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Risk, Ambiguity, and the Separation of Utility and Beliefs[†]

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Abstract

We introduce a general model of static choice under uncertainty, arguably the weakest model achieving a separation of cardinal utility and a unique representation of beliefs. Most of the non-expected utility models existing in the literature are special cases of it. Such separation is motivated by the view that tastes are constant, whereas beliefs change with new information. The model has a simple and natural axiomatization.

Elsewhere (forthcoming) we show that it can be very helpful in the characterization of a notion of ambiguity aversion, as separating utility and beliefs allows to identify and remove aspects of risk attitude from the decision maker's behavior. Here we show that the model allows to generalize several results on the characterization of risk aversion in betting behavior. These generalizations are of independent interest, as they show that some traditional results for subjective expected utility preferences can be formulated only in terms of binary acts.

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Introduction

In this paper, we introduce and characterize axiomatically a general model of static choice under uncertainty, whose main interest lies in being arguably the most general model that

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achieves a separation between *cardinal* utility and a unique representation of beliefs. Though very general, the model has enough structure to prove some interesting results on the formal representation of economic behavior. We illustrate this by showing three characterizations of an important aspect of risk aversion. In a companion paper (forthcoming), we show how the model can also be used to provide a very general characterization of a notion of ambiguity aversion.

The motivation for looking at models of choice under uncertainty that generalize the classical *subjective expected utility* (SEU) model of Savage (1954) is well known. There is a wealth of evidence as to the descriptive limitations of the SEU model. The two most popular problems are the *Allais and Ellsberg paradoxes*, respectively due to Allais (1953) and Ellsberg (1961). Besides showing that the strong state separability enjoyed by expected utility is not descriptively accurate, these 'paradoxes' raise the issue of whether it is normatively compelling. A large number of extensions of SEU have been developed that weaken state separability and rationalize some of these violations. There are models that rationalize the 'ambiguity averse' behavior of the Ellsberg paradox, like the *Choquet expected utility* (CEU) model of Schmeidler (1989), or the *maxmin expected utility* (MEU) model of Gilboa and Schmeidler (1989). There are models that rationalize the choices in the Allais paradox, like the subjective *rank-dependent expected utility* (RDEU) model based on Quiggin (1982) and Yaari (1987), Gul (1991)'s *disappointment aversion* model, or the more general model of *probabilistically sophisticated* (PS) preferences of Machina and Schmeidler (1992). Finally, there are models that rationalize additional features of observed behavior, like Tversky and Kahneman (1992)'s *cumulative prospect theory* (CPT).

Though most of these models share some features, such are the differences between them that so far there have been few attempts at finding nontrivial results on economic behavior that hold for most, if not all, of them. Doing so requires finding a 'common denominator' model with enough structure to impose meaningful restrictions. For instance, just assuming that every preference is a weak order does not allow us to make any formal statements about beliefs or risk attitudes.

The first principal objective of this paper to is introduce a model in this spirit, that we call the 'biseparable preferences' model, in Savage's fully subjective decision setting. A preference relation is *biseparable* (short for 'binary separable') if it can be represented by a functional V on acts that respects state-by-state dominance, takes a 'generalized expected utility' form on binary acts, and is otherwise unconstrained. By generalized expected utility we mean the following: If u denotes the restriction of V to constants, there is a capacity (i.e., a monotone set-function) ρ such that

$$V(f) = u(x)\,\rho(A) + u(y)\,(1 - \rho(A)), \tag{1}$$

when f is an act which pays x if event A obtains and y otherwise, with x preferred to y, that we could call a *bet on* A. Such V is essentially unique whenever there is at least one event with nontrivial ρ weight and a standard continuity assumption holds. By 'essentially unique' we mean that V is determined up to a positive affine transformation, or is *cardinal*. This implies that u is also cardinal and that ρ is unique.

Thus, a decision maker (DM) with biseparable preferences evaluates consequences by a cardinal state-independent *utility index* and evaluates bets by a unique capacity, that represents his *willingness to bet*. When evaluating more complex nonbinary acts, though, he is only restricted to satisfying monotonicity. We also look at the special setting with 'horse races and lottery wheels' of Anscomb and Aumann (1963) and discuss a variant of the biseparable preference model — called the *c-linearly biseparable* preference model — in which the DM's utility index is linear on lotteries.

The biseparable preferences model, or its c-linear variant, is easily seen to encompass all the decision models listed above, with the exception of the PS and CPT models (see the discussion below). In fact, we argue that it is the most general model in which the DM's cardinal utility index and his willingness to bet are separated and univocally identified. Clearly, the interest in such separation is motivated by the view that tastes are constant, whereas beliefs change with new information.

In order to better understand the type of behavioral restrictions that the model entails, we present a simple axiomatic characterization for it. We show that the only significant restriction is a very weak version of the traditional independence axiom (based on the technique for constructing 'subjective mixtures' due to Nakamura (1990) and Gul (1992)), where independence is only imposed in comparisons among certain pairs of binary acts. Thus, the biseparable preferences model lends itself to simple experimental verification.

The second principal objective of this paper is to convince the reader that the biseparable preferences model has enough structure to enable us to prove some interesting results about economic behavior. A compelling example of the usefulness of the biseparable preferences model is our study of ambiguity and ambiguity attitudes in (forthcoming). In that paper we propose an extended notion of ambiguity aversion and related notions of ambiguity for acts and events. We then provide their characterizations for any biseparable preference. For instance, we show that if a biseparable preference is ambiguity averse in the sense we propose there, then its willingness to bet ρ is dominated state-by-state by a probability. We also show that the set of events which are unambiguous for an ambiguity averse (or loving) biseparable preference is the collection of all the A's such that $\rho(A) + \rho(A^c) = 1$. Biseparable preferences play a key role in the analysis there because they provide the most general model for which it is possible to cleanly separate cardinal risk attitude from ambiguity attitude in the broad sense described there, so as to avoid confusing the two.

Here, we provide another example of the advantages of using a 'common denominator' model like biseparable preferences. We show that thanks to the cardinality of utility, it is possible to extend to biseparable preferences several results on the characterization of risk aversion for SEU preferences, as long as we limit our attention to betting behavior: (1) a natural 'more risk averse than' relation between preferences is characterized by the existence of a concave transformation between their utility indices; (2) the classical notion of risk aversion as preference for the expected value (duly transposed to a purely subjective framework) is characterized by the concavity of the utility index u; (3) a type of preference for diversification in betting is also characterized by the concavity of u.

In assessing these results, the reader should be aware that they describe only *an aspect* of the complex phenomenon of risk aversion. First, because we mostly (with the exception of Subsection 4.3) restrict our attention to a DM's behavior over bets. Second, because unlike the traditional definitions of risk aversion, we use a fully subjective setting (i.e., we do not assume the existence of 'known' probabilities) and do not assume that DMs have PS preferences in the sense of Machina and Schmeidler (1992). Therefore, we cannot capture the aspects of risk aversion which conceptually hinge on those assumptions. The aspect modelled here is called *cardinal risk aversion* (in the sense of 'aversion to cardinal risk'), and it is what explains the differences in betting behavior of two biseparable preferences with the *same* willingness to bet. The results mentioned above show that cardinal risk aversion is characterized by the concavity of a biseparable preference's utility index. Other aspects of a DM's risk attitude, that we do not model here, could for instance be reflected in his willingness to bet — as it is the case for what is usually called the DM's 'probabilistic risk attitude' (see the discussion in (forthcoming)). On the other hand, these two limitations do not persist in the case of SEU preferences, where any behavioral trait that we associate with risk averse behavior is equivalent to the concavity of utility.

Indeed, with SEU preferences our results acquire additional interest, as they show that some well-known traits can be equivalently formulated in the much simpler world of bets and certain consequences. For instance, rather than the complex notion of mean-preserving spread *à la* Rothschild and Stiglitz (1970), we show that it is enough to look at 'binary mean-preserving spreads' of bets.

A final aspect of interest of a model as general as the present one is that it provides a general foundation for many of the models of multiattribute utility theory (see Miyamoto (1988) for details), which enjoys the advantage of being detached from the many of the descriptive difficulties that are associated with the DM's reaction to uncertainty, like ambiguity aversion. Since we obtain cardinal utility and still allow a variety of departures from EU maximization, any multiattribute utility model built on biseparable preferences has wide applicability and it can also be easily tested experimentally.

The Related Literature

The paper which is closest to the present one is Miyamoto (1988), which shares our double objective of providing a model which encompasses many non-SEU models, and showing its potential usefulness. His 'generic utility theory' uses a von Neumann-Morgenstern setting with 'known' probabilities and — while it obtains cardinal utility — it does not deliver a representation of the DM's beliefs over the state space. His axiomatization is also not as simple as the one we propose.

There are then several axiomatic papers presenting decision models whose relation to biseparable preferences deserves a brief comment.

Machina and Schmeidler (1992)'s PS preferences model is comparable to ours in generality. While the intersection between the two models is nontrivial, including for instance the RDEU preferences, they embody different rationality restrictions on preferences and have a different scope. The PS model provides a little structure on the 'functional' representation of preferences over all acts, whereas we are very precise on the functional representation, but *only* over the binary acts. This is because our objective is obtaining a cardinal utility and a unique representation of beliefs so that, for instance, cardinal risk and ambiguity attitude can be clearly separated. We remark that our model could be used to provide a choice-theoretic derivation of subjective probability different from Machina-Schmeidler's. In fact, it is easy to show (see Section 5) that a simple and natural reinforcement of the axioms that characterize biseparable preferences yields a DM with *probabilistic beliefs*: His willingness to bet ρ is a probability measure (i.e., it is additive). Such DM is not necessarily a SEU maximizer, and may not even be PS in the sense of Machina and Schmeidler.

Another preference model not encompassed by ours is Tversky and Kahneman (1992)'s CPT, where a DM has a given reference point and his willingness to bet can be different depending on whether he considers bets with 'gains' (consequences preferred to the reference point), or bets with 'losses' (consequences to which the reference point is preferred). Clearly, all the results proved here hold for CPT when we look at gains or losses only, as then CPT collapses to CEU. Moreover, it would be straightforward to introduce a reference point in our analysis to obtain what could be called the 'cumulative' biseparable preferences model. Luce (2000, Chapter 3) has a model in this spirit, which is logically independent of ours, since he uses a dynamic decision framework that deviates from the traditional Savage setting we employ. Moreover, the representation he obtains has more structure than that entailed by biseparable preferences.

The axiomatics in this paper are related to those in some papers on the characterization of CEU and MEU preferences, in particular to Nakamura (1990), Chew and Karni (1994), and Casadesus-Masanell, Klibanoff and Ozdenoren (2000). Of course, the main innovation in our contribution is the realization that the preferences which only satisfy Eq. (1) on binary acts can be useful and interesting.

Finally, a part of our discussion of cardinal risk aversion is related to Yaari (1969)'s paper on the characterization of relative risk aversion for a large class of preferences.[1] Roughly, the class of preferences he considers is wider than the class of the biseparable preferences. However, our results have the advantage of avoiding differentiability assumptions — whose behavioral characterization is not straightforward.

Organization

The paper is organized as follows. Section 1 introduces some required notation and terminology. Section 2 introduces formally the class of the biseparable and c-linearly biseparable preferences, it shows some properties that they possess, and provides some examples. Section 3 provides the axiomatizations of the two preference models, and shows how to obtain probabilistic beliefs. Section 4 contains the results on cardinal risk aversion. Section 5 offers some concluding remarks. The Appendix contains the proofs for all the results in the paper.

[1] For brevity, we do not discuss the papers which analyze risk aversion in the context of a specific decision model.

1 Set-Up and Preliminaries

The Savage-style setting we use consists of a set S of **states of the world**, an algebra Σ of subsets of S called **events**, and a set X of **consequences**. We denote by $\mathcal{F}$ the set of all the **simple acts**: finite-valued functions $f : S \to X$ which are measurable with respect to Σ. For $x \in X$ we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. So, with the usual slight abuse of notation, we identify X with the subset of the constant acts in $\mathcal{F}$. Moreover, $x\,A\,y$ denotes the **bet** f such that $f(s) = x$ for $s \in A$, and $f(s) = y$ for $s \notin A$, where $A \in \Sigma$.

We model the DM's preferences on $\mathcal{F}$ by a binary relation $\succcurlyeq$. A functional $V : \mathcal{F} \to \mathbb{R}$ **represents** $\succcurlyeq$ if $V(f) \geq V(g)$ if and only if $f \succcurlyeq g$. Clearly, a necessary condition for $\succcurlyeq$ to have a representation is that it be a **weak order** — a complete and transitive relation — so that, as customary, we can denote by $\sim$ and $\succ$ its symmetric and asymmetric components, respectively. A representation V is **monotonic** if $V(f) \geq V(g)$ whenever $f, g \in \mathcal{F}$ are such that $f(s) \succcurlyeq g(s)$ for all $s \in S$; it is **nontrivial** if $V(f) > V(g)$ for some $f, g \in \mathcal{F}$.

Given a weak order $\succcurlyeq$, acts $f, g \in \mathcal{F}$ are called **comonotonic** if there are no $s, s' \in S$ such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$. Finally, an event $A \in \Sigma$ is **null** (resp. **universal**) for a weak order $\succcurlyeq$ if $y \sim x\,A\,y$ (resp. $x \sim x\,A\,y$) for all $x, y \in X$ such that $x \succ y$, while A is **essential** for $\succcurlyeq$ if for some $x, y \in X$ we have $x \succ x\,A\,y \succ y$. We remark that our notion of null event is less demanding than that in Savage (Savage 1954).

1.1 Capacities and Choquet Integrals

A set-function ρ on (S, Σ) is called a **capacity** if it is monotone and normalized, that is: if for $A, B \in \Sigma$, $A \subseteq B$, then $\rho(A) \leq \rho(B)$; $\rho(\emptyset) = 0$ and $\rho(S) = 1$. A capacity is called a **probability measure** if it is **(finitely) additive**: For all $A, B \in \Sigma$ such that $A \cap B = \emptyset$, $\rho(A \cup B) = \rho(A) + \rho(B)$.

The notion of integral used for capacities is the **Choquet integral**, due to Choquet (1953). For a given bounded Σ-measurable function $\varphi : S \to \mathbb{R}$, the Choquet integral of φ with respect to a capacity ρ is defined as follows:

$$\int_S \varphi\, d\rho = \int_0^{+\infty} \rho(\{s \in S : \varphi(s) \geq \alpha\})\, d\alpha + \int_{-\infty}^0 [\rho(\{s \in S : \varphi(s) \geq \alpha\}) - 1]\, d\alpha, \tag{2}$$

where the integrals are taken in the sense of Riemann. When ρ is additive, (2) is equal to a standard (additive) integral. In general, Choquet integrals are seen to be monotonic, positive homogeneous and **comonotonic additive**: If φ, ψ are comonotonic functions from S into $\mathbb{R}$, then $\int (\varphi + \psi)\, d\rho = \int \varphi\, d\rho + \int \psi\, d\rho$.

2 Biseparable Preferences

The following definition is central to the paper. Given a representable binary relation, it singles out a subset of its representations. We call these representations 'canonical', as they are of special interest due to their separability properties:

Definition 1 *Let $\succcurlyeq$ be a binary relation. We say that a representation $V : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ is* ***canonical*** *if it is nontrivial and monotonic and there is a set-function $\rho_V : \Sigma \to [0,1]$ such that, if we let $u(x) \equiv V(x)$ for all $x \in X$, for all consequences $x \succcurlyeq y$ and all $A \in \Sigma$ we have:*

$$V(x\,A\,y) = u(x)\,\rho_V(A) + u(y)\,(1 - \rho_V(A)). \tag{3}$$

The next result clarifies the roles of the functions u and ρ_V.

Proposition 2 *Let $\succcurlyeq$ be a binary relation with a canonical representation. For all its canonical representations V and all $x, y \in X$,*

$$x \succcurlyeq y \Longleftrightarrow u(x) \geq u(y). \tag{4}$$

Moreover, for all $x \succ y$ and all $A, B \in \Sigma$ we have

$$x\,A\,y \succcurlyeq x\,B\,y \Longleftrightarrow \rho_V(A) \geq \rho_V(B). \tag{5}$$

That is, the index u given by the restriction of V to X is a (state-independent) utility function, so that we call it the **canonical utility index** of $\succcurlyeq$. The set-function ρ_V — the unique set-function that satisfies Eq. (3) for a given V — is a numerical representation of the DM's 'likelihood' relation, that we call the DM's **willingness to bet**. Roughly, $\rho_V(A)$ is the number of euros the DM is willing to exchange for a bet which pays 1 euro if $A \in \Sigma$ obtains, and nothing otherwise. The next simple result presents some properties of the willingness to bet function ρ_V. In particular, it is worth noting that ρ_V is a monotone set-function, i.e., a capacity.

Proposition 3 *Let V be a canonical representation of a binary relation $\succcurlyeq$. The set-function ρ_V has the following properties:*

(i) *An event $A \in \Sigma$ is essential iff $\rho_V(A) \in (0,1)$.*

(ii) *An event $A \in \Sigma$ is not essential iff $\rho_V(A) \in \{0,1\}$. In particular, $\rho_V(A) = 0$ iff A is null, and $\rho_V(A) = 1$ iff A is universal.*

(iii) *If $A, B \in \Sigma$ are such that $A \subseteq B$, then $\rho_V(A) \leq \rho_V(B)$.*

Remark 4 The proposition also shows that an event must be either null, universal, or essential. Moreover, it shows that for a binary relation with a canonical representation, A is essential if and only if $x \succ x\,A\,y \succ y$ for *every* $x \succ y$.

In light of these results we can say that a DM whose preference $\succcurlyeq$ has a canonical representation chooses among *binary* acts as if he was maximizing the (Choquet) 'expectation' of the canonical utility u with respect to the willingness to bet ρ_V. But his preferences over non-binary acts are not constrained to a specific functional form.

So far, we have not said anything about the uniqueness properties of canonical representations V, and hence of the canonical utilities u and willingness to bet functions ρ_V. That is, given two canonical representations V and V' of the same binary relation $\succcurlyeq$, are they related by more than just an increasing transformation? Clearly, if V' is a positive affine transformation of V (i.e., there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $V' = \alpha V + \beta$), it is a canonical representation. So canonical representations can at most be *cardinal scales*. In general, though, there is no reason to expect that they be cardinal. First of all, if $\succcurlyeq$ has no essential events, then Definition 1 does not impose any restriction on the structure of V beyond monotonicity: Any increasing transformation of V is also canonical (cf. Remark 8 below). Second, even if there are essential events it is simple to construct examples with finite S and X where V and V' are canonical representations of $\succcurlyeq$ which are not related by an affine transformation (they might even be expected utility functionals).

On the other hand, one of our main objectives is to achieve a proper separation of utility from willingness to bet. The examples just mentioned show that such separation could fail when V is not cardinal. As it turns out, ruling out this possibility suffices to reach our objective. In fact, we presently show in Proposition 6 that when V is cardinal the willingness to bet function is univocally determined. In turn, this uniqueness yields the separation of u and ρ. We thus restrict our attention to the following subclass of preferences:

Definition 5 *A binary relation $\succcurlyeq$ is called a* ***biseparable preference*** *if it admits a canonical representation, and moreover such representation is unique up to positive affine transformations when $\succcurlyeq$ has at least one essential event.*

We remark that the class of preferences with a canonical representation but not biseparable does not seem to contain many examples of interest. In fact, we show later (Proposition 10) that cardinality of the canonical representation is guaranteed for any preference satisfying a weak continuity condition.

Given a biseparable preference, its u is clearly cardinal. Cardinality is quite helpful, as it allows us to discuss u's concavity (but see Theorem 17 below) or any other of its cardinal properties (cf. Miyamoto (1988)). We now show that its willingness to bet is also unique:

Proposition 6 *Let $\succcurlyeq$ be a biseparable preference. Then ρ is unique: $\rho_V = \rho_{V'}$ for all canonical representations V and V' of $\succcurlyeq$.*

Because of this result, we shall henceforth write ρ instead of ρ_V to denote the willingness to bet of a biseparable preference.

Before showing that most of the decision models mentioned in the Introduction describe biseparable preferences, we look at a more special but popular decision setting, and present a variant of biseparability for that setting.

2.1 Constant Linearity and the Anscombe-Aumann Setting

An important special case of the decision setting we use is the one in which X has a vector structure; precisely, it is a convex subset of a vector space. For instance, this is the case if X is the set of all the lotteries on a set of prizes, as it happens in the classical setting of Anscombe and Aumann (1963). In this framework, it is natural to consider the preferences satisfying the following condition — where as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $\alpha f + (1-\alpha) g$ denotes the act which pays $\alpha f(s) + (1-\alpha) g(s) \in X$ for every $s \in S$.

Definition 7 *Let X be a convex subset of a vector space. A canonical representation V of a binary relation $\succcurlyeq$ is* ***constant linear (c-linear****, for short) if for all binary $f \in \mathcal{F}$, $x \in X$, and $\alpha \in [0,1]$,*

$$V(\alpha\, f + (1-\alpha)\, x) = \alpha\, V(f) + (1-\alpha)\, V(x).$$

A relation is called a ***c-linearly biseparable preference*** *if it admits a c-linear canonical representation.*

It is easy to verify that if a binary relation has a c-linear canonical representation, such representation is unique up to positive affine transformations. Therefore, for this class of preferences we do not have to specifically add a uniqueness requirement. It is important to observe that c-linearly biseparable preferences are not necessarily biseparable in the sense of Definition 5: They may have two canonical representations which are not related by a positive affine transformation (of course only one of them can be c-linear). As their name suggests, they are biseparable *in the c-linear class.*

2.2 Examples

Here we partially substantiate our earlier claim that many preference models are biseparable by showing that they all have a canonical representation. When X is a connected and separable topological space and each u function is continuous this suffices to show that they are biseparable (see Proposition 10 below). In the Anscombe-Aumann setting, c-linear biseparability follows immediately if we assume that each u function is affine on X.

(i) A relation $\succcurlyeq$ is a **CEU ordering** if there exist a utility $u : X \to \mathbb{R}$ and a capacity ν on (S, Σ) such that $\succcurlyeq$ can be represented by the functional $V : \mathcal{F} \to \mathbb{R}$ defined as follows:

$$V(f) \equiv \int_S u(f(\cdot))\, d\nu, \tag{6}$$

where the integral is taken in the sense of Choquet (notice that $V(f)$ is finite because $f \in \mathcal{F}$ is finite-valued). A special type of CEU preferences are the **RDEU orderings**, for which the capacity ρ is a 'distortion' $g(P)$ of some probability P. A **SEU ordering** corresponds to the special case of RDEU in which g is the identity. An axiomatization of CEU (and SEU) preferences in the Anscombe-Aumann setting is found in Schmeidler (1989); one in the Savage setting is found, e.g., in Gilboa (1987), Wakker (1989), Nakamura (1990), and Chew and Karni (1994).

If $\succcurlyeq$ is a CEU ordering with a nonconstant u, it is immediate to check that V is a canonical representation of $\succcurlyeq$, and u is a canonical utility index of $\succcurlyeq$. In fact, monotonicity is well known, and Eq. (3) holds with $\rho(A) = \nu(A)$ for all $A \in \Sigma$.

(ii) A popular generalization of CEU is the **CPT** of Tversky and Kahneman (1992). In CPT some consequence is established to be the DM's *reference point*. The consequences which are better than the reference point are called 'gains', and those which are worse are called 'losses'. The preferences over $\mathcal{F}$ are represented as follows: Given a nonconstant utility function u, normalized so that the reference point has utility 0, every act f is split into its 'gain' part f^+ (of the payoffs with positive utility) and its 'loss' part f^- (of the payoffs with negative utility). $V(f)$ is the sum of the Choquet integral of $u(f^+)$ w.r.t. a capacity ν^+ and the Choquet integral of $u(f^-)$ w.r.t. another capacity ν^-. (CEU corresponds to the special case in which $\nu^- = \nu^+$.) A CPT preference has a canonical representation only if it is also CEU. However, a CPT preference has a canonical representation on the sets of acts which only yield gains (or only losses). Also, it would be easy to generalize the notion of biseparable preference to allow willingness to bet to be different depending on whether gains or losses are considered.

(iii) Using a von Neumann-Morgenstern setting with 'known' probabilities, Gul (1991) presents a model in which preferences over lotteries are represented by a functional V obtained roughly as follows: Given a utility index $u : X \to \mathbb{R}$ and a lottery p, find its certainty equivalent, and write p as a mixture $\alpha q + (1-\alpha)r$ — where q (resp. r) is the sublottery obtained by taking the payoffs of p which are better (resp. worse) than the certainty equivalent of p. Then let $V(p) = \gamma(\alpha)\, U(q) + (1-\gamma(\alpha))\, U(r)$, where $U(q)$ denotes the expectation of u w.r.t. q (and analogously for $U(r)$) and for any $\alpha \in [0,1]$, there is $\beta \in (-1,\infty)$ such that $\gamma(\alpha) = \alpha/(1+(1-\alpha)\beta)$. In other words, the DM 'distorts' the probability α of getting a positive result because of the parameter β, that Gul calls an index of 'disappointment aversion'. In the case of binary lotteries, the functional V can be explicitly written as follows: If we let $p = \alpha\, x + (1-\alpha)\, y$, with $x \succcurlyeq y$, then

$$V(p) = \frac{\alpha}{1+(1-\alpha)\beta}\, u(x) + \frac{(1-\alpha)(1+\beta)}{1+(1-\alpha)\beta}\, u(y).$$

That is, the ordering over binary lotteries has an RDEU representation (though not over more general lotteries). Therefore it follows from our discussion of example (i) above that Gul's model, once adapted to the Savage setting, also describes preferences in the biseparable class.

(iv) A relation $\succcurlyeq$ is a **MEU ordering** if there exist a utility index u and a unique nonempty, (weak*)-compact and convex set C of probabilities on (S,Σ), such that $\succcurlyeq$ can be represented by the functional $V : \mathcal{F} \to \mathbb{R}$ defined as follows:

$$V(f) \equiv \min_{P \in C} \int_S u(f(\cdot))\, dP. \tag{7}$$

Again, SEU corresponds to a special case of MEU: That in which $C = \{P\}$ for some probability measure P. An axiomatization of MEU preferences in the Anscombe-Aumann setting is in Gilboa and Schmeidler (1989); one in the Savage setting is in Casadesus-Masanell, Klibanoff and Ozdenoren (2000).

If $\succcurlyeq$ is a MEU ordering with a nonconstant utility u, it is easy to check that V is a canonical representation of $\succcurlyeq$, and u is a canonical utility index of $\succcurlyeq$. In fact, if we let $\rho(A) = \min_{P \in C} P(A)$ for all $A \in \Sigma$ we see that $V(f) = \int u(f)\, d\rho$ for every *binary* f (notice that the equality does not hold necessarily for general f), so that Eq. (3) holds.

More generally, consider an 'α-MEU' preference $\succcurlyeq$ such that there is $\alpha \in [0, 1]$ for which $\succcurlyeq$ is represented by the functional

$$V(f) = \alpha \min_{P \in C} \int_S u(f(s))\, P(ds) + (1 - \alpha) \max_{P \in C} \int_S u(f(s))\, P(ds).$$

This includes the case of a 'maximax' DM, who has $\alpha = 0$. V is a canonical representation of $\succcurlyeq$: Eq. (3) holds with ρ defined for every $A \in \Sigma$ by

$$\rho(A) = \alpha \min_{P \in C} P(A) + (1 - \alpha) \max_{P \in C} P(A).$$

3 Axiomatization

Here we provide the axiomatic characterization of biseparable preferences. We begin with three necessary axioms, which are also sufficient when there are no essential events. Next, we look at the characterization of c-linearly biseparable preferences in the Anscombe-Aumann setting in which X is a convex subset of a vector space. Finally, we provide the characterization of biseparable preferences in the more general Savage setting.

3.1 Three Necessary Axioms

The following simple behavioral properties are satisfied by any preference with a canonical representation, *a fortiori* by a biseparable one. First of all, any such preference is a nontrivial weak order.

B1 (Preference Relation) *(a) For all $f, g \in \mathcal{F}$, $f \succcurlyeq g$ or $g \succcurlyeq f$. (b) For all $f, g, h \in \mathcal{F}$, if $f \succcurlyeq g$ and $g \succcurlyeq h$, then $f \succcurlyeq h$. (c) There are $f, g \in \mathcal{F}$ such that $f \succ g$.*

Then, a preference with a canonical representation satisfies two mild monotonicity axioms. Both axioms are widely used in the literature, and imply a form of state independence. The first one is the behavioral equivalent of monotonicity:

B2 (Dominance) *For every $f, g \in \mathcal{F}$, if $f(s) \succcurlyeq g(s)$ for every $s \in S$, then $f \succcurlyeq g$.*

The last necessary axiom, which is a weak version of Savage's P3 (1954), is a converse to B2 for some binary acts.

B3 (Eventwise Monotonicity) *For every non-null $A \in \Sigma$ and every $x, y \succcurlyeq z \in X$,*

$$x \succ y \Longrightarrow x\,A\,z \succ y\,A\,z.$$

For every non-universal $A \in \Sigma$ and every $x, y \preccurlyeq z \in X$,

$$x \succ y \Longrightarrow z\,A\,x \succ z\,A\,y.$$

Recalling Remark 4, axiom B1 immediately implies that an event can be *at most one* of null, essential or universal. Axiom B3 can be used to show directly that if an event A is non-null and non-universal then $x \succ x\,A\,y \succ y$ for *all* $x \succ y$, so that A is essential. Therefore, under axioms B1 and B3, the sets of the null, universal and essential events form a partition of Σ.

Remark 8 It is simple to show that if $\succcurlyeq$ is a weak order with no essential events, axioms B1–B3 are *sufficient* as well as necessary for $\succcurlyeq$ to be a biseparable preference, provided it admits a representation. More precisely, let $\succcurlyeq$ be a binary relation satisfying axioms B1–B3. If $\succcurlyeq$ has no essential events, every functional $V : \mathcal{F} \to \mathbb{R}$ representing $\succcurlyeq$ has the following (Choquet) integral representation: for every $f \in \mathcal{F}$,

$$V(f) = \int_S u(f(s))\,\rho(ds), \tag{8}$$

where ρ is the $\{0,1\}$-valued capacity on Σ defined by $\rho(A) = 0$ if A is null, $\rho(A) = 1$ if A is universal, and $u : X \to \mathbb{R}$ is defined by $u(x) \equiv V(x)$ for every $x \in X$. Thus, every representation of $\succcurlyeq$ is CEU, so that Definition 5 implies that $\succcurlyeq$ is a biseparable preference.

3.2 The Anscombe-Aumann Case

Consider the following axioms, some of which exploit the vector structure of X:

A1 (Certainty Equivalents) *For all $f \in \mathcal{F}$, there is $x \in X$ such that $x \sim f$.*

A2 (Weak Certainty Independence) *For all binary $f, g \in \mathcal{F}$ and all $x \in X$, if $\alpha \in (0,1)$ then*

$$f \succ g \Longleftrightarrow \alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x.$$

A3 (Archimedean Axiom) *For all $x, y, z \in X$, if $x \succ y \succ z$ then there exist $\alpha, \beta \in (0,1)$ such that*

$$\alpha x + (1-\alpha)z \succ y \succ \beta y + (1-\beta)z.$$

These axioms are mild behavioral assumptions. Axioms A1 and A3 are standard and play mostly a technical role. Axiom A2 is a very weak and compelling version of the independence axiom: It only requires that independence holds whenever we are comparing binary acts, and we are mixing them with a constant act. It is a weakening of the 'certainty independence' axiom introduced by Gilboa and Schmeidler (1989) as the cornerstone of their axiomatization of MEU in the Anscombe-Aumann setting. As it turns out, alongside B1–B2, these three axioms characterize c-linearly biseparable preferences. (Notice that axiom B3 is not used, as it is implied by the other axioms in the Anscombe-Aumann setting.)

Theorem 9 *Let X be a convex subset of a vector space, and let $\succcurlyeq$ be a binary relation on $\mathcal{F}$. Then the following statements are equivalent:*

(*i*) *$\succcurlyeq$ satisfies axioms B1–B2 and A1–A3.*

(*ii*) *There exist a nontrivial monotonic representation $V : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ and a capacity $\rho : \Sigma \to [0,1]$ such that:*

- *for all $x \succcurlyeq y$ in X, all $A \in \Sigma$, letting $u(x) \equiv V(x)$ for all $x \in X$, we have*

$$V(x\,A\,y) = u(x)\,\rho(A) + u(y)\,(1 - \rho(A));$$

- *for all binary $f \in \mathcal{F}$ and $x \in X$, and all $\alpha \in [0,1]$,*

$$V(\alpha\, f + (1-\alpha)\, x) = \alpha\, V(f) + (1-\alpha)\, V(x).$$

Moreover, the representation V is unique up to positive affine transformations and the capacity $\rho : \Sigma \to [0,1]$ is unique.

3.3 The Savage Case

We now come to the characterization in the more general Savage setting. Since the case of a $\succcurlyeq$ with no essential events is dealt with in Remark 8, we directly assume that $\succcurlyeq$ has at least one essential event.

Endow X with a topology τ. In turn, τ induces the product topology on the set X^S of all functions from S into X. In this topology, a net $\{f_\alpha\}_{\alpha\in D} \subseteq X^S$ converges to $f \in X^S$ if and only if $f_\alpha(s) \xrightarrow{\tau} f(s)$ for all $s \in S$ (remember that S is arbitrary). For this reason it also called the topology of **pointwise convergence**. We now have:

S1 (Continuity) *Let $\{f_\alpha\}_{\alpha\in D} \subseteq \mathcal{F}$ be a net that pointwise converges to $f \in \mathcal{F}$, and such that all f_α's and f are measurable with respect to the same finite partition. If $f_\alpha \succcurlyeq g$ (resp. $g \succcurlyeq f_\alpha$) for all $\alpha \in D$, then $f \succcurlyeq g$ (resp. $g \succcurlyeq f$).*

S1 is a standard continuity axiom. It is very weak, as the clause that all acts in the net used in the axiom are measurable w.r.t. the same partition significantly limits its demands. In particular, it is substantially weaker than requiring continuity w.r.t. to the product topology.

It is easy to see (Lemma 29 in Appendix B) that any binary relation satisfying B1, B2 and S1 on a connected X has certainty equivalents. That is, it satisfies axiom A1 above. Granted this, we henceforth denote by c_f an arbitrarily chosen certainty equivalent of $f \in \mathcal{F}$.

Another consequence of S1 is the following result, that was anticipated in Section 2.

Proposition 10 *Let X be a connected and separable topological space, and let $\succcurlyeq$ be a binary relation with a canonical representation. Then $\succcurlyeq$ is biseparable if it satisfies S1.*

That is, when continuity holds, canonical representations are unique up to positive affine transformations (in the presence of an essential event). As mentioned at the beginning of Subsection 2.2, this result can be used to show that the preferences in examples (i)–(iv) are biseparable.

The characterization of biseparable preferences requires another axiom with more substantial empirical content, a separability property. To introduce it, we begin with the observation (due to Nakamura (1990) and Gul (1992)) that even if we do not have access to a randomizing device, we can still construct a 'subjective mixture' of two acts f and h as follows: Fix some event B, and then construct *state by state* the act which for every state s yields the certainty equivalent of the bet $f(s)\, B\, h(s)$. Formally, the **statewise B-mixture of f and h** is the act $s(f\, B\, h) \in \mathcal{F}$ defined as follows: For all $s \in S$,

$$s(f\, B\, h)(s) \equiv c_{(f(s) B h(s))}.$$

If h is statewise dominated by f, the constructed act's payoffs are all indifferent to bets *on* the event B, so that it is analogous to a mixture in the Anscombe-Aumann setting. We are now ready to state the axiom (where $\{x, y\} \succcurlyeq z$ is a short-hand for $x \succcurlyeq z$ and $y \succcurlyeq z$):

S2 (Binary Comonotonic Act Independence) *For every essential $A \in \Sigma$, every $B \in \Sigma$, and for all $f, g, h \in \mathcal{F}$ such that $f = x\, A\, y$, $g = x'\, A\, y'$ and $h = x''\, A\, y''$. If f, g, h are pairwise comonotonic, and $\{x, x'\} \succcurlyeq x''$ and $\{y, y'\} \succcurlyeq y''$ (or $x'' \succcurlyeq \{x, x'\}$ and $y'' \succcurlyeq \{y, y'\}$), then*

$$f \succcurlyeq g \Longrightarrow s(f\, B\, h) \succcurlyeq s(g\, B\, h).$$

S2 is a weak version of the well known independence axiom of EU, and of Schmeidler's 'comonotonic independence' (1989) axiom:[2] In the spirit of the latter, S2 requires that 'subjective mixing' with a *comonotonic* act h does not affect the ranking of the two acts f and g. The additional requirement that h be either state by state better, or worse, than the other two acts guarantees that the 'mixing' operation is performed correctly, i.e., without involving bets on different events (B and B^c). However, S2 generalizes Schmeidler's axiom by only applying to bets measurable with respect to the algebra $\Sigma_A \equiv \{\emptyset, A, A^c, S\}$.

We can now state the representation result for the essential case. We say that the preference functional $V : \mathcal{F} \to \mathbb{R}$ is **sub-continuous** if $\lim_\alpha V(f_\alpha) = V(f)$ whenever $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$

[2] Precisely, it is an extension of the 'comonotonic act-independence' axiom of Chew and Karni (1994), in turn based on earlier work of Gul (1992) and Nakamura (1990).

is a net that pointwise converges to $f \in \mathcal{F}$, and such that all f_α's and f are measurable with respect to the same finite partition. Notice that this implies that u is τ-continuous.

Theorem 11 *Let X be a connected and separable topological space and let $\succcurlyeq$ be a binary relation on $\mathcal{F}$. Then the following statements are equivalent:*

(*i*) *$\succcurlyeq$ satisfies B1–B3 and S1–S2 and it has some essential event.*

(*ii*) *There exist a sub-continuous nontrivial monotonic representation $V : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ and a capacity $\rho : \Sigma \to [0,1]$ such that: for all $f \in \mathcal{F}$, all $x \succcurlyeq y$ in X, all $A \in \Sigma$, letting $u(x) \equiv V(x)$ for all $x \in X$,*

$$V(x\,A\,y) = u(x)\,\rho(A) + u(y)\,(1 - \rho(A)).$$

Moreover, the representation V is unique up to positive affine transformations and the capacity $\rho : \Sigma \to [0,1]$ is unique.

Summing up, axioms B1–B3 and S1–S2 are sufficient for showing that a binary relation is biseparable when the space of consequences has some topological structure (whether or not there is an essential event). Axioms B1–B3 and S2 are also necessary, while axiom S1 is necessary if one of the canonical representations is sub-continuous.

4 Risk Aversion and the Canonical Utility Function

In this section, we show that some well-known characterizations of risk aversion — that have been proved for SEU or some other preference models that belong to the biseparable class — can be extended to biseparable preferences. Though we do not explicitly repeat it every time, identical results are obtained with c-linearly biseparable preferences. Moreover, symmetric results characterize cardinal risk loving.

We present three different approaches. We start with a general comparative approach that applies to an arbitrary consequence space, explaining how such approach can be used to formalize the absolute notion of 'cardinal' risk aversion that was described in the Introduction. Then, we look at the case of monetary consequences and show that cardinal risk aversion can be equivalently stated in terms of preference for diversification when betting, or in terms of dislike of binary mean-preserving spreads.

4.1 The Comparative Approach

In the comparative approach, we depart from a comparison of the relative risk aversion of two DMs in the spirit of Yaari (1969):

Definition 12 *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two weak orders. We say that $\succcurlyeq_2$ is* ***a more conservative bettor than*** *$\succcurlyeq_1$ if: For all $x \in X$ and all* binary *$f \in \mathcal{F}$, both*

$$x \succcurlyeq_1 f \Longrightarrow x \succcurlyeq_2 f \tag{9}$$

and

$$x \succ_1 f \Longrightarrow x \succ_2 f. \tag{10}$$

That is, DM 1 uniformly undervalues every bet from DM 2's perspective.

A feature of this ranking is that it can reflect also possible differences in beliefs of the two DMs, that we want to remove from the comparison. We thus apply this ranking only to pairs of preferences with the same willingness to bet. This is analogous to what is usually done in the SEU case (cf. also Yaari (1969, Remark 1), noticing that his subjective probability is conceptually identical to our willingness to bet), where a comparative ranking of this sort is applied only to preferences with identical beliefs, possibly because they reflect a 'known' probability.

Definition 13 *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two biseparable preferences. We say that $\succcurlyeq_2$ is* ***more cardinal risk averse than*** *$\succcurlyeq_1$ whenever both the following conditions hold:*

1. *$\succcurlyeq_2$ is a more conservative bettor than $\succcurlyeq_1$;*

2. *$\succcurlyeq_1$ and $\succcurlyeq_2$ have a common willingness to bet ρ.*

It is important to stress that it is theoretically possible to obtain condition 2 in a purely behavioral fashion. That is, one can formulate a condition (in the same spirit as 'cardinal symmetry' in (forthcoming)) in terms of the two DMs' preferences which implies that two biseparable preferences have the same ρ. We refrain from doing so in the sake of brevity.

We now show that 'more cardinal risk averse' in this sense is equivalent to 'more concave'. Precisely:

Theorem 14 *Let X be a connected topological space, and let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two biseparable preferences with continuous canonical utility indices u_1 and u_2 respectively. Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ have a common willingness to bet ρ. Then, $\succcurlyeq_2$ is a more conservative bettor than $\succcurlyeq_1$* if and only if *u_2 is an increasing concave transformation of u_1.*

In other words, relative cardinal risk aversion is equivalent to relative concavity in biseparable preferences with the same willingness to bet. When both preferences are SEU, Theorem 14 generalizes the traditional result by showing that it is enough that Eqs. (9) and (10) hold for every x and *binary* f for relative concavity to obtain (see also Definition 24 below).

Remark 15 Unlike the mentioned SEU result, we do not assume any order or metric structure on X, or differentiability of the utility functions. It is well-known (de Finetti (1952), Arrow (1974), Pratt (1964)) that in the case in which $X \subseteq \mathbb{R}$ and the utility functions have differentiability properties, the fact that u_2 is an increasing concave transformation of u_1 is furthermore equivalent to pointwise dominance of the Arrow-Pratt coefficients of absolute risk aversion. Yaari (1969) uses differentiability assumptions to obtain a different characterization of relative cardinal risk aversion (for a different but general class of preferences).

Given the relative notion, it is theoretically possible to provide an *absolute* notion of cardinal risk aversion: Establish arbitrarily that certain benchmark preferences are to be classified as 'cardinal risk neutral', and then say that a preference is **cardinal risk averse** if it is more cardinal risk averse than some cardinal risk neutral preference. A problem with this latter step is that in general there may be no obvious choice of such benchmark, so that we cannot give a general characterization of such notion. However, when consequences are monetary, there is a natural way to obtain the absolute notion of cardinal risk aversion, so we now turn our attention to this important special case.

4.2 Monetary Payoffs

Assume that consequences are monetary, i.e., $X \subseteq \mathbb{R}$, and that our DM is greedy. In this case it is natural to define 'cardinal risk neutral' any preference for which u is increasing and affine (so that on binary acts, it maximizes Choquet expected value). It then follows from Theorem 14 that a biseparable preference is cardinal risk averse if and only if its canonical utility index is increasing and concave. We now show that some other natural properties of a biseparable preference are equivalent to cardinal risk aversion.

4.2.1 Preference for Diversification

The following is a weak version of properties studied by Dekel (1989) and then by Chateauneuf and Tallon (1998), which enjoys the advantage over the other notions presented in this section of being easily definable in terms of primitives.

Definition 16 *A weak order $\succcurlyeq$ exhibits **preference for bet diversification** if, for every essential event $A \in \Sigma$ and $x, x', y, y' \in X$, with $x \succ y$ and $x' \succ y'$, we have for every $\alpha \in [0,1]$,*

$$x\,A\,y \sim x'\,A\,y' \Longrightarrow [\alpha x + (1-\alpha)x']\,A\,[\alpha y + (1-\alpha)y'] \succcurlyeq x\,A\,y.$$

A DM exhibiting preference for bet diversification prefers bets with a smoother payoff profile, when betting on the same essential event. Intuitively, this is a feature that we would relate to the DM's risk aversion. The following result — generalizing a result for CEU preferences of Chateauneuf and Tallon (1998, Theorem 3.3) — shows that it is equivalent to the concavity of u:

Theorem 17 *Let X be an interval of $\mathbb{R}$, and let $\succcurlyeq$ be a binary relation with a canonical representation whose canonical utility function u is continuous. Then, $\succcurlyeq$ exhibits preference for bet diversification* if and only if *u is concave.*

4.2.2 'Subjective' Expected Values and Mean Preserving Spreads

The two most popular definitions of risk aversion in the literature hinge crucially on the existence of an external notion of 'objective' probability. The first one is the traditional 'direct' definition, that says that a preference is risk averse if it prefers the expected value of a lottery to the lottery itself. The second definition — which can also be applied to non-EU preferences — defines risk averse a preference which between any pair of lotteries, such that one of the two is a mean-preserving spread (MPS) of the other, prefers the less dispersed one. Clearly, in general the latter definition is more demanding than the former. In the EU case, they are both characterized by concavity of u.

In the Anscombe-Aumann setting, where we interpret the convexity of X as arising from the existence of an external randomizing device, similar results hold for c-linearly biseparable preferences, with identical definitions and characterization. In contrast, formulating such notions in a purely subjective setting is not as straightforward, since it is not clear how to obtain expected values and mean-preserving spreads. However, when dealing with biseparable preferences, we can use the DM's subjective beliefs to obtain 'subjective' expected values and MPSs for *binary acts.* (Clearly, we do not have any basis for calculating the subjective expected value or MPSs of nonbinary acts.)

We thus have the following version of the first definition of risk aversion, where for any biseparable preference with willingness to bet ρ and act $f = x\,A\,y$ with $x \succcurlyeq y$, we let $EV(f) \equiv x\,\rho(A) + y(1-\rho(A)) \in X$, the **subjective expected value (SEV)** of f.

Definition 18 *Given a biseparable preference $\succcurlyeq$ with willingness to bet ρ, we say that $\succcurlyeq$ has a* ***restricted preference for the SEV*** *if for every $f = x\,A\,y$, we have that*

$$EV(f) \succcurlyeq f.$$

To introduce a version of the MPS definition of risk aversion, we first need to define a (binary) subjective MPS of a given binary act.

Definition 19 *Given a biseparable preference $\succcurlyeq$ with willingness to bet ρ, and a bet $x\,A\,y$ with $x \succcurlyeq y$, we say that $f \in \mathcal{F}$ is a* ***binary subjective mean-preserving spread (SMPS) of*** *$x\,A\,y$* ***(for $\succcurlyeq$)*** *if there are $x', y' \in \mathcal{X}$ with $x' \succcurlyeq x$ and $y \succcurlyeq y'$, and $B \in \Sigma$ with $\rho(A) = \rho(B)$, such that $f = x'\,B\,y'$ and*

$$x'\rho(B) + y'(1-\rho(B)) = x\rho(A) + y(1-\rho(A)). \tag{11}$$

Clearly, both the subjective notions of expected value and MPS coincide with their traditional counterparts when there is an external probability, and ρ coincides with it. We can now reinforce Definition 18 as follows:

Definition 20 *A biseparable preference $\succcurlyeq$ is* ***averse to binary SMPSs*** *if it weakly prefers $x\,A\,y$ (with $x \succcurlyeq y$) to every one of its binary SMPSs.*

Once again we remark that, even though we simplify our exposition by explicitly using the canonical representation in Definitions 18 and 20, both could conceptually be stated in behavioral terms by only using the DM's preferences. In fact, it is possible to precisely elicit the number $\rho(A)$ just by looking at a subset of the preference relation on binary acts.

The following straightforward result shows that under a weak assumption both these behavioral features are equivalent to the concavity of the canonical utility index:

Proposition 21 *Let X be an interval of $\mathbb{R}$, and let $\succcurlyeq$ be a biseparable preference. Consider the following statements:*

(i) *The canonical utility index u is concave;*

(ii) *$\succcurlyeq$ is averse to binary SMPSs;*

(iii) *$\succcurlyeq$ has a restricted preference for the SEV.*

We have $(i) \Rightarrow (ii) \Rightarrow (iii)$, while $(iii) \Rightarrow (i)$ holds whenever $\rho(\Sigma) = [0, 1]$.

4.3 Extensions to Nonbinary Acts

The results of the previous subsections hold true to our promise of applying to any biseparable preference. However, their generality comes with a cost: the only predictions that we are allowed to draw are on betting behavior.

Clearly, more can be said if we impose further structure to the preferences under study. For instance, it immediately follows from Proposition 21 that a SEU preference is averse to binary MPSs if and only if it is averse to the standard MPSs of Rothschild and Stiglitz (1970). Similarly, it follows from Theorem 17 and the results of Chateauneuf and Tallon (1998) that a CEU preference exhibits a preference for bet diversification if and only if it exhibits a preference for 'comonotonic' diversification, that is, the extension of our property to pairs of comonotonic *nonbinary* acts.

We now show that a similar generalization of our comparative result holds for a much larger class of preferences, which includes *both* the CEU and MEU models. As this class is obtained by adding just a little structure to the biseparable preferences, this result is a further illustration of the potential usefulness of the biseparable preferences model.

To introduce such class, we depart from a simple and general observation. Let $B_0(\Sigma)$ denote the set of all real-valued Σ-measurable finite-valued functions, and say that a functional $I : B_0(\Sigma) \to \mathbb{R}$ is **monotonic** if $\phi \geq \psi$ implies $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in B_0(\Sigma)$. We have:

Lemma 22 *Let $\succcurlyeq$ be a binary relation represented by a functional $V : \mathcal{F} \to \mathbb{R}$. Then, V is monotonic* if and only if *there exists a monotonic functional $I_V : B_0(\Sigma) \to \mathbb{R}$ such that $V(f) = I_V(u(f))$, where $u(x) \equiv V(x)$ for all $x \in X$.*

In particular, the lemma implies that for any canonical representation V of a biseparable preference, we can find a functional I_V which 'generates' V from u. For instance, if $\succcurlyeq$ is a CEU preference with utility u and capacity ν, for the canonical representation $V(f) = \int u(f)\, d\nu$

we have $I_V(\phi) = \int \phi \, d\nu$. We use the functional I_V to identify the subset of the biseparable preferences that we are interested in:

Definition 23 *A biseparable preference $\succcurlyeq$ has the* ***Jensen property*** *if for every canonical representation V of $\succcurlyeq$, the functional I_V satisfies*

$$\psi(I_V(\phi)) \geq I_V(\psi(\phi)) \tag{12}$$

for all $\phi \in B_0(\Sigma)$ and all increasing concave $\psi \in B_0(\Sigma)$.

The Jensen property of Eq. (12) is satisfied by a number of preference functionals of interest, including the CEU and the MEU preference functionals.

The comparative ranking that we are interested in is the following:

Definition 24 *Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two weak orders. We say that $\succcurlyeq_2$ is* ***more uncertainty averse than*** *$\succcurlyeq_1$ if Eqs. (9) and (10) hold for all $x \in X$ and all (possibly nonbinary) $f \in \mathcal{F}$.*

We show below that the comparative property of Definition 12 is equivalent to the coarser ranking above on biseparable preferences with the Jensen property. The 'equality of beliefs' requirement in this case takes a stronger form than equality of ρ:

Definition 25 *Two biseparable preferences $\succcurlyeq_1$ and $\succcurlyeq_2$ have a* ***common canonical functional*** *if there exist two canonical representations V_1 and V_2, of $\succcurlyeq_1$ and $\succcurlyeq_2$ respectively, such that $V_1 = I^*(u_1)$ and $V_2 = I^*(u_2)$ for the* same *monotone functional $I^* : B_0(\Sigma) \to \mathbb{R}$.*

This condition — in principle verifiable using only behavioral data — has two important roles. First of all, it guarantees that the two biseparable preferences belong to the same 'class'; e.g., they are both SEU, CEU, or MEU. Second, it implies that the preferences must agree on anything except the specific profile of utilities that an act delivers. For instance, if $\succcurlyeq_1$ and $\succcurlyeq_2$ are both CEU (resp. both MEU), they have a common canonical functional iff $\nu_1 = \nu_2$ (resp. $C_1 = C_2$).

We now state the promised result:

Proposition 26 *Let X be a connected topological space and let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two biseparable preferences which satisfy the Jensen property and induce continuous canonical utility indices u_1 and u_2 respectively. Then, if $\succcurlyeq_1$ and $\succcurlyeq_2$ have a common canonical functional, $\succcurlyeq_2$ is a more conservative bettor than $\succcurlyeq_1$* if and only if *$\succcurlyeq_2$ is more uncertainty averse than $\succcurlyeq_1$.*

It follows that if $\succcurlyeq_1$ and $\succcurlyeq_2$ are both CEU (resp. MEU) and satisfy $\nu_1 = \nu_2$ (resp. $C_1 = C_2$), $\succcurlyeq_2$ is more uncertainty averse than $\succcurlyeq_1$ if and only if u_2 is an increasing concave transformation of u_1.

5 Concluding Remarks

5. 1 The results in Section 3 can be modified to find conditions that deliver a biseparable preference with probabilistic beliefs, i.e., one whose willingness to bet ρ is a probability measure. One straightforward possibility is to obtain finite additivity of ρ by adding a preference version of the classical necessary and sufficient condition for representability of a qualitative probability. Another possibility is to strengthen axiom S2 so that it applies to any triple of binary acts, thus obtaining an axiom of 'binary act independence'.This delivers a ρ which is such that for all essential $A \in \Sigma$, $\rho(A) + \rho(A^c) = 1$. Such property can be easily extended to all the events by an axiom that requires that if A is null (resp. universal), then A^c is non-null (resp. non-universal). It can then be seen that if $\succcurlyeq$ is ambiguity averse or loving in the sense of our (forthcoming), ρ is a probability measure.

A biseparable preference with probabilistic beliefs does not have to be SEU: it is possible to show by example that it does not even have to be 'probabilistically sophisticated' in the sense of Machina and Schmeidler (1992).[3] Conversely, there are PS preferences which are not even biseparable (see Machina and Schmeidler (1992) for an example). Thus the biseparable preference model with probabilistic beliefs and the PS model are logically independent. $\diamond$

5. 2 In the Introduction we claimed that the biseparable preferences model is the most general model in which the state-independent utility function and the willingness to bet are separated and univocally identified. Clearly, the separation of utility and 'beliefs' in biseparable preferences is not full if one takes the latter term to mean a stronger concept than the DM's willingness to bet. In fact, even if we know ρ (and u) we cannot necessarily predict how the DM will choose among two non-binary acts. Moreover, ρ is not the unique capacity representing the DM's likelihood ordering (any increasing and normalized transformation of ρ works as well). However, it is easy to convince oneself that if what is sought are unique u and ρ that satisfy Eq. (1), then any more general model will not achieve the desired separation *and* uniqueness properties. For instance, Miyamoto's generic utility theory (which requires Eq. (1) to hold for only one event) obtains cardinal u, but not a unique ρ. $\diamond$

5. 3 In conclusion, in this paper we present a general decision model that only imposes structure on how DMs evaluate binary acts and constants, and encompasses some of the most popular non-SEU models. We also provide a simple and intuitive axiomatization for such model, both for the Savage and Anscombe-Aumann settings. In order to illustrate the potential interest of this model, we look at three facets of a DM's risk attitude — that have been studied in the context of the SEU model or of some non-SEU models narrower than the one presented here — and we show that they can be simply characterized also for biseparable preferences, if we restrict our attention to bets.

Other results on biseparable or c-linearly biseparable preferences are already available. First of all, as mentioned earlier, in the companion paper (forthcoming) we use these models

[3] An example proving this claim, as well as details on the claims in the previous paragraph, are available from the authors upon request.

to characterize an extended notion of ambiguity attitude, as well as a notion of ambiguity for acts and events. Applying the latter notions, we show in (2001) that a seemingly harmless technical condition (range convexity of the willingness to bet) can force ambiguity averse biseparable preferences to have probabilistic beliefs. Finally, Ozdenoren (2000) presents some results on auction theory that hold for any biseparable preference. $\diamond$

Appendix

A Proofs for Section 2

Proof of Proposition 2: Let V be a canonical representation of $\succcurlyeq$. We have $x \succcurlyeq y$ iff $x\,A\,x \succcurlyeq y\,A\,y$ iff $u(x) \geq u(y)$. Moreover, if $x \succ y$, we have

$$\begin{aligned} x\,A\,y \succcurlyeq x\,B\,y \quad &\iff \quad V(x\,A\,y) \geq V(x\,B\,y) \\ &\iff \quad u(x)\,\rho_V(A) + u(y)\,(1-\rho_V(A)) \geq u(x)\,\rho_V(B) + u(y)\,(1-\rho_V(B)) \\ &\iff \quad [u(x)-u(y)]\,\rho_V(A) \geq [u(x)-u(y)]\,\rho_V(B) \\ &\iff \quad \rho_V(A) \geq \rho_V(B), \end{aligned}$$

as wanted. ∎

Proof of Proposition 3: Suppose that $A \in \Sigma$ is essential. By definition, there exist $x \succ y$ such that $x \succ x\,A\,y \succ y$. Hence, $u(x) > u(x)\,\rho_V(A) + u(y)\,[1-\rho_V(A)] > u(y)$, which implies $u(x)\,(1-\rho_V(A)) > u(y)\,(1-\rho_V(A))$ and $u(x)\,\rho_V(A) > u(y)\,\rho_V(A)$. Hence, $\rho_V(A) \in (0,1)$. As to the converse, $\rho_V(A) \in (0,1)$ implies, for all $x \succ y$,

$$u(x) > u(x)\,\rho_V(A) + u(y)\,[1-\rho_V(A)] > u(y),$$

and so $x \succ x\,A\,y \succ y$.

Next, suppose that A is null. By definition, there exist $x, y \in X$ such that $x \succ y$ and $y \sim x\,A\,y$. Then, for all canonical representation V of $\succcurlyeq$ we have $u(y) = u(x)\,\rho_V(A) + u(y)\,[1-\rho_V(A)]$, and so $\rho_V(A) = 0$ since $u(x) \neq u(y)$. The converse holds. In fact, suppose that $\rho_V(A) = 0$. For any $x \succ y$ and any canonical representation V we have $V(x\,A\,y) = u(y)$, and so $x\,A\,y \sim y$, which implies that A is null. A similar argument shows that $\rho_V(A) = 1$ iff A is universal.

We conclude that (i) and (ii) hold. As to (iii), given any $x \succ y$, $A \subseteq B$ implies $(xBy)(s) \succcurlyeq (xAy)(s)$, and

$$\rho_V(A) = \frac{V(x\,A\,y) - u(y)}{u(x) - u(y)} \leq \frac{V(x\,B\,y) - u(y)}{u(x) - u(y)} = \rho_V(B),$$

using the monotonicity of V. ∎

Proof of Proposition 6: If there are no essential events, then the result follows immediately from part (ii) of Proposition 3. Suppose that there exists some essential event. Then if V and V' are two canonical representations of $\succcurlyeq$ there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $V = \alpha V' + \beta$. Let $x \succ y$. By Eq. (3), for every essential $A \in \Sigma$,

$$\rho_V(A) = \frac{V(x\,A\,y) - u(y)}{u(x) - u(y)} = \frac{(\alpha V(x\,A\,y) + \beta) - (\alpha\,u(y) + \beta)}{(\alpha\,u(x) + \beta) - (\alpha\,u(y) + \beta)} = \rho_{V'}(A).$$

If $A \in \Sigma$ is not essential, $\rho_V(A) = \rho_{V'}(A)$ follows again from (ii) of Proposition 3. ∎

B Proofs for Section 3

For expositional reasons that will become clear as we proceed, the proofs for this section are not presented in the same order as they appear there. Precisely, we prove the main representation results (Theorems 9 and 11) *before* Proposition 10.

B.1 Theorem 9

We start by proving two lemmas, in which $\succcurlyeq$ is a binary relation satisfying B1–B2, A1–A3.

Lemma 27 *There exists $V : \mathcal{F} \to \mathbb{R}$, unique up to a positive affine transformation, such that for all $f, g \in \mathcal{F}$,*

$$f \succcurlyeq g \text{ if and only if } V(f) \geq V(g) \tag{13}$$

and for all binary $f \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$,

$$V(\alpha f + (1-\alpha)x) = \alpha V(f) + (1-\alpha)V(x). \tag{14}$$

Proof: By axioms B1 and A2–A3 and the von Neumann-Morgenstern theorem, there exists $u : X \to \mathbb{R}$, unique up to positive affine transformations, such that for every $x, y \in X$, and every $\alpha \in [0,1]$,

$$u(\alpha\, x + (1-\alpha)\, y) = \alpha\, u(x) + (1-\alpha)\, u(y).$$

Given an act $f \in \mathcal{F}$, let $c_f \in X$ be one of its certainty equivalents (which exist by axiom A1). Define $V : \mathcal{F} \to \mathbb{R}$ by $V(f) = u(c_f)$. By A2, $f \sim c_f \in X$ implies $\alpha\, f + (1-\alpha)\, x \sim \alpha\, c_f + (1-\alpha)\, x$, for all $x \in X$ and all binary $f \in \mathcal{F}$. Then, for all binary $f \in \mathcal{F}$ and $x \in X$,

$$\begin{aligned} V(\alpha\, f + (1-\alpha)\, x) &= V(\alpha\, c_f + (1-\alpha)\, x) \\ &= \alpha\, V(f) + (1-\alpha)\, V(x). \end{aligned}$$

Moreover, let V' be another representation satisfying (14). Since u' is a positive affine transformation of u there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that:

$$\begin{aligned} V'(f) &= u'(c_f) = \alpha\, u(c_f) + \beta \\ &= \alpha V(f) + \beta, \end{aligned}$$

as wanted. ∎

Lemma 28 *Let V be the functional provided by Lemma 27. Then, for every $x^* \succ x_*$ and essential $A \in \Sigma$, if we normalize V so that $V(x^*) = 1$, $V(x_*) = 0$, we have for all $x, y \in X$*

such that $x \succcurlyeq y$*,*

$$V(x\,A\,y) = V(y) + (V(x) - V(y))V(x^*\,A\,x_*).$$

Proof: First notice that B2 implies that for all events $A \in \Sigma$, and all $x, y, z \in X$,

$$x \sim y \Longrightarrow x\,A\,z \sim y\,A\,z. \tag{15}$$

We now consider the case $x^* \succcurlyeq x \succ y \succcurlyeq x_*$. Let $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ be such that

$$x \sim \alpha x^* + (1-\alpha)y \text{ and } y \sim \beta x^* + (1-\beta)x_*.$$

W.l.o.g., set $V(x_*) = 0$ and $V(x^*) = 1$. Hence, $V(y) = \beta$, while $V(x) = \alpha + (1-\alpha)V(y)$ implies that

$$V(x) - V(y) = \alpha[1 - V(y)] = \alpha(1-\beta).$$

Therefore, using (13), (14) and (15) we obtain:

$$\begin{aligned}
V(x\,A\,y) &= V((\alpha x^* + (1-\alpha)y)\,A\,y) = V(\alpha(x^*\,A\,y) + (1-\alpha)y) \\
&= \alpha V(x^*\,A\,y) + (1-\alpha)V(y) \\
&= \alpha V(x^*\,A\,(\beta x^* + (1-\beta)x_*)) + (1-\alpha)V(y) \\
&= \alpha V(\beta x^* + (1-\beta)(x^*\,A\,x_*)) + (1-\alpha)V(y) \\
&= \alpha\,[\beta V(x^*) + (1-\beta)V(x^*\,A\,x_*)] + (1-\alpha)V(y) \\
&= \alpha\beta V(x^*) + \alpha(1-\beta)V(x^*\,A\,x_*) + (1-\alpha)V(y) \\
&= \alpha V(y) + \alpha(1-\beta)V(x^*\,A\,x_*) + (1-\alpha)V(y) \\
&= V(y) + \alpha(1-\beta)V(x^*\,A\,x_*) \\
&= V(y) + (V(x) - V(y))V(x^*\,A\,x_*)
\end{aligned}$$

as wanted. To conclude the proof, we have to show the result in the remaining three cases: (i) $x \succcurlyeq x^*$, $y \succcurlyeq x_*$; (ii) $x^* \succ x$, $x_* \succ y$; (iii) $x \succ x^*$, $x_* \succ y$. As the proof for each case is analogous to the one above, we omit it. ∎

Proof of Theorem 9: The $(ii) \Rightarrow (i)$ part is immediate. As to the $(i) \Rightarrow (ii)$ part, let V be the functional provided by Lemma 27. Fix $x^* \succ x_*$ and, taking if necessary a positive affine transformation of V, suppose that $V(x^*) = 1$ and $V(x_*) = 0$. For all essential $A \in \Sigma$ set $\rho(A) = V(x^*\,A\,x_*)$, and for all null (resp. universal) $A \in \Sigma$, set $\rho(A) = 0$ (resp. $\rho(A) = 1$). By Lemma 28 and the definitions of null and universal, we then have $V(x\,A\,y) = u(x)\,\rho(A) + u(y)\,(1 - \rho(A))$, so that V and ρ are the required functions.

As for the uniqueness statement, it is clear that any positive affine transformation of V satisfies the representation with the ρ constructed above. Suppose that V' is another representation and ρ' is another capacity which represent $\succcurlyeq$. From Lemma 27 it follows that V' must be a positive affine transformation of V. This fact allows us to show that

$\rho(A) = \rho'(A)$ for all $A \in \Sigma$ along the same lines as the proof of Proposition 6. Thus, ρ is unique and V is unique up to a positive affine transformation. ∎

B.2 Theorem 11

The proof of Theorem 11 involves several lemmas. We start with a simple solvability result, whose proof is given for completeness. Given any $A \in \Sigma$, $x \in X$ and $g \in \mathcal{F}$, $x\,A\,g$ denotes the act such that $(x\,A\,g)(s) = x$ if $s \in A$, and $(x\,A\,g)(s) = g(s)$ if $s \notin A$.

Lemma 29 *Let $\succcurlyeq$ be a binary relation satisfying B1, B2 and S1. If X is connected, then*

- (a) *for every $f \in \mathcal{F}$ there exists $x \in X$ such that $f \sim x$.*
- (b) *for every $x, y \in X$, $A \in \Sigma$, and $f, g \in \mathcal{F}$, if $x\,A\,f \succ g \succ y\,A\,f$, there exists $x' \in X$ such that $x'\,A\,f \sim g$.*

Proof: We want to show that the set $\{x \in X : x\,A\,f \succcurlyeq g\}$ is τ-closed in X for all $f, g \in \mathcal{F}$. Let $\{x_\alpha\}_{\alpha \in D} \subseteq \{x \in X : x\,A\,f \succcurlyeq g\}$ be a net such that $x_\alpha \xrightarrow{\tau} x_0$. Then $x_\alpha\,A\,f$ converges pointwise to $x_0\,A\,f$, and all acts are measurable w.r.t. the same partition. By S1, if $x_\alpha\,A\,f \succcurlyeq g$ for all $\alpha \in D$, then $x_0\,A\,f \succcurlyeq g$. Hence, $x_0 \in \{x \in X : x\,A\,f \succcurlyeq g\}$, which is therefore a τ-closed set in X. In particular, if A is universal, this implies that the set $\{x \in X : x \succcurlyeq g\}$ is τ-closed in X. Symmetric arguments show that both sets $\{x \in X : x\,A\,f \preccurlyeq g\}$ and $\{x \in X : x \preccurlyeq g\}$ are τ-closed in X. Both (a) and (b) now follow from the connectedness of X. ∎

Presenting the next lemma requires introducing some terminology and notation. Given $m'' \succ m'$ and an essential event $A \in \Sigma$, we define an **increasing standard sequence with mesh** (m', m'') **and carrier** A a sequence $\{x^0, x^1, x^2, \dots\} \subseteq X$ such that $x^0 \succ m''$ and for every $n \geq 0$, $x^n\,A\,m' \sim x^{n+1}\,A\,m''$ for all n such that x^n is not the last element of the sequence. Symmetrically, we define a **decreasing standard sequence with mesh** (m', m'') **and carrier** A a sequence $\{x^0, x^1, x^2, \dots\} \subseteq X$ such that $m' \succ x^0$ and for every $n \geq 0$, $m''\,A\,x^{n+1} \sim m'\,A\,x^n$ for all n such that x^n is not the last element of the sequence. Finally, we say that $\{x^0, x^1, x^2, \dots\}$ is a **standard sequence** if there are a mesh and a carrier which make it into an increasing or decreasing standard sequence with respect to that mesh and carrier. A standard sequence $\{x^0, x^1, x^2, \dots\}$ is said to be **strictly bounded** if there are $x_*, x^* \in X$ such that $x^* \succ x_n \succ x_*$ for every $n \geq 0$.

We can now state and prove:

Lemma 30 *Let $\succcurlyeq$ be a binary relation satisfying B1–B3, and S1. If X is a connected and separable topological space, every strictly bounded standard sequence is finite.*

Proof: Since X is connected and separable, there exists a continuous representation $\phi : X \to \mathbb{R}$ of the restriction of $\succcurlyeq$ to X. Moreover, B1–B3 imply that for all essential events $A \in \Sigma$, and all $x, y, z \in X$ such that $x, y \succcurlyeq z$,

$$x \succcurlyeq y \Longleftrightarrow x\,A\,z \succcurlyeq y\,A\,z. \tag{16}$$

Define $V : \mathcal{F} \to \mathbb{R}$ by $V(f) \equiv \phi(c_f)$, where c_f is the certainty equivalent of f. Clearly, $V(x) = \phi(x)$ for all $x \in X$. Let $x, y \in X$ be such that $x \succ y \succcurlyeq z$, and let $\alpha \in \mathbb{R}$ be such that $V(x\,A\,z) > \alpha > V(y\,A\,z)$. Let $c' \sim x\,A\,z$ and $c'' \sim y\,A\,z$. Then $\phi(c') > \alpha > \phi(c'')$. Being ϕ continuous, $\phi(X)$ is an interval, and so there is $c \in X$ such that $\phi(c) = \alpha$. Hence, $x\,A\,z \succ c \succ y\,A\,z$, and so, by Lemma 29, there is $x' \in X$ such that $x'\,A\,z \sim c$, which implies $V(x'\,A\,z) = \alpha$.

Next, let $\{x_n\}_{n\geq 1}$ be a decreasing sequence with $x_n \succcurlyeq x_{n+1} \succcurlyeq z$ for all $n \geq 1$. If $\lim_{n\to\infty} \phi(x_n) = \phi(x)$, then $\lim_{n\to\infty} V(x_n\,A\,z) = V(x\,A\,z)$. In fact, suppose *per contra* that $\lim_{n\to\infty} V(x_n\,A\,z) = \alpha > V(x\,A\,z)$. By what we proved above, there exists $x' \in X$ such that $V(x'\,A\,z) = \alpha$, so that $V(x_n\,A\,z) \geq V(x'\,A\,z) > V(x\,A\,z)$ for all $n \geq 1$. By (16), this implies that $x_n \succcurlyeq x' \succ x$, which contradicts $\lim_{n\to\infty} \phi(x_n) = \phi(x)$. We conclude that $\lim_{n\to\infty} V(x_n\,A\,z) = V(x\,A\,z)$.

Let $\{x^n\}_{n\geq 0}$ be an increasing *standard* sequence with the essential event A as carrier and mesh (m', m''). By definition, $x^n \succ m'' \succ m'$ for all $n \geq 0$. Suppose $\{x^n\}_{n\geq 0}$ is strictly bounded by $x_*, x^* \in X$, i.e., $x^* \succ x^n \succ x_*$ for all $n \geq 0$. For all $n \geq 0$, we have

$$\begin{aligned} V(x^n\,A\,m'') &= V(x^{n+1}\,A\,m'), \\ \phi(x^n) &\in [\phi(x_*), \phi(x^*)]. \end{aligned}$$

Since $[\phi(x_*), \phi(x^*)]$ is compact in $\mathbb{R}$, w.l.o.g. (taking subsequences if needed) we can assume that the sequence $\phi(x^n)$ converges to some $\alpha \in [\phi(x_*), \phi(x^*)]$. Since ϕ is continuous and $[\phi(x_*), \phi(x^*)] \subseteq u(X)$, there is $x' \in X$ such that $\phi(x') = \alpha$, and so $\lim_{n\to\infty} \phi(x^n) = \phi(x')$. By what we proved above, if we take respectively $z = m'$ and $z = m''$, we get $\lim_{n\to\infty} V(x^n\,A\,m') = V(x'\,A\,m')$ and $\lim_{n\to\infty} V(x^n\,A\,m'') = V(x'\,A\,m'')$. Since $V(x'\,A\,m') < V(x'\,A\,m'')$, this contradicts $V(x^n\,A\,m'') = V(x^{n+1}\,A\,m')$ for n large enough. A symmetric argument holds for decreasing standard sequences.s. ∎

Next, we provide two useful results on the representation of a binary relation satisfying axioms B1–B3, S1 and S2 when there are essential events. The first result shows that such a preference has a 'locally canonical' representation, holding for every act which is measurable w.r.t. the algebra generated by an essential event. Henceforth, for any $A \in \Sigma$ we let $\mathcal{F}_A$ be the set of (binary) acts which are measurable w.r.t. the algebra Σ_A generated by A.

Lemma 31 *Let $\succcurlyeq$ be a binary relation satisfying axioms B1–B3, S1, S2 with some essential event. For any essential $A \in \Sigma$ there is a representation $V_A : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ which satisfies: There exist a unique capacity $\rho_A : \Sigma_A \to [0,1]$ and a function $u_A : X \to \mathbb{R}$, unique up to a positive affine transformation, such that $V_A(f) = u_A(c_f)$ for every $f \in \mathcal{F}$ and for all $x \succcurlyeq y$ and all $B \in \Sigma_A$,*

$$V_A(x\,B\,y) = u_A(x)\rho_A(B) + u_A(y)(1 - \rho_A(B)). \tag{17}$$

Moreover such V_A is sub-continuous (hence u_A is τ-continuous).

Notice that the properties of ρ_A and u_A and (17) imply that V_A is unique only up to positive affine transformations and that $\rho_A(A) \in (0,1)$.

Proof: We first show that for any essential A, all hypotheses of Theorem 1 of Chew and Karni (1994, henceforth CK) are satisfied on the restriction $\succcurlyeq$ to the acts in $\mathcal{F}_A$. CK's axioms 1' and 5' are obviously satisfied. Their axiom 3' is also satisfied: It is easy to see that an event A is 'null' in the CK-sense if and only if A^c is universal, while A is 'universal' in the CK-sense if and only if A^c is null. Hence, our eventwise monotonicity axiom is equivalent to their axiom 3'. Also, if A is essential, then $A^c \in \Sigma_A$ is neither null nor universal in the CK-sense. Axioms 2 and 6 are satisfied by Lemma 29. Axiom 7 by Lemma 30. We can thus conclude that all of CK's axioms are satisfied.

By Theorem 1 of CK, there is a unique ρ_A on Σ_A and a function $u_A : X \to \mathbb{R}$, unique up to positive affine transformations, such that the functional $V_A : \mathcal{F}_A \to \mathbb{R}$ defined by (17) represents $\succcurlyeq$ on $\mathcal{F}_A$. We extend V_A to all of $\mathcal{F}$ by letting $V_A(f) = u_A(c_f)$ for any $f \in \mathcal{F}$. It is immediate to check that, thus defined, V_A represents $\succcurlyeq$.

We now show that V_A is sub-continuous:[4] if $\{f_\alpha\}_{\alpha \in D}$ is a net converging to f such that each f_α and f are measurable w.r.t. the same partition, then $V_A(f_\alpha) \to V_A(f)$. Fix $\epsilon > 0$. Suppose that, for every $\alpha \in D$, there exists $\beta \geq_D \alpha$ such that $V_A(f_\beta) \geq V_A(f) + \epsilon$. Fix any such f_β, and set

$$z_1 = c_{[c_{f_\beta} A c_f]} \quad \text{and} \quad z_{n+1} = c_{[z_n A c_f]},$$

for every $n > 1$. Then for every $n \geq 1$ we have

$$u_A(z_n) = \rho_A(A)^n V_A(f_\beta) + (1 - \rho_A(A)^n) V_A(f).$$

By choosing a sufficiently large n, we thus have $V_A(f) + \epsilon > u_A(z_n) > V_A(f)$.

Let $E = \{\beta \in D : V_A(f_\beta) \geq V_A(f) + \epsilon\}$. It is straightforward to see that $\{f_\beta\}_{\beta \in E}$ is a subnet of $\{f_\alpha\}_{\alpha \in D}$ with respect to the inclusion mapping. Moreover $f_\beta \succ z_n \succ f$ for all $\beta \in E$. Since $f_\alpha \to f$, $f_\beta \to f$ as well. Invoking the continuity axiom yields a contradiction.

We conclude that there exists γ such that $\alpha \geq_D \gamma$ implies $V_A(f_\alpha) < V_A(f) + \epsilon$. Similarly, there exists γ' such that $\alpha \geq_D \gamma'$ implies $V_A(f_\alpha) > V_A(f) - \epsilon$. Consider now a $\gamma'' \in D$ such that $\gamma'' \geq_D \{\gamma, \gamma'\}$. For all $\alpha \geq_D \gamma''$ we have $V_A(f_\alpha) \in (V_A(f) - \epsilon, V_A(f) + \epsilon)$. Therefore $V_A(f_\alpha) \to V_A(f)$. ∎

To conclude the proof of Theorem 11, we need to show that the representations V_A that we obtained for all the essential events A are **essentially identical**; i.e., they are all equal subject to a common normalization. Proving this is easy because we can again exploit a previous result, this time of Nakamura (1990). First, however, we need to show that our axiom S2 is equivalent to the restriction of Nakamura's A6 axiom to binary acts:

[4] We are grateful to Fabio Maccheroni and Marciano Siniscalchi for their help in developing this argument.

Lemma 32 *Under axioms B1–B3 and S1, axiom S2 holds* if and only if *the following property holds: For every essential $A \in \Sigma$, every $B \in \Sigma$, and for all $f, g \in \mathcal{F}$ such that $f = x\,A\,y$, $g = x'\,A\,y'$. If f, g are comonotonic, and $x \succcurlyeq x'$ and $y \succcurlyeq y'$, then*

$$[c_{(xAy)}\,B\,c_{(x'Ay')}] \sim [c_{(xBx')}\,A\,c_{(yBy')}].$$

Proof: The result is proved immediately using the same argument that CK use in their Lemma 3 to show that their axiom 5' is equivalent to axiom A6 in Nakamura (1990). (In our case things are even simpler because we have to apply CK's Lemma 2 only once.) ∎

Proof of Theorem 11: The 'only if' part is immediate. As to the 'if' part, fix an essential $B \in \Sigma$, and a representation V_B obtained by Lemma 31. We first show that for any other essential $A \in \Sigma$, $u_A = u_B$ under a common normalization. To see this, we invoke Lemma 32 to argue that the restriction to binary acts of Nakamura's axiom A6 holds. The claim is now shown by repeating an argument in Nakamura's proof of his Proposition 1 (1990, pp. 356–359), that we omit.

Given that $u_A = u_B$, we can let $V \equiv V_B$. We then define $\rho : \Sigma \to [0, 1]$ as follows: For every essential (resp. null, universal) A, we let $\rho(A) = \rho_A(A)$ (resp. $\rho(A) = 0$, $\rho(A) = 1$).[5] Thus, for all $C \in \Sigma$ such that $C \in \Sigma_A$ for an essential $A \in \Sigma$ and all $x \succcurlyeq y$,

$$V(x\,C\,y) = V_A(x\,C\,y) = u_A(x)\,\rho_A(C) + u_A(y)\,(1 - \rho_A(C)) = u(x)\,\rho(C) + u(y)\,(1 - \rho(C)).$$

If C does not belong to any such Σ_A, it is easy to check that (3) holds by construction. The cardinality and sub-continuity of V follows immediately from the properties of the representation V_B from Lemma 31. The uniqueness of ρ is then proved as in Theorem 9. ∎

B.3 Proposition 10

Suppose that $\succcurlyeq$ is a relation with a canonical representation which satisfies S1. In the case in which $\succcurlyeq$ has no essential event, there is nothing to prove, so assume that there is an essential event A. It is easy to verify that since it has a canonical representation V, $\succcurlyeq$ satisfies axioms B1–B3 and S2. Given that by assumption $\succcurlyeq$ also satisfies S1, we can follow the steps of the proof of Lemma 31 to show that we can apply Theorem 1 of CK to $\succcurlyeq_A$, the restriction of $\succcurlyeq$ to $\mathcal{F}_A$. Thus, $\succcurlyeq_A$ admits a CEU representation V_A which is unique only up to positive affine transformations. Since also the restriction of V to $\mathcal{F}_A$ is a CEU representation of $\succcurlyeq_A$, on $\mathcal{F}_A$ the functionals V and V_A are positive affine transformations of each other. Given $x^* \succ x_*$, if we impose the normalization $u(x_*) = u_A(x_*) = 0$ and $u(x^*) = u_A(x^*) = 1$, we thence have $V_A(f) = V(f)$ for all $f \in \mathcal{F}_A$. In turn, this implies $\rho(B) = \rho_A(B)$ for all $B \in \Sigma_A$ and $u(x) = u_A(x)$ for all $x \in X$. Finally, for every $f \in \mathcal{F}$,

$$V_A(f) = u_A(c_f) = u(c_f) = V(f).$$

[5] Notice that if A and A^c are both essential then $\rho_A = \rho_{A^c}$ and (with the common normalization) $u_A = u_{A^c}$. This is due to the uniqueness properties of the representation in Lemma 31, and to the fact that it is constructed for the algebra Σ_A, rather than the single events A or A^c.

This shows that for all essential $A \in \Sigma$, $V_A = V$ after a common normalization. Since this is true for every canonical representation V, they are all positive affine transformations of each other. Hence $\succcurlyeq$ is biseparable.

C Proofs for Section 4

Proof of Theorem 14: 'Only if': We start by showing that $\succcurlyeq_2$ is a more conservative bettor than $\succcurlyeq_1$ implies that $\succcurlyeq_1$ and $\succcurlyeq_2$ are ordinally equivalent, and for any pair $x \succ y$, if we renormalize the utilities u_1 and u_2 so that they take identical values on x and y, we have for every $z \in X$ such that $x \succcurlyeq z \succcurlyeq y$, $u_2(z) \geq u_1(z)$. Ordinal equivalence follows immediately. As for the second part of the statement, let $x \succ y$ be any two consequences and take the common normalization $u_1(x) = u_2(x) = 1$ and $u_1(y) = u_2(y) = 0$. If $\rho(A) \in \{0, 1\}$ for all $A \in \Sigma$, it is w.l.o.g. to assume that $u_1 = u_2$, so there is nothing to prove. So assume that there is one $A \in \Sigma$ such that $\rho(A) \equiv \rho \in (0, 1)$.

Consider the subset of X thus defined:

$$I \equiv \{z \in X : y \preccurlyeq_1 z \preccurlyeq_1 x\} = \{z \in X : y \preccurlyeq_2 z \preccurlyeq_2 x\},$$

where the equality again follows from ordinal equivalence. We now construct a 'grid' of points on I, that we label G. Let $G_0 \equiv \{y, x\}$. Find the certainty equivalent $c^1_{(xAy)}$ of $x\,A\,y$ for $\succcurlyeq_1$ and let $G_1 \equiv \{y, c^1_{(xAy)}, x\}$. Inductively, define $G_i \equiv \{c^1_{(x'Ax'')} : x', x'' \in G_{i-1}\}$ for $i = 2, 3, \ldots$, and let $G \equiv \lim_{i\to\infty} G_i = \cup_{i=0}^{\infty} G_i$ (notice that $G_{i-1} \subseteq G_i$).

We claim that $u_2(z) \geq u_1(z)$ for every $z \in G$. To prove this claim, we use induction on i. Consider G_1, and let $z = c^1_{(xAy)}$. By (i) we have that

$$z \sim_1 x\,A\,y \Rightarrow z \succcurlyeq_2 x\,A\,y,$$

which, in terms of the representations, is written $u_2(z) \geq \rho = u_1(z)$. Suppose now that the claim holds for every $z \in G_j$, for $j \leq i - 1$. Consider $z \in G_i$. There are $x', x'' \in G_{i-1}$ such that $z = c^1_{(x'Ax'')}$, and we know that $u_2(x') \geq u_1(x')$ and $u_2(x'') \geq u_1(x'')$. Hence,

$$u_2(x')\,\rho + u_2(x'')\,(1-\rho) \geq u_1(x')\,\rho + u_1(x'')\,(1-\rho) = u_1(z).$$

Since from (i) we also have $z \succcurlyeq_2 x'\,A\,x''$, we thus find $u_2(z) \geq u_1(z)$, as required. This concludes the induction step, and proves the claim.

Next, consider a $z \in I$. It is immediate to use the fact that $\rho \in (0, 1)$ to show that $u_1(G)$ is a dense subset of the interval $[u_1(y), u_1(x)]$. Hence, there is a sequence $\{z_n\}_{n=1}^{\infty} \subseteq G$ for which $\lim_{n\to\infty} u_1(z_n) = u_1(z)$. We use ordinal equivalence to show that $\lim_{n\to\infty} u_2(z_n) = u_2(z)$ as well. Start by observing that it is w.l.o.g. to assume that either $u_1(z_n) \uparrow u_1(z)$ or $u_1(z_n) \downarrow u_1(z)$. By contradiction, suppose that $\lim_{n\to\infty} u_2(z_n) = \alpha \neq u_2(z)$, in particular that $\alpha > u_2(z)$. If $u_1(z_n) \uparrow u_1(z)$, we immediately have a contradiction, since eventually

$u_2(z_n) > u_2(z)$. So, suppose that $u_1(z_n) \downarrow u_1(z)$. Since $\rho \in (0,1)$, we can find $\epsilon > 0$ such that

$$(\alpha + \epsilon)\rho + u_2(z)(1-\rho) < \alpha - \epsilon.$$

Take N large enough so that for every $n \geq N$, $|\alpha - u_2(z_n)| < \epsilon$, and let $f = z_N \, A \, z$. From our choices of N and ϵ it follows that for every $n \geq N$,

$$u_2(z_n) > \alpha - \epsilon > u_2(c_f) > u_2(z),$$

implying $z_n \succ_2 c_f$. On the other hand, $u_1(z_N) > u_1(c_f) > u_1(z)$, so that there is N' such that $u_1(z_n) < u_1(c_f)$ for $n \geq N'$. Thus, for $n \geq \max[N, N']$ we have $z_n \succ_2 c_f$ and $z_n \prec_1 c_f$, a contradiction. The case in which $\alpha < u_2(z)$ is dealt with symmetrically.

This shows that for all $z \in I$, there is a sequence $\{z_n\} \subseteq G$ such that $u_i(z_n) \to u_i(z)$ for $i = 1, 2$. Since $u_2(z_n) \geq u_1(z_n)$ for all $n \geq 1$, the continuity of the u_i's on X implies that for every $z \in I$, $u_2(z) \geq u_1(z)$.

Next, we show that the fact just proved implies that u_2 is an increasing concave transformation of u_1. Let $x, y \in X$ be as above. By ordinal equivalence, there exists an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $u_2(z) = \phi(u_1(z))$ for all $z \in X$. We want to show that ϕ is concave. For each $\alpha \in (0,1)$ there exists a $z_\alpha \in X$ with $x \succ z \succ y$ and such that $u_1(z_\alpha) = \alpha u_1(x) + (1-\alpha) u_1(y)$. Then

$$\begin{aligned}
\phi(\alpha u_1(x) + (1-\alpha)u_1(y)) = \phi(u_1(z_\alpha)) &= u_2(z_\alpha) \\
&\geq u_1(z_\alpha) = \alpha u_2(x) + (1-\alpha)u_2(y) \\
&= \alpha\phi(u_1(x)) + (1-\alpha)\phi(u_1(y))
\end{aligned}$$

and so ϕ is concave.

'If': Suppose that $u_2 = \phi(u_1)$ where $\phi : u_1(X) \to \mathbb{R}$ is increasing and concave. For $x \in X$ and $f = y \, A \, z$ with $u_1(y) \geq u_1(z)$, suppose that $x \succcurlyeq_1 f$. This implies $u_1(x) \geq V_1(f) = u_1(z)\rho(A) + u(z)(1-\rho(A))$, and so, being ϕ increasing, $\phi(u_1(x)) \geq \phi(V_1(f))$. But, since Choquet integrals satisfy Jensen's inequality for increasing concave transformations, we have

$$u_2(x) = \phi(u_1(x)) \geq \phi(V_1(f)) \geq \phi(u_1(y))\rho(A) + \phi(u_1(z))(1-\rho(A)) = V_2(f),$$

and so $x \succcurlyeq_2 f$. A similar argument shows that $x \succ_1 f \Longrightarrow x \succ_2 f$. We can thus conclude that $\succcurlyeq_2$ is a more conservative bettor than $\succcurlyeq_1$. ∎

Proof of Theorem 17: Suppose $\succcurlyeq$ exhibits preference for bet diversification. For every $z, z' \in X$ and $\alpha \in [0,1]$, let $z \, \alpha \, z' \equiv \alpha z + (1-\alpha) z'$. We first show that

$$x \, A \, y \succcurlyeq x' \, A \, y' \Longrightarrow [x \, \alpha \, x'] \, A \, [y \, \alpha \, y'] \succcurlyeq x' A \, y', \tag{18}$$

for all $\alpha \in [0,1]$ and essential $A \in \Sigma$.[6] *Per contra*, suppose that for some $\alpha^* \in [0,1]$ we have

[6] For u increasing, this is proved in Proposition 3.1 of Chateauneuf and Tallon (1998).

$x\,A\,y \succ x'\,A\,y'$ and $[x\,\alpha^*\,x']\,A\,[y\,\alpha^*\,y'] \prec x'Ay'$. Let V be the canonical representation of $\succcurlyeq$ whose utility index u is continuous. For all $\alpha \in [0,1]$, set

$$G(\alpha) \equiv V([x\,\alpha\,x']\,A\,[y\,\alpha\,y']) = u(x\,\alpha\,x')\,\rho_V(A) + u(y\,\alpha\,y')\,(1-\rho_V(A)).$$

Since u is continuous and $\rho_V(A) \in (0,1)$, $G(\cdot)$ is continuous, so that $I \equiv \{G(\alpha) : \alpha \in [\alpha^*, 1]\}$ is an interval. I contains $V(x\,A\,y)$ (for $\alpha = 1$) and $V([x\,\alpha^*\,x']\,A\,[y\,\alpha^*\,y'])$, and also $V(x'\,A\,y')$ by the assumption. Thus, there exists $\alpha' \in [\alpha^*, 1]$ such that $G(\alpha') = V(x'\,A\,y')$, so that $[x\,\alpha'\,x']\,A\,[y\,\alpha'\,y'] \sim x'\,A\,y'$. Choose $\beta \in [0,1]$ such that $\alpha'\beta = \alpha^*$. Then, by preference for bet diversification,

$$\beta([x\,\alpha'\,x']\,A\,[y\,\alpha'\,y']) + (1-\beta)(x'\,A\,y') \not\succcurlyeq x'\,A\,y',$$

which contradicts the assumption and proves (18).

Following (1998), we now apply a result of Debreu (1982). Let $X^o = (x_*, x^*)$ be the interior of the interval X. For $\alpha \in X^o$, set $X^\alpha \equiv \{x \in X^o : x < \alpha\}$ and $X_\alpha \equiv \{x \in X^o : x > \alpha\}$. Define $F : X_\alpha \times X^\alpha \to \mathbb{R}$ by $F(x,y) = V(x\,A\,y)$. By (18), F is quasi-concave. Moreover, F is separable since $F(x,y) = u(x)\,\rho_V(A) + u(y)\,(1-\rho_V(A))$. It follows that u is concave on X_α or X^α. Now, let

$$X^* \equiv \bigcup_{\{\alpha \in X^o:\ u \text{ is concave on } X^\alpha\}} X^\alpha \quad \text{and} \quad X_* \equiv \bigcup_{\{\alpha \in X^o:\ u \text{ is concave on } X_\alpha\}} X_\alpha.$$

Since for each $\alpha \in X^o$, u is either concave on X^α or on X_α, we have $X^o \subseteq X^* \cup X_*$. As X^o is connected and X^* and X_* are open, $X^* \cap X_*$ is non-empty and open, so there are $z, z' \in X^* \cap X_*$ with $z < z'$. It follows that u is concave on both (x_*, z') and (z, x^*). Hence, D^+u is non-increasing on both (x_*, z') and (z, x^*), so that it is non-increasing on X^o. Since u is continuous on X^o, this implies that u is concave on X^o (see, e.g., Royden (1988, p. 114)). Being u continuous on X, in turn this implies the concavity of u on X.

As for the converse, suppose that u is concave on X. For essential $A \in \Sigma$ and $x, x', y, y' \in X$ such that $x \succ y$ and $x' \succ y'$, suppose that $x\,A\,y \sim x'\,A\,y'$. Then,

$$\begin{aligned} V([x\,\alpha\,x']\,A\,[y\,\alpha\,y']) &= u(x\,\alpha\,x')\rho_V(A) + u(y\alpha\,y')(1-\rho_V(A)) \\ &\geq \alpha\,[u(x)\,\rho_V(A) + u(y)\,(1-\rho_V(A))] \\ &\quad +(1-\alpha)\,[u(x')\,\rho_V(A) + u(y')(1-\rho_V(A))] \\ &= \alpha\,V(x\,A\,y) + (1-\alpha)\,V(x'\,A\,y') = V(x'\,A\,y'), \end{aligned}$$

as wanted. ∎

Proof of Proposition 21: (*i*) ⇒ (*ii*): Suppose that u is concave on X, and set $\rho(A) = \rho(B) = p \in (0,1)$. Since $px + (1-p)y = px' + (1-p)y'$, we have

$$y' - y = \frac{p}{1-p}(x - x'). \tag{19}$$

We have $y' \leq y \leq x \leq x'$. Suppose that both $x' \neq x$ and $y \neq y'$. By concavity,

$$u(y') \leq u(y) + \frac{u(x') - u(x)}{x' - x}(y' - y). \tag{20}$$

Using (19) and (20), we can write:

$$\begin{aligned}
pu(x') + (1-p)u(y') &\leq pu(x') + (1-p)\left[u(y) + \frac{u(x') - u(x)}{x' - x}(y' - y)\right] \\
&= pu(x') + (1-p)u(y) + (1-p)\frac{u(x') - u(x)}{x' - x}(\frac{p}{1-p})(x - x') \\
&= pu(x') + (1-p)u(y) + p(u(x) - u(x')) \\
&= pu(x) + (1-p)u(y),
\end{aligned}$$

so that $x\,A\,y \succcurlyeq x'\,B\,y'$. If $x = x'$, then $x\,A\,y \succcurlyeq x'\,B\,y'$ follows by B2. The same if $y = y'$.

The $(ii) \Rightarrow (iii)$ statement is immediate. As for the $(iii) \Rightarrow (i)$ statement, we prove the contrapositive. Assume that u is not concave: There are $x, y \in X$ and $\alpha \in (0,1)$ such that

$$u(\alpha x + (1-\alpha)y) < \alpha u(x) + (1-\alpha)u(y).$$

W.l.o.g., assume that $u(x) > u(y)$. Since $\rho(\Sigma) = [0,1]$, there is $A \in \Sigma$ such that $\rho(A) = \alpha$. Consider the act $f = x\,A\,y$ and the constant act $EV(f) = \alpha\,x + (1-\alpha)\,y$. Then

$$u(EV(f)) = u(\alpha x + (1-\alpha)y) < \alpha u(x) + (1-\alpha)u(y) = V(f).$$

This proves that $\succcurlyeq$ has a preference for the subjective expected value. ∎

Proof of Lemma 22: The 'if' part is obvious. As to the 'only if' part, let $f, g \in \mathcal{F}$ be such that $u(f(s)) = u(g(s))$ for all $s \in S$. Then, $f(s) \sim g(s)$ for all $s \in S$, i.e., for all $s \in S$ we have both $f(s) \succcurlyeq g(s)$ and $f(s) \preccurlyeq g(s)$. By monotonicity, in turn this implies that $f \succcurlyeq g$ and $f \preccurlyeq g$, and so $f \sim g$. Since V represents $\succcurlyeq$, this implies $V(f) = V(g)$. Hence, there exists $I' : u(\mathcal{F}) \to \mathbb{R}$ such that $V(f) = I'(u(f))$ for all $f \in \mathcal{F}$, where $u(\mathcal{F}) = \{u(f) : f \in \mathcal{F}\}$. We now show that I' is monotone on $u(\mathcal{F})$. Suppose that $f, g \in \mathcal{F}$ are such that $u(f(s)) \geq u(g(s))$ for all $s \in S$. Then, $f(s) \succcurlyeq g(s)$ for all $s \in S$, and so, by monotonicity, $V(f) \geq V(g)$, which proves that I' is monotone on $u(\mathcal{F})$. We now want to extend I' from $u(\mathcal{F})$ to $B_0(\Sigma)$.

Suppose that $u(X)$ is bounded below: there exists a positive integer M such that $-M < u(x)$ for all $x \in X$. Set $m = \inf\{I'(\psi) : \psi \in u(\mathcal{F})\}$. Since all $\psi \in u(\mathcal{F})$ are finite-valued, we have $m \geq -M$. Given $\phi \in B_0(\Sigma)$, let $L_\phi = \{\psi \in u(\mathcal{F}) : \psi \leq \phi\}$. Define $I : B_0(\Sigma) \to \mathbb{R}$ as follows:

$$I(\phi) = \begin{cases} \sup\{I'(\psi) : \psi \in L_\phi\} & \text{if } L_\phi \neq \emptyset \\ m & \text{if } L_\phi = \emptyset. \end{cases}$$

We show that $I(\phi) \in \mathbb{R}$ for all $\phi \in B_0(\Sigma)$. Suppose that $u(X)$ is bounded above, so that there exists a positive constant M' such that $M' > u(x)$ for all $x \in X$. Since $u(\mathcal{F})$ consists of simple functions, for each $\psi \in u(\mathcal{F})$ there exists $x \in X$ such that $u(x) \geq \psi$. Hence, $M' > \psi$ for all $\psi \in u(\mathcal{F})$, and so $I(\phi) \leq M'$. Suppose that $u(X)$ is not bounded above. Let $\phi \in B_0(\Sigma)$. Since ϕ is finite-valued and $u(X)$ is unbounded, there exists $x \in X$ such that $u(x) > \phi$. Hence, $u(\mathcal{F}) \ni \psi \leq \phi$ implies $\psi < u(x)$, and so $I(\phi) \leq u(x)$. In both cases, $I(\phi) \in \mathbb{R}$. Let $\phi, \phi' \in B_0(\Sigma)$ be such that $\phi \geq \phi'$. Clearly, $L_{\phi'} \subseteq L_\phi$. Suppose that $L_{\phi'} \neq \emptyset$. Then $L_\phi \neq \emptyset$, and it is easy to check that $I(\phi) \geq I(\phi')$. Next suppose that $L_{\phi'} = \emptyset$. By definition, $I(\phi') = m$. If $L_\phi = \emptyset$, $I(\phi) = m$ as well. If $L_\phi \neq \emptyset$, then there is some $\psi \in u(\mathcal{F})$ such that $I(\phi) \geq I'(\psi) \geq m$. In both cases, $I(\phi) \geq I(\phi')$, thus showing that I is monotone.

Suppose now that $u(X)$ is not bounded below. Then $L_\phi \neq \emptyset$ for all $\phi \in B_0(\Sigma)$. In fact, since ϕ is finite-valued, there exists m'' such that $m'' \leq \phi$. But, for each such m'' there exists $x \in X$ with $u(x) < m''$, so that $u(x) \in L_\phi$. We can define $I : B_0(\Sigma) \to \mathbb{R}$ by $\sup\{I'(\psi) : \psi \in L_\phi\}$. Proceeding as above, such I is shown to be a monotone functional extending I'. This completes the proof of the 'only if' part. ∎

Proof of Proposition 26: The 'if' is immediate. As to the 'only if', by Theorem 14 it suffices to show that if $u_2 = \phi(u_1)$ for some increasing concave $\phi : u_1(X) \to \mathbb{R}$, $\succcurlyeq_2$ is more uncertainty averse than $\succcurlyeq_1$. For $x \in X$ and $f \in \mathcal{F}$, suppose that $x \succcurlyeq_1 f$. This implies $u_1(x) \geq I^*(u_1(f))$, and so, being ϕ increasing, $\phi(u_1(x)) \geq \phi(I^*(u_1(f)))$. But, by the Jensen property, we have

$$u_2(x) = \phi(u_1(x)) \geq \phi(I^*(u_1(f))) \geq I^*(\phi(u_1(f))) = I^*(u_2(f)),$$

and so $x \succcurlyeq_2 f$. A similar argument shows that $x \succ_1 f \Longrightarrow x \succ_2 f$. We can thus conclude that $\succcurlyeq_2$ is more uncertainty averse than $\succcurlyeq_1$. ∎

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