X(q - \|X - X_n\|) - \overline{G}_X(q + \|X - X_n\|) \le \delta^{-1}(2\|X - X_n\|). \end{aligned}$$

Summing up, we have

$$\|Y_n - Y\| \le \|X_n - X\| + 4\delta^{-1}(\|X - X_n\|) + (3/2)\,\delta^{-1}(2\|X - X_n\|),$$

and we conclude that $\|Y_n - Y\| \to 0$ as $\|X_n - X\| \to 0$.

We have now to prove that $D\nu(Y_n) \xrightarrow{L_1} D\nu(Y) = D\nu(X)$. We first show that the convergence occurs $\overline{P}$-a.e.. Since $G^i_{Y_n} \circ Y_n \to G^i_Y \circ Y$ for all i uniformly, we have, $\overline{P}$-a.e.,

$$\nabla f(G_{Y_n} \circ Y_n) \cdot dP/d\overline{P} \to \nabla f(G_Y \circ Y) \cdot dP/d\overline{P}.$$

Hence, $D\nu(Y_n) \xrightarrow{\overline{P}\text{-a.e.}} D\nu(Y) = D\nu(X)$. To prove that this sequence converges in L_1, it suffices to prove that it is uniformly integrable. But, this is true because this sequence belongs to *core* (ν), which is $\sigma(L_1, L_\infty)$-compact. This completes the proof of the theorem. ■

6.7 Proposition 12

As $n = 1$, pick f'_+ as f'. We know that f'_+ is continuous over $[0,1]$, except at most a countable set C. As G_X is continuous, $G_X \circ X$ is measure preserving. Therefore, the set $(G_X \circ X)^{-1}(C)$ is P-negligible. Hence, $f'(G_{X_n} \circ X_n) \to f'(G_X \circ X)$ almost surely. ¿From now on, the proof is similar to that we just used for the general case $n \geq 1$. ■

6.8 Theorem 14

Consider two functions as in Lemma 25:

$$\begin{aligned} Y(\omega) &= X(\omega) + \int_{\mathbb{R}} Y_q(\omega)\, d\overline{F}_X(q), \\ \widehat{Y}(\omega) &= X(\omega) + \int_{\mathbb{R}} \widehat{Y}_q(\omega)\, d\overline{F}_X(q). \end{aligned}$$

In the first one, we use a function $Z \in B(\Sigma)$, while in $\widehat{Y}$ we use $-Z$. Lemma 25 implies that $D\nu(Y), D\nu\left(\widehat{Y}\right) \in \partial\nu(X)$. It remains to show that $D\nu(Y) \neq D\nu\left(\widehat{Y}\right)$.

By assumption, there exists some q_1 such that $\overline{P}\{X = q_1\} > 0$. Let $\overline{\omega} \in \{X = q_1\}$ and compute $Y(\overline{\omega})$ and $\widehat{Y}(\overline{\omega})$. In view of (23), we have

$$\begin{aligned} Y(\overline{\omega}) &= \widehat{q}_1 + (1/2)\, z(\overline{\omega})\, \overline{P}\{X = q_1\} \\ \widehat{Y}(\overline{\omega}) &= \widehat{q}_1 - (1/2)\, z(\overline{\omega})\, \overline{P}\{X = q_1\} \end{aligned}$$

where

$$\widehat{q}_1 = q_1 + n - 2\left|G_X\right|_1(q_1) + \overline{P}\{X = q_1\}$$

We now calculate the distributions $G_Y^i(q)$ and $G_{\widehat{Y}}^i(q)$, whenever $q = Y(\overline{\omega})$ and $q = \widehat{Y}(\overline{\omega})$ respectively. Some tedious algebra leads to:

$$\begin{aligned}
G_Y^i \circ Y(\overline{\omega}) &= P_i\{\{z \geq z(\overline{\omega})\} \cap \{X = q_1\}\} + P_i\{X > q_1\}, \\
G_{\widehat{Y}}^i \circ \widehat{Y}(\overline{\omega}) &= P_i\{\{z \leq z(\overline{\omega})\} \cap \{X = q_1\}\} + P_i\{X > q_1\} \\
&= P_i\{X \geq q_1\} - P_i\{\{z \geq z(\overline{\omega})\} \cap \{X = q_1\}\}
\end{aligned}$$

Using the vector notation:

$$\begin{aligned}
G_X(q_1) &= (P_i\{X \geq q_1\})_{i=1}^n, \\
G_X\left(q_1^+\right) &= (P_i\{X > q_1\})_{i=1}^n, \\
h(\omega) &= (P_i\{\{z \geq z(\overline{\omega})\} \cap \{X = q_1\}\})_{i=1}^n,
\end{aligned}$$

we can write

$$\begin{aligned}
G_Y \circ Y(\overline{\omega}) &= G_X\left(q_1^+\right) + h(\overline{\omega}), \\
G_{\widehat{Y}} \circ \widehat{Y}(\overline{\omega}) &= G_X(q_1) - h(\overline{\omega}),
\end{aligned}$$

where $G_X\left(q_1^+\right) \leq G_X(q_1)$ and $G_X\left(q_1^+\right) \neq G_X(q_1)$. It is easy to see that $G_{\widehat{Y}} \circ \widehat{Y}(\overline{\omega}) \geq G_Y \circ Y(\overline{\omega})$ over a subset of $\{X = q_1\}$ of positive measure. Let $I \subseteq \{1, 2, ..., n\}$ be the non-empty set of i such that $P_i\{X > q_1\} > 0$. We have $G_{\widehat{Y}}^i \circ \widehat{Y}(\overline{\omega}) > G_Y^i \circ Y(\overline{\omega})$ for $i \in I$, and $G_{\widehat{Y}}^i \circ \widehat{Y}(\overline{\omega}) = G_Y^i \circ Y(\overline{\omega})$ for $i \notin I$. This over a set of positive measure. By strict monotonicity of order 2, we have

$$\frac{\partial f}{\partial x_i}\left(G_{\widehat{Y}} \circ \widehat{Y}(\overline{\omega})\right) > \frac{\partial f}{\partial x_i}\left(G_Y \circ Y(\overline{\omega})\right)$$

for $i \in I$. It is easy to see that $dP_i/d\overline{P} = 0$ over $\{X = q_1\}$ for $i \notin I$. Clearly, $\sum_{i=1}^n dP_i/d\overline{P} = 1$ $\overline{P}$-a.e., and so

$$\nabla f\left(G_{\widehat{Y}} \circ \widehat{Y}(\omega)\right) \cdot dP/d\overline{P} > \nabla f\left(G_Y \circ Y(\omega)\right) \cdot dP/d\overline{P}$$

for all ω of a set of positive probability. This means that $D\nu\left(\widehat{Y}\right) \neq D\nu(Y)$, as desired. ■

6.9 Proof of Theorem 16

We know that $core(\nu) \subseteq L^1\left(\Sigma, \overline{P}\right) \subseteq ca(\Sigma)$. It is easy to see that the weak* topology of $ca(\Sigma)$ agrees over $L^1\left(\Sigma, \overline{P}\right)$ with the weak topology $\sigma(L^1, L^\infty)$. Eq. (10) of Theorem 11 can be written as

$$\begin{aligned} \langle D\nu(X), Y\rangle &= \sum_{i=1}^{n} \int \frac{\partial f}{\partial x_i}(G_X \circ X)\, Y dP_i \\ &= \int \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\overline{P}}\right) Y d\overline{P}. \end{aligned}$$

Hence, by Theorem 7, we have

$$core(\nu) = \overline{co}^{\sigma\left(L^1, L^\infty\right)} \left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(G_X \circ X) \frac{dP_i}{d\overline{P}} : X \in BI(\Sigma) / \sim\right\}$$

On the other hand, it is well known that the closed convex hull in the norm and in weak topology coincide, and so the first claim follows. The second formula is just Theorem 7. ■

6.10 Proof of Corollary 17

Let $g \in \overline{co}^s \{\mathcal{D}\}$. By definition, there exists a sequence $\{g_n\}_n \subseteq co(\mathcal{D})$ such that $g_n \to g$ in L^1. This implies that $g_n \to g$ in measure, which in turn implies the existence of a subsequence g_{n_k} such that, $\overline{P} - a.s.$, $\lim_k g_{n_k} = g$. Conversely, let $g = \overline{P} - \lim_{n\to\infty} g_n$ for a sequence $\{g_n\}_n \subseteq co(\mathcal{D})$. This implies that $g_n \to g$ in measure. Moreover, as $\{g_n\}_n \subseteq core(\nu)$, by the $\sigma(L^1, L^\infty)$-compactness of $core(\nu)$ the sequence $\{g_n\}_n$ is uniformly integrable (see Corollary IV.8.11 of Dunford and Schwartz 1957). Hence, by the Vitali Converge Theorem (see Theorem IV.10.9 of Dunford and Schwartz 1957), we have $g_n \to g$ in L_1. ■

6.11 Proof of Proposition 21

Suppose $X \in B(\Sigma)$ has a continuous distribution function. By what established in proving Lemma 24, $G_X \circ X$ is a measure preserving function. Conversely, let $s : \Omega \to [0,1]$ be any measure preserving map. Consider the map $s_1 = 1 - s$, which is m.p. as well. It is easy to check that $G_{s_1}(q) = 1 - q$, and so $G_{s_1} \circ s_1 = 1 - s_1 = s$. This proves the proposition. ■

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