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DRIVEN BY MULTIFACTOR MARKET POINT PROCESSES

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Change of Numéraire for Affine Arbitrage Pricing Models Driven by Multifactor Marked Point Processes

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Abstract

We derive a general formula for the change of numéraire in multifactor affine arbitrage free models driven by marked point processes. As a complement, we present both affine structures and change of measures in the general setting of jump diffusions. This provides for a comprehensive view on the subject.

Contents

Interest-rate derivatives can be expressed as contingent claims on zero-coupon bonds. Thus, one may wish to build an arbitrage-free bond-market model where discount bonds have a predefined convenient form that facilitates computations for pricing. Several particular short rate models generate bonds of exponential affine form. On the one hand, they are solution of the fundamental pricing equation with final value identically equal to 1: that is, for such models, the backward equation can be explicitly solved and has an exponential affine solution. On the other hand, such a solution, the bond price, enters the pay-off function of interest rate European contingent claims; if these function is linear in the bond price, one intuitively looks for a solution of the backward equation for the bond-option of the exponential affine form. In any case, one may look for the characteristic function of the underlying interest rate process. This is itself a solution of the backward equation with boundary condition an exponential complex function. Still based on a purely intuitive ground, one may still have a strong confidence to find a solution of this problem, that is the characteristic function, of exponential affine form.

Several example of particular short rate models give rise to exponential affine bond prices and let one successfully accomplish the above stated program delivering explicit expressions for bond-options and for the characteristic function of the underlying short rate.

This arises the issue of finding sufficient and necessary conditions for short rate process to give rise to an exponential affine bond market.

In this section the problem of determining such conditions is solved in a pretty general case. Specifically we work within arbitrage pricing theory, in a short rate framework, where a model is defined by a short rate process of Markovian type, that is the discount bond price is generated by formula:

$$P_T(t,\omega) = \mathbb{E}_t^{\mathbb{P}^*}\left(e^{-\int_t^T r(s)ds}\right)(\omega) = F(t, r(t,\omega); T),$$

where the last equality hold true for some function F since Markovianity of r implies that conditional expectations with respect to $\mathcal{F}_t$ (i.e.e the cumulated information up to t) coincide with the σ-algebra $\mathcal{G}_{r(t)}$ generated by the random variable $r(t,\omega)$ (i.e.e the information generated by the only observation of the short rate at time t) and thus are functions of the short rate at time t. We work in a multifactor setting, in that $r(t)$ is generated by a multidimensional marked point Markovian

Brownian diffusion:

$$dX(t) = \boldsymbol{\mu}(t, X(t))\, dt + \boldsymbol{\Sigma}(t, X(t)) \cdot d\mathbf{W}(t) + \int_E \mathbf{m}(t, X(t), y)\, \mu(dt, dy; \omega) \tag{0.1}$$

according to:

$$r(t, \omega) \overset{def}{=} R(t, X(t, \omega)). \tag{0.2}$$

Here X is in $\mathbb{R}^n$and $\boldsymbol{\mu}, \boldsymbol{\Sigma}, m$ are deterministic functions with values respectively in $\mathbb{R}^n$, $\mathbb{M}(n \times d)$, $\mathbb{R}^n$; $\mathbf{W}$ is a d-dimensional Brownian motion; μ is a Random measure on a Lusin space E, with compensator given by $\nu(t, X(t), dy)\, dt$ for a suitable deterministic function ν. Notice that the compensator is absolutely continuous with respect to the Lebesgue measure on the time axis and the randomness is exclusively through the current state of the factor process.

In synthesis, a model is specified by $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{m}, \nu; R)$. Indeed one might redefine μ by setting its compensator as $\tilde{\nu} \overset{def}{=} \mathbf{m}\nu$ and specify a model by $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \widetilde{\nu}; R)$, that is a tern defining the semi-martingale factor and a short rate process generated from it. Since in concrete models one usually takes for a Poisson measure and uses $\mathbf{m}$ to model the impact factor on X stemming from μ, it is more convenient to work with a model specification of form $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{m}, \nu; R)$.

A bond market is exponential affine, or simpler exp-affine, if F has form:

$$F(t, r; T) = \exp(\alpha(t, T) - \boldsymbol{\beta}(t, T) \cdot X(t)). \tag{0.3}$$

The minus sign in front of $\boldsymbol{\beta}$ is conventional: if this coefficient is strictly positive, then one realize the appealing property that bond prices are decreasing in underlying short rates. Our aim is to find necessary and sufficient conditions on the model $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{m}, \nu; R)$ such that the resulting bond market is exp-affine.

To warm up, we begin with the easiest case of a single factor short rate process of continuous diffusion type. Then we proceed with identification of sufficient conditions in the general case by studying a more general problem that will let us to apply Fourier transform analysis in order to come up to valuation formulae for European derivative assets with exponential pay-off function. Finally we attack the much harder problem of identifying necessary conditions for exp-affine bond markets. The basic tool we use will be Ito calculus, a part from the point concerning necessary conditions in the general case of multidimensional marked point diffusions.

1. Affine term-structures driven by one-factor continuous diffusions

Let r be given by:

$$dr(t) = \mu(t, r(t))\, dt + \sigma(t, r(t))\, dW(t). \tag{1.1}$$

This corresponds to setting $(0.1), (0.2)$ with $R(t,r) = X \in \mathbb{R}$ and $m = 0$.

Fix a maturity $T > t_0$ defining a discount bond P_T. Suppose this bond has an exp-affine form $P_T(t) = \exp(\alpha(t,T) - \beta(t,T) r(t))$. In other words, this is a solution of the backward equation $\left[\partial_t + \mu\partial_r + \frac{1}{2}\sigma^2\partial^2_{rr} - r\cdot\right] F(t,r) = 0$ on $[t_0, T] \times \mathbb{R}$, with terminal condition $F(T,r) \equiv 1$, computed at $(t, r(t))$. Plugging partial derivatives, this is equivalent to state that functions α and β satisfy:

$$\partial_t\alpha(t,T) - r\partial_t\beta(t,T) - \mu(t,r)\beta(t,T) + \frac{1}{2}\sigma^2(t,T)\beta^2(t,T) - r = 0 \tag{1.2}$$

for all $t \in [t_0, T]$. We want to find sufficient condition for this to hold, that is functions μ and σ such that (1.1) admits a unique solution and there exist functions α and β satisfying the equation above with terminal conditions:

$$\alpha(T,T) = 0 \tag{1.3}$$
$$\beta(T,T) = 0, \tag{1.4}$$

which derive from $\exp(\alpha(T,T) - \beta(T,T)r) = P_T(T) = 1$ (note that $a + bx$ is identically zero for all x in $\mathbb{R}$ iff both a and b are zero: this is the linear matching principle). In other words we look for a short rate of interest process r, solution of (1.1), generating an arbitrage-free bond market model of exp-affine form. Notice that problem (1.2),(1.3) , (1.4) is equivalent to simultaneously imposing both an exp-affine bond price and the absence of arbitrage opportunities in that an exp-affine function satisfies the backward equation for a discount arbitrage-free bond price.

Equation (1.2) is linear in r. Thus, one argues that functions μ and σ^2 linear in r let equation (1.2) preserve its linear structure to which one may apply the linear matching principle and come up to a system of two ordinary differential equation in α and β. More precisely, if $\mu(t,r) = k_0(t) + k_1(t)r$ and $\sigma(t,r) = \sqrt{h_0(t) + h_1(t)r}$ equation (1.2) becomes:

$$\begin{aligned} 0 = {} & \left[\partial_t\alpha(t,T) - k_0(t)\beta(t,T) + \frac{1}{2}h_0\beta^2(t,T)\right] \\ & + \left[-\partial_t\beta(t,T) - k_1\beta(t,T) + \frac{1}{2}h_1\beta^2(t,T) - 1\right] r \end{aligned}$$

for each fixed T, identically for any $t \in [t_0, T]$. This happens iff, for each T, both of the coefficient of this linear relation in r are identically zero along $t \in [t_0, T]$, namely:

$$\begin{cases} -\partial_t\beta(t,T) - k_1\beta(t,T) + \frac{1}{2}h_1\beta^2(t,T) - 1 = 0 & t_0 \le t \le T \\ \beta(T,T) = 0 & \end{cases}$$
$$\begin{cases} \partial_t\alpha(t,T) - k_0(t)\beta(t,T) + \frac{1}{2}h_0\beta^2(t,T) = 0 & t_0 \le t \le T \\ \alpha(T,T) = 0. & \end{cases} \tag{1.5}$$

Conclusion: if, for each $T > t_0$, the short rate process is given by:

$$dr(t) = [k_0(t) + k_1(t) r(t)] dt + \sqrt{h_0(t) + h_1(t) r(t)} dW(t),$$

for deterministic functions k_0,k_1,h_0,h_1 of t, then the bond prices are of exp-affine form $P_T(t) = F(t, r; T) = \exp(\alpha(t, T) - \beta(t, T) r(t))$ where $\beta(\cdot, T)$ is the solution of the Riccati equation (1.5) and $\alpha(\cdot, T)$ is the solution of the degenerate linear equation obtained by substituting solution $\beta(t, T)$ in expression (1.5).

In the next section, this sufficient condition will be generalized for multifactor short rate models driven by marked point diffusion factor dynamics. The technique employed will be similar to that here shown. In the following section we will prove the converse for multifactor short rate models driven by continuous diffusions. The result holds for single-factor short rate models driven by marked point diffusions. Next, a counterexample will show that for multidimensional factor processes of marked point diffusion type, one may well have an affine term structure without factor processes possessing linear coefficients.

2. Affine term-structures driven by multi-factor marked point diffusions

We work within the setting defined by (0.1), (0.2) and look for sufficient conditions on $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{m}, \nu; R)$ such that the resulting bond price dynamics $\mathbb{E}_t^{\mathbb{P}^*}\left(e^{-\int_t^T R(s,X(s))ds}\right)$ has an exp-affine form (0.3). We indeed study a slight variation of the problem which actually accommodates the Fourier transform analysis method for derivative pricing that will be dealt with in the last section of this paragraph.

Theorem 2.1.
Let a multi-factor short rate model be given by:

$$\begin{aligned} r(t, \omega) &\overset{def}{=} R(t, X(t, \omega)) \\ dX(t) &= \boldsymbol{\mu}(t, X(t)) dt + \boldsymbol{\Sigma}(t, X(t)) \cdot d\mathbf{W}(t) + \int_E \mathbf{m}(t, X(t), y) \mu(dt, dy; \omega), \end{aligned}$$

where $\mathbf{W}$ is a d-dimensional Brownian motion under $\mathbb{P}^*$, μ is a random measure on $[t_0, T] \times E$, for any $T > t_0$. X is thus an n-dimensional marked point diffusion process. Suppose that:

- ***(affinity)*** *drift, squared volatility, compensator, short rate function are all affine in the factor state and the marked point coefficient is independent of the factor process, that is the* *determining quintuple* $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{m}, \nu; R)$ *has form:*

$$\begin{array}{lcl} \boldsymbol{\mu}(t, \mathbf{x}) & \overset{def}{=} & \mathbf{k}_0(t) + \mathbf{K}_1(t) \cdot \mathbf{x} \\ \boldsymbol{\Sigma}(t, \mathbf{x}) \boldsymbol{\Sigma}(t, \mathbf{x})^\intercal & \overset{def}{=} & \mathbf{H}_0(t) + \mathbb{H}_1(t) \cdot \mathbf{x} \\ \mathbf{m}(t, \mathbf{x}, y) & \overset{def}{=} & \mathbf{m}(t, y) \\ \nu(t, \mathbf{x}, dy)\, dt & \overset{def}{=} & l_0(t, dy) + \mathbf{l}_1(t, dy) \cdot \mathbf{x} \\ R(t, \mathbf{x}) & \overset{def}{=} & \rho_0(t) + \boldsymbol{\rho}_1(t) \cdot \mathbf{x}, \end{array} \tag{2.1}$$

where $\mathbb{H}(t) : \mathbb{R}_n \to \mathcal{M}_S(n, d)$, *the class of symmetric positive definite matrices* n *by* d.

- ***(ode-s)*** *coefficients are such that solutions* $\boldsymbol{\beta}$ *and* α *to the following system of ordinary differential equations exist:*

$$\begin{cases} \partial_t \boldsymbol{\beta}(t, T) = -\boldsymbol{\rho}_1(t) - \mathbf{K}_1(t)^\intercal \boldsymbol{\beta}(t, T) + \frac{1}{2} \boldsymbol{\beta}(t, T)^\intercal \mathbb{H}_1(t) \boldsymbol{\beta}(t, T) \\ \qquad\qquad + \int_E \left[e^{-\boldsymbol{\beta}(t,T)^\intercal \mathbf{m}(t,y)} - 1 \right] \mathbf{l}_1(t, dy) \\ \boldsymbol{\beta}(T, T) = \mathbf{u} \end{cases}$$
$$\begin{cases} \partial_t \alpha(t, T) = \rho_0(t) + \mathbf{k}_0(t) \cdot \boldsymbol{\beta}(t, T) - \frac{1}{2} \boldsymbol{\beta}(t, T)^\intercal \mathbf{H}_0(t) \boldsymbol{\beta}(t, T) \\ \qquad\qquad - \int_E \left[e^{-\boldsymbol{\beta}(t,T)^\intercal \mathbf{m}(t,y)} - 1 \right] l_0(t, dy) \\ \alpha(T, T) = 0. \end{cases} \tag{2.2}$$

- ***(regularity)*** *local martingales are true martingales.*

Then, for each $u \in \mathbb{R}_n$, *the process:*

$$\psi(t, \omega; u) \overset{def}{=} \mathbb{E}_t \left(e^{-\int_t^T R(s, X(s)) ds} e^{\mathbf{u} \cdot X(T)} \right) \tag{2.3}$$

has exponential affine form in X, *namely:*

$$\psi(t, \omega; u) = \exp(\alpha(t, T) - \boldsymbol{\beta}(t, T) \cdot X(t)).$$

Corollary 2.2. *Under the above mentioned conditions, the generated arbitrage-free bond-market model is exp-affine. Just take* $\mathbf{u} = \mathbf{0}$.

Proof.
1. The problem consists in solving equation:

$$\exp\left(\alpha\left(t,T\right)-\boldsymbol{\beta}\left(t,T\right)^{\mathsf{T}}X\left(t\right)\right)=\mathbb{E}_{t}\left(e^{-\int_{t}^{T}R(X(s))ds}e^{\mathbf{u}\cdot X(T)}\right)$$

in α and $\boldsymbol{\beta}$, for given $T>t_0$ and any $t\in[t_0,T]$. Throughout the proof, T is fixed.
2. The above expression reminds the martingale property if not for the time dependence of the expression inside the conditional expectation. To get rid of it, one may multiply both of sides by $\exp\left(-\int_{t_0}^{t}R\left(s,X\left(s\right)\right)ds\right)$ and use a well-known property of conditional expectations to obtain the equivalent equation:

$$M\left(t\right):=e^{-\int_{t_0}^{t}r(X(s))ds}\exp\left(\alpha\left(t,T\right)-\boldsymbol{\beta}\left(t,T\right)^{\mathsf{T}}X\left(t\right)\right)=\mathbb{E}_{t}\left(e^{-\int_{t_0}^{T}r(X(s))ds}e^{\mathbf{u}\cdot X(T)}\right)\tag{2.4}$$

This equation holds true for $t\in[t_0,T]$ iff: 1) the left-hand-side is a martingale and 2) both of sides have equal final value. This follows from the a.s.s-$\mathbb{P}^*$ unicity of the conditional expectation operator.
3. The latter condition is:

$$e^{-\int_{t_0}^{T}R(s,X(s))ds}\exp\left(\alpha\left(T,T\right)-\boldsymbol{\beta}\left(T,T\right)^{\mathsf{T}}X\left(t\right)\right)=e^{-\int_{t_0}^{T}R(s,X(s))ds}e^{\mathbf{u}\cdot X(T)},$$

that is coefficients α and $\boldsymbol{\beta}$ must satisfy:

$$\begin{aligned}\alpha\left(T,T\right)&=0\\ \boldsymbol{\beta}\left(T,T\right)&=\mathbf{u}.\end{aligned}$$

4. By applying Ito formula to M and imposing the resulting drift to be zero, one obtains equations for α and $\boldsymbol{\beta}$ as in the statement of the theorem. In other words, one imposes the above exponential to satisfy the backward equation relative to a discount bond in the considered market. Driftless Ito processes are local martingales. Regularity conditions ensure the true martingale property for M. Notice that one is allowed to apply Ito formula to M because it is a product of a finite variation process times an exponential which, by

(2.4), is itself a product of a finite variation process and a local martingale; thus M is a semimartingale.
5. To simplify notation, set: $R=R\left(t,X\left(t\right)\right)$, $M=M\left(t\right)$, $\alpha=\alpha\left(t,T\right)$, $\boldsymbol{\beta}=\boldsymbol{\beta}\left(t,T\right)$, $X_-=X_-\left(t\right)$, $\Sigma=\boldsymbol{\Sigma}\left(t,X\left(t\right)\right)$, $dl_0=l_0\left(t,dy\right)$, $d\mathbf{l}_1=\mathbf{l}_1\left(t,dy\right)$, $d\mu=\mu\left(dt,dy;\omega\right)$ and similarly for other measures and time dependent variables. By Ito formula, we have:

$$\begin{aligned}d_tM\left(t\right)=-RMdt+e^{-\int_{t_0}^{T}Rds}e^{\alpha-\boldsymbol{\beta}\cdot X}\left(\partial_t\alpha dt-\partial_t\boldsymbol{\beta}Xdt-\boldsymbol{\beta}d_tX^c\right)+\tfrac{1}{2}\mathrm{Tr}\left[\mathrm{He}\left[e^{\alpha-\boldsymbol{\beta}\cdot X}\right]\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathsf{T}}\right]dt\\ +\int_E\left[e^{\alpha-\boldsymbol{\beta}[X_-+\mathbf{m}(t,X(t),y)]}-e^{\alpha-\boldsymbol{\beta}\cdot X_-}\right]d\mu,\end{aligned}$$

that is:

$$\begin{aligned}d_tM\left(t\right)=-RMdt+e^{-\int_{t_0}^{T}Rds}e^{\alpha-\boldsymbol{\beta}\cdot X_-}\left(\partial_t\alpha dt-\partial_t\boldsymbol{\beta}Xdt-\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\mu}dt-\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\Sigma}dW\right)+\tfrac{1}{2}\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathsf{T}}\boldsymbol{\beta}dt\\ +\int_E\left[e^{-\boldsymbol{\beta}\mathbf{m}(t,X(t),y)}-1\right]d\left(\nu+\overline{\mu}\right),\end{aligned}$$

where $\overline{\mu}$ is the compensated martingale measure under the underlying probability measure $\mathbb{P}^*$. By substituting expressions (2.1) and setting the finite variation term to zero one obtains:

$$\begin{aligned} 0 &= -R + \partial_t \alpha - \partial_t \boldsymbol{\beta} \cdot X - \boldsymbol{\beta} \cdot \boldsymbol{\mu} + \frac{1}{2} \boldsymbol{\beta}^\mathsf{T} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^\mathsf{T} \boldsymbol{\beta} + \int_E \left[e^{-\boldsymbol{\beta} \cdot \mathbf{m}(t,y)} - 1 \right] d\nu \\ &= -\rho_0 - \boldsymbol{\rho}_1 X + \partial_t \alpha - \partial_t \boldsymbol{\beta} X - \boldsymbol{\beta} \cdot \mathbf{k}_0 - \boldsymbol{\beta} \mathbf{K}_1 X + \frac{1}{2} \boldsymbol{\beta}^\mathsf{T} \mathbf{H}_0 \overline{\boldsymbol{\beta}} + \frac{1}{2} \boldsymbol{\beta}^\mathsf{T} \mathbb{H}_1 X \boldsymbol{\beta} + \int_E \left[e^{-\boldsymbol{\beta} \mathbf{m}} - 1 \right] (dl_0 + d\mathbf{l}_1 X). \end{aligned}$$

By gathering terms homogeneous of degree 1 in the factor process, one has:

$$0 = -\rho_0 + \partial_t \alpha - \boldsymbol{\beta} \cdot \mathbf{k}_0 + \frac{1}{2} \boldsymbol{\beta}^\mathsf{T} \mathbf{H}_0 \overline{\boldsymbol{\beta}} + \int_E \left[e^{-\boldsymbol{\beta} \cdot \mathbf{m}} - 1 \right] dl_0 + \left[-\boldsymbol{\rho}_1 - \partial_t \boldsymbol{\beta} - \mathbf{K}_1^\mathsf{T} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta}^\mathsf{T} \mathbb{H}_1 \boldsymbol{\beta} + \int_E \left[e^{-\boldsymbol{\beta} \cdot \mathbf{m}} - 1 \right] d\mathbf{l}_1 \right] X.$$

Finally, the principle of affinity matching gives (2.2).

■

An example of regularity conditions referred to in condition 3. are:

1. $\mathbb{E} \left(\int_{t_0}^T \left| M(t) \int_E \left[e^{-\boldsymbol{\beta}(t,T) \mathbf{m}(t,y)} - 1 \right] (l_0(t,dy) + \mathbf{l}_1(t,dy) X(t)) \right| dt \right) < \infty$
2. $\mathbb{E} \left[\left(\int_{t_0}^T \boldsymbol{\beta}(t,T)^\mathsf{T} \boldsymbol{\Sigma}(t, X(t)) \boldsymbol{\beta}(t,T)^\mathsf{T} \boldsymbol{\Sigma}(t, X(t)) \, dt \right)^{\frac{1}{2}} \right] < \infty$
3. $\mathbb{E}(|M(T)|) < \infty$,

where M may be computed in terms of $\alpha, \boldsymbol{\beta}$ and R by its very definition. That is, the compensator of the marked point term is in $L^1(\Omega \times [t_0, T], \mathbb{P}^* \times \mathcal{L})$, the volatility is in $L^2(\Omega \times [t_0, T], \mathbb{P}^* \times \mathcal{L})$ and the final discounted gain is an integrable r.v.
In practical applications, one decomposes the compensating measure as:

$$\nu(t, \mathbf{x}, dy) \, dt \stackrel{def}{=} \widetilde{\nu}(t, \mathbf{x}, dy) \lambda(t, \mathbf{x}) \, dt,$$

where $\lambda(t, \mathbf{x}) = \int_E \nu(t, \mathbf{x}, dy)$ interprets a measure of arrival rate and $\widetilde{\nu}$ is a conditional probability density of jumping in points belonging to E given that a jump has occurred.. Further one assumes $\widetilde{\nu}$ to be independent of $\mathbf{x}$. In this context, the affine condition for ν translates to λ being affine in $\mathbf{x}$.
Suppose one is given an affine bond market driven by a multidimensional factor process X generating an affine short rate process $r(t) = R_0 + \mathbf{R}_1 \cdot X(t)$. Consider a contingent claim on X whose profile is given by a final cash amount of $h(X(T))$ Euros. The time t value of this claim is:

$$V(t, X(t)) = \mathbb{E}_t^{\mathbb{P}^*} \left(e^{\int_t^T r(X(u)) du} h(X(T)) \right).$$

In section (5) we will see that the general theorem above stated lets a direct pricing of such claims in the case of either affine or exponential affine pay-off functions in the factor X. If this is not the case, one may proceed as follows. First, redefine the state variable as:

$$X^*(s)^{\mathsf{T}} \overset{def}{=} \left[X(s)^{\mathsf{T}} \middle| \int_t^s r(X(u))\,du \right]$$

and the pay-off functions as:

$$h^*(\mathbf{x}^*) \overset{def}{=} h\left(\left(\mathbf{x}^*_{(1)}, ..., \mathbf{x}^*_{(n)}\right)^{\mathsf{T}}\right)$$
$$g^*(s, \mathbf{x}^*) \overset{def}{=} g\left(s, \left(\mathbf{x}^*_{(1)}, ..., \mathbf{x}^*_{(n)}\right)^{\mathsf{T}}\right),$$

for $\mathbf{x}^* \in \mathbb{R}_{n+1}$. A $\mathbf{x}^*$ indicates a vector in $\mathbb{R}_{n+1}$, whereas $\mathbf{x}$ identifies an element in $\mathbb{R}_n$. Secondly, write the pricing expectation as:

$$V(t, X(t)) = \mathbb{E}_t^{\mathbb{P}^*}\left(e^{X^*_{(n+1)}(T)} h^*(X^*(T))\right).$$

Thirdly, compute the Green function $\psi(s, \mathbf{y}^*; t, \mathbf{x})$of $X^*(s)$ given t, that is the conditional density at point $\mathbf{y}^*$ of the random variable $X^*(s)$ given $X^*(t) = \mathbf{x}^*$. Finally compute:

$$V(t, X(t)) = \int_{D^*} \left[e^{\mathbf{y}^*_{(n+1)}} h^*(\mathbf{y}^*)\right] \psi(T, \mathbf{y}^*; t, \mathbf{x}^*)\, d\mathbf{y}^*,$$

where D^* is the support of the process X^*. Since pay-off do not depend on $\mathbf{y}^*_{(n+1)}$, one can decuple the integral above in two parts, the former over the support D_1 of $\mathbf{y}^*_{(n+1)}$, the latter on the support D of X:

$$\begin{aligned} V(t, X(t)) &= \int_{D_1 \times D} \left[e^{\mathbf{y}^*_{(n+1)}} h\left(\left(\mathbf{y}^*_{(1)}, ..., \mathbf{y}^*_{(n)}\right)^{\mathsf{T}}\right)\right] \psi(T, \mathbf{y}^*; t, \mathbf{x})\, d\mathbf{y}^* \\ &= \int_D \left[\int_{D_1} e^{\mathbf{y}^*_{(n+1)}} \psi(T, \mathbf{y}^*; t, \mathbf{x}^*)\, d\mathbf{y}^*_{(n+1)}\right] h(\mathbf{y})\, d\mathbf{y} \\ &\overset{def}{=} \int_D \psi(T, \mathbf{y}; t, \mathbf{x})\, h(\mathbf{y})\, d\mathbf{y}, \end{aligned}$$

where $\mathbf{x} := \left(\mathbf{x}^*_{(1)}, ..., \mathbf{x}^*_{(n)}\right)^{\mathsf{T}}$. The problem is to evaluate this green function. By definition:

$$\psi(T, \mathbf{y}^*; t, \mathbf{x}^*) = \mathbb{E}_t^{\mathbb{P}^*}\left(e^{-i\mathbf{y}^* \cdot X^*(T)}\right).$$

Since the considered bond-market model is affine, this expectation is recovered by the explicit formula in the theorem, where $\mathbf{u} = -i\mathbf{y}^*$, $\rho_0 = 0$, $\boldsymbol{\rho}_1 = 0$ and the dynamics of the first n components of X^* equates that of X, while the $(n+1)$-th component is given by:

$$d_t X^*_{(n+1)}(t) = d_t \left(\int_{t_0}^{t} r(s)\, ds \right) = r(t)\, dt = R_0 + \mathbf{R}_1 \cdot X(t)$$

and thus it is affine too.

3. Necessary conditions for exponential-affine bond markets

We have seen that affinity conditions on the model coefficients imply exp-affine bond prices. This holds for short rate dynamics driven by marked point diffusions. Turning the other way around, the exp-affinity of a bond market does not in general imply affinity in coefficients. Indeed, under pretty weak assumption, one can prove the reverse side claim in the case of short rate dynamics driven by multidimensional continuous diffusions. However, a simple counterexample shows that this is not the case once one turns to more general processes. Moreover, even the sufficient conditions are subject to some degree of incertitude about the regularity conditions required in the theorem. In particular, L^1 and L^2 conditions are often not very easy to check since explicit solutions of the two ode-s are pretty rarely available (recall the Riccati equations are not analytically solvable unless one already knows one particular solution) and this is necessary for having an analytical expression for M to plug into regularity conditions. In view of these remarks, one might legitimately argue that Ito calculus is not the best analytical tool to deal with such Markov processes. From a purely theoretical ground, it can be shown that classical tool for studying Markov processes such as generators and resolvents, deliver nicer results in term of necessary and sufficient conditions. Yet, from a practical side, this method has not much more to offer than Ito calculus.

In this section we investigate necessary conditions for exp-affinity of a bond market, provide the stated counterexample and summarize recent results obtained using the classical tool-kit from the theory of Markov processes.

Suppose an exp-affine bond market is given in a multidimensional continuous diffusion setting. By applying Ito formula to the exponential-affine function representing the bond price and the underlying factor process, setting the resulting drift to 0 and supposing that the short rate is affine in the facto process, one comes up to:

$$0 = -\rho_0 - \boldsymbol{\rho}_1 \cdot \mathbf{x} + \partial_t \alpha - \partial_t \boldsymbol{\beta} \cdot \mathbf{x} - \boldsymbol{\beta} \cdot \boldsymbol{\mu} + \frac{1}{2} \boldsymbol{\beta}^{\mathsf{T}} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^{\mathsf{T}} \bar{\boldsymbol{\beta}}.$$

This follows immediately by replacing R with $\rho_0 + \boldsymbol{\rho}_1 \cdot \mathbf{x}$ and delete the marked point term. This condition is necessary for a exp-affine bond prices. We look at it as an equation in the unknowns $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Set $\mathbf{A} := \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathsf{T}}$. After gathering the first four summand into a unique affine function in $\mathbf{x}$, say $-d(t,T,\mathbf{x})$, the previous equation takes form:

$$\underbrace{\left[\begin{array}{cccccccc} \boldsymbol{\beta}^{\mathsf{T}}, & \frac{\boldsymbol{\beta}^{(1)2}}{2}, & \dots & \frac{\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(n)}}{2}, & \frac{\boldsymbol{\beta}^{(2)}\boldsymbol{\beta}^{(1)}}{2}, & \dots, & \frac{\boldsymbol{\beta}^{(n)2}}{2} \end{array}\right]}_{\overset{def}{=}\mathbf{c}^{\mathsf{T}}(t,T)} \cdot \underbrace{\left[\begin{array}{c} \boldsymbol{\mu} \\ \mathbf{A}^{11} \\ \dots \\ \mathbf{A}^{1n} \\ \mathbf{A}^{21} \\ \dots \\ \mathbf{A}^{nn} \end{array}\right]}_{\overset{def}{=}\mathbf{h}(t,\mathbf{x})} = d(t,T,\mathbf{x}),$$

or:

$$\mathbf{c}(t,T) \cdot \mathbf{h}(t,\mathbf{x}) = d(t,T,\mathbf{x}),$$

where vectors are columnwise expressed and $\mathbf{c}$ has dimension $N := n + \frac{n^2-n}{2} + n$ (i.e.e dimension of $\boldsymbol{\beta}$ + number of elements in the upper-east triangular part of $\mathbf{A}$ + number of elements in the main diagonal of $\mathbf{A}$). For each $T > t_0$ identifying a bond, last equation must hold identically for $t \in [t_0, T]$ and $\mathbf{x} \in \mathbb{R}_n$.

Condition I: for each $t > t_0$, there exist times $T_1, ..., T_n$ greater than t, such that matrix $\mathbf{C}(t, T_1, ..., T_n)$, defined by:

$$\mathbf{C}(t, T_1, ..., T_n) \overset{def}{=} \left[\begin{array}{c} \mathbf{c}(t,T_1) \\ \dots \\ \mathbf{c}(t,T_n) \end{array}\right],$$

is invertible.

Suppose that condition I holds. Then, for each time t, one has:

$$\mathbf{h}(t,\mathbf{x}) = \mathbf{C}(t, T_1, ..., T_n)^{-1} \left[\begin{array}{c} d(t,T_1,\mathbf{x}) \\ \dots \\ d(t,T_n,\mathbf{x}) \end{array}\right].$$

Since all d are affine in $\mathbf{x}$, then the components of $\mathbf{h}$ are affine in $\mathbf{x}$ too.

Theorem 3.1. *If a short rate is affine in a factor process driven by a multidimensional continuous diffusion satisfying condition I above stated, then the corresponding bond market is exp-affine only if drift and squared diffusion coefficients of the factor dynamics are all affine in the factor.*

We now turn to applications involving affine term structures.

4. A method for identifying factors

Multi-factor models serves the scope of driving bond dynamics by several imperfectly correlated noises underlying a given market. Once such noises are detected in their cardinality and impact over the discount function, the model can be used to price exotic interest-rate derivatives whose pay-off function acts on several points on the discount function, either directly or through interest-rates. It is just the asynchronous movement of these points (discount factors or rates) that reflects the presence of several underlying factors. Intuitively, one may argue that a good property of a multi-factor model is the identification of these noises with market observables, such as rates or yields. If this is possible, one would be given a short rate dynamics as a deterministic function of several market values. In principle one could estimate these dynamics, impose a structure and find and estimate of the prices of risk for all of these factors and come up with a risk-neutral dynamics of the short rate in terms of observable quantities. this can be used to price derivatives.

In this section we identify factor with some observables and give some examples of concrete affine term-structure models. We present a method for changing variables in a given affine model. Suppose an affine structure has already been determined in that $P_T(t) = \exp(\alpha(t,T) - \boldsymbol{\beta}(t,T) \cdot X(t))$, for an n-dimensional factor process X. For each $i = 1, ..., n$, let F_i be a function of X, possibly representing observable values in the market place.

For instance, F_i may be the yield in τ_i years, that is:

$$F_i(t, X(t)) \overset{def}{=} -\frac{\lg P_{t+\tau_i}(t)}{\tau_i} = \frac{-\alpha(t, t+\tau_i) + \boldsymbol{\beta}(t.t+\tau_i) \cdot X(t)}{\tau_i}.$$

Let $Y_i(t) \overset{def}{=} F_i(t, X(t))$ be the components of the observable vector-valued process $Y(t)$. If $\mathbf{F} : (F_1, ..., F_n)$ is invertible with respect to X, then one can identify factors $X_1, .., X_n$ with observable values $Y_1, ..., Y_n$ according to the relation:

$$X(t) = \mathbf{F}^{-1}(t, Y(t)).$$

In general, under the new vector of factors, the model is no more affine since:

$$P_T(t) = \exp\left(\alpha(t,T) - \boldsymbol{\beta}(t,T) \cdot \mathbf{F}^{-1}(t, Y(t))\right).$$

If $\mathbf{F}^{-1}$ is affine, then the bond market stay exp-affine. This is the case for the example above. Indeed:

$$\mathbf{F}^{-1}(t, Y(t)) = \mathbf{K}^{-1}(t)\left[Y(t) + \mathbf{k}(t)\right],$$

with:

$$\mathbf{k}(t) \stackrel{def}{=} \begin{bmatrix} \tau_1^{-1}\alpha(t, t+\tau_1) \\ \dots \\ \tau_n^{-1}\alpha(t, t+\tau_n) \end{bmatrix}, \ \mathbf{K} \stackrel{def}{=} \begin{bmatrix} \tau_1^{-1}\boldsymbol{\beta}(t.t+\tau_1)^{\mathsf{T}} \\ \dots \\ \tau_n^{-1}\boldsymbol{\beta}(t.t+\tau_1)^{\mathsf{T}} \end{bmatrix}$$

and thus:

$$P_T(t) = \exp\left(\alpha(t,T) - \boldsymbol{\beta}(t,T)\cdot\mathbf{K}^{-1}(t)\left[Y(t) + \mathbf{k}(t)\right]\right) = \exp\left(\alpha'(t,T) - \boldsymbol{\beta}'(t,T)\cdot Y(t)\right).$$

Of course, apart from the invertibility condition of $\mathbf{F}$ and the affinity requirement, further constraints are imposed by the redemption at par property of discount bonds:

$$\alpha'(t,t) = 0, \ \boldsymbol{\beta}'(t,t) = \mathbf{0}$$

and by conditions of the particular instance. For the case of yield factor, one must have:

$$Y_i(t) = -\frac{\lg P_{t+\tau_i}(t)}{\tau_i} = \frac{-\alpha'(t,t+\tau_i) + \boldsymbol{\beta}'(t.t+\tau_i)\cdot Y(t)}{\tau_i},$$

which holds iff:

$$\begin{aligned} &\alpha'(t,t+\tau_i) = 0, \ i = 1,...,n \\ &\boldsymbol{\beta}'_j(t.t+\tau_i) = 0, \ j \neq i \\ &\boldsymbol{\beta}'_i(t.t+\tau_i) = 1. \end{aligned}$$

This tells us nothing else but our factors are just yields, in that:

$$P_{t+\tau_i}(t) = \exp\left(\alpha'(t,t+\tau_i) - \boldsymbol{\beta}'(t,t+\tau_i)\cdot Y(t)\right) = \exp\left[-Y_t(t)\right].$$

If $\mathbf{F}$ is in $\mathcal{C}^{1,2}$, then the new factor dynamics is easily computed by Ito formula. In the case of yield factors, if:

$$dX(t) = \boldsymbol{\mu}dt + \boldsymbol{\Sigma}d\mathbf{W}(t),$$

one has:

$$dY(t) = d_t\mathbf{F}(t, X(t)) = \mathbf{K}\boldsymbol{\mu}\left(\mathbf{K}^{-1}(Y(t) + \mathbf{k})\right)dt + \mathbf{K}\boldsymbol{\Sigma}\left(\mathbf{K}^{-1}(Y(t) + \mathbf{k})\right)d\mathbf{W}(t).$$

5. Pricing via Fourier transform methods

We can use the general result in theorem (2.1) in order to compute characteristic function of distributions and then numerically invert them so to recover the corresponding density functions.

A typical application of this method is the following. Consider a call option on a function g of the underlying factor expiring at T and striken at price c. For any monotonically increasing function h one has:

$$\begin{aligned}
V^{call}(t) &= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds}\left(g\left(X\left(T\right)\right)-c\right)_+\right) \\
&= \mathbb{E}_t^*\left(e^{-\int_t^T Rds} g\left(X\left(T\right)\right)\mathbb{I}_{\{-h\circ g(X(T))\leq -h(c)\}}\right) - c\mathbb{E}_t^*\left(e^{-\int_t^T Rds}\chi_{\{-h\circ g(X(T))\leq -h\circ c\}}\right),
\end{aligned}$$

where R is a shorthand for $R\left(X\left(s\right)\right)$. By now, h is arbitrarily fixed. Later we will see it has to chosen according to the functional form of the pay-off function g.

The function:

$$f_g^h\left(\mathbf{x}\right) \stackrel{def}{=} \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds} g\left(X\left(T\right)\right)\chi_{\{-h\circ g(X(T))\leq x\}}\right)$$

is not easy to compute, especially due to the presence of the indicator function. Yet, it is non-decreasing in x and thus it can be seen as a distribution function on $\mathbb{R}$ so that we can calculate its Fourier transform:

$$\begin{aligned}
F\left[f_g\right]\left(\mathbf{v}\right) &= \int_{\mathbb{R}} e^{ivx} d_{\mathbf{x}}\mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds} g\left(X\left(T\right)\right)\chi_{\{-h\circ g(X(T))\leq x\}}\right) \\
&= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds} g\left(X\left(T\right)\right)\int_{\mathbb{R}} e^{ivx} d_x\chi_{\{-h\circ g(X(T))\leq x\}}\right) \\
&= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds} g\left(X\left(T\right)\right)e^{iv(-h\circ g(X(T)))}\right), \qquad (5.1)
\end{aligned}$$

what gets rid of the indicator. Once $F\left[f_g\right]\left(\mathbf{v}\right)$ is computed, one can numerically invert it and recover the values of f. The price of the call is then:

$$V^{call}(t) = f_g\left(-h\left(c\right)\right) - cf_1\left(-h\left(c\right)\right),$$

where 1 indicates the constant function equal to 1.

For a given g, the idea is to choose h and eventually express g conveniently, such that the conditional expectation (5.1) is of form (2.3). We examine two cases. Throughout R is affine X. Suppose that $g\left(\mathbf{x}\right) = e^{\mathbf{a}\cdot\mathbf{x}+d}$ for some $\mathbf{a}$. Setting $h\left(x\right) = \lg\left(x\right)$, one has:

$$F\left[f_g\right]\left(v\right) = e^d\mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds}e^{(1-iv)\mathbf{a}^\intercal X(T)}\right),$$

which is given by constant e^d times the exp-affine function in X provided by theorem (2.1), where $\mathbf{u}$ has been replaced by $\left(1-iv\right)\mathbf{a}$. An explicit expression for this expectation is given by theorem (2.1).The following instances provide concrete derivatives in this class.

- Bond options. The time t value of a call option written on a bond P_S in an affine term structure model, striken at c and expiring at $T < S$, is:

$$V^{call}(t) = f_g(-h(c)) - cf_1(-h(c)),$$

with $h = \lg$. This follows as $P_T(t) = \exp(\alpha(t,T) + \boldsymbol{\beta}(t,T) \cdot X(t))$ and thus $g(\mathbf{x}) = \exp(\alpha(T,T) + \boldsymbol{\beta}(T,T) \cdot \mathbf{x}(t))$.

- Quanto options. A quanto option on an asset S whose value is expressed in unit of numéraire N_1, say dollars, pays out a functional of S many unities of another numéraire N_2, say Euros. If the underlying asset S is exponential affine in the factor X, say $S(t) = \exp(\alpha(t) + \boldsymbol{\beta}(t) \cdot X(t))$, and the conversion value of N_1 into N_2 can be expressed as an exponential affine function of the state variable, that is each unity of N_1 at T is financially equivalent to $\exp(\mathbf{a}'(T) \cdot X(T) + e'(T))$ units of N_2 and we wish to evaluate a quanto call option striken at c on the $S(T)$ unities of N_2, expiring at T, the pay-off is:

$$(\exp(\alpha(t) + \boldsymbol{\beta}(t) \cdot X(t)) \exp(\mathbf{a}'(T) \cdot X(T) + e'(T)) - c)_+$$

and the time t value of this derivative is given by same formula as above, with $\mathbf{a} = \mathbf{a}' + \boldsymbol{\beta}$ and $e = e' + \alpha$. An example is a foreign bond option.

A slight extension occurs when the pay-off function g and the term determining the moneyness of the option are different exp-affine functions, that is $g(\mathbf{x}) = e^{\mathbf{a} \cdot \mathbf{x} + c}$ and the indicator function is $\chi_{\{-\lg m(X(T)) \le x\}}$ with $m(\mathbf{x}) = e^{\mathbf{b} \cdot \mathbf{x} + d}$. In this case:

$$F[f_g](v) = e^{c+d} \mathbb{E}_t^* \left(e^{-\int_t^T R(X(s))ds} e^{\left(\mathbf{a} - iv\mathbf{b}^\mathsf{T}\right) X(T)} \right)$$

and $\mathbf{u} = \mathbf{a} - iv\mathbf{b}^\mathsf{T}$.This expression lets us computes the value of trigger options.

- Exchange-assets options. An option on the exchange of two assets has pay-off:

$$g := \max(S_1(T), S_2(T)) = S_1(T) \chi_{\{S_1(T) > S_2(T)\}} + S_2(T) \chi_{\{S_1(T) \le S_2(T)\}}$$

and has time t value decomposable as the sum of the arbitrage-free value of each of the pay-off components. For the first one, let $X = (\lg S_1, \lg S_2)$, $\mathbf{a} = (1, 0)$, $\mathbf{b} = (+1, -1)^\mathsf{T}$, $c = d = 0$ and $h = \exp$. Similarly for the other summand.

- Trigger bond option. Let a contingent claim pay-off at time T the time T value of a bond P_S if the time T value of a second bond P_U is above a certain ceiling d. Suppose the underlying bond-market model is affine. The pay-off is:

$$\begin{aligned} g\left(P_S\left(T\right), P_U\left(T\right)\right) &= P_S\left(T\right) \chi_{\{P_U(T)>d\}} \\ &= \exp\left(\alpha\left(T,S\right)+\boldsymbol{\beta}\left(T,S\right)\cdot X\left(T\right)\right) \chi_{\{-\alpha(T,U)+\boldsymbol{\beta}(T,U)\cdot X(T)\leq -\lg d\}}, \end{aligned}$$

so that: $c=\alpha\left(T,S\right)$, $d=\boldsymbol{\beta}\left(T,S\right)$, $\mathbf{a}=\boldsymbol{\beta}\left(T,S\right)$ and $\mathbf{b}=\boldsymbol{\beta}\left(T,U\right)$ work out the case.

Consider pay-off functionals of the form:

$$g\left(X\right) \overset{def}{=} \alpha_1 \mathbf{k}_1^{\intercal} X\left(T\right) + \alpha_2 \int_t^T \mathbf{k}_2^{\intercal}\left(s\right) X\left(s\right) ds = \partial_\theta \partial_\upsilon e^{\theta \alpha_1 \mathbf{k}_1^{\intercal} X(T) + \upsilon \alpha_2 \int_t^T \mathbf{k}_2^{\intercal}(s) X(s) ds}\Big|_{\theta=0=\upsilon},$$

where X denotes the whole trajectory of the underlying factor over $[t,T]$. Then:

$$\begin{aligned} F\left[f_g\right]\left(\mathbf{v}\right) &= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds} g\left(X\right) e^{iv(-g(X))}\right) \\ &= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds}\, \partial_\theta \partial_\upsilon e^{\theta \alpha_1 \mathbf{k}_1^{\intercal} X(T) + \upsilon \alpha_2 \int_t^T \mathbf{k}_2^{\intercal}(s) X(s) ds}\Big|_{\theta=0=\upsilon} \right. \\ &\quad \left. \times\, e^{iv\left(-\alpha_1 \mathbf{k}_1^{\intercal} X(T) - \alpha_2 \int_t^T \mathbf{k}_2^{\intercal}(s) X(s) ds\right)}\right) \\ &= \partial_\theta \partial_\upsilon \mathbb{E}_t^*\left(e^{-\int_t^T \left[R(X(s)) + (\upsilon - iv)\alpha_2 \mathbf{k}_2^{\intercal}(s) X(s)\right] ds} e^{(\theta - iv)\alpha_1 \mathbf{k}_1^{\intercal} X(T)}\right)\Big|_{\theta=0=\upsilon}. \end{aligned}$$

If the model is affine, one redefines the rate function as:

$$R\left(\mathbf{x}\right) = \rho_0 + \left(\boldsymbol{\rho}_1 + \left(\upsilon - iv\right)\alpha_2 \mathbf{k}_2^{\intercal}\left(s\right)\right)\mathbf{x}$$

and applies theorem (2.1) with $\boldsymbol{\rho}_1' = \boldsymbol{\rho}_1 + \left(\upsilon - iv\right)\alpha_2 \mathbf{k}_2\left(s\right)$ and $\mathbf{u} = \left(\theta - iv\right)\alpha_1 \mathbf{k}_1$ to obtain an explicit solution of the expectation inside differentiation. Taking partial derivatives gives the characteristic function.

- Asian options. The pay-off is $\left(\frac{1}{T}\int_0^T S\left(u\right) du - c\right)$, with S exp-affine, let $X\left(t\right) = \lg S\left(t\right)$.

In both of the cases just considered, we tried to represent the transform induced by the pay-off as a the conditional expectation of the theorem. That is the argument inside expectation must be represented as an exponential affine function of the factor. One has two degrees of freedom. One can use linear operators switching with conditional expectation in order to represent the real term in exp-affine form. Also the function h is free to accommodate an exp-affine form for the imaginary term. When the pay-off is exp-affine, one already has an exp-affine real term; thus function h has to be used in order for the imaginary part to be represented as an exp-affine function. In the case of an affine pay-off, the imaginary part is already of the required form; thus h is useless a tool and one has to play with linear operators to represent the real part as an exponential affine object. In this case we represented an affine function as partial derivatives of an exponential affine function with respect to dummy variables computed at 0. If we leave these variables at unspecified values, the exponential does not delete and one is left with:

$$\left[\theta\alpha_1\mathbf{k}_1^\intercal X(T)+\upsilon\alpha_2\int_t^T \mathbf{k}_2^\intercal(s)X(s)\,ds\right]e^{\theta\alpha_1\mathbf{k}_1^\intercal X(T)+\upsilon\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)ds}. \tag{5.2}$$

This may arise the suspect that the transform method can be applied to such kinds of pay-off.. Unfortunately this is not the case. Indeed at look at expression (5.1) show immediately that for this kind of mixed affine-exponential pay-off, no function h exists such that $h\circ g(X(T))$ is affine too, what ought to be verified for the theorem to apply. Therefore one cannot use this technique to value vanilla options, such as calls and puts, one underlying assets having time T value of form (5.2). Yet the discussion just made is not useless. Indeed, any contingent claim on an asset whose dynamics is either of affine-integral form $\theta\alpha_1\mathbf{k}_1^\intercal X(T)+\upsilon\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)\,ds$ or of exponential affine-integral form $\exp\left(\theta\alpha_1\mathbf{k}_1^\intercal X(T)+\upsilon\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)\,ds\right)$ and whose payoff is of mixed affine-exponential form (5.2) (possibly with constants α_1, α_2 different than those at the exponent and $\mathbf{k}_1^\intercal$, $\mathbf{k}_2^\intercal$ multiple of the ones at the exponent) can be evaluated with the transform method above. If suffices to take h equal to the identity in the former case, equal to lg in the latter and conveniently adjust the values of variables θ and υ according to the values of the other constants and deterministic functions in the pay-off expression. We summarize the two cases just mentioned.
If the pay-off is:

$$g(X)=\left[\alpha_1\mathbf{k}_1^\intercal X(T)+\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)\,ds\right]e^{\alpha_1\mathbf{k}_1^\intercal X(T)+\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)ds}\chi_D,$$

where $D\overset{def}{=}\left\{-\left(\alpha_1'\mathbf{k}_1'\cdot X(T)+\alpha_2'\int_t^T\mathbf{k}_2'\cdot(s)X(s)\,ds\right)\leq -c\right\}$, then the Fourier transform for $h=1$ is:

$$\begin{aligned}
F[f_g](\mathbf{v}) &= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds}g(X)\,e^{iv(-g(X))}\right)\\
&= \mathbb{E}_t^*\left(e^{-\int_t^T R(X(s))ds}\partial^2_{\theta\upsilon}e^{\theta\alpha_1\mathbf{k}_1^\intercal X(T)+\upsilon\alpha_2\int_t^T\mathbf{k}_2^\intercal(s)X(s)ds}e^{iv\left(-\alpha_1\mathbf{k}_1'X(T)-\alpha_2\int_t^T\mathbf{k}_2'(s)X(s)ds\right)}\right)\\
&= \partial_\theta\partial_\upsilon\mathbb{E}_t^*\left(e^{-\int_t^T\left[R(X(s))+\left[\upsilon\alpha_2\mathbf{k}_2(s)-iv\alpha_2\mathbf{k}_2'(s)\right]X(s)\right]ds}e^{\left[\theta\alpha_1\mathbf{k}_1-iv\alpha_1\mathbf{k}_1'\right]X(T)}\right)\Big|_{\theta,\upsilon},
\end{aligned}$$

which is a double partial derivative of an exponential affine function determined by theorem (2.1). This is also the expression for the Fourier transform for $h = \lg$ in the case of a pay-off of form:

$$\left[\theta\alpha_1\mathbf{k}_1 \cdot X(T) + \upsilon\alpha_2 \int_t^T \mathbf{k}_2(s) \cdot X(s)\, ds\right] e^{\theta\alpha_1\mathbf{k}_1 \cdot X(T) + \upsilon\alpha_2 \int_t^T \mathbf{k}_2(s)\cdot X(s) ds} \chi_D,$$

where $D \overset{def}{=} \left\{\exp\left(\theta\alpha_1\mathbf{k}_1' \cdot X(T) + \upsilon\alpha_2 \int_t^T \mathbf{k}_2'(s) \cdot X(s)\, ds\right)\right\}$.

6. Change of numéraire

6.1. Forward contract and measure construction in a lognormal market

At time t, two counterparts trade for a zero spot price a security, called forward contract, for which the holder will obtain at a subsequent time T a financial payoff h from the short party in exchange of a price $f_T^h(t)$, fixed at t once for ever: this trade is compulsory for both of the counterparts.. A mathematical definition runs as follows:

Definition 6.1 (forward price and forward contract). *Let an $\mathcal{F}_t$-measurable random variable $h(\omega)$ represent the time T value of a given security and let prices be expressed in terms of a given currency \$. The \$-denominated time t forward price for delivery of h at maturity T is the $\mathcal{F}_t$-measurable number $f_T^h(t)$ representing the price decided at time t (and then kept constant up to T) such that the time t value of the time t set up security, called forward contract, defined by its time T payoff $h - f_T^h(t)$, is zero.*

Evidently the forward price is not a spot price of any security; its time t numerical value is simply the price then contracted for a trade on h to be performed at a future time T; once fixed, it stays fixed up to T; on the other hand the forward contract corresponding to the forward price $f_T^h(t)$ is an actual security denoted by $fwd(T,h)$, defined on $[t,T]$ and whose time s value is the spot price $V^{fwd(T,h)}(s)$.

In order to fully specify a forward contract one needs a date of issue t, a maturity T, an underlying financial quantity h available at time T and a currency \$: this information coupled with the above contractual structure fully determine the corresponding forward price. By 'contractual structure' we mean the statement according to which the *value of the forward contract* is 0 at t and $h - f$ at the delivery time T:

$$V^{fwd(T,h)}(t) := 0$$

$$V^{fwd(T,h)}(T) := h - f_T^h(t)$$

The forward contract value $V^{fwd(T,h)}(s)$ for $0 < s < T$ is not a priori specified, yet its two endpoints values above provide sufficient information to have the forward price be well defined and thus to determine such intermediate values by arbitrage pricing. Indeed:

$$0 := V^{fwd(T,h)}(t) = E^{P^*}\left(e^{-\int_t^T r(s)ds} \left[h(\cdot) - f_T^h(t) \right] \Big|_{\mathcal{F}_t} \right) = V^h(t) - f_T^h(t)\, E^{P^*}\left(e^{-\int_t^T r(s)ds} \Big|_{\mathcal{F}_t} \right)$$

implies:

$$f_T^h(t) = \frac{V^h(t)}{P^T(t)}$$

Therefore, for each $0 < s < T$, the time s forward contract value is:

$$V^{fwd(T,h)}(s) = V^{h - f_T^h}(s) = V^h(s) - P^T(s)\, f_T^h(t) \tag{6.1}$$

which needs not be 0 such as at time t, because V^h and P^T vary over $[t,T]$, but f is held fixed from t on.

A scheme summarizes the former results:

$$\left.\begin{array}{ll} \text{data} & \left\{ \begin{array}{l} t \\ T \\ h \\ \$ \end{array} \right. \\ \begin{array}{l} \text{contractual} \\ \text{structure} \end{array} & \left\{ \begin{array}{l} V(t) := 0 \\ V(T) := h - f_T^h(t) \end{array} \right. \end{array} \right\} \longrightarrow \begin{array}{l} f_T^h(t) \\ \text{fixed} \\ \text{on } [t,T] \end{array} \longrightarrow \left\{ V^{fwd(T,h)}(s) \right\}_{t \le s \le T}$$

Formula (6.1) states also that a long forward contract is replicated by a short position of as many T-maturing bonds as f and a long position in one unity of the security defined by h. We now turn to the construction of a forward measure in the case of a lognormal market.

Given a market $\left\{ M, S_1, ..., S_n, P^T \right\}$ with risk neutral dynamics for the rolled over money market account M, time T maturing zero coupon bond P^T and risky assets S_i given by:

$$\begin{aligned} dS_i(t) &= S_i(t)\left(r(t)\,dt + \sigma_i(t) \cdot dW(t) \right) & (6.2) \\ dP^T(t) &= P^T(t)\left(r(t)\,dt + \sigma^T(t) \cdot dW(t) \right) & (6.3) \\ dM(t) &= M(t)\, r(t)\, dt & (6.4) \end{aligned}$$

the forward price $f_T^{S_i}$ is not a martingale under the risk neutral measure P^M. By applying Ito formula to the function $\lg x$, processes S_i and P^T one finds explicit solutions to (6.2) and (6.3); thus:

$$f_T^{S_i}(s) := \frac{S_i(s)}{P^T(s)} = \frac{S_i(0)}{P^T(0)} e^{\int_0^s -\frac{1}{2}\left[\|\sigma_i(t)\|^2 - \left\|\sigma^T(t)\right\|^2 \right] dt + \int_0^s \left[\sigma_i(t) - \sigma^T(t) \right] \cdot dW(t)}$$

under P^M. By completing the square:

$$-\frac{1}{2}\left[\left\|\sigma_i(t)\right\|^2-\left\|\sigma^T(t)\right\|^2\right]=-\frac{1}{2}\left\|\sigma_i(t)-\sigma^T(t)\right\|^2-\sigma^T\cdot\left[\sigma_i(t)-\sigma^T(t)\right],$$

we see that:

$$f_T^{S_i}(s)=f_T^{S_i}(0)\,e^{\int_0^s-\frac{1}{2}\left\|\sigma_i(t)-\sigma^T(t)\right\|^2dt+\int_0^s\left[\sigma_i(t)-\sigma^T(t)\right]\cdot d\left(W(t)-\int_0^t\sigma^T(u)du\right)} \tag{6.5}$$

According to the Girsanov theorem, under the measure P^{P^T} defined by:

$$\left.\frac{dP^{P^T}}{dP^M}\right|_{\mathcal{F}_t}:=\mathcal{E}\left(\int_0^{\cdot}c'\cdot dW(u)\right)_t \tag{6.6}$$

with c' such that c'·fwd volatility=drift$_{P^{P^T}}$-drift$_{P^M}$, namely:

$$\begin{aligned}c'\cdot\left[\sigma(t)-\sigma^T(t)\right] &= 0-\left[-\sigma^T\cdot\left(\sigma(t)-\sigma^T(t)\right)\right] &(6.7)\\ c' &= \sigma^T &(6.8)\end{aligned}$$

all the processes $f_T^{S_i}$ satisfy a zero drift s.d.e.:

$$df_T^{S_i}(s)=f_T^{S_i}(s)\left[\sigma_i(t)-\sigma^T(t)\right]dW^T(t) \tag{6.9}$$

where:

$$W^T(t):=W(t)-\int_0^t\sigma^T(u)\,du$$

is an n-dimensional Brownian motion under P^{P^T}.
(6.9) states that the forward price process is the exponential martingale:

$$f_T^{S_i}(s)=f_T^{S_i}(0)\,\mathcal{E}\left(\int_0^{\cdot}\left[\sigma_i(t)-\sigma^T(t)\right]\cdot dW^T(t)\right)_s$$

We have created a probability measure P^{P^T} under which the P^T discounted prices $\frac{S_i(s)}{P^T(s)}$, which are also forward prices relative to assets S_i, are martingales: this is a martingale measure for the *numeraire* P^T, in contrast with P^M, which is a martingale measure for the money market account M.

Remark 1. *The new measure is built up from the process $\sigma^T(t)$ which formally represents $\frac{d\left\langle \lg P^T(\cdot)\right\rangle_t}{dt}$.*

Remark 2. *Instead of P^T we could have had any lognormally distributed asset: taking the T-maturing bond implies that under the new measure forward prices are martingales.*

The martingale defining the Radon-Nikodym (6.6)can be put in terms of asset processes; indeed:

$$d\frac{P^T(t)}{M(t)} = \frac{P^T(t)}{M(t)}\sigma^T(t)\,dW(t) \tag{6.10}$$

follows from the product rule applied to (6.3) and (6.4). Its solution it just the right hand side of (6.6) with $c' = \sigma^T$:

$$\left.\frac{dP^{P^T}}{dP^M}\right|_{\mathcal{F}_t} = \frac{P^T(t)}{M(t)}\frac{M(0)}{P^T(0)} \tag{6.11}$$

The factor $\frac{M(0)}{P^T(0)}$ is the normalizing constant ensuring $P^{P^T}(\Omega) = 1$. Since in a lognormal market any asset value V per unity of money market account M is a martingale satisfying an equation of form (6.10), we argue that the transformation from the risk neutral to a martingale measure for the V-discounted prices is obtained by a Radon-Nikodym transformation of form (6.11) where V substitutes P^T, namely:

$$\left.\frac{dP^V}{dP^M}\right|_{\mathcal{F}_t} = \frac{V(t)}{M(t)}\frac{M(0)}{V(0)} \tag{6.12}$$

This formula will be proved to hold independently from the lognormal assumption to which we owe the nice representation of the derivative (6.11) in exponential form (6.6)and this, in turns, lets us determine the drift modification in the underlying processes due to change of measure. In concrete applications a Radon-Nikodym of form (6.12) is chosen according to the user's need, then a representation in terms of stochastic exponential is looked for in order to determine the drift modifications occurring in the underlying processes. We now turn to a financial interpretation of the result above. Formula (6.5)shows that the quantity $\sigma^T \cdot \left(\sigma_i(t) - \sigma^T(t)\right)$ is the defect or excess of percentage instantaneous mean return of all the T-forward prices $f_T^{S_i}$, $i = 1, ..., n$ in the risk neutral world over the T-maturing bond P^T. Expression (6.7) states that the Girsanov coefficient $c' = \sigma^T$ is *the same for all the assets* and represents that excess or defects per unit of instantaneous volatility $\sigma_i(t) - \sigma^T(t)$ of the relative prices $\frac{S_i}{P^T}$.

It is convenient to draw a parallel between the risk neutral world and the forward risk adjusted. Arbitrage pricing theory characterizes absence of arbitrage opportunities by the possibility of carrying an *objective* probability measure P to a *risk neutral* one P^M under which each asset grows in percentage, instantaneously and on average (p.i.a.) as the money market account; this amounts to find a Girsanov coefficient λ satisfying:

$$\lambda \cdot \text{i-th asset volatility} = \text{risk neutral drift} - \text{objective drift}$$

Since the money market account is thought of as *the* riskless security, the world where all assets' p.i.a. growth equates the money market account's can be said *risk neutral.* The vector λ is easily interpreted as the market price of risk because it represents the excess or defects of p.i.a. return of any traded asset over the risk free asset's per unit of p.i.a. volatility. On the other hand it is difficult to be inferred from the market.

Alternatively, if we think of the bond P^T as the riskless asset, then the world where all the assets' values relative to the bond P^T do not p.i.a. grow is the 'risk neutral world'. Since this latter expression is by convention attached to the case where the money market account is the riskless asset, in our situation a new label is needed: commonly adopted terms are '*forward neutral*' [17],[18] and '*forward risk adjusted*' measure [19]. The meaning is the same.

We turn to the interpretation of Girsanov coefficients as prices of risk.

If one passes from the objective measure P to the forward risk adjusted P^{P^T} and interprets P^T as the riskless asset, then the corresponding Girsanov coefficient $\lambda + c'$ is worth to be called 'market price of risk' or, in order to avoid confusion with the case where the money market is the riskless asset, 'market price of forward risk'. Actually we start from the risk neutral world, not the objective one: the gain is that the Girsanov coefficient is surely determined by market dynamics: it is the T-maturing bond volatility σ^T, which is independent of the asset whose forward is considered. The drawback is that we cannot interpret it as a market price of risk because we pass from a fictitious (not real) world to another fictitious environment.

Finally, it is inconsistent to label more than one asset (here more than one bond) as *the riskless asset*: in fact for each P^T there is a corresponding forward neutral world. This is because c' depends on T.

Summarizing, *in a lognormal market*, the T-forward prices are 1) stochastic exponentials 2) of an Ito integral 3) of the difference between the unitary diffusion coefficients of the asset price and the T maturing bond price 4) with respect to a standard Brownian motion. Since nothing formally distinguished the asset P^T from the others in terms of dynamics, any lognormal asset can be taken as numeraire.

6.2. General change of numeraire

A market model is a finite collection of semimartingales $\{N_0, N_1, ..., N_n\}$ each one representing the stochastic behavior of a corresponding security price.

Given i such that N_i is strictly positive, a probability measure P^{N_i} is a N_i-martingale measure for the market if all the relative prices $\frac{N_j}{N_i}$, $j = 0, 1, ..., n$, are martingales w.r.t. P^{N_i}. It turns out that this property also holds for all the value process V of admissable replicable securities (see [**?**] sections 2 and 3)

If $N_i = e^{\int_0^t r(u)du}$, where r is the instantaneous interest rate rolling over in the money market, then the N_i-martingale measure is called risk neutral probability.

Problem: given P^{N_i}, find the Radon-Nykodim derivative $\frac{dP^{N_j}}{dP^{N_i}}$ carrying to P^{N_j}.
Solution: Abstract Bayes formula (see Appendix) gives P^{N_j} conditional expectation in terms of the ones w.r.t. P^{N_i} :

$$E^{P^{N_j}}\left(\frac{V(T)}{N_j(T)}\bigg|\,\mathcal{F}_t\right)=\frac{E^{P^{N_i}}\left(\left(\frac{dP^{N_j}}{dP^{N_i}}\right)_{\mathcal{F}_T}\frac{V(T)}{N_j(T)}\bigg|\,\mathcal{F}_t\right)}{E^{P^{N_i}}\left(\left(\frac{dP^{N_j}}{dP^{N_i}}\right)_{\mathcal{F}_T}\bigg|\,\mathcal{F}_t\right)} \tag{6.13}$$

where the subscript $\mathcal{F}_s$, for $0\leq s\leq T$, selects the restriction of $\frac{dP^{N_j}}{dP^{N_i}}(\omega)$ to the σ-algebra $\mathcal{F}_s$. In order to exploit the martingale property for the N_i-discounted prices under the measure P^{N_i}, it seems reasonable to try with:

$$\left(\frac{dP^{N_j}}{dP^{N_i}}\right)_{\mathcal{F}_t}:=k\frac{N_j(t)}{N_i(t)} \tag{6.14}$$

so that the left hand side of (6.13) becomes equal to:

$$\frac{E^{P^{N_i}}\left(k\frac{V(T)}{N_i(T)}\mid\mathcal{F}_t\right)}{k\frac{N_j(t)}{N_i(t)}} \tag{6.15}$$

and then:

$$E^{P^{N_j}}\left(\frac{V(T)}{N_j(T)}\mid\mathcal{F}_t\right)=\frac{V(t)}{N_j(t)} \tag{6.16}$$

If we prove that the measure P^{N_j} is indeed a probability measure, then (6.16) implies that (6.14) defines the required measure.

$$\begin{aligned}
1\overset{imposed}{=}\int_\Omega dP^{N_j} &= \int_\Omega\left(\frac{dP^{N_j}}{dP^{N_i}}\right)_{\mathcal{F}_T}dP^{N_i}=\\
&= \int_\Omega k\frac{N_j(T)}{N_i(T)}dP^{N_i}=\\
&= kE^{P^{N_i}}\left(\frac{N_j(t)}{N_i(t)}\right)=k\frac{N_j(0)}{N_i(0)}
\end{aligned}$$

imposes a value for k :

$$k=\frac{N_i(0)}{N_j(0)}$$

The new martingale measure is assigned by the derivative:

$$\left(\frac{dP^{N_j}}{dP^{N_i}}\right)_{\mathcal{F}_t} := \frac{N_i(0)}{N_j(0)}\frac{N_j(t)}{N_i(t)} \tag{6.17}$$

Since any replicable security V can be added to the market model without altering its properties of completeness and absence of arbitrage, if V is strictly positive it can take the place of N_j in (6.17), giving rise to the V-martingale measure. In the following example, the time-T zero coupon bond martingale measure will be considered.

Example 6.2 (The forward risk adjusted probability measure). *Definition* (6.17) *sets up the process R_t described in section (3.2). If the dynamics of processes N_i and N_j are available, it is possible to compute the differential equation for R_t by making use of product rule for stochastic differentials. As an example, let N_i be the money market account, that is:*

$$N_i(t) := M(t) = e^{\int_0^t r(u)du} \tag{6.18}$$

and N_j represent the time-T maturing zero coupon bond price, namely:

$$N_j(t) := P^T(t). \tag{6.19}$$

Then we define the T-forward risk adjusted probability measure P^{P^T} on $\mathcal{F}_T$ by its Radon-Nykodim derivative with respect to the risk neutral probability P^* according to formula (6.17)*:*

$$\left(\frac{dP^{P^T}}{dP^*}\right)_{\mathcal{F}_T} := \frac{1}{P^T(0)}\frac{P^T(T)}{e^{\int_0^T r(u)du}} = \frac{e^{-\int_0^T r(u)du}}{P^T(0)} \tag{6.20}$$

Under P^{P^T} all the relative price processes $\frac{V(t)}{P^T(t)}$ are martingales.
Therefore:

$$\frac{V(t)}{P^T(t)} = E^{P^{P^T}_{t,r}}\left(\frac{V(T)}{P^T(T)}\right)$$

If the final payoff $V(T)$ is assigned by a deterministic function h of the underlying state variable r_t, one gets to:

$$V(t) = P^T(t)\,E^{P^{P^T}_{t,r}}\left(h(r_T)\right)$$

in contrast to the more awkward risk neutral valuation formula:

$$V(t) = E^{P^*_{t,r}}\left(e^{-\int_0^t r(u)du}h(r_T)\right)$$

Let $M(t)$ *denote the time* t *value of the money market account and the new numeraire have* P*-dynamics assigned by:*

$$dP^T(t) = P^T(t)\left(m^T(t)\,dt + \sigma^T(t)\,dW_t\right);$$

since under P^M *the relative price* $\frac{P^T(t)}{M(t)}$ *is a martingale, we have:*

$$d\left(\frac{P^T(t)}{M(t)}\right) = \sigma^T(t)\,\frac{P^T(t)}{M(t)}dW_t \tag{6.21}$$

which, together with (6.20), *leads to a differential expression for* R_t*:*

$$dR_t = \sigma^T(t)\,R_t dW_t \tag{6.22}$$

whose unique solution is:

$$R_t = e^{\int_0^t \sigma^T(u)dW_u - \frac{1}{2}\int_0^t \left(\sigma^T\right)^2(u)du} \tag{6.23}$$

which recovers (**??**) *with* $c' := \sigma^T$.

7. Extension to markets driven by marked point processes

7.1. General results

Results presented for general change of numeraire are directly applicable to the present context by simply performing computations of the case. We recall that changing the measure from $\mathbb{P}_{N_1}$, under which discounted prices $\frac{P_T}{N_1}$ are all martingales, to $\mathbb{P}_{N_2}$, under which $\frac{P_T}{N_2}$ are martingales for all maturities T, is possible by setting:

$$\left.\frac{d\mathbb{P}_{N_1}}{d\mathbb{P}_{N_2}}\right|_{\mathcal{F}_t} := \frac{N_1(t_0)}{N_2(t_0)}\frac{N_2(t)}{N_1(t)}$$

Let X be any Ito process in the canonical form under the original measure $\mathbb{P}_{N_1}$, say:

$$\begin{aligned} X(t) &= \mathbf{x} + \int_{t_0}^t \boldsymbol{\kappa}(s)\,ds + \int_{t_0}^t \int_E \mathbf{f}(s,x)\,\nu^{\mathbb{P}_{N_1}}(ds,dx) \\ &\quad + \int_{t_0}^t \boldsymbol{\Sigma}(s)\,d\mathbf{W}^{\mathbb{P}_{N_1}}(s) + \int_{t_0}^t \int_E \mathbf{f}(s,x)\,\overline{\mu}^{\mathbb{P}_{N_1}}(ds,dx), \end{aligned} \tag{7.1}$$

where coefficients $\boldsymbol{\kappa}$, $\boldsymbol{\Sigma}$ and $\mathbf{f}$ are suitable adapted stochastic process. In view of the preceding results on change of equivalent measure for marked point diffusion, if one can represent this process as a discontinuous Doléan exponential martingale $\mathcal{E}'(\mathbf{c}, d-1)$, that is a process Z satisfying:

$$dZ(t) = Z(t)\left[c(s)\, d\mathbf{W}^{\mathbb{P}_{N_1}}(t) + \int_E [d(t,x) - 1]\, \overline{\mu}^{\mathbb{P}_{N_1}}(dt, dx)\right],$$

then one automatically has the following change of the finite variation term in the stochastic differential of X:

$$\boxed{\begin{array}{c} \boldsymbol{\kappa}(s)\, ds + \int_E \mathbf{f}(s,x)\, \nu^{\mathbb{P}_{N_1}}(ds, dx) \\ \downarrow \\ \boldsymbol{\kappa}(s)\, ds + \mathbf{c}(s)\, \boldsymbol{\Sigma}(s) + \int_E \mathbf{f}(s,x)\, d(s,x)\, \nu^{\mathbb{P}_{N_1}}(s, dx) \end{array}} \tag{7.2}$$

that is drift changes from $\boldsymbol{\kappa}$ to $\boldsymbol{\kappa} + \mathbf{c}^{\mathsf{T}}\Sigma$ and the compensator from $\nu^{\mathbb{P}_{N_1}}$ to d times $\nu^{\mathbb{P}_{N_1}}$. The resulting dynamics of X under the new measure $\mathbb{P}_{N_2}$ is:

$$\begin{aligned} X(t) &= \mathbf{x} + \int_{t_0}^{t} [\boldsymbol{\kappa}(s) + \mathbf{c}(s)\, \boldsymbol{\Sigma}(s)]\, ds + \int_E \mathbf{f}(s,x)\, d(s,x)\, \nu^{\mathbb{P}_{N_1}}(s, dx) \\ &\quad + \int_{t_0}^{t} \boldsymbol{\Sigma}(s)\, d\mathbf{W}^{\mathbb{P}_{N_2}}(s) + \int_{t_0}^{t}\int_E \mathbf{f}(s,x)\, \overline{\mu}^{\mathbb{P}_{N_2}}(ds, dx) \end{aligned} \tag{7.3}$$

7.2. Forward-risk-adjusted measure

We recall that changing the measure from the risk neutral $\mathbb{P}^*$, under which discounted prices $\frac{P_S}{B}$ are all martingales, to the forward risk adjusted $\mathbb{P}_T$, under which $\frac{P_S}{P_T}$ are martingales for all maturities S, is possible by setting:

$$\left.\frac{d\mathbb{P}_T}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} := \frac{1}{P_T(t_0)} \frac{P_T(t)}{B(t)} =: Z(t)$$

Notation: superscript * denotes the correspondent process under $\mathbb{P}^*$, while superscript ^{T} stands for "under $\mathbb{P}_T$".

Under $\mathbb{P}^*$ we know bond dynamics are:

$$\frac{dP_T(t)}{P_T(t-)} = r(t)\, dt + \mathbf{v}_T(t) \cdot d\mathbf{W}^*(t) + \int_E n_T(t,x)\, \overline{\mu}^*(dt, dx) \tag{7.4}$$

where all of the measure dependencies have been specified for clarity. Also $dB(t) = B(t)\, r(t)\, dt$. Either by direct computation using the product formula rule or by use of $\mathcal{E}[X] / \mathcal{E}[Y] = \mathcal{E}[X - Y] - [X, Y]$ one arrives at:

$$\frac{dZ(t)}{Z(t)} = \mathbf{v}_T(t) \cdot d\mathbf{W}^*(t) + \int_E n_T(t, x)\, \overline{\mu}^*(dt, dx)$$

which matches (**??**) for:

$$\boxed{\mathbf{c} := \mathbf{v}_T} \tag{7.5}$$

as in the continuous diffusions case, and:

$$\boxed{d = n_T + 1} \tag{7.6}$$

With these coefficients, one can compute drift and compensator changes of any process of form (7.1) by use of (7.3). The marked point diffusion for a process X under $\mathbb{P}^T$ is:

$$\begin{aligned} X(t) &= \mathbf{x} + \int_{t_0}^{t} \left[\boldsymbol{\kappa}^*(s) + \mathbf{v}_T(s)^{\mathsf{T}}\, \boldsymbol{\Sigma}(s) \right] ds + \int_{t_0}^{t} \int_E \mathbf{f}(s, x)\, (n_T(s, x) + 1)\, \nu^*(s, dx) \\ &\quad + \int_{t_0}^{t} \boldsymbol{\Sigma}(s) \cdot d\mathbf{W}^T(s) + \int_{t_0}^{t} \int_E \mathbf{f}(s, x)\, \overline{\mu}^T(ds, dx) \end{aligned} \tag{7.7}$$

Example 7.1 (Forward-risk-adjustment in a whole-yield-curve model). *In a whole-yield-curve model in terms of instantaneous forward rates one has* $n_T(t, x) = \exp\left\{ -\int_t^T \xi_S(t, x)\, dS \right\} - 1$, *where* ξ_T *is the marked point coefficient of the forward rate process under the risk neutral measure* $\mathbb{P}^*$. *Thus* $d = \exp\left\{ -\int_t^T \xi_S(t, x)\, dS \right\}$ *and the finite variation term of the forward rate dynamics changes of an amount:*

$$\mathbf{v}_T(s)\, \boldsymbol{\Sigma}(s) + \int_E \mathbf{f}(s, x)\, e^{-\int_t^T \xi_S(t,x) dS} \nu^*(s, dx)$$

for a marked point diffusion factor process with instantaneous matrix volatility $\boldsymbol{\Sigma}$ *and marked point coefficient* $\mathbf{f}$.

8. Application to multifactor affine models driven by marked point processes

In a general multifactor bond market model one is given a factor dynamics under $\mathbb{P}^*$ of form:

$$X(t) = \mathbf{x} + \int_{t_0}^{t} \boldsymbol{\kappa}^*(s)\, ds + \int_{t_0}^{t} \boldsymbol{\Sigma}(s)\, d\mathbf{W}^*(s) + \int_{t_0}^{t}\int_{E} \mathbf{f}(s,x)\, \mu^*(ds,dx)$$

and a bond price process $P_T(t,\omega) := P_T(t, X(t,\omega))$, with $P_T \in \mathcal{C}^{1,2}[\mathbb{R}_+ \times \mathbb{R}_n]$. Alternatively, any of the quantities equivalent to a full specification of that term-structure of interest-rate dynamics can be imposed to be of this form, that is a deterministic function of time and factor's value. In the discount function formulation, we first have to find the bond dynamics by Ito formula (**??**) in the form $d_t P_T = P_T \times Ito\ differential$ and then substitute the obtained volatility function and marked point coefficient into (7.7).

$$\begin{aligned} \frac{d_t P_T(t, X(t))}{P_T(t-, X(t-))} &= \left(\frac{\partial_t P_T + \nabla_x P_T \cdot \boldsymbol{\kappa}^* + \frac{1}{2}\mathrm{Tr}\left[\mathrm{He} P_T \cdot \boldsymbol{\Sigma}^\top \cdot \boldsymbol{\Sigma}\right]}{P_T(t-, X(t-))} \right) dt \\ &\quad + \nabla_x \lg P_T(t-, X(t-)) \cdot \boldsymbol{\Sigma} dW^*(t) \\ &\quad + \int_E \left(\frac{P_T(t, X(t-) + \mathbf{f}(t,x))}{P_T(t, X(t-))} - 1 \right) \mu^*(dt, dx) \end{aligned}$$

There is no need to compute the above drift: since dynamics is under $\mathbb{P}^*$, then it must be equal to $r(t)$. Comparing this expression with (7.4) we see that instantaneous volatility and marked point coefficients under $\mathbb{P}^*$ are given by:

$$\begin{aligned} \mathbf{v}_T &= \nabla_x \lg P_T(t-, X(t-)) \cdot \boldsymbol{\Sigma} \\ n_T &= \frac{P_T(t, X(t-) + \mathbf{f}(t,x))}{P_T(t, X(t-))} \end{aligned}$$

and the change of the finite variation term of the dynamics for X is:

$$\begin{gathered} \boldsymbol{\kappa}^*(s)\, ds + \textstyle\int_E \mathbf{f}(s,x)\, \nu^*(ds,dx) \\ \downarrow \\ \boldsymbol{\kappa}^*(s)\, ds + \nabla_x \lg P_T(t-, X(t-))^\top \boldsymbol{\Sigma}(s)^\top \boldsymbol{\Sigma}(s)\, ds \\ \qquad + \textstyle\int_E \mathbf{f}(s,x)\, \frac{P_T(t, X(t-) + \mathbf{f}(t,x))}{P_T(t, X(t-))} \nu^*(ds,dx), \end{gathered}$$

that is:

$$\boxed{\begin{array}{l} \boldsymbol{\kappa}^{T}(s)=\boldsymbol{\kappa}^{*}(s)+\nabla_{x}\lg P_{T}(t-,X(t-))^{\mathsf{T}}\boldsymbol{\Sigma}(s)^{\mathsf{T}}\boldsymbol{\Sigma}(s) \\ \nu^{T}(ds,dx)=\frac{P_{T}(t,X(t-)+\mathbf{f}(t,x))}{P_{T}(t,X(t-))}\nu^{*}(ds,dx) \end{array}} \tag{8.1}$$

Let X be a multidimensional marked point process representing a factor process inducing an exponential affine bond market. This may be accomplished by the following set up:

$$\begin{aligned} r(t,\omega) &:= R(t,X(t)), \\ dX(t) &= \boldsymbol{\kappa}^{*}(t,X(t))\,dt+\boldsymbol{\Sigma}(t,X(t))\cdot d\mathbf{W}(t)+\int_{E}\mathbf{f}(t,X(t),y)\,\mu(dt,dy;\omega), \end{aligned}$$

under $\mathbb{P}^{*}$, with:

$$\begin{aligned} \boldsymbol{\kappa}^{*}(t,\mathbf{x}) &= \mathbf{k}_{0}^{*}(t)+\mathbf{K}_{1}^{*}(t)\,\mathbf{x} \\ \boldsymbol{\Sigma}(t,\mathbf{x})\,\boldsymbol{\Sigma}(t,\mathbf{x})^{\mathsf{T}} &= \mathbf{H}_{0}(t)+\mathbb{H}_{1}(t)\,\mathbf{x} \\ \mathbf{f}(t,\mathbf{x},y) &= \mathbf{f}(t,y) \\ \nu^{*}(t,\mathbf{x},dy)\,dt &= l_{0}^{*}(t,dy)+\mathbf{l}_{1}^{*}(t,dy)\cdot\mathbf{x} \\ R(t,\mathbf{x}) &= \rho_{0}(t)+\boldsymbol{\rho}_{1}(t)\cdot\mathbf{x} \end{aligned}$$

where ν is the compensator of μ under $\mathbb{P}^{*}$ and r is the short rate process.

Example 8.1. *The bond price is given by:*

$$P_{T}(t)=\exp\left(\alpha(t,T)-\boldsymbol{\beta}(t,T)\cdot X(t)\right)$$

for suitable processes α and $\boldsymbol{\beta}$.

Formulae (8.1) give the forward-risk-adjusted coefficients of the dynamics of the factor process.

$$\begin{aligned} \boldsymbol{\kappa}^{T}(s,X(s)) &= \boldsymbol{\kappa}^{*}(s,X(s))+\nabla_{x}\lg P_{T}(t-,X(t-))^{\mathsf{T}}\boldsymbol{\Sigma}(s)^{\mathsf{T}}\boldsymbol{\Sigma}(s) \\ &= \mathbf{k}_{0}^{*}(t)+\mathbf{K}_{1}^{*}(t)\,X(t)+\boldsymbol{\beta}(t,T)^{\mathsf{T}}\left(\mathbf{H}_{0}(t)+\mathbb{H}_{1}(t)\,X(t)\right) \\ &= \left[\mathbf{k}_{0}^{*}(t)+\boldsymbol{\beta}(t,T)^{\mathsf{T}}\mathbf{H}_{0}(t)\right]+\left[\mathbf{K}_{1}^{*}(t)+\boldsymbol{\beta}(t,T)^{\mathsf{T}}\mathbb{H}_{1}(t)\right]X(t) \end{aligned}$$

$$\begin{aligned} \nu^{T}(ds,dx) &= \frac{P_{T}(t,X(t-)+\mathbf{f}(t,x))}{P_{T}(t,X(t-))}\nu^{*}(ds,dx) \\ &= \exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right)\nu^{*}(ds,dx) \\ &= \exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right)\left(l_{0}^{*}(t,dy)+\mathbf{l}_{1}^{*}(t,dy)\cdot X(t)\right) \\ &= \left[\exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right)l_{0}^{*}(t,dy)\right] \\ &\quad+\left[\exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right)\mathbf{l}_{1}^{*}(t,dy)\right]\cdot X(t) \end{aligned}$$

Under $\mathbb{P}_T$, an affine model is defined by:

$$
\begin{aligned}
r(t,\omega) &: = R(t, X(t)),\\
dX(t) &= \boldsymbol{\kappa}^T(t, X(t))\,dt + \boldsymbol{\Sigma}(t, X(t)) \cdot d\mathbf{W}(t) + \int_E \mathbf{f}(t, X(t), y)\,\mu(dt, dy; \omega),
\end{aligned}
$$

with:

$$
\begin{array}{lcl}
\boldsymbol{\kappa}^T(t,\mathbf{x}) & = & \mathbf{k}_0^T(t) + \mathbf{K}_1^T(t)\,\mathbf{x} \\
 & & \textit{with } \mathbf{k}_0^T(t) := \left[\mathbf{k}_0^*(t) + \boldsymbol{\beta}(t,T)^\intercal\,\mathbf{H}_0(t)\right] \\
 & & \textit{and } \mathbf{K}_1^T(t) := \left[\mathbf{K}_1^*(t) + \boldsymbol{\beta}(t,T)^\intercal\,\mathbb{H}_1(t)\right] \\
\boldsymbol{\Sigma}(t,\mathbf{x})\,\boldsymbol{\Sigma}(t,\mathbf{x})^\intercal & = & \mathbf{H}_0(t) + \mathbb{H}_1(t)\,\mathbf{x} \\
\mathbf{f}(t,\mathbf{x},y) & = & \mathbf{f}(t,y) \\
\nu^*(t,\mathbf{x},dy)\,dt & = & l_0^T(t,dy) + \mathbf{l}_1^T(t,dy)\cdot\mathbf{x} \\
 & & \textit{with } l_0^T(t,dy) := \exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right) l_0^*(t,dy) \\
 & & \textit{and } \mathbf{l}_1^T(t,dy) := \exp\left(-\boldsymbol{\beta}(t,T)\cdot\mathbf{f}(t,x)\right) \mathbf{l}_1^*(t,dy) \\
R(t,\mathbf{x}) & = & \rho_0(t) + \boldsymbol{\rho}_1(t)\cdot\mathbf{x}
\end{array}
$$

where ν is the compensator of μ under $\mathbb{P}^*$ and r is the short rate process.

9. A guide to affine models

This final section is intended as a guide to the rapidly growing literature on affine arbitrage pricing models. The problem of characterization of affine term structures is natural once it is realized that two mostly used short rate models, namely the Gaussian and square-root models, share the common nice feature of delivering explicit bond price processes which are affine in the underlying state variable. Indeed all the studies of those models led to the system of ode-s identifying the coefficient of the bond price. The first studies in this direction are those of Vasicek [20] and, respectively, Cox, Ingersoll and Ross [6]. Beaglehole and Tenney [2] investigated the direct problem of finding sufficient conditions in several dynamical instances for both affine and quadratic structures. El Karoui, Myneni and Viswanatan [12] investigated the direct problem for both affine and quadratic structures in the case of general Gaussian multifactor models. They also carried out a complete study on the pricing of interest-rate derivatives, accompanied by a rigorous treatment of convexity adjustments for futures forward price relation. El Karoui and Lacoste [11] embodied these results into a whole-yield curve model where factors are identified with either instantaneous forward rates or successive partial derivatives of one single rate. Frachot and Lesne [14] and Duffie and Kan [8][9] independently studied the case of affine structures driven by more general diffusions, where volatility is allowed to be dependent on the factor process. The former authors gave a detailed econometric analysis of the model for estimation in actual markets (see also Frachot, Lesne and Renault [15]). The latter studied the inverse problem of finding necessary conditions for affinity of rates; they also provided for explicit examples where numerical method are employed in order to obtain solution to bond option. Instances of affine structures where factor are identified with interest-rate mean and volatility treated as parameters have been pursued by Balduzzi, Das, Foresi and Sundaram [1], Beaglehole and Tenney [2], Chen [5] among others. Extension to one-factor jump-diffusion is sketched by Duffie and Kan [9], to one-factor marked point processes by Björk, Kabanov and Runggaldier [3], to multifactor jump-diffusion processes by Duffie, Pan and Singleton [10]. Our detailed derivation in a multidimensional marked point diffusion setting is new and embraces all of the other cases, except for a few pathological instances that cannot be dealt with by Ito calculus and require a study based upon methods from the general theory of Markov processes. This point has been recently illustrated in the nice technical study by Filipovič [13]. The method of transform has been developed by Duffie, Pan and Singleton [10] and applied to various pay-off functions by Chacko and Das [4]. Our development unifies all of the possible cases treated in literature..

References

[1] Balduzzi, Pierluigi, Sanjiv Das, Silverio Foresi and Rangarajan K. Sundaram (1996): "A Simple Approach to Three-Factor Affine Term Structure Models", J. Fixed Income, 6, 43-53

[2] Beaglehole, D. and Michael Tenney (1991): " General Solutions of Some Interest Rate Contingent Claim Pricing Equations", J. Fixed Income, 1, 69-83

[3] Björk, Tomas, Yuri Kabanov and Wolfgang Runggaldier (1997): "Bond Market Structure in the Presence of Marked Point Processes", Math. Fin., 7(2), 211-239

[4] Chacko, George and Sanjiv Das (1999): "Pricing Interest Rate Derivatives: A General Approach", Working Paper, Harvard Business School, Boston

[5] Chen, Liu (1995): Interest Rate Derivatives, Lecture Notes in Economics, Springer-Verlag

[6] Cox, John C., John E. Ingersoll and Stephen Ross (1985): "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408

[7] Dai, Qiang and Kenneth Singleton (1999): "Specification Analysis of Affine Term Structure Models", Working Paper, New York University and Stanford University

[8] Duffie, Darrell and Rui Kan (1993): "A Yield-Factor Model of Interest Rates", Working Paper, Graduate School of Business, Stanford University, Stanford

[9] Duffie, Darrell and Rui Kan (1996): "A Yield-Factor Model of Interest Rates", Math. Fin., 6(4), 379-406

[10] Duffie, Darrell, Jun Pan and Kenneth Singleton (1999): "Transform Analysis and Asset Pricing for Affine Jump-Diffusions", Graduate School of Business, Stanford University, Stanford

[11] El Karoui, Nicole and Vincent Lacoste (1992): "Multifactor Models of the term Structure of Interest Rates", AFFI Conference Proceedings, Paris

[12] El Karoui, Nicole, Ravi Myneni and Ravi Viswanatan (1990)(1992): "Arbitrage Pricing and Hedging of Interest Rate Claims with State Variables I: Theory", AFFI Congrès de Louvain 1991 and AFFI Congrès de Paris 1992 Proceedings and Working Paper 1990-1992, Laboratoire de Probabilités, Université de Paris VI

[13] Filipovič, Damir (1999): "A General Characterization of Affine Term Structure Models", Working Paper, ETH, Zürich

[14] Frachot, Antoine and Jean Paul Lesne (1993): "Econometrics of Linear factor Models of Interest Rates", Working Paper, Banque de France

[15] Frachot, Antoine, Jean Paul Lesne and Eric Renault (1995): "Indirect Inference Estimation of Factor Models of the Term Structure of Interest Rates", Working Paper, Banque de France, Unversité de Cergy and GREMAQ

[16] Foresi, Silverio and Régis J. Van Steenkiste (1999): "Arrow-Debreu Prices for Affine Models", Working Paper, Goldman Sachs Asset Management and Salomon Smith Barney, Inc., New York

[17] Geman, Hélyette (1989): The Importance of the Forward Neutral Probability in a Stochastic Approach of Interest Rates, Working Paper, ESSEC.

[18] Geman, Hélyette, Nicole El Karoui, and Jean Charles Rochet (1995): Changes of Numéraire, Changes of Probability Measures and Option Pricing, Journal of Applied Probability **32**, 443-458

[19] Jamshidian, Farshid (1987): Pricing of Contingent Claims in the One-Factor Term Structure Model, Merril Lynch, Working Paper.

[20] Vasiček, Oldrich A. (1977): " An Equilibrium Characterization of the Term Structure", J. Fin. Econ., 5, 177-188

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