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Certainty Independence and the Separation of Utility and Beliefs*

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Abstract

Economists often operate under an implicit assumption that the tastes of a decision maker are constant, while his beliefs change with the availability of new information. It is therefore customary to seek representations of preferences which cleanly separate the taste component, called 'utility,' from the beliefs component.

We show that a complete separation of utility from the other components of the representation is possible only if the decision maker's preferences satisfy a mild but not completely innocuous condition, called 'certainty independence.' We prove that the preferences that obtain such separation are a subset of the biseparable preferences.

Introduction

Economists often operate under an implicit assumption that the tastes of a decision maker are constant, while his beliefs change with the availability of new information. It is therefore customary to seek representations of preferences which cleanly separate the taste component, called 'utility,' from the beliefs component. In this note, we argue that a complete separation of utility from the other components of the representation is possible only if the decision

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maker's preferences satisfy a mild but not completely innocuous condition, called 'certainty independence.'

This result completes the work of Ghirardato and Marinacci [4, henceforth GM], who as we shall argue achieve the desired separation only in a limited sense. Clearly, a full understanding of the conditions under which beliefs can be modelled independently of utility is material to any analysis of 'belief' formation and revision.

To illustrate the issues and difficulties connected with a complete separation of utility from the other components of the preference representation, we begin with a simple result, stated below as Lemma 1: Let $\succcurlyeq$ be a monotonic preference over a set of acts (maps from a state space S to a consequence set X) which has certainty equivalents, and let u be a utility function on X.[1] Then, there exists a monotonic and normalized functional I_u such that $f \succcurlyeq g$ if and only if

$$I_u(u \circ f) \geq I_u(u \circ g).$$

That is, it is *as if* the decision maker used the following procedure to construct his preferences over $\mathcal{F}$: Using a specific utility function u, he first transforms every act f into a *utility profile* $u \circ f$, and then uses the functional I_u to find f's place in the ranking. This procedure is implicitly followed by many well-known decision rules. For instance, suppose that $\succcurlyeq$ is the subjective expected utility (SEU) preference described by $\int u \circ f \, dP$, for given u and P. Then, it is the case that $I_u(\cdot) = \int (\cdot) \, dP$. That is, it is as if an act's evaluation was found by calculating its utility profile and plugging it into the expectation with respect to P.

Clearly, u is a representation of the decision maker's tastes. It is tempting to intepret I_u as representing his 'beliefs.' However, as it is well-known, the function u is not unique. Consider for instance a positive affine transformation $v = au + b$. By the result above, there is a corresponding functional I_v. The question is whether $I_v = I_u$. Interpreting I_u as 'beliefs' intuitively requires that such equality holds. 'Beliefs' should not depend on mathematical aspects of the representation of tastes devoid of behavioral content. The following example shows that, unfortunately, such dependence may well occur.

Example 1 *Let $S = \{s_1, s_2, s_3\}$, $X = [0,1]$, $u : [0,1] \to [0,1]$ be the identity function, and P the uniform probability $P(s_i) = 1/3$ for all $i = 1, 2, 3$. Denote by $\mathbb{B}^3$ the subset of $\mathbb{R}^3$ consisting of vectors taking only two values, and define $J, I_u : \mathbb{R}^3 \to \mathbb{R}$ as follows*

$$J(\varphi) \equiv \max\left\{ \int_S \psi \, dP : \psi \in \mathbb{B}^3, \psi \leq \varphi \right\},$$

[1] See Section 1 for a definition of these standard properties, as well as those of monotonic and normalized functionals. Notice that monotonicity implies state independence of tastes. State dependence is not considered in this note.

$$I_u(\varphi) \quad \equiv \quad \frac{e^{\varphi(s_1)}}{1+e^{\varphi(s_1)}} \int \varphi \, dP + \frac{1}{1+e^{\varphi(s_1)}} J(\varphi).$$

It can be seen that both J and I_u are monotonic and normalized functionals. Therefore, the preference $\succcurlyeq$ induced on the set $\mathcal{F}$ of all the acts by

$$f \succcurlyeq g \Leftrightarrow I_u(u \circ f) \geq I_u(u \circ g)$$

is monotonic and has certainty equivalents. Let $f = [21/100, 21/100, 21/100]$ and $g = [0, 1/4, 1/2]$. It can be calculated that $I_u(u \circ f) > I_u(u \circ g)$, implying $f \succ g$. It can also be calculated that $I_u(u \circ f + 1/2) < I_u(u \circ g + 1/2)$.

On the other hand, consider the alternative utility function $v = u + 1/2$. Since v represents $\succcurlyeq$ on X, there exists a monotonic and normalized I_v such that $I_v(v \circ \cdot)$ also represents $\succcurlyeq$. It follows that $I_v(v \circ f) > I_v(v \circ g)$, or $I_v(u \circ f + 1/2) > I_v(u \circ g + 1/2)$. This proves that it cannot be that case that $I_v = I_u$.

It should be noticed that the example shows a preference with a representation for which $I_u(u \circ f) > I_u(u \circ g)$, but

$$I_u(a\, u \circ f + b) < I_u(a\, u \circ g + b)$$

for some $a > 0$ and b. For future use, we also remark that the preference described in the example is a 'biseparable preference' in the sense of GM (see Definition 1 below). In fact, it is easy to see that on binary acts $\succcurlyeq$ is represented by $\int u \circ f \, dP$.

The obvious solution of the problem illustrated by the example is to impose constraints on the preference relation beyond monotonicity and the existence of certainty equivalents, which deliver the independence of I_u from u — to the extent that the latter is possible. For instance, it is clear that such independence obtains when $\succcurlyeq$ is a SEU preference (e.g., Anscombe and Aumann [1]), or more generally when it satisfies the Choquet expected utility (CEU) and maxmin expected utility (MEU) models of Schmeidler [6] and Gilboa and Schmeidler [5] respectively.

This note shows that independence of u and I_u, and with it a complete separation of 'utility' and 'beliefs,' obtains under more general circumstances than those described by these preference models. In particular, we show that the property that needs to be added to obtain separation is the 'certainty independence' axiom (Gilboa and Schmeidler [5]), which requires that independence holds when mixing with constant acts.

1 Preliminaries

Consider a set S of **states of the world**, an algebra Σ of subsets of S called **events**, and a set X of **consequences**. We denote by $\mathcal{F}$ the set of all the **simple acts**: finite-valued functions $f : S \to X$ which are measurable with respect to Σ. For $x \in X$ we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. So, with the usual slight abuse of notation, we identify X with the subset of the constant acts in $\mathcal{F}$. Whenever X has a mixture set structure (e.g., in the 'Anscombe-Aumann' setting [1] where consequences are lotteries on a randomizing device), for every $f, g \in \mathcal{F}$ and $\lambda \in [0, 1]$ we denote by $\lambda f + (1 - \lambda)g$ the act in $\mathcal{F}$ which yields $\lambda f(s) + (1 - \lambda)g(s) \in X$ for every $s \in S$.

We model the DM's preferences on $\mathcal{F}$ by a binary relation $\succcurlyeq$, and as customary we denote by $\sim$ and $\succ$ its symmetric and asymmetric components, respectively. Given $f \in \mathcal{F}$, a **certainty equivalent** of f is an element $c_f \in X$ such that $f \sim c_f$. A functional $U : \mathcal{F} \to \mathbb{R}$ **represents** $\succcurlyeq$ if $U(f) \geq U(g)$ if and only if $f \succcurlyeq g$. Clearly, a necessary condition for $\succcurlyeq$ to have a representation is that it be a **weak order**; i.e., a complete and transitive relation. A binary relation $\succcurlyeq$ is **nontrivial** if $f \not\succcurlyeq g$ for some $f, g \in \mathcal{F}$; it is **monotonic** if $f(s) \succcurlyeq g(s)$ for every $s \in S$ implies $f \succcurlyeq g$. Whenever U represents $\succcurlyeq$, both these properties translate into obvious properties of U.

Given a set K of real numbers, we denote by $B_0(\Sigma, K)$ the set of all Σ-measurable simple functions taking values in K. When $K = \mathbb{R}$, we just write $B_0(\Sigma)$. Given a functional $I : B_0(\Sigma, K) \to \mathbb{R}$, we say that I is **monotonic** if $\varphi \geq \psi$ implies $I(\varphi) \geq I(\psi)$ for all $\varphi, \psi \in B_0(\Sigma, K)$, while it is **normalized** if $I(k) = k$ for all $k \in K$. When K is a nontrivial interval, I is **constant affine** if $I(\alpha\varphi + (1 - \alpha)k) = \alpha I(\varphi) + (1 - \alpha)k$ for all $\varphi \in B_0(\Sigma, K)$, all $\alpha \in [0, 1]$ and all $k \in K$. It is easy to see that a constant affine functional $I : B_0(\Sigma) \to \mathbb{R}$ is **constant linear**; that is, $I(a\varphi + b) = aI(\varphi) + b$ for all $a, b \in \mathbb{R}$, $a \geq 0$. Moreover, any constant affine functional $I : B_0(\Sigma, K) \to \mathbb{R}$ admits a unique constant linear extension to $B_0(\Sigma)$ (see Lemma 4 in the Appendix).

2 Separating Beliefs and Utility

The following is the formal statement of the result mentioned in the Introduction.

Lemma 1 *Let $\succcurlyeq$ be a monotonic binary relation on $\mathcal{F}$ with certainty equivalents, and suppose that $u : X \to \mathbb{R}$ is a representation of $\succcurlyeq$ on X. Then there exists a unique monotonic normalized functional $I_u : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that*

$$f \succcurlyeq g \Longleftrightarrow I_u(u \circ f) \geq I_u(u \circ g).$$

Moreover, if a functional $U : \mathcal{F} \to \mathbb{R}$ represents $\succcurlyeq$ and $U_{|X} = u$, then $U(f) = I_u(u \circ f)$ for all $f \in \mathcal{F}$.

We remark that, given a preference $\succcurlyeq$ satisfying the assumptions of this lemma, any representation U of $\succcurlyeq$ is uniquely determined by its restriction u to X; that is, there is a one-to-one mapping between U and u.

As we argued in the Introduction, we are interested in a factorization of 'utility' from the preference representation such that everything that is tied to the specific identity of the payoffs is described by the function u, and the residual aspects of the representation, that we earlier called 'beliefs,' are independent of u, in particular of its possible transformations. That is, we look for a functional $I : B_0(\Sigma) \to \mathbb{R}$ such that $I_u = I_{|B_0(\Sigma,u(X))}$ for all the 'admissible' transformations of u. The following remark illustrates that including all *increasing* transformations makes for too strong a requisite.

Remark 1 Suppose that $\succcurlyeq$ is a monotonic preference with certainty equivalents, and assume that $\succcurlyeq$ has at least one essential event $A \in \Sigma$. We recall from GM that an event $A \in \Sigma$ is **essential for** $\succcurlyeq$ if $x \succ x\,A\,y \succ y$ for some $x \succ y$ (this clearly implies that $\succcurlyeq$ is nontrivial).[2] We show that there is no functional I such that $I_u = I_{|B_0(\Sigma,u(X))}$ for all the utilities u that represent $\succcurlyeq$ on X.

Per absurdum, suppose that such I existed, and consider a utility u such that (with the x and y just mentioned) $u(x) = 1$ and $u(y) = 0$. The utility profile that corresponds to the act $x\,A\,y$ is then 1_A, the characteristic function of A. As A is essential,

$$1 = I(u(x)) > I(1_A) > I(u(y)) = 0. \tag{1}$$

Next, consider the alternative utility $v = u^3$. We have that $v(x) = 1$ and $v(y) = 0$, so that $v \circ (x\,A\,y) = 1_A$. Since $\succcurlyeq$ has certainty equivalents, for every act $f \in \mathcal{F}$ we must have that

$$I[(u \circ f)^3] = I(v \circ f) = v(c_f) = [u(c_f)]^3 = [I(u \circ f)]^3.$$

In particular, $I(1_A) = I[(1_A)^3] = [I(1_A)]^3$. Yet, the only numbers that satisfy this equality are 0 and ± 1, and Eq. (1) rules that out. This provides the sought contradiction. $\square$

The remark shows that a preference can have 'beliefs' which are independent of any monotonic transformation of the utility function only if it has no essential events. Such preferences are very special and of limited interest in a context of uncertainty, as they basically treat every act as a constant (see Remark 8 in GM).

[2] $x\,A\,y$ denotes the act which yields x for every $s \in A$ and y otherwise.

In addition, it should be observed that in many models X has a mixture set structure with respect to which u is affine. Clearly, arbitrary monotonic transformations of u are less interesting, as they do not satisfy this property.

In view of these considerations, we say that there is 'complete' separation of utility from 'beliefs' when there exists a utility function $u : X \to \mathbb{R}$ that represents $\succcurlyeq$ on X and a functional $I : B_0(\Sigma) \to \mathbb{R}$ such that $I_v = I_{|B_0(\Sigma, v(X))}$ for every positive affine transformation v of u. The next result provides necessary and sufficient conditions for such complete separation for a large subclass of the preferences used in Lemma 1.

Theorem 2 *Let $\succcurlyeq$ be a nontrivial monotonic binary relation on $\mathcal{F}$ with certainty equivalents, and let $u : X \to \mathbb{R}$ be a function representing $\succcurlyeq$ on X such that $u(X)$ is an interval. The following statements are equivalent:*

(*i*) *There exists $I : B_0(\Sigma) \to \mathbb{R}$ such that $I_v = I_{|B_0(\Sigma, v(X))}$ for all positive affine transformations v of u.*

(*ii*) *There exists a positive affine transformation v of u such that I_v is constant affine.*

Whenever it exists, I is unique, monotonic and constant linear. If in addition there is a mixture set structure X with respect to which u is affine, then (*i*) *and* (*ii*) *are equivalent to*

(*iii*) *(Certainty Independence) If $f, g \in \mathcal{F}$, $x \in X$, and $\lambda \in (0, 1)$, then*

$$f \succcurlyeq g \iff \lambda f + (1 - \lambda)x \succcurlyeq \lambda g + (1 - \lambda)x.$$

It should be remarked that, under the conditions of the theorem, the requirement that there is a mixture set structure on X (with respect to which the utility is affine) is less demanding than it appears to be. For, given a utility u whose range $u(X)$ is an interval, it is always possible to derive such mixture set structure. Given consequences x and y and weight λ, it suffices to pick a consequence z satisfying

$$u(z) = \lambda\, u(x) + (1 - \lambda)\, u(y),$$

using that as $\lambda\, x + (1 - \lambda)\, y$. The problem is that such construction in general requires knowledge of the function u. (For a fully behavioral approach that does not, see Ghirardato, Maccheroni, Marinacci and Siniscalchi [3].)

Theorem 2 can thus be restated as follows: complete separation of utility and 'beliefs' is possible if and only if the preference satisfies certainty independence. The theorem does *not* provide a full axiomatic characterization of such preferences, as it assumes the existence of a convex-ranged utility function representing $\succcurlyeq$ on X. In the next section, we briefly describe two possible axiomatizations, after observing that the preferences of interest are a subset of GM's 'biseparable' preferences.

3 Invariant Biseparable Preferences

As we mentioned in the Introduction, the separation of utility and beliefs is also an objective of the work in GM. They study a class of preferences which also achieve a form of separation, in the following sense:

Definition 1 *A binary relation $\succcurlyeq$ on $\mathcal{F}$ is a* ***biseparable preference*** *if there is a representation $U : \mathcal{F} \to \mathbb{R}$ of $\succcurlyeq$ which is nontrivial, monotonic and convex-ranged on X, and for which there exists $\rho : \Sigma \to [0,1]$ such that for all consequences $x \succcurlyeq y$ and all $A \in \Sigma$, if we let $u \equiv U_{|X}$,*

$$U(x\,A\,y) = u(x)\,\rho(A) + u(y)\,(1 - \rho(A)). \tag{2}$$

Eq. (2), which motivates the term 'biseparable' (short for *binary separable*), describes a separation between the utility function u and 'beliefs' as represented by the set-function ρ. However, such separation only holds in the evaluation of binary acts. And indeed, Example 1 illustrates that it is possible for a preference to satisfy Eq. (2) and yet not to achieve complete separation in the sense of Theorem 2.

While biseparability is not sufficient for complete separation, the next result shows that it is necessary:

Proposition 3 *Let $\succcurlyeq$ be a binary relation on $\mathcal{F}$ satisfying the assumptions of Theorem 2, as well as its condition (i). Then $\succcurlyeq$ is a biseparable preference, with*

$$\rho(A) = I(1_A) \quad \textit{for all } A \in \Sigma.$$

This result shows that the preferences which satisfy the assumptions of Theorem 2 and achieve complete separation of utility and 'beliefs' are a subset of the biseparable preferences. For this reason, we call them **invariant biseparable preferences** (invariant refers to the fact that the functional I is invariant under transformations of u).

Since a biseparable preference $\succcurlyeq$ satisfies by definition all the assumptions of Theorem 2, it follows from that result and Proposition 3 that a biseparable preference is invariant biseparable if and only if it satisfies certainty independence (with mixtures defined as explained after Theorem 2). This brings us back to the question of the axiomatization, which can now be simply addressed.

If, for instance, X is a connected separable topological space, it is straightforward to axiomatize invariant biseparable preferences by building on GM's characterization of biseparable preferences, using the 'subjective' mixtures of Ghirardato, Maccheroni, Marinacci and Siniscalchi [3] in the certainty independence axiom. If, on the other hand, X has an 'objective' mixture set structure (as in the Anscombe-Aumann setting), then the well-known axioms of Gilboa and Schmeidler [5] minus their 'uncertainty aversion' axiom yield invariant biseparable preferences (cf. Ghirardato, Maccheroni and Marinacci [2], which heavily builds on the separation of utility and beliefs in this model to analyse the presence of ambiguity in the decision problem).

Appendix

A Proof of Lemma 1

We report it just for the sake of completeness. Notice that $f \in \mathcal{F}$ iff there exist $x_1, \ldots, x_N \in X$ and a partition $\{A_1, A_2, \ldots, A_N\}$ of S in Σ such that $f(s) = x_i$ if $s \in A_i$, $i = 1, \ldots, N$. Analogously, $\varphi \in B_0(\Sigma, u(X))$ iff there exist $u(y_1), \ldots, u(y_M) \in u(X)$ and a partition $\{B_1, B_2, \ldots, B_M\}$ of S in Σ such that $\varphi(s) = u(y_i)$ for all $s \in B_i$, $i = 1, \ldots, M$. We conclude that $B(\Sigma, u(X)) = \{u \circ f : f \in \mathcal{F}\}$.

Since $\succcurlyeq$ is monotonic, if $\varphi = u \circ f = u \circ g$ for $f, g \in \mathcal{F}$, then $u(f(s)) = u(g(s))$ for all $s \in S$ and $f(s) \sim g(s)$ for all $s \in S$, whence $f \sim g$ and $c_f \sim c_g$. For all $\varphi \in B(\Sigma, u(X))$, set $I_u(\varphi) = u(c_f)$ if $\varphi = u \circ f$ for some $f \in \mathcal{F}$. I_u is well defined, normalized (for all $k = u(x) \in u(X)$, $I_u(k) = I_u(u \circ x) = u(x) = k$), and $f \succcurlyeq g$ iff $u(c_f) \geq u(c_g)$ iff $I_u(u \circ f) \geq I_u(u \circ g)$. Therefore, $I_u(u \circ \cdot)$ represents $\succcurlyeq$, and I_u is thus monotonic. Suppose that J_u is a normalized monotonic functional on $B_0(\Sigma, u(X))$ such that $J_u(u \circ \cdot)$ represents $\succcurlyeq$. Given $\varphi \in B_0(\Sigma, u(X))$, let $f \in \mathcal{F}$ be such that $\varphi = u \circ f$, and let c_f be a certainty equivalent of f. It follows that

$$J_u(\varphi) = J_u(u \circ f) = J_u(u \circ c_f) = u(c_f) = I_u(\varphi).$$

Finally, if U represents $\succcurlyeq$ and $U_{|X} = u$, then

$$U(f) = U(c_f) = u(c_f) = I_u(u \circ f).$$

B Proof of Theorem 2

Lemma 4 *Let K be a nontrivial interval. Any constant affine functional I on $B_0(\Sigma, K)$ admits a unique constant linear extension to $B_0(\Sigma)$. Moreover, I is monotonic iff its extension is monotonic.*

Proof: Let $J : B_0(\Sigma) \to \mathbb{R}$ be a constant linear extension of I and $k \in \operatorname{int} K$. For all $\varphi \in B_0(\Sigma)$, there exists $\alpha \in (0,1]$ such that $\alpha\varphi + (1-\alpha)k \in B_0(\Sigma, K)$. It follows that

$$\alpha\, J(\varphi) + (1-\alpha)\, k = J(\alpha\, \varphi + (1-\alpha)\, k) = I(\alpha\, \varphi + (1-\alpha)\, k).$$

That is,

$$J(\varphi) = \frac{I(\alpha\, \varphi + (1-\alpha)\, k) - (1-\alpha)\, k}{\alpha}. \tag{3}$$

Therefore if a constant linear extension exists, it is unique and it is given by Eq. (3).

We next show that there is a constant linear extension of I. Assume first that $0 \in \operatorname{int} K$. Taking $k = 0$ in the definition of constant affinity we obtain $I(\alpha\varphi) = \alpha I(\varphi)$ for all $\varphi \in B_0(\Sigma, K)$ and all $\alpha \in [0,1]$. That is, I is positively homogeneous. As suggested by Eq. (3), for all $\varphi \in B_0(\Sigma)$ we take

$$J(\varphi) = \frac{I(\alpha\varphi)}{\alpha}$$

if $\alpha \in (0,1]$ and $\alpha\varphi \in B_0(\Sigma, K)$. If $\alpha, \beta \in (0,1]$ and $\alpha\varphi, \beta\varphi \in B_0(\Sigma, K)$, w.l.o.g. we can assume $\beta \geq \alpha > 0$, so that

$$I(\alpha\varphi) = I\left(\frac{\alpha}{\beta}\beta\varphi\right) = \frac{\alpha}{\beta} I(\beta\varphi).$$

Therefore, $I(\alpha\varphi)/\alpha = I(\beta\varphi)/\beta$, whence J is well defined and it obviously extends I. In particular, $J(0\varphi) = 0$ for all $\varphi \in B_0(\Sigma)$. Given $\varphi \in B_0(\Sigma)$, let $\beta \in (0,1]$ be such that $\beta\varphi \in B_0(\Sigma, K)$. Then, for any $\alpha \in (0,1)$, $\beta\alpha\varphi \in B_0(\Sigma, K)$ and

$$J(\alpha\varphi) = \frac{I(\beta\alpha\varphi)}{\beta} = \alpha\frac{I(\beta\varphi)}{\beta} = \alpha J(\varphi).$$

That is, J is positively homogeneous.[3]

[3] For $\alpha > 1$ and $\varphi \in B_0(\Sigma)$,

$$J(\varphi) = J\left(\frac{1}{\alpha}\alpha\varphi\right) = \frac{1}{\alpha}J(\alpha\varphi).$$

Let $\varphi \in B_0(\Sigma)$, $a > 0$, and $b \in \mathbb{R}$. Choose $\alpha \in (0,1]$ such that $\alpha\varphi, \alpha(b/a) \in B_0(\Sigma, K)$. Then $(1/2)\alpha\varphi, (1/2)\alpha(b/a) \in B_0(\Sigma, K)$ and

$$\begin{aligned} J(a\varphi + b) &= aJ\left(\varphi + \frac{b}{a}\right) = \frac{2a}{\alpha} I\left(\frac{1}{2}\alpha\varphi + \frac{1}{2}\alpha\frac{b}{a}\right) \\ &= \frac{2a}{\alpha}\left(\frac{1}{2} I(\alpha\varphi) + \frac{1}{2}\alpha\frac{b}{a}\right) \\ &= aJ(\varphi) + b. \end{aligned}$$

Finally, given $b \in \mathbb{R}$ choose $\alpha \in (0,1]$ such that $\alpha b \in K$. Then,

$$J(b) = \frac{1}{\alpha} I(\alpha b) = \frac{1}{\alpha} I\left((1-\alpha)\, 0 + \alpha b\right) = b.$$

We conclude that J is constant linear.

Next, suppose that $0 \notin \operatorname{int} K$. Take $k_0 \in \operatorname{int} K$ and notice that $0 \in \operatorname{int}(K - k_0)$. For all $\psi \in B_0(\Sigma, K - k_0)$, set

$$G(\psi) = I(\psi + k_0) - k_0.$$

Given any $\psi \in B_0(\Sigma, K - k_0)$, $c \in K - k_0$ and $\alpha \in [0,1]$,

$$\begin{aligned} G(\alpha\psi + (1-\alpha)c) &= I(\alpha\psi + (1-\alpha)c + k_0) - k_0 \\ &= I(\alpha(\psi + k_0) + (1-\alpha)(c + k_0)) - k_0 \\ &= \alpha I(\psi + k_0) + (1-\alpha)(c + k_0) - k_0 \\ &= \alpha(I(\psi + k_0) - k_0) + (1-\alpha)c \\ &= \alpha G(\psi) + (1-\alpha)c. \end{aligned}$$

That is, G is constant affine on $B_0(\Sigma, K - k_0)$. Let J be its constant linear extension to $B_0(\Sigma)$, whose existence we proved above. If $\varphi \in B_0(\Sigma, K)$, $\varphi - k_0 \in B_0(\Sigma, K - k_0)$ and

$$J(\varphi) = J(\varphi - k_0) + k_0 = G(\varphi - k_0) + k_0 = I(\varphi - k_0 + k_0) - k_0 + k_0 = I(\varphi).$$

The monotonicity statement is trivial. *Q.E.D.*

Assume that there exists a positive affine transformation v of u such that I_v is constant affine. Let I be the unique constant linear extension of I_v to $B_0(\Sigma)$ given by Lemma 4. Since I_v is monotonic, I is monotonic as well. If w is another positive affine transformation of u, then there exist $a > 0$ and $b \in \mathbb{R}$ such that $w = av + b$ and

$$\begin{aligned} I_w(w \circ f) &= w(c_f) = av(c_f) + b = aI_v(v \circ f) + b \\ &= aI(v \circ f) + b = I(av \circ f + b) = I(w \circ f). \end{aligned}$$

Conversely, assume there exists $I : B_0(\Sigma) \to \mathbb{R}$ such that $I_v = I_{|B_0(\Sigma, v(X))}$ for all positive affine transformations v of u. For all $\varphi \in B_0(\Sigma)$, there exists a positive affine trasformation v of u such that $\varphi = v \circ f \in B_0(\Sigma, v(X))$, and consider any $a > 0$ and $b \in \mathbb{R}$

$$\begin{aligned} I(a\varphi + b) &= I(av \circ f + b)) = I_{av+b}(av \circ f + b) = av(c_f) + b \\ &= aI_v(v \circ f) + b = aI(\varphi) + b; \end{aligned}$$

If $a = 0$ and $b \in \mathbb{R}$ there exists a positive affine trasformation v such that $b = v(x)$, then

$$I(b) = I(v \circ x) = I_v(v \circ x) = v(x) = b.$$

We conclude that I is constant linear. *A fortiori*, for all positive affine transformations v of u, $I_v = I_{|B_0(\Sigma, v(X))}$ is constant affine. This proves the equivalence of (i) and (ii) (notice that the last equality also guarantees uniqueness of I).

When X has a mixture set structure and u is affine w.r.t. such structure, the equivalence of (ii) and (iii) is proved mimicking the arguments of Gilboa and Schmeidler [5].

C Proof of Proposition 3

Let I denote the functional described in statement (i) of Theorem 2 and u the convex-ranged function representing $\succcurlyeq$. Let $U(f) \equiv I(u \circ f)$. We now show that U satisfies the conditions in the definition of biseparable preference.

Clearly U is a representation of $\succcurlyeq$ and it is nontrivial, monotonic and convex-ranged on X. Set $\rho(A) = I(1_A)$ for all $A \in \Sigma$. Consider $x \succcurlyeq y$, $A \in \Sigma$. Recalling that I is constant linear, we have

$$\begin{aligned} U(x\,A\,y) &= I[u \circ (x\,A\,y)] \\ &= I[(u(x) - u(y))1_A + u(y)] \\ &= (u(x) - u(y))\,I(1_A) + u(y) \\ &= u(x)\rho(A) + u(y)(1 - \rho(A)). \end{aligned}$$

This concludes the proof.

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