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Multivariate option pricing with copulas

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Abstract

In this paper we suggest the adoption of copula functions in order to price multivariate contingent claims. Copulas enable us to imbed the marginal distributions extracted from vertical spreads in the options markets in a multivariate pricing kernel. We prove that such kernel is a copula fucntion, and that its super-replication strategy is represented by the Fréchet bounds. As applications, we provide prices for binary digital options, options on the minimum and options to exchange one asset for another. For each of these products, we provide no-arbitrage pricing bounds, as well as the values consistent with independence of the underlying assets. As a final reference value, we use a copula function calibrated on historical data.

Keywords: option pricing, basket options, copula functions, non-normal returns.

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1. Introduction

One of the open issues in contingent claim pricing is the extension of the well-known risk-neutral valuation technique to the multivariate case, that is the case of contingent claims written on "baskets" of several underlying assets, rather than on a single security.

Multivariate contingent claims have been "traditionally" represented by options on futures, whenever the underlying entails a delivery grade or quality option. Options on the maximum, minimum or options to exchange one asset for another are other "traditional" examples of multivariate claims. Recently, binary digital options have been added to the list. More generally, one can envisage a multivariate feature in every OTC contract, due to the joint riskiness of the underlying and the counterpart business (Klein, 1996).

As the existing literature demonstrates, the pricing problem of multivariate contingent claims is not elementary, whenever the simplifying – and sometimes unrealistic – hypothesis of independence or perfect correlation between the underlying risks is abandoned. It becomes even more involved when another questionable assumption is dropped, that of joint normality.

In this paper we abandon both the linear correlation and gaussian assumptions. We explore copula functions as pricing devices. Copulas allow us not only to separate the impact on the joint distribution of the marginals and the association, but also to exploit non-parametric measures of the latter.

The paper is structured as follows: section 2 presents the multivariate pricing problem in detail. Section 3 introduces the notion of copula, reviews some of its properties and introduces the relationship with well-known non-parametric association measures. Section 4 explains how to use copulas to represent the multivariate pricing kernel of the economy. Section 5 presents an extensive application to four market indices (Mib30, S&P500, FTSE, DAX), for which we price binary digital options, options on the minimum and options to exchange one asset for another. Section 6 concludes.

2. Multivariate Contingent Claims

In mathematical terms, the multivariate feature of a contingent claim shows up in a pay-off, which in general can be written as

$$g(f(S_i(T), T; i = 1, 2...n))$$

where $g(.)$ is a univariate pay-off function which identifies the derivative contract, $f(.)$ is a multivariate function which describes how the n underlying securities determine the final cash-flow, S_i denotes the price of the $i-th$ underlying security and T is the contract maturity. In what follows, for notational convenience, we skip the reference to T whenever it is not needed.

As an example, in the case of rainbow call options g is the familiar payoff function

$$g(f) = \max(f - K, 0)$$

where K is the strike price, while $f(\cdot)$ may select the minimum (or maximum or some kind of average) of n assets:

$$f(S_i(T), T; i = 1, 2...n) = \min(S_i(T); i = 1, 2...n)$$

The option on the minimum (or maximum) was first studied by Stulz (1982) in the lognormal case. In the same setting, Margrabe (1978) had already studied the most specific instance of the option to exchange.

Another example, which is easier to address, is the multivariate digital option. In this case the g function is simply a multiplicative constant; f spots the event that each underlying is greater than or equal to some corresponding threshold K_i

$$f(S_i(T), T; i = 1, 2...n) = I_{\{(S_1(T) \geq K_1) \cap .. \cap (S_n(T) \geq K_n)\}}$$

where I is the indicator function.

The multi-asset feature has a long standing history in fairly standard and mature derivative markets. The most standard case is offered by the market of options on futures: as typically the standardization feature of futures contracts entails a delivery grade option (or quality option), the option on the futures contract may be seen as an option written on a basket of deliverable products. The quality option issue, however, has not posed much of a problem as for the pricing techniques: in fact, the standardization feature of futures contracts also implies that most of the products eligible for delivery were chosen in such a way as to grant a high degree of similarity. In other words, the products chosen as deliverable under the contract are typically strongly correlated and considering them perfectly correlated is not much of a mistake. This is what is done when using a one factor model of the term structure in order to evaluate the quality option on long term interest futures and options .

It cannot be ignored that the perfect correlation assumption always leads to an approximation of the problem. Even in the case of the quality option in interest rate futures we know that this approach is not able to take into account some market anomalies, such as coupon or seasoning effects, that may play a relevant role in the determination of the cheapest asset to deliver. Besides this, there are also cases in which the imperfect correlation issue cannot be avoided, as it represents the main motivation which inspires the product. In fact, the pricing task gets more involved for products in which the multiasset feature is meant to provide diversification. The most straightforward example is offered by call options written on the minimum or maximum among some market indices. In these cases, ignoring imperfect dependence among the markets may lead to substantial mispricing of the products, as well as to inaccurate hedging policies and unreliable risk evaluations.

While the multiasset pricing problem may be already complex in a standard gaussian world, the evaluation task is compounded by the well known evidence of departures from normality. Following the stock market crash in October 1987, departures from normality have shown up in the well known effects of smile and term structure of volatility and have become common ground of work of academics and traders.

Of course, jointly taking into account non-normality of yields and their dependence structure makes the two problems far more involved. As a simple example of this complexity, just consider the fact that the standard linear correlation figure that we use to measure dependency cannot be safely used in a non-normal world, as it may not be able to take values in the whole range between -1 and +1: the effect is that in a non-gaussian world we may well observe a correlation figure lower than 1 in a case in which there is perfect dependence between the variables. So, the correlation figure may lead to mis-representation of the degree of diversification in a portfolio. By the same token, in a non-gaussian world a trader may pursue the task of modifying the correlation figure of his portfolio to a value which is simply impossible to reach under the real multivariate distribution in the data.

A possible strategy to address the problem of dependency under non-normality is to separate the two issues, i.e. working with non-gaussian marginal probability distributions and using some technique to combine these distributions in a multivariate setting. This is the approach followed by Rosenberg (1999), who uses the set of Plackett distributions to price bivariate contingent claims consistently with given marginals. In the sequel we generalize his approach using copula functions, of which the Plackett family is only a specific case. The main advantage of the copula approach to pricing is to write the multivariate pricing kernel as a function of univariate pricing functions. This enables us to carry out sensitivity analysis with respect to the dependence structure of the underlying assets, separately from that on univariate prices.

3. Mathematical background

In what follows we give the definition of copula function and some of its basic properties, while we refer the reader interested in a more detailed treatment to Nelsen (1999) and Joe (1997). Here we stick to the bivariate case: nonetheless, all the results carry over to the general multivariate setting.

Definition 3.1. *A two-dimensional copula C is a real function defined on $I^2 \stackrel{d}{=} [0,1] \times [0,1]$, with range $I \stackrel{d}{=} [0,1]$, such that*
for every (v, z) of $I^2, C(v, 0) = 0 = C(0, z)$, $C(v, 1) = v, C(1, z) = z$;
for every rectangle $[v_1, v_2] \times [z_1, z_2]$ in I^2,with $v_1 \leq v_2$ and $z_1 \leq z_2$, $C(v_2, z_2) - C(v_2, z_1) - C(v_1, z_2) + C(v_1, z_1) \geq 0$

As such, it can represent the joint distribution function of two standard uniform random variables U_1, U_2:

$$C(u_1, u_2) = \Pr(U_1 \leq u_1, U_2 \leq u_2)$$

We can use this feature in order to re-write via copulas the joint distribution function of two (even non-uniform) random variables. The most interesting fact about copulas in this sense is Sklar's theorem:

Theorem 3.2 (Sklar (1959)). *Let $F(x, y)$ be a joint distribution function with continuous marginals $F_1(x)$ and $F_2(y)$. Then there exists a unique copula such that*

$$F(x, y) = C(F_1(x), F_2(y)) \tag{3.1}$$

Conversely, if C is a copula and $F_1(x)$, $F_2(y)$ are continuous univariate distributions, $F(x, y) = C(F_1(x), F_2(y))$ is a joint distribution function with marginals $F_1(x)$, $F_2(y)$.

The theorem suggests then to represent the multiplicity of joint distributions consistent with given marginals through copulas.

Three specific copulas are worth mentioning: the *product* copula, the *minimum* and the *maximum* copulas. Families of copulas which encompass all of these copulas are called *comprehensive*. As for the first, the copula representation of a distribution F degenerates into the so-called product copula, $C(v,z) = v \cdot z$, if and only if X and Y are independent. As for the others, they derive from the well-known Fréchet-Hoeffding result in probability theory, stating that every joint distribution function is constrained between the bounds

$$\max(F_1(x) + F_2(y) - 1, 0) \leq F(x,y) \leq \min(F_1(x), F_2(y)) \tag{3.2}$$

As a consequence of Sklar's theorem, the Fréchet-Hoeffding bounds exist for copulas too:

$$\max(v + z - 1, 0) \leq C(v,z) \leq \min(v,z)$$

In correspondence of the extreme copula bounds, there is perfect positive and negative dependence between the variables, and every variable can be obtained as a deterministic function of the other (see Embrechts, McNeil and Straumann,1999 for a proof). We can state that

Theorem 3.3 (Hoeffding (1940), Frechet (1957)). *If the continuous random variables X and Y have the copula $\min(v,z)$, then there exists a monotonically increasing function U such that*

$$Y = U(X) \qquad U = F_2^{-1}(F_1)$$

where F_2^{-1} is the generalized inverse of F_2. If instead they have the copula $\max(v+z-1,0)$, then there exists a monotonically decreasing function L such that

$$Y = L(X) \qquad L = F_2^{-1}(1 - F_1)$$

The converse of the previous results holds too.

In the first case, X and Y are called *comonotonic*, while in the second they are deemed *countermonotonic*.

Copulas are linked to non-parametric association measures by useful relationships. As an example, Kendall's τ may be proved to be

$$\tau = 4 \int\int_{I^2} C(v,z)dC(v,z) - 1$$

while Spearman's ρ measure is given by

$$\rho = 12 \int\int_{I^2} C(v,z)dvdz - 3 \tag{3.3}$$

As a consequence of these results, it can be proved that the bounds of τ and ρ are always -1 and +1, while this is not true for the linear correlation coefficient, whose value depends on the specific shape of the marginal distribution functions, as can be glanced from the relationship

$$cov(X,Y) = \int\int_D (F(x,y) - F_1(x)F_2(y))\, dxdy \tag{3.4}$$

where D is the cartesian product of X and Y's domains.

A particular class of copulas, so-called Archimedean, is particularly easy to handle and will be used in this paper (see Genest and MacKay, 1986, Genest and Rivest, 1993).

Archimedean copulas may be constructed using a generating fuynction $\phi : [0,1] \to [0,\infty]$ continuous, strictly decreasing, convex and such that $\phi(1) = 0$. Given such a function ϕ a copula may be generated computing

$$C(v,z) = \phi^{[-1]}\ (\phi(v) + \phi(z)) \tag{3.5}$$

where $\phi^{[-1]}$ is the pseudo-inverse of ϕ.

Among Archimedean copulas, we are going to consider only the one-parameter ones, which are constructed using a generator $\varphi_\alpha(t)$, indexed by the parameter α. The table below describes well known Archimedean copulas and their generators (for a complete list see Nelsen, 1999):

Family	$\phi_\alpha(t)$	range for α	$C(v,z)$
Gumbel (1960)	$(-\ln t)^\alpha$	$[1,+\infty)$	$\exp\left\{-[(-\ln v)^\alpha + (-\ln z)^\alpha]^{1/\alpha}\right\}$
Clayton (1978)	$\frac{1}{\alpha}(t^{-\alpha} - 1)$	$[-1,0) \cup (0,+\infty)$	$\max\left[(v^{-\alpha} + z^{-\alpha} - 1)^{-1/\alpha}, 0\right]$
Frank (1979)	$-\ln \frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}$	$(-\infty,0) \cup (0,+\infty)$	$-\frac{1}{\alpha}\ln\left(1 + \frac{(\exp(-\alpha v)-1)(\exp(-\alpha z)-1)}{\exp(-\alpha)-1}\right)$

Table 3: some Archimedean copulas

The second and third are particularly interesting since their are *comprehensive* according to the definition above. The Gumbel family gives the product copula if $\alpha = 1$ and the upper Fréchet bound $\min(v,z)$ for $\alpha \to +\infty$: it describes positive association only. The Clayton family gives the product copula if $\alpha \to 0$, the lower

Fréchet bound $\max(v+z-1,0)$ when $\alpha=-1$, and $\min(v,z)$ for $\alpha \to +\infty$. To end up with, the Frank's family, which is discussed at length in Genest (1987), reduces to the product copula if $\alpha \to 0$, and reaches the lower and upper Fréchet bounds for $\alpha \to -\infty$ and $\alpha \to +\infty$, respectively.

4. Pricing multivariate contingent claims

As a first step to price multivariate contingent claims with copulas we may focus on the case of digital binary options. Every practitioner would agree that this case is not simply of an academic interest, as this kind of derivative is present in some widely known structured finance products such as digital binary notes, that is debt instruments promising to pay a fixed coupon if the prices of two assets are above some predefined strike levels at some future date and zero otherwise. In order to price and hedge products like these one has to find a replicating strategy for the digital binary option. As there is not a market for them, this task would lead us straight into the problem of market incompleteness. One could even argue that the problem is further complicated by the fact that replicating a digital note written on a single underlying asset would also be involved, as a market of digital options is not available for each and every strike: while this is true, we may assume that products like these may be satisfactorily approximated using vertical spreads, as suggested in the seminal work by Breeden and Litzenberger (1978) and in all the more recent literature drawing from that idea (Shimko 1993, Rubinstein 1994, Derman and Kani 1994a,b to quote a few).

In order to focus on the multivariate feature of the pricing problem, we assume that we may replicate and price two single digital options with the same exercise date T written on the underlying markets S_1 and S_2 for strikes K_1 and K_2 respectively. Our problem is then to use these products to replicate a double binary option which pays 1 if $X_1 \geq K_1$ and $X_2 \geq K_2$ and zero otherwise. Let us first break the sample space, which coincides with the positive hortant, in the four relevant regions

	State H	State L
State H	$S_1 \geq K_1, S_2 \geq K_2$	$S_1 \geq K_1, S_2 < K_2$
State L	$S_1 < K_1, S_2 \geq K_2$	$S_1 < K_1, S_2 < K_2$

Table 1: breaking down the sample space for the digital option.

The binary digital option pays 1 unit only if both of the assets are in state H, that is in the upper left cell of the table. The single digital options written on assets 1 and 2 pay in the first row and the first column respectively. In table 2 below we sum up the payoffs of these different assets and we make clear which prices are observed in the market. We assume that the risk free rate be zero and we denote with P_1, P_2 the prices of the single digital options

	Price	HH	HL	LH	LL
Digital option asset 1	P_1	1	1	0	0
Digital option asset 2	P_2	1	0	1	0
Risk free asset	1	1	1	1	1
Digital binary option	?	1	0	0	0

Table 2: Prices and payoffs for digital options

Our problem is to use no-arbitrage arguments to recover the price of the binary digital option. Some interesting no arbitrage implications can be easily obtained by comparing its pay-off with that of portfolios of the single digital options and the risk free asset. Furthermore, as we face a pricing problem in an incomplete market, we may expect to find super-replication strategies leading to pricing bounds for the binary asset. The following proposition states such bounds for the price.

Proposition 4.1. *The no-arbitrage price $P(S_1 \geq K_1, S_2 \geq K_2)$ of a digital binary option is bounded by the inequality*

$$\max(P_1 + P_2 - 1, 0) \leq P(S_1 \geq K_1, S_2 \geq K_2) \leq \min(P_1, P_2) \tag{4.1}$$

Proof: assume first that the right side of the inequality is violated: say that, without loss of generality, it is $P(S_1 \geq K_1, S_2 \geq K_2) > P_1$; in this case selling the binary digital option and buying the single digital option would allow a free lunch in the state $[S_1 \geq K_1, S_2 < K_2]$. As for the left side of the inequality, it is straightforward to see that P must be non-negative. There is also a bound $P_1 + P_2 - 1$: assume in fact that $P_1 + P_2 - 1 > P(S_1 \geq K_1, S_2 \geq K_2)$; in this case buying the binary digital option and a risk-free asset and selling the two single digital options would allow a free lunch in the state $[S_1 < K_1, S_2 < K_2]\square$.

The proposition exploits a static super-replication strategy for the binary digital option: the lower and upper bounds have a direct financial meaning, as they describe the pricing bounds for long and short positions in the double binary options. We may take one step further and investigate the features of a pricing function $P(S_1 \geq K_1, S_2 \geq K_2) = C(P_1, P_2)$. It may be easily checked that the following requirements amount to rule out arbitrage opportunities.

Proposition 4.2. *The no-arbitrage pricing function $C(v, z)$ fulfills the following requirements:*

- it is defined in $I^2 = [0, 1] \times [0, 1]$ and takes values in $I = [0, 1]$;
- for every v and z of $I^2, C(v, 0) = 0 = C(0, z)$, $C(v, 1) = v, C(1, z) = z$;
- for every rectangle $[v_1, v_2] \times [z_1, z_2]$ in I^2,with $v_1 \leq v_2$ and $z_1 \leq z_2$,

$$C(v_2, z_2) - C(v_2, z_1) - C(v_1, z_2) + C(v_1, z_1) \geq 0$$

Proof: The first condition, that the price of the single digital options and that of the digital binary option should be constrained in the unit interval, is trivial, and is the standard no-arbitrage condition that applies to Arrow-Debreu prices. As for the second condition, it follows directly from the no-arbitrage inequality (4.1), by substituting the values 0 and 1 for $v = P_1$ or $z = P_2$. As for the last requirement, consider taking two different strike prices $K_{11} > K_{12}$ for the first security, and $K_{21} > K_{22}$ for the second. Denote with v_1 the price of the first digital corresponding to the strike K_{11}; with v_2 that of the first digital for the strike K_{12} and use an analogous notation for the second security. Then, the third condition above can be re-written as

$$P(S_1 \geq K_{12}, S_2 \geq K_{22}) - P(S_1 \geq K_{12}, S_2 \geq K_{21}) +$$

$$-P(S_1 \geq K_{11}, S_2 \geq K_{22}) + P(S_1 \geq K_{11}, S_2 \geq K_{21}) \geq 0$$

As such, it implies that a spread position in binary options paying one unit if the two underlying assets end in the region $[K_{12}, K_{11}] \times [K_{22}, K_{21}]$ cannot have negative value[1]$\square$.

Matching the two propositions above with the mathematical definitions given in the previous paragraph we may restate the main results of our analysis as

Proposition 4.3. *The arbitrage-free pricing kernel of a multivariate contingent claim is a copula function and the corresponding super-replication strategies are represented by the Fréchet bounds.*

[1] This property is akin to the requirement, in the one dimensional pricing problem, that the option price be decreasing and convex in the strike.

It must be stressed that in order to prove the result we did not rely on any assumption concerning the probabilistic nature of the arguments of the pricing function: these are only required to be no-arbitrage prices of single digital options. In this respect, the may also be viewed as capacities, i.e. non-additive functionals, rather than probability measures, and this enables us to extend the analysis to the presence of frictions or to the case in which the market for options written on any single underlying asset is incomplete. In the case in which the market for single digital options is complete, so that these options can be exactly replicated and the prices are the risk-neutral probabilities

$$P_1 = \Pr\left(S_1 \geq K_1\right), P_2 = \Pr\left(S_2 \geq K_2\right)$$

we are allowed to give a probabilistic interpretation of the price of a digital binary option, resorting directly to Sklar's theorem.

First, the price is a function taking as arguments the two single digital options prices: a crucial result is that the restrictions which rule out arbitrage opportunities are the same that define a copula function in multivariate statistics. Second, as the arguments of such function are digital options prices, the risk neutral valuation principle implies that also the price of the multivariate contingent claim is a distribution function under the risk-neutral probability measure:

$$C(P_1, P_2) = \Pr(S_1 \geq K_1, S_2 \geq K_2) \tag{4.2}$$

Finally, Sklar's theorem implies, not surprisingly, that the prices of digital binary options are multivariate distribution functions, as well as the price of univariate digital options represents the corresponding marginal.

The strength of these results is that they enable us to break the multivariate pricing kernel of the economy into a function of marginal univariate kernels: we may then extract separately marginal pricing kernels from vertical spreads and the multivariate pricing kernel of the economy from the dependence structure in the data.

It may be easily checked that the two no-arbitrage bounds in (4.1), which we interpreted as the buyer and seller's prices, represent the financial application of the minimal and maximal copulas of the Fréchet-Hoeffding inequality. The financial meaning of the two bounds becomes even more relevant if we exploit the result in theorem 3.3, that associates the lower and upper bounds to the case of perfect negative and positive dependence between the two underlying assets. Intuitively, this means that the value of any of the two assets can be recovered exactly once the value of the other is known. This result is particularly useful since it enables us to reduce the multivariate pricing problem to a standard univariate problem, at least as far as super-replication strategies are concerned, and can be usefully exploited to recover pricing bounds for any bivariate derivative contract.

If we now have to recover the pricing bounds of a generic bivariate derivative contract with payoff $g(f(S_1, S_2))$ we may obtain

Proposition 4.4. *If two underlying assets $S_i, i = 1, 2$ have marginal pricing kernels $P_i, i = 1, 2$ and $g(f(S_1, S_2))$ denotes the payoff of a bivariate derivative contract whose price is $G(S_1, S_2)$ we must have*

$$E[g_l(S_1)] \leq G(S_1, S_2) \leq E[g_u(S_1)]$$

where

$$g_l(S_1) \overset{d}{=} g\left(f\left(S_1, P_2^{-1}\left(1 - P_1(S_1)\right)\right)\right)$$

$$g_u(S_1) \overset{d}{=} g\left(f\left(S_1, P_2^{-1}\left(P_1(S_1)\right)\right)\right)$$

and the expectation is computed under the probability measure P_1

The evaluation of the two pricing bounds typically would require numerical integration. The numerical evaluation of the pricing bounds could exploit the algorithm used in Cherubini and Esposito (1995): if for example we would like to find the upper bound we could partition the domain of S_1using the fixed points of the function $P_2^{-1}\left(P_1(S_1)\right)$ and integrate the relevant pay-off over each interval.

An interesting question is how wide is the range between the two pricing bounds. It may be the case that assuming perfect dependence leads us to overlook the most relevant feature of a multivariate contingent claim. In this case, relying on some figure representing the dependence structure in the data may help to yield a more precise evaluation of the contingent claim. Indeed, knowing the dependence structure in the data may help to characterize the copula function used in order to price the multivariate contingent claim, as we are going to show in the next sections.

5. Empirical applications

In this section we apply copulas to the pricing problem of multivariate derivatives written on four indices: MIB30, S&P500, FTSE, DAX.

We first derive the risk-neutral marginal density function of each index, using the technique of Shimko (1993)[2]. We then obtain the lower and upper Fréchet bounds for their joint distribution functions, which are the pricing kernel bounds. In order to pick out a single value between these bounds, one would need to observe at least one price of a multivariate claim. Since typically these products are not traded on organized markets, we present a sensitivity analysis of the multivariate option values with respect to the dependence structure of the underlying assets. As a reference case, we choose a price consistent with the dependence statistics estimated on historical data using a non parametric procedure.

5.1. Descriptive statistics and implied marginal densities

The data used for implied volatilities, time to maturity, dividend points and risk-free rate were downloaded from Bloomberg on March 27, 2000 and refer to European calls closing prices with June expiration and different strikes. As for the non-parametric dependence estimation, we used the time series of the corresponding indices, from January 2, 1999 to March 27, 2000.

We estimate the risk-neutral marginal distribution of each index (MIB30, S&P500, FTSE, DAX) on the cross section data on European calls. As a result, given a quadratic smile function $\sigma(K) = A_0 + A_1 K + A_2 K^2$, where K is the strike price, we recover the risk-neutral distribution function F_S of the underlying S:

$$F_S(s) = 1 + sn(D_2(s))\sqrt{T-t}(A_1 + 2A_2 s) - N(D_2(s)) \tag{5.1}$$

$$D_2(s) = \frac{\ln \frac{X}{sB(T-t)}}{\sigma(s)\sqrt{T-t}} - \frac{1}{2}\sigma(s)\sqrt{T-t}$$

where $T-t$ is the time to maturity, $n(\cdot)$ and $N(\cdot)$ are respectively the density and the distribution of the standard normal, X is the current value of the underlying, less the discounted dividends, and $B(T-t)$ is the risk-free discount factor for the same maturity of the option. Figure 6.1. reports the marginal risk-neutral density functions obtained from the data.

Insert here figure 6.1

[2] The estimation technique for the marginals can be changed without modifying the core of our procedure.

Using the four marginals so obtained and (3.4), we numerically compute the lower and upper Fréchet bounds for the joint distributions. Substituting them in (3.3) we recover the correlation bounds in figure 6.2 below.

Insert here figure 6.2

The Fréchet bounds could be exploited for super-replication pricing of multivariate claims. Before doing that, we present the results of copula estimation using historical data, that will serve as a reference point for pricing: the copula estimation procedure is described in Frees and Valdez (1998) and based on Genest and Rivest (1993).

In figure 6.3 we report the estimated parameters α for the families of Archimedean copulas described in table 3, as well as the standard non parametric statistics (Kendall's τ and Spearman's ρ).

Insert here figure 6.3

In what follows we will use Frank's copula, both on the ground that it turned out to provide a better fit in all cases - as measured visually and by the mean square error - than the Clayton, and on the fact that the Gumbel copula does not allow for negative dependence.

5.2. Option pricing

Using the Frank's copula and the estimation of the marginal densities described above we get the following joint distribution for each couple of indices

$$F(s_1, s_2) = -\frac{1}{\alpha} ln \left(1 + \frac{\left(\exp\left(-\alpha F_{S_1}(s_1)\right) - 1\right)\left(\exp\left(-\alpha F_{S_2}(s_2)\right) - 1\right)}{\exp\left(-\alpha\right) - 1} \right) \quad (5.2)$$

where $F_{S_1}(s_1)$, $F_{S_2}(s_2)$ are defined according to (5.1). This is the basic tool for option pricing, once the proper parameter estimates are plugged in. As an example, in figure 6.4. below we present the joint distribution for the DAX/FTSE case (5.2), together with its level curves[3]. The distribution was computed using the historical estimate of α.

Insert here figure 6.4

[3] Level curves can be defined in terms of risk-neutral joint distributions or in terms of copulas. With respect to the latter they are defined as usual as

$$\left\{(v, z) \in I^2 \mid C(v, z) = t\right\}$$

5.2.1. Digital binary option

First of all, using the joint distribution (5.2) we are able to price digital binary options. In order to stress the immediate relationship between copulas and digital binary prices, we assume a zero interest rate and use proposition 4.3. above. In order to take into consideration a deterministic, non-zero assumption on interest rates, it would be sufficient to scale the prices presented below with the dicount factor $B(T-t)$.

Figure 6.5. below presents some examples, again with α estimated from historical data, for the cases in which the corresponding single digital options are out-of-the-money (OTM), nearly at-the money (ATM) and in-the money (ITM). The digital binary options in the figure are then evaluated under different moneyness combinations for each underlying. The maturity is that of the corresponding marginals, i.e. three months.

Incidentally, the reader can easily notice that the familiar behavior of prices with respect to moneyness is respected: prices are decreasing, for every couple of indices, from top left towards bottom right and, ceteris paribus, from left to right and from top to bottom.

Insert here figure 6.5

By letting the strikes vary on a finer grid, we obtain the whole pricing surface for the three-month digital option on DAX and FTSE, which corresponds to the risk-neutral probability that both underlying assets be above the corresponding strikes. In figure 6.6 below we present the pricing surface corresponding to the historically calibrated copula and to the product copula.

Insert here figure 6.6

In figure 6.7 we present a numerical example of the sensitivity analysis shown graphically above. The example refers to the case in which both of the strikes are either ITM or nearly ATM. We analyze how the value of the option changes with respect to the parameter α, as we move from perfect dependence to independence

In the Archimedean case they become

$$z = \varphi^{[-1]}\left(\varphi(t) - \varphi(v)\right)$$

of the underlying assets. By letting α reach its bounds the binary digital option prices converge to the Fréchet bounds, while the independence case ($\alpha = 0$) is simply the product copula.

Insert here figure 6.7

5.2.2. Option on the minimum of two assets

We turn now to the pricing of the option on the minimum between two risky assets, which has been priced, in the Black-Scholes (lognormal) framework, by Stulz (1982). We assume deterministic, non zero interest rates.

The payoff of the call option on the minimum, with maturity T, is

$$\max(\min(S_1(T), S_2(T)) - K, 0)$$

where K is the strike price. Provided only that interest rates are non-stochastic, its price is

$$B(T-t)\left[\int_K^{+\infty} qg(q)dq - K(1 - G(K))\right] \tag{5.3}$$

where $g(q)$ is the risk-neutral density of the minimum, while $G(K)$ is the corresponding distribution function evaluated at K. In turn, since G can be computed to be

$$G(q) = F_{S_1}(q) + F_{S_2}(q) - c(F_{S_1}(q), F_{S_2}(q))$$

where c is the copula density, the density $g(q)$ is

$$g(q) = f_{S_1}(q) + f_{S_2}(q) - c_1(F_{S_1}(q), F_{S_2}(q))f_{S_1}(q) - c_2(F_{S_1}(q), F_{S_2}(q))f_{S_2}(q)$$

where c_1 and c_2 are the partial derivatives of c with respect to its arguments, while f_{S_i}, $i = 1, 2$, are the densities corresponding to F_{S_i}. These densities turn out to be

$$f_{S_i}(q) = n(D_{2i}(q))\left[\mathcal{D}_{2i}(q) - (A_{1i} + 2A_{2i}q)\sqrt{T-t}(1 - D_{2i}(q)\mathcal{D}_{2i}(q)) - 2A_{2i}q\sqrt{T-t}\right] \tag{5.4}$$

where

$$\mathcal{D}_{2i}(q) = -\frac{1}{q\sigma_i(q)\sqrt{T-t}} - \frac{D_{1i}(q)(A_{1i} + 2A_{2i}q)}{\sigma_i(q)}$$

$$D_{1i}(q) = D_{2i}(q) + \sigma_i(q)\sqrt{T-t}$$

and the functions $D_{2i}(q)$ and $\sigma_i(q)$ are defined as in section 5.1.

Figure 6.8 presents the prices for ITM, ATM, OTM three-months call options on the minimum between DAX and FTSE, computed according to (5.3) and using our estimate for the risk-neutral joint distribution function.

Insert here figure 6.8

In the first column we present the strike prices, corresponding to ITM (5653), ATM (6653) and OTM (7653) options. Moneyness is defined with respect to the FTSE index, whose current value on March 27, 2000, was 6653. From left to right, we present the prices corresponding to the historical association between the two indices ($\alpha = 4.469$), to an hypothetical quasi perfect positive association ($\alpha = 100$), to independency ($\alpha = .0001$) and to quasi perfect negative association ($\alpha = -100$). The reader can easily notice that, ceteris paribus, option prices are increasing with the association between the underlying assets, as the prices get closer to the super-replication values. They are decreasing with the strike, as usual.

5.2.3. Option to exchange one asset for another

The last line of the table in figure 6.8 presents the price of the option to exchange DAX for FTSE, corresponding to different α values. In order to understand the pricing technique used, let us recall that the price of the option to exchange one asset for another, originally derived – for lognormal distributions – by Margrabe (1978), can be obtained as a portfolio of one underlying and a zero-strike option on the minimum. In fact, the payoff of the exchange option is

$$\max(S_1(T) - S_2(T), 0)$$

which can be rewritten as

$$S_1 - \max(\min(S_1, S_2), 0)$$

Recalling that the risk-neutral expected value of the underlying at maturity is the forward price, it follows that its price is the current value of the first underlying minus the price of the option on the minimum, with strike equal to zero. Using this device, under the assumption of deterministic, non zero interest rates, we obtain the exchange option value as

$$S_1(t) - B(T - t) \int_0^{+\infty} qg(q)dq$$

which is the formula used in figure 6.8.

6. Conclusions

In this paper we suggest a strategy to address the joint issues of non-normality of returns and dependence in the multivariate contingent pricing problem. It is well known that under non-normality of returns the linear correlation is not an useful indicator of dependence, as it is not bounded in the unit range. We suggest to resort to the concept of copula in order to account for this problem: using copula functions enables to "decouple" the pricing problem, addressing the specification of marginal distributions and the dependence problem separately. A relevant advantage of this approach is that it is directly amenable to concrete applications: as a matter of fact, we have very well developed markets for many products, and the marginal distributions can be directly computed from the prices of vertical spreads of plain vanilla options. The use of copula functions enables us to link these marginal distributions in a multivariate pricing kernel.

As an example of the power of the approach, we present an application to four stock market indices: for each market we use vertical spreads and the interpolation technique due to Shimko to recover the implied marginal probability distributions from the market. We provide prices for a set of multivariate contingent claims, such as binary digital options, which represent the basis of prices for all of the bivariate contingent claims in the economy, options on the minimum and options to exchange one asset for another. For each of these products, we provide super-replication prices, as well as the values consistent with independence of the underlying assets. As a final reference value, we show how to price the multivariate options using a copula function calibrated on historical data.

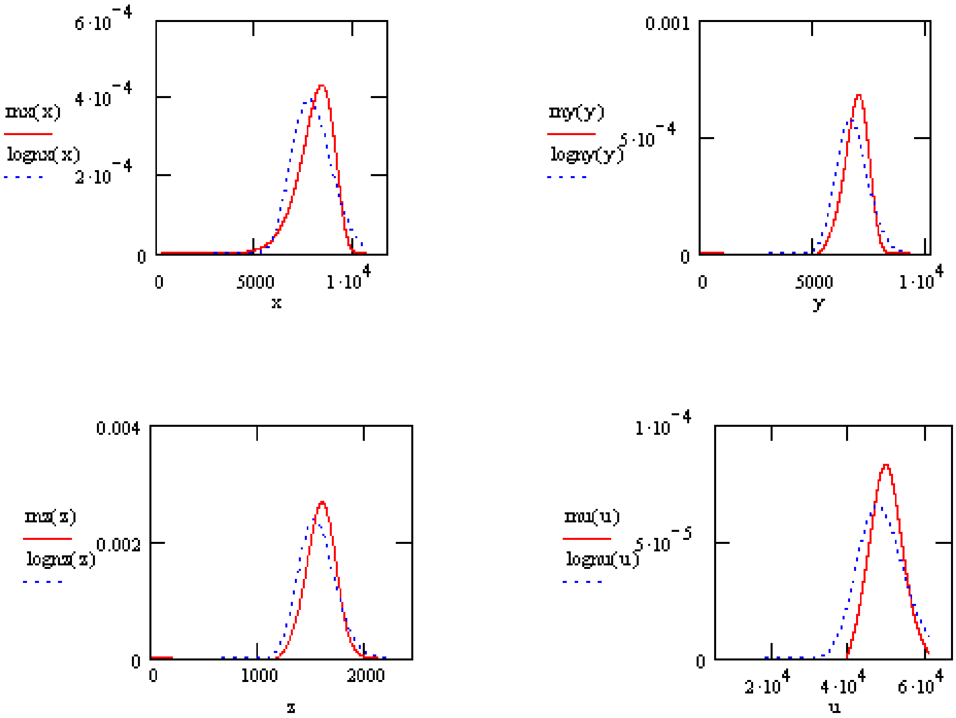


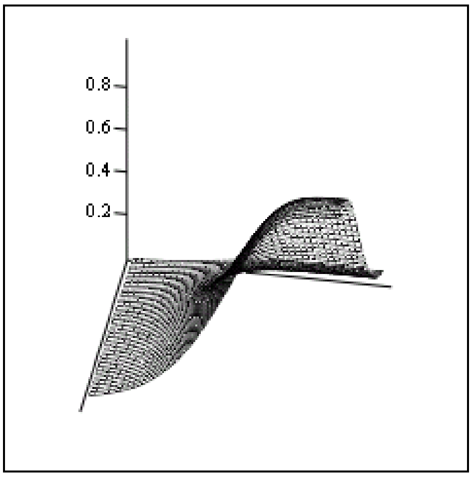
Figure 6.1: Solid lines: risk-neutral densities of DAX (x), FTSE (y), S&P500 (z), MIB30 (u), computed according to (5.4). Dotted lines: corresponding lognormal densities.

Mib-S&P	Mib-FTSE	Mib-DAX	S&P-FTSE	S&P-DAX	FTSE-DAX
-0.673	-0.642	-0.678	-0.868	-0.9	-0.865
0.73	0.732	0.735	0.997	0.985	0.984

Figure 6.2: Linear correlation bounds

	Sample Size	Kendall's tau	Spearman's rho	Gumbel's alpha	Clayton's alpha	Frank's alpha
Mib-S&P	306	0.372	0.548	1.593	1.185	3.789
Mib-FTSE	308	0.351	0.508	1.540	1.080	3.518
Mib-DAX	311	0.433	0.580	1.765	1.530	4.642
S&P-FTSE	304	0.581	0.772	2.387	2.774	7.445
S&P-DAX	306	0.646	0.846	2.828	3.657	9.317
FTSE-DAX	311	0.406	0.582	1.683	1.367	4.469

Figure 6.3: Estimated Kendall's τ, Spearman's R, and α for MIB30, S&P500, FTSE, DAX,1/2/99-3/27/00



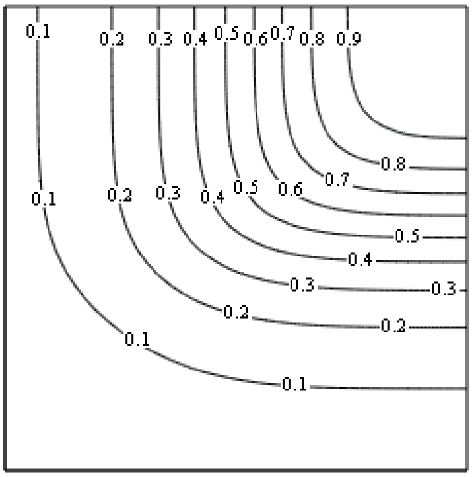


Figure 6.4: Joint distribution and level curves, DAX-FTSE

S&P-Mib		ITM	ATM	OTM
	Strike	38450	49702	60953
ITM	1218	0.9254	0.4344	0.0000
ATM	1649	0.2684	0.1999	0.0000
OTM	2079	0.0000	0.0000	0.0000

Mib-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	38450	0.9268	0.4982	0.0000
ATM	49702	0.4341	0.3177	0.0000
OTM	60953	0.0000	0.0000	0.0000

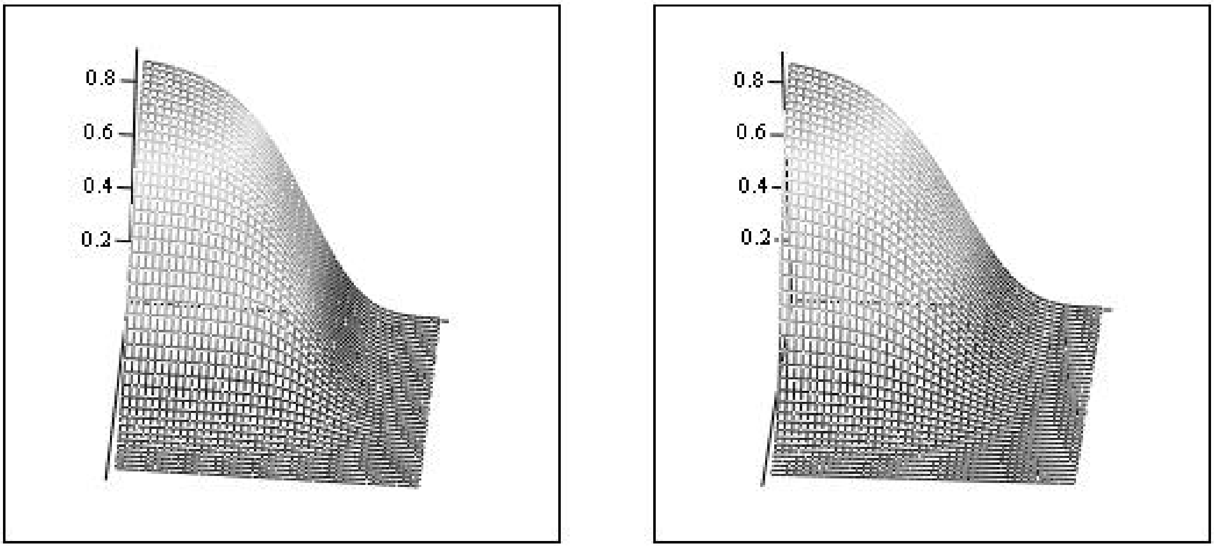
DAX-Mib		ITM	ATM	OTM
	Strike	38450	49702	60953
ITM	6287	0.9010	0.4335	0.0000
ATM	8166	0.4630	0.3235	0.0000
OTM	10045	0.0000	0.0000	0.0000

S&P-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	1218	0.9261	0.5028	0.0000
ATM	1649	0.2698	0.2515	0.0000
OTM	2079	0.0000	0.0000	0.0000

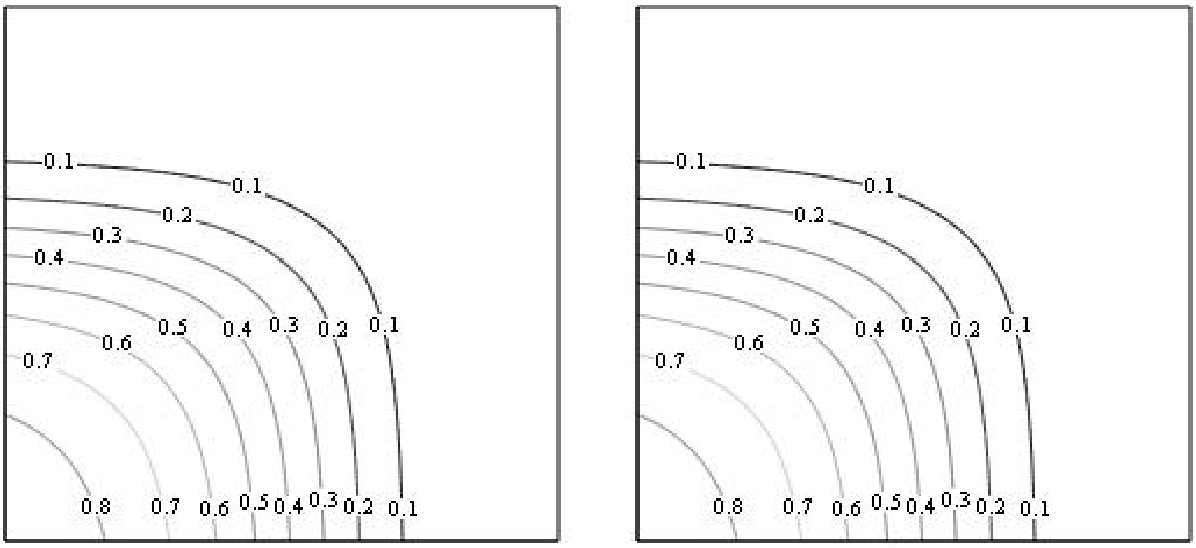
S&P-DAX		ITM	ATM	OTM
	Strike	6287	8166	10045
ITM	1218	0.9039	0.4656	0.0000
ATM	1649	0.2699	0.2553	0.0000
OTM	2079	0.0000	0.0000	0.0000

DAX-FTSE		ITM	ATM	OTM
	Strike	5322	6851	8379
ITM	6287	0.8977	0.4958	0.0000
ATM	8219	0.4398	0.3390	0.0000
OTM	10152	0.0000	0.0000	0.0000

Figure 6.5: Three-months binary digital option prices, selected strikes



historic independence



historic independence

Figure 6.6: Binary digital option prices on DAX and FTSE, strikes 6300-10700,5300 9300,computed with: the historically calibrated copula and the product one (indipendence).

DAX strike	FTSE strike	α -4.469	α --100	α -0.0001	α -100
6287	5322	0.8977	0.8864	0.8894	0.9278
6287	6851	0.4958	0.4313	0.4671	0.5034
8219	5322	0.4398	0.4016	0.4247	0.4430
8219	6851	0.3390	0.0000	0.2230	0.4430

Figure:6.7: Sensitivity analysis for the digital binary option, DAX-FTSE.

K	$\alpha=4.469$	$\alpha=-100$	$\alpha=0.0001$	$\alpha=100$
5653	996.0	927.4	972.4	1021.0
6653	257.1	154.6	227.2	264.9
7653	6.6	0.0	4.4	6.9
opt.to exchange	1075.0	679.3	892.5	1187.0

Figure 6.8: Prices of the call option on the minimum and option to exchange DAX for FTSE

References

[1] Breeden, D.T. and Litzenberger, R.H., 1978, Prices of State-Contingent Claims Implicit in Option Prices, *Journal of Business*, 51, 621-51.

[2] Cherubini, U. and Esposito, M., 1995, Options in and on interest rate futures contracts: results from martingale pricing theory, *Applied Mathematical Finance*, 2, 1-16.

[3] Derman, E. and Kani, I., 1994a, The volatility Smile and its Implied Tree, Quantitative Strategies Publication, Goldman-Sachs.

[4] Derman, E. and Kani, I., 1994b, Riding on a smile, *Risk*, 7, 32-39.

[5] Embrechts, P., McNeil, A. and Straumann, D., 1999, Correlation and dependence in risk management: properties and pitfalls, ETHZ working paper.

[6] Fréchet, M., 1957, Les tableaux de corrélations dont les marges sont données, *Annales de l'Université de Lyon, Sciences Mathématiques at Astronomie*, Série A, 4, 13-31.

[7] Frees, E.,W. and Valdez, E., 1998, Understanding relationships using copulas, *North American Actuarial Journal*, 2, 1-25.

[8] Genest, C., 1987, Frank's family of bivariate distributions, *Biometrika*, 74, 549-555.

[9] Genest, C. and MacKay, J., 1986, The joy of copulas: bivariate distributions with uniform marginals, *The American Statistician*, 40, 280-283.

[10] Genest, C. and Rivest, L.P., 1993, Statistical inference procedures for bivariate Archimedian copulas, *Journal of Amer. Statist. Assoc.*, 88, 423, 1034-1043.

[11] Hoeffding, W., 1940, Massstabinvariante Korrelationstheorie, *Schriften des Mathematischen Seminars und des Instituts fur Angewandte Mathematik der Universitat Berlin*, 5, 181-233.

[12] Joe, H., 1997, *Multivariate models and dependence concepts*, Chapman and Hall, London.

[13] Klein, P., 1996, Pricing Black-Scholes options with correlated credit risk, *Journal of Banking and Finance,* 20, 1211-1230.

[14] Margrabe, W., 1978, The value of an option to exchange one asset for another, *Journal of Finance*, 33, 177-186.

[15] Nelsen, R.B., 1999, *An introduction to copulas*, Springer, New York.

[16] Rosenberg, J.V., 1999, Semiparametric pricing of multivariate contingent claims, NYU- Stern School of Business Working Paper.

[17] Rubinstein, M., 1994, Implied Binomial Trees, *Journal of Finance*, 49, 771-818.

[18] Shimko, D.C., 1993, Bounds of probability, *Risk*, 6, 33-37.

[19] Sklar, A., 1959, Fonctions de repartition à n dimensions et leurs marges, Publication Inst. Statist. Univ. Paris 8, 229-231.

[20] Stulz, R.M., 1982, Options on the minimum or the maximum of two risky assets: Analysis and applications, J*ournal of Financial Economics*, 10, 161-185

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