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Working Paper no. 25/2003
July 2003

APPLIED MATHEMATICS
WORKING PAPER SERIES

Archimedean Copulae and Positive Dependence*

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June 2003

*Partially supported by the Vigoni Program and MIUR-COFIN
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Abstract

In the first part of the paper we consider positive dependence properties of Archimedean copulae. Especially we characterize the Archimedean copulae that are multivariate totally positive of order 2 (MTP_2) and conditionally increasing in sequence. In the second part we investigate conditions for binary sequences to admit an Archimedean copula.

AMS 2000 Subject Classification: Primary 60E15, Secondary 60G09, 62H10, 62H20.

Keywords: Conditionally increasing, MTP_2, positive lower orthant dependent, exchangeability, binary sequences.

1 Introduction

Archimedean copulae have recently received increasing attention for their computational tractability and their flexibility to model dependencies between exchangeable random variables. They have been introduced by Kimberling (1974). Some basic contributions have been given by Genest and MacKay (1986), Marshall and Olkin (1988), Ballerini (1994), and Oakes (1994). More specific results about Archimedean copulae have been given, among others, by Genest and Rivest (1993), who studied statistical inference, Bagdonavičius et al. (1999), who provided characterizations, and Juri and Wüthrich (2002) proved convergence theorems for tail events. Cuculescu and Theodorescu (2003) investigate unimodality of bivariate Archimedean copulas. In recent years these copulae have been used more and more as a tool for modelling dependence in many diverse areas. Frees and Valdez (1998) demonstrated their usefulness for problems in actuarial sciences. In the literature on financial mathematics they have been considered in Embrechts et al. (2003) and Hennessy and Lapan (2002).

In this paper we will characterize several well known notions of positive dependence in the case of Archimedean copulae. For the case of positive orthant dependence such a result can be found in Joe (1997). Some results for the bivariate case can also be found in Averous and Dortet-Bernadet (2000). Bassan and Spizzichino (2002) study dependence and aging concepts for bivariate distributions in terms of an object that they call Archimedean semi-copula.

In this paper we deal with the multivariate case. In particular we derive necessary and sufficient conditions for the generator of an Archimedean copula to yield a random vector which is MTP_2 or conditionally increasing in sequence. For these two important notions of positive dependence the reader is referred for instance to Barlow and Proschan (1975) and Karlin and Rinott (1980).

Moreover, we will show that any infinitely exchangeable binary sequence of random variables admits a representation by an Archimedean copula. This demonstrates the flexibility of Archimedean copulae for modelling dependence.

The rest of the paper is organized as follows. In Section 2 we first give the basic definitions on copulae and notions of positive dependence, and then state the main results characterizing these concepts of dependence for Archimedean copulae. In Section 3 we investigate the relation between exchangeable binary sequences and Archimedean copulae.

2 Copulae and Positive dependence

The notion of copula has been introduced by Sklar (1959), and studied, among others, by Kimeldorf and Sampson (1975), under the name of uniform representation, and by Deheuvels (1978), under the name of dependence function. The copula is one of the most useful tools for handling multivariate distributions with dependent components. Formally, given a distribution function F with marginals $F_1, ..., F_d$, there exists a function $C : [0,1]^d \to [0,1]$ such that, for all $\mathbf{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)). \tag{2.1}$$

The function C is unique on $\times_{i=1}^d \operatorname{Ran}(F_i)$, the product of the ranges of F_i, $i = 1, \ldots, d$. Therefore, if F is continuous, then C is unique and can be constructed as follows

$$C(\mathbf{u}) = F[F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_d^{-1}(u_d)], \quad \mathbf{u} = (u_1, \ldots, u_d) \in [0,1]^d. \tag{2.2}$$

Here the (generalized) inverse G^{-1} of a univariate distribution function G is defined as

$$G^{-1}(u) = \sup\{x : G(x) \leq u\}, \quad u \in [0,1].$$

Otherwise C can be extended to $[0,1]^d$ in such a way that it is (the restriction to $[0,1]^d$ of) a distribution function with uniform marginals on $[0,1]$. Any such extension is called copula of F. The construction of a particularly interesting extension is shown in detail in Schweizer and Sklar (1983). Most of the multivariate dependence structure properties of F are in the copula, which does not depend on the marginals, and is often easier to handle than the original F. More details about copulae can be found in Joe (1997), Nelsen (1999), and Roncalli (2001).

An interesting class of copulae was introduced by Kimberling (1974), and studied, among others by Genest and MacKay (1986).

Definition 2.1. A function $\psi : \mathbb{R}_+ \to [0,1]$ is called *d-alternating* if $(-1)^k \psi^{(k)} \geq 0$ for $k \in \{1, \ldots, d\}$. A function which is d-alternating for all $d \in \mathbb{N}$ is called *completely monotone.*

Definition 2.2. A copula C_ψ is called *Archimedean* if it has the form

$$C_\psi(x_1, \ldots, x_d) = \psi\left(\sum_{i=1}^d \psi^{-1}(x_i)\right), \tag{2.3}$$

where $\psi : \mathbb{R}_+ \to [0,1]$ is a d-alternating function such that $\psi(0) = 1$, and $\lim_{x\to\infty} \psi(x) = 0$. The function ψ is called *generator* of the copula.

Notice that other authors call generator the function ψ^{-1} (see e.g. Nelsen (1999)).

Remark 2.3. We indicate by $C^{\perp}$ the copula of a distribution with independent components: $C^{\perp}(x_1, \ldots, x_d) = \prod_{i=1}^{d} x_i$. It is easy to see that $C^{\perp}$ is a particular case of Archimedean copula with $\psi(x) = \exp(-x),\ x \geq 0$.

Remark 2.4. Let $Y_1, \ldots, Y_d$ be i.i.d non-negative random variables, independent of the random variable Z. Then the vector

$$\mathbf{X} = (Y_1^{1/Z}, \ldots, Y_d^{1/Z})$$

has an Archimedean copula whose generator is given by the Laplace transform of Z, $\psi(t) = E\exp(-tZ),\ t \geq 0$. This result is due to Marshall and Olkin (1988).

Whenever we consider a subset $I \subset \{1, \ldots, d\}$ of the coordinates of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$, we will write $\mathbf{X}_I = (X_i,\ i \in I)$, and throughout the paper we will assume the existence of regular conditional probabilities $P(\mathbf{X}_I \in A | \mathbf{X}_J = \mathbf{x}_J)$ for all $A \in \mathbb{R}^{|I|},\ \mathbf{x}_J \in \mathbb{R}^{|J|},\ I, J \subset \{1, \ldots, d\}$. Assuming the existence of the corresponding derivatives, it holds for an arbitrary random vector $\mathbf{X}$ with distribution function F that

$$P(X_{i+1} \leq t | X_1 = x_1, \ldots, X_i = x_i) \;=\; \frac{\dfrac{\partial^i}{\partial x_1 \ldots \partial x_i} F(x_1, ..., x_i, t, 1, ..., 1)}{\dfrac{\partial^i}{\partial x_1 \ldots \partial x_i} F(x_1, ..., x_i, 1, 1, ..., 1)}. \tag{2.4}$$

When $F = C_\psi$ equation (2.4) implies

$$P(X_i \leq t | \mathbf{X}_J = \mathbf{x}_J) \;=\; \frac{\dfrac{\partial}{\partial \mathbf{x}_J} \psi\left(\sum_{j \in J} \psi^{-1}(x_j) + \psi^{-1}(t)\right)}{\dfrac{\partial}{\partial \mathbf{x}_J} \psi\left(\sum_{j \in J} \psi^{-1}(x_j)\right)}. \tag{2.5}$$

We recall now some well known concepts of positive dependence. To do that we need the concept of *supermodular function.* A function $f : \mathbb{R}^d \to \mathbb{R}$ is supermodular, if

$$f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \;\geq\; f(\mathbf{x}) + f(\mathbf{y}) \quad \text{for all } \mathbf{x} \text{ and } \mathbf{y}, \tag{2.6}$$

where the lattice operators $\wedge$ and $\vee$ are defined as

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\}),$$

and

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \ldots, \max\{x_d, y_d\}).$$

A twice differentiable function is supermodular if and only if all mixed derivatives are non-negative, i.e.

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \;\geq\; 0 \quad \text{for all } \mathbf{x} \quad \text{ and all } \; 1 \leq i < j \leq n.$$

Definition 2.5. Given two copulae C, K, we say that $C \geq_{\text{LO}} K$ if $C(x_1, \ldots, x_d) \geq K(x_1, \ldots, x_d)$, for all $(x_1, \ldots, x_d) \in [0,1]^d$.

Definition 2.6. A copula C is

(a) PLOD (Positive Lower Orthant Dependent) if $C \geq_{\text{LO}} C^{\perp}$,

(b) CIS if $\mathbf{X} \sim C$ and X_i is stochastically increasing in $(X_1, \ldots, X_{i-1})$ for all $i \in \{2, \ldots, d\}$, i.e. if $P(X_i > t | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is increasing in $x_1, \ldots, x_{i-1}$ for all t.

(c) CI if $\mathbf{X} \sim C$ and X_i is stochastically increasing in $\mathbf{X}_J$ for all $i \notin J$ and all $J \subset \{1, \ldots, d\}$,

(d) MTP_2 if C has a density which is log-supermodular, i.e. if

$$\log \frac{\partial^d}{\partial x_1 \cdots \partial x_d} C(x_1, \ldots, x_d) \tag{2.7}$$

is supermodular.

For the concepts of PLOD and CIS we refer to Lehmann (1966) and Barlow and Proschan (1975). The idea of MTP_2 was studied by Karlin and Rinott (1980). CI was introduced by Müller and Scarsini (2001).

The following result can be found in Joe (1997), page 109.

Theorem 2.7. *The Archimedean copula C_ψ is PLOD iff $\psi \circ \exp(\cdot)$ is superadditive.*

The following theorem establishes conditions for CI and CIS of an Archimedean copula.

Theorem 2.8. *For an Archimedean copula C_ψ the following conditions are equivalent:*

(a) *C_ψ is CIS,*

(b) *C_ψ is CI,*

(c) *$(-1)^k \psi^{(k)}(\cdot)$ is log-convex for $k \in \{1, \ldots, d-1\}$,*

(d) $(-1)^{d-1}\psi^{(d-1)}(\cdot)$ *is log-convex.*

In the proof of Theorem 2.8 we need the following Lemma.

Lemma 2.9. *If $g:\mathbb{R}_+\to\mathbb{R}_+$ is decreasing and log-convex, and $f(x)=\int_x^\infty g(t)\,\mathrm{d}t$ is finite, then f is log-convex, too.*

Proof. It is known that the set of log-convex functions is a convex cone closed under the taking of limits (see Roberts and Varberg (1973), Theorem F, page 19). Moreover if g is log-convex, then $x\mapsto g(x+\alpha)$ is log-convex for all $\alpha>0$. The result then follows from the fact that

$$f(x)=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{\infty}g\left(x+\frac{i}{n}\right)\right).$$

$\square$

Proof of Theorem 2.8. The equivalence of CI and CIS is due to exchangeability of the Archimedean copula. By (2.5), C_ψ is CI iff

$$\frac{\frac{\partial}{\partial\mathbf{x}_J}\psi\left(\sum_{j\in J}\psi^{-1}(x_j)+\psi^{-1}(t)\right)}{\frac{\partial}{\partial\mathbf{x}_J}\psi\left(\sum_{j\in J}\psi^{-1}(x_j)\right)}$$

is decreasing in $\mathbf{x}_J$ for all $t\in[0,1]$. Since ψ is decreasing (and therefore ψ^{-1} is decreasing), the above expression is decreasing in $\mathbf{x}_J$ if it is increasing in $\sum_{j\in J}\psi^{-1}(x_j)$ on $(0,\infty)$. So we want

$$\frac{\psi^{(k)}(y+z)}{\psi^{(k)}(y)}$$

to be increasing in y for all $z\in(0,\infty)$, for $k=|J|$,. This holds iff

$$\log\left((-1)^k\psi^{(k)}(y+z)\right)-\log\left((-1)^k\psi^{(k)}(y)\right)$$

is increasing in y for all $z\in(0,\infty)$, namely, $\log((-1)^k\psi^{(k)})$ is convex for $k\in\{1,\dots,d-1\}$.

By Lemma 2.9 this is true iff $(-1)^{d-1}\psi^{(d-1)}$ is log-convex, since

$$(-1)^k\psi^{(k)}(x)=\int_x^\infty(-1)^{k+1}\psi^{(k+1)}(t)\,\mathrm{d}t\qquad\text{for }k=1,\dots,d-2.$$

$\square$

Remark 2.10. In the bivariate case the condition for an arbitrary copula to be CI is concavity in each direction, while the other is held fixed (see Nelsen (1999)). An easy calculation shows that this is equivalent to log-convexity of $-\psi'(\cdot)$, namely, condition (d) of Theorem 2.8.

Theorem 2.11. *The Archimedean copula C_ψ is MTP_2 iff $(-1)^d\psi^{(d)}(\cdot)$ is log-convex.*

Proof. Let C_ψ be MTP_2. For its density we get the expression

$$f(\mathbf{x}) = \frac{\partial^d}{\partial x_1 \cdots \partial x_d}\psi\left(\sum_{i=1}^d \psi^{-1}(x_i)\right) = \psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right) \cdot \prod_{i=1}^d (\psi^{-1})'(x_i).$$

Therefore ψ is MTP_2 if and only if

$$\log f(\mathbf{x}) = \log\left((-1)^d\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right) + \sum_{i=1}^d \log\left((-\psi^{-1})'(x_i)\right)$$

is supermodular, which is equivalent to

$$\log\left((-1)^d\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right)$$

being supermodular, that is its mixed derivatives must be non-negative. In case of the first two coordinates this leads to the condition

$$\frac{(-1)^{d+2}\psi^{(d+2)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)(\psi^{-1})'(x_1)(\psi^{-1})'(x_2)(-1)^d\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)}{\left(\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right)^2}$$
$$- \frac{\left((-1)^{d+1}\psi^{(d+1)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right)^2(\psi^{-1})'(x_1)(\psi^{-1})'(x_2)}{\left(\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right)^2} \geq 0$$

for all $x_1, \ldots, x_d$, and to similar conditions in the other cases. Since $(\psi^{-1})'(x_1)(\psi^{-1})'(x_2) \geq 0$, we need

$$(-1)^{d+2}\psi^{(d+2)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)(-1)^d\psi^{(d)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right) \geq \left((-1)^{d+1}\psi^{(d+1)}\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)\right)^2$$

for all $x_1, \ldots, x_d$, or equivalently,

$$\psi^{(d+2)}(y)\psi^{(d)}(y) \geq \left(\psi^{(d+1)}(y)\right)^2$$

for all y, which corresponds to $(-1)^d\psi^{(d)}$ being log-convex. $\square$

Remark 2.12. The condition for MTP_2 ($(-1)^d\psi^{(d)}$ log-convex) is *strictly* stronger than the condition for CI ($(-1)^{d-1}\psi^{(d-1)}$ log-convex). The following example shows a copula which is CI, but not MTP_2. Let Φ be the distribution function of a standard normal

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \, \mathrm{d}t.$$

Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be such that $\psi'(x) = -c\exp\{-g(x)\}$ with

$$g(x) = \begin{cases} 2\Phi(x) - 1, & x \leq a \\ \alpha x + \beta, & x > a, \end{cases}$$

where $c^{-1} = \int_0^\infty \exp\{-g(x)\} \, \mathrm{d}x$, where a is a large constant, and α and β are such that g is continuously differentiable in a, and therefore $g : \mathbb{R}_+ \to \mathbb{R}$ is concave.

Then

$$\psi''(x) = c\exp\{-g(x)\}g'(x).$$

We see that $\log(-\psi'(x)) = \log c - g(x)$ is convex, but

$$\begin{aligned} \log(\psi''(x)) &= \log c - g(x) + \log(g'(x)) \\ &= \begin{cases} \log c - (2\Phi(x) - 1) + \log 2 - \dfrac{1}{2}\log(2\pi) - \dfrac{x^2}{2}, & x \leq a, \\ \log c - (\alpha x + \beta) + \log \alpha, & x > a, \end{cases} \end{aligned}$$

is clearly not convex for large enough a.

Therefore $C_\psi : [0,1]^2 \to [0,1]$ is CI, but not MTP_2.

3 Binary random variables

In Section 2 we have shown how positive dependence concepts can be characterized for Archimedean copulae. In this section we will try to determine how general is the assumption of an Archimedean copula for binary random variables.

Theorem 3.1. *For every infinite exchangeable binary sequence $\{Y_n\}_{n\in\mathbb{N}}$ there exists a generator ψ of an Archimedean copula such that $\psi\left(\sum_{i=1}^d \psi^{-1}(x_i)\right)$ is a copula of $(Y_1, \ldots, Y_d)$.*

Proof. Given $d \in \mathbb{N}$, we know by de Finetti's theorem that there exists a random variable Θ with distribution μ_Θ supported in $[0,1]$ such that

$$P(Y_i = 1, \ i \in I; \ Y_j = 0, \ j \in J) = \int \theta^{|I|}(1-\theta)^{|J|} \, \mu_\Theta(\mathrm{d}\theta).$$

Consider a random vector $(X_1, \ldots, X_d)$ of i.i.d. random variables, uniformly distributed on $[0, 1]$.

By Corollary 2.2 of Marshall and Olkin (1988) we know that, if Z is a random variable, then the vector $(X_1^{1/Z}, \ldots, X_d^{1/Z})$ has an Archimedean copula C_ψ with

$$\psi(t) = E[\exp\{-tZ\}]. \tag{3.1}$$

Define

$$Z = \frac{\ln(1-\Theta)}{\ln(1/2)}, \qquad \widetilde{Y}_i = \mathbf{1}_{[X_i^{1/Z} \geq 1/2]}, \quad i \in \mathbb{N}.$$

Then for any partition I, J of $\{1, \ldots, d\}$ we have

$$\begin{aligned} P(\widetilde{Y}_i = 1,\ i \in I;\ \widetilde{Y}_j = 0,\ j \in J) &= P\left(X_i^{1/Z} \geq \frac{1}{2},\ i \in I;\ X_j^{1/Z} < \frac{1}{2},\ j \in J\right) \\ &= P(X_i \geq 1 - \Theta,\ i \in I;\ X_j < 1 - \Theta,\ j \in J) \\ &= \int P(X_i \geq 1 - \theta,\ i \in I;\ X_j < 1 - \theta,\ j \in J)\ \mu_\Theta(\mathrm{d}\theta) \\ &= \int \theta^{|I|}(1-\theta)^{|J|}\ \mu_\Theta(\mathrm{d}\theta). \end{aligned}$$

Thus $\{Y_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{Y}_n\}_{n\in\mathbb{N}}$ are stochastically equal, and since $\widetilde{Y}_i$ is a increasing function of $X_i^{1/Z}$, the vector $(Y_1, \ldots, Y_d)$ therefore has a common copula with $(X_1^{1/Z}, \ldots, X_d^{1/Z})$. □

Remark 3.2. The above result fails for finite exchangeable sequences, as the following example shows. Let $\mathbf{X} = (X_1, X_2, X_3)$ be exchangeable and such that

$$P(\mathbf{X} = (0,0,1)) = P(\mathbf{X} = (0,1,0)) = P(\mathbf{X} = (1,0,0)) = \frac{1}{3}. \tag{3.2}$$

Therefore $P(X_1 = 1) = 1/3$ and any copula C of (X_1, X_2, X_3) is such that

$$C\left(\frac{2}{3}, \frac{2}{3}, 1\right) = \frac{1}{3}, \qquad C\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = 0. \tag{3.3}$$

If there existed an Archimedean copula that satisfies (3.3), then there would exist a 3-alternating function ψ such that $\psi^{-1}(1) = 0$, and

$$\begin{aligned} \psi\left(\psi^{-1}\left(\frac{2}{3}\right) + \psi^{-1}\left(\frac{2}{3}\right) + \psi^{-1}(1)\right) &= \frac{1}{3} \\ \psi\left(\psi^{-1}\left(\frac{2}{3}\right) + \psi^{-1}\left(\frac{2}{3}\right) + \psi^{-1}\left(\frac{2}{3}\right)\right) &= 0. \end{aligned}$$

This implies

$$
\begin{aligned}
2\psi^{-1}\left(\frac{2}{3}\right) &= \psi^{-1}\left(\frac{1}{3}\right)\\
3\psi^{-1}\left(\frac{2}{3}\right) &= \psi^{-1}(0)\,.
\end{aligned}
$$

Hence ψ^{-1} is affine on $[0,1]$. So ψ has a kink in $\psi^{-1}(0)$, and therefore it cannot be 3-alternating.

The counterexample holds also for small perturbations of (3.2).

An exchangeable random vector $(X_1,\ldots,X_d)$ is said k-extendible ($k > d$) if it is the initial part of a k-dimensional exchangeable random vector.

It is clear that an exchangeable random vector with an Archimedean copula C_ψ is k-extendible iff ψ is k-alternating.

Theorem 3.1 shows that there always exists a completely monotone function ψ which is the generator of the copula of an infinitely extendible binary exchangeable random vector. Frey and McNeil (2001) have proved the reverse implication, namely, they have shown that if $(X_1,\ldots,X_d)$ have an Archimedean copula C_ψ with ψ completely monotone, then the indicators $(Y_i = \mathbf{1}_{[X_i\le t]})$ form an exchangeable Bernoulli sequence.

Remark 3.2 shows that k-extendibility is not enough to ensure the existence of a k-alternating ψ with the same property.

Remark 3.3. It is possible for arbitrary dimension d to have an Archimedean copula of a binary exchangeable random vector $\mathbf{X}$ such that $P(\mathbf{X}=\mathbf{0})=0$. For instance take

$$
C(x_1,\ldots,x_d) = \left(\sum_{i=1}^{d}(x_i)^{\frac{1}{d-1}} - (d-1)\right)_+^{d-1}
$$

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