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Adriana Castaldo and Massimo Marinacci

RANDOM CORRESPONDENCES AS BUNDLES
OF RANDOM VARIABLES

Working Paper no. 12/2001
April 2001

APPLIED MATHEMATICS
WORKING PAPER SERIES

Random Correspondences as Bundles of Random Variables

Adriana Castaldo Massimo Marinacci*

April 2001

Abstract

We prove results that relate random correspondences with their measurable selections, thus providing a foundation for viewing random correspondences as "bundles" of random variables.

1 Introduction

Since the seminal works of Dempster (1967, 1968), Kendall (1974), and Matheron (1975), random correspondences have been widely used as a generalization of standard random variables. Given two measurable spaces (S, Σ) and $(X, \mathcal{B})$, while random variables associate to elements of S single elements of X, random correspondences relax this assumption by associating nonempty subsets of X to elements of S. This added flexibility turned out to be useful in several areas and we refer the interested reader to the original works, as well as to the recent surveys in Stoyan, Kendall, and Mecke (1995) and Barndorff-Nielsen, Kendall, and van Lieshout (1999). For example, building on Dempster's works, random correspondences have been recently used in Bayesian decision theory to model unforeseen contingencies by Mukerji (1997) and Ghirardato (2000).

*Castaldo is at Dipartimento di Matematica e Statistica, Università di Napoli, Italy, acastald@unina.it; Marinacci is at Dipartimento di Statistica e Matematica Applicata, Università di Torino, Italy, and ICER; massimo@econ.unito.it. We wish to thank Fabio Maccheroni for valuable discussions. The financial support of MURST is gratefully acknowledged.

Given a probability $P : \Sigma \to [0,1]$, a suitably measurable random correspondence $F : S \to 2^X$ induces a lower distribution ν and upper distribution $\overline{\nu}$ on X as follows:

$$\begin{aligned} \nu(A) &= P(\{s : F(s) \subseteq A\}), \\ \overline{\nu}(A) &= P(\{s : F(s) \cap A \neq \emptyset\}) \end{aligned}$$

for all subsets $A \in \mathcal{B}$. In the special case of a random variable $f : S \to X$, we have $\nu(A) = \overline{\nu}(A)$ for all $A \in \mathcal{B}$ and ν reduces to the standard probability distribution P_f induced by f.

The distributions ν and $\overline{\nu}$ therefore generalize the usual probability distributions induced by random variables. The purpose of our work is to study these distributions and in particular their relationships with the standard probability distributions induced by the measurable selections of the random correspondence F.

Specifically, let $S(F)$ be the set of all measurable selections of the random correspondence F, that is, the set of all random variables $h : S \to X$ such that $h(s) \in F(s)$ for all $s \in S$. Each selection $h \in S(F)$ induces a probability distribution P_h on X defined by $P_h(A) = P(\{s : h(s) \in A\})$ for all $A \in \mathcal{B}$. Our purpose is to relate the distributions ν and $\overline{\nu}$ with the set $\{P_h : h \in S(F)\}$ of the standard probability distributions induced by the selections of F. In this we follow Aumann (1965)'s lead, who showed that a fruitful way to look at correspondences is as "bundles" of their selections, a standpoint that makes it possible to relate correspondences with the more familiar single-valued functions. In a probabilistic setting, we adopt a similar view and we provide a connection between the generalized distributions ν and $\overline{\nu}$ and the standard probability distributions $\{P_h : h \in S(F)\}$ that are naturally associated with a random correspondence F.

We have two main results. Consider a real-valued function $u : X \to \mathbb{R}$ defined on the space X, that in applications will be in general the space of interest – e.g., a space of consequences. Since ν and $\overline{\nu}$ are non-additive set functions, we have to consider their Choquet integrals $\int u d\nu$ and $\int u d\overline{\nu}$, defined in the next section. Our first result, Theorem 1, shows that $\int u d\nu$ and $\int u d\overline{\nu}$ are, respectively, the lower and upper envelopes of the sets of the standard integrals $\left\{\int u dP_h : h \in S(F)\right\}$. That is, we prove that

$$\int u d\nu = \inf \left\{ \int u dP_h : h \in S(F) \right\},$$

$$\int u d\overline{\nu} \;=\; \sup \left\{ \int u dP_h : h \in S(F) \right\},$$

provided X is Polish and F compact-valued, conditions often satisfied in applications.

Our second main result, Corollary 1, considers $core(\nu)$, the set of all countably additive probability measures that setwise dominate ν. This set is often associated with the distribution ν (see, e.g., Huber and Strassen, 1973), and Corollary 1 shows that it is nothing but the weak*-closed convex hull of the set $\{P_h : h \in S(F)\}$ of induced probability distributions. That is,

$$core(\nu) = \overline{co}^{w^*}(\{P_h : h \in S(F)\}).$$

These two results (as well as Corollary 2) show that there exists a tight connection between the generalized distributions ν and $\overline{\nu}$ and the standard probability distributions $\{P_h : h \in S(F)\}$ that are naturally associated with them. In this way, we can relate these generalized notions with more familiar standard notions and offer a novel perspective on random correspondences as "bundles" of random variables.

We close by mentioning that in our derivation we obtain two results of some independent interest: Theorem 2 provides a change of variable formula for the Aumann integral, while Lemma 7 generalizes the classic Lusin Theorem to Choquet capacities.

The paper is organized as follows. Section 2 contains notation and some preliminary results. Section 3, which is the heart of the paper, states our main results. Section 4 contains two subsections. Subsection 4.1 contains change of variable formula for the Aumann integral, while subsection 4.2 extends our results to random sets. Finally, section 5 contains the proofs and related material.

2 Termini technici and preliminary results

Let Σ be an event σ-algebra of a state space S, and X a metric space with Borel σ-algebra $\mathcal{B}$. As usual in Probability Theory, we will often assume that X is a Polish space, i.e., a separable and complete metric space.

We denote by $ca(\mathcal{B})$ the set of all countably additive measures on $\mathcal{B}$ that are bounded with respect to the variation norm. Probabilities are the positive and normalized elements of $ca(\mathcal{B})$ that take on value 1 on X. On

$ca(\mathcal{B})$ we use the weak*-topology induced by $C_b(X)$, the set of all continuous and bounded functions $f : X \to \mathbb{R}$. In particular, a net of probabilities $\{p_\alpha\}_\alpha \subseteq ca(\mathcal{B})$ weak*-converges to a $p \in ca(\mathcal{B})$ if $\lim_\alpha \int f dp_\alpha = \int f dp$ for all $f \in C_b(X)$.

A capacity $\nu : \mathcal{B} \to [0,1]$ is a set function such that:

1. $\nu(\emptyset) = 0$ and $\nu(X) = 1$;

2. $\nu(A) \leq \nu(B)$ for all $A, B \in \mathcal{B}$ such that $A \subseteq B$.

The capacity ν is *convex* if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all $A, B \in \mathcal{B}$. We denote by $core(\nu)$ the core of a capacity ν, i.e., the set $\{p \in ca(\mathcal{B}) : p(X) = 1 \text{ and } p(A) \geq \nu(A) \text{ for all } A \in \mathcal{B}\}$ of all probabilities that setwise dominate the capacity. If ν is convex, then $core(\nu)$ is nonempty.

The notion of integral associated with capacities is the Choquet integral. Given a measurable real-valued function $u : X \to \mathbb{R}$, the Choquet integral $\int u d\nu$ is defined by

$$\int u d\nu = \int_0^{+\infty} \nu(u \geq t)\, dt + \int_{-\infty}^{0} [\nu(u \geq t) - 1]\, dt,$$

where the right hand side is a Riemann integral, which is well defined since $\nu(u \geq t)$ is a monotone function in t.

A correspondence $F : S \to 2^X$ associates nonempty subsets of X to states of S. The strong inverse F^{-1} of F is defined by $F^{-1}(A) = \{s : F(s) \subseteq A\}$ for all sets $A \subseteq X$, while the weak inverse F^w is defined by $F^w(A) = \{s : F(s) \cap A \neq \emptyset\}$. Since $F^w(A) = S - F^{-1}(A^c)$ for all $A \subseteq X$, in general it will be enough to focus on F^{-1}.

Using a standard notion of measurability, we now introduce random correspondences.

Definition 1 *A correspondence $F : S \to 2^X$ is $\mathcal{B}$-measurable if $F^{-1}(A) \in \Sigma$ for all Borel sets $A \in \mathcal{B}$. A correspondence $F : S \to 2^X$ which is $\mathcal{B}$-measurable is called a random correspondence.*

If the values of F are singletons, random correspondences reduce to standard random variables. Random correspondences are closely related to random sets, the only difference being in the notion of measurability used. The main reason why we use random correspondences is to have the distribution

ν defined on the entire σ-algebra $\mathcal{B}$. In any event, in section 4 we show that our main result holds for random sets as well.

Given a probability measure $P : \Sigma \to [0,1]$ on the state space, a random variable $f : S \to X$ induces a distribution $P_f : \mathcal{B} \to [0,1]$ defined by $P_f(A) = P(\{s : f(s) \in A\})$ for all $A \in \mathcal{B}$. In a similar way, random correspondences induce a lower distribution $\nu : \mathcal{B} \to [0,1]$ and an upper distribution $\overline{\nu} : \mathcal{B} \to [0,1]$ defined by:

$$\begin{aligned} \nu(A) &= P\left(F^{-1}(A)\right) = P(\{s : F(s) \subseteq A\}), \text{ and} \\ \overline{\nu}(A) &= P(F^{w}(A)) = P(s : F(s) \cap A \neq \emptyset) \end{aligned}$$

for all $A \in \mathcal{B}$. Since $\overline{\nu}(A) = 1 - \nu(A^c)$ for all $A \in \mathcal{B}$, there is a simple duality between the two distributions, and in the sequel we will mostly focus on ν.

Unlike the distributions P_f, the set function ν is in general non-additive. However, it is totally monotone, i.e.,

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right)$$

for all $\{A_i\}_{i\in\{1,\dots,n\}} \subseteq \mathcal{B}$ (see, e.g., Nguyen, 1978). Besides total monotonicity, the distributions ν have other important properties. Following Kuratowski (1966) we say that a monotone increasing sequence $\{A_n\}_{n\geq 1}$ of Borel sets is strictly monotone if $A_n \subseteq int(A_{n+1})$ for all $n \geq 1$ (e.g., all sets A_n are open).

Proposition 1 *Let $\nu : \mathcal{B} \to [0,1]$ be the distribution induced by a random correspondence. Then:*

- *(i)* $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcap_{n\geq 1} A_n\right)$ *for all non-increasing sequences of Borel sets.*

  If, in addition, the correspondence has compact values, then:

- *(ii)* $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n\geq 1} A_n\right)$ *for all non-decreasing strictly monotone sequences of Borel sets.*

Point (i) is easy to check (see, e.g., Nguyen, 1978). Point (ii) is in general false if the sequence is not strictly monotone, as the following example shows.

Example. Let $X = [0,1]$ and $K = [1/2, 1]$. Consider the multifunction $F : S \to 2^X$ defined by $F(s) = K$ for all $s \in S$. Then ν is $\{0,1\}$-valued and

$\nu(A) = 1$ if and only if $K \subseteq A$.[1] Set $A_n = [0, 1 - 1/n] \cup \{1\}$. The sequence $\{A_n\}_{n \geq 1}$ is not strictly monotone and so Proposition 1 does not apply. We have $A_n \uparrow X$, but $\lim_n \nu(A_n) \neq \nu(X)$. In fact, $\lim_n \nu(A_n) = 0$.

The next result shows some regularity properties of the distribution ν. Parts of this result are more or less known, though we did not find a reference for the result in this generality.[2] For this reason we provide a proof.

Proposition 2 *The distribution ν induced by a compact-valued random correspondence is regular, i.e.,*

$$\begin{aligned}\nu(A) &= \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\} \\ &= \inf\{\nu(G) : A \subseteq G \text{ and } G \text{ open}\}\end{aligned}$$

for all Borel sets A. If, in addition, X is Polish, then ν is tight, i.e., $\nu(A) = \sup\{\nu(K) : K \subseteq A \text{ and } K \text{ compact}\}$ for all Borel sets A.

3 Main results

In this section we characterize the random correspondence $F : S \to 2^X$ via the probability distributions induced by its measurable selections. The first result, which is our main result, shows that the Choquet integral relative to ν can be expressed in terms of the standard integrals associated with the probability distributions induced by the measurable selections of F.

Before stating the result we introduce a class of functions.

Definition 2 *A real-valued function $u : X \to \mathbb{R}$ is lower (upper, resp.) Weierstrass if it attains its infimum (supremum, resp.) on all compact sets of X.*

The class of lower Weierstrass functions is broad and it includes:

(i) all lower semicontinuous functions $u : X \to \mathbb{R}$;

(ii) all finite-valued functions $u : X \to \mathbb{R}$.

[1] In other words, ν is the unanimity game u_K.

[2] For example, the Polish space part is an immediate consequence of an unproved observation on p. 253 of Huber and Strassen (1973).

On the other hand, continuous functions and finite-valued functions are examples of functions that are both lower and upper Weierstrass.

We can now state our main result. Recall that $S(F)$ is the set of measurable selections of F.

Theorem 1 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$\int u d\nu = \inf_{h \in S(F)} \int u dP_h \tag{1}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (1) we have a min *instead of an* inf*.*

Remark. A dual version of Theorem 1 holds, where ν, inf, and lower Weierstrass are replaced respectively with $\overline{\nu}$, sup, and upper Weierstrass.

From a probabilistic standpoint, the set of induced probability distributions $\{P_h : h \in S(F)\}$ is a very important subset of $core(\nu)$ since it has a direct connection with the random correspondence F. It would be therefore desirable that $\{P_h : h \in S(F)\}$ were also a mathematically important subset of $core(\nu)$. In general, $\{P_h : h \in S(F)\}$ is not a convex set and so in general $core(\nu) \neq \{P_h : h \in S(F)\}$. For example, let $S = X = [0,1]$ and let $F(s) = \{0, s\}$ for all $s \in S$. It can be checked that $\{P_h : h \in S(F)\}$ is not convex.

However, the next result – based on Theorem 1 – shows that $\{P_h : h \in S(F)\}$ is still an important subset of $core(\nu)$. As a matter of fact, $core(\nu)$ is nothing but the weak*-closed convex hull $\overline{co}^{w^*}(\cdot)$ of $\{P_h : h \in S(F)\}$.

Corollary 1 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$core(\nu) = \overline{co}^{w^*}(\{P_h : h \in S(F)\}),$$

and $ext(core(\nu)) \subseteq \overline{\{P_h : h \in S(F)\}}^{w^*}$, i.e., all extreme points of $core(\nu)$ belong to the weak-closure of $\{P_h : h \in S(F)\}$.*

We close with a simple but useful consequence of Theorem 1, that further shows the importance of the set $\{P_h : h \in S(F)\}$ for ν.

Corollary 2 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then:*

(i) for each finite chain $\{A_i\}_{i=1}^n$ of Borel sets there exists $h \in S(F)$ such that $\nu(A_i) = P_h(A_i)$ for each $i = 1, ..., n$.

(ii) for each infinite chain $\{G_i\}_{i \in [0,1]}$ of open sets with

(a) $G_i \supseteq G_j$ if $i \geq j$ and $G_0 = \emptyset$,

(b) $\bigcup_{j<i} G_j = G_i$,

there exists $h \in S(F)$ such that $P_h(G_i) = \nu(G_i)$ for all $i \in [0,1]$.

(iii) for each infinite chain $\{C_i\}_{i \in [0,1]}$ of closed sets with

(a) $C_i \subseteq int(C_j)$ if $i \geq j$,

(b) $\bigcap_{j<i} C_j = C_i$,

there exists $h \in S(F)$ such that $P_h(C_i) = \nu(C_i)$ for all $i \in [0,1]$.

4 Additional results

4.1 A change of variable formula for the Aumann integral

Given a correspondence $G : S \to 2^{\mathbb{R}}$, let $\widetilde{S}(G)$ be the set of all P-a.e. measurable selections, i.e., $h \in \widetilde{S}(G)$ if it is measurable and P-a.e. $h(s) \in G(s)$. The Aumann integral $\int GdP$ is then defined as the set

$$\left\{ \int hdP : h \text{ integrable and } h \in \widetilde{S}(G) \right\}.$$

Consider the correspondence $u \circ F : S \to 2^{\mathbb{R}}$, the composition of the function $u : X \to \mathbb{R}$ with the correspondence $F : S \to 2^X$. For all $s \in S$, $(u \circ F)(s) = \{u(x) : x \in F(s)\}$.

Lemma 1 *Let $F : S \to 2^X$ be a compact-valued random correspondence. If X is a Polish space, then*

$$\int (u \circ F)\, dP = \left\{ \int u dP_h : h \in S(F) \right\} \tag{2}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$.

Along with Theorem 1, this lemma delivers a change of variable formula for the Aumann integral. Our result complements Theorem 5 of Hildenbrand (1974), that considers the composition of a correspondence with a function; in contrast, we consider the composition of function with a correspondence.

Theorem 2 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space, then*

$$\int u d\nu = \inf \int (u \circ F)\, dP \tag{3}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (3) we have a min *instead of an* inf*.*

Using Lemma 1 we can get some further information about the set of induced distributions $\{P_h : h \in S(F)\}$ by using some well-known properties of the Aumann integral. First of all, if u is continuous and bounded or finite-valued, the correspondence $u \circ F$ is closed-valued and integrably bounded provided F is compact-valued. Hence, $\int (u \circ F)\, dP$ is a compact subset of $\mathbb{R}$ (see, e.g., Proposition 7 p. 73 of Hildenbrand, 1974), and we conclude that the set $\left\{ \int u dP_h : h \in S(F) \right\}$ is a compact subset of $\mathbb{R}$ provided F is compact-valued and u is continuous and bounded or finite-valued.

Another interesting property of the Aumann integral is that it is convex when P is non-atomic. Along with Theorem 1 and Lemma 1, this immediately implies the following useful result.

Proposition 3 *Let ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is a Polish space and P is non-atomic, then*

$$\left(\int u d\nu, \int u d\overline{\nu} \right) \subseteq \left\{ \int u dP_h : h \in S(F) \right\} \subseteq \left[\int u d\nu, \int u d\overline{\nu} \right]$$

for all measurable functions $u : X \to \mathbb{R}$*. If, in addition,* u *is lower Weierstrass, then*

$$\left[\int ud\nu, \int ud\overline{\nu}\right) \subseteq \left\{\int udP_h : h \in S(F)\right\}$$

while, if u *is upper Weierstrass, then*

$$\left(\int ud\nu, \int ud\overline{\nu}\right] \subseteq \left\{\int udP_h : h \in S(F)\right\}.$$

4.2 The results for random sets

As mentioned before, the difference between random sets and random correspondences lies in the notion of measurability used.

Definition 3 *A correspondence* $F : S \to 2^X$ *is measurable if* $F^{-1}(G) \in \Sigma$ *for all open sets* $G \subseteq X$*. A correspondence* $F : S \to 2^X$ *which is measurable is called a random set.*

Remark. In σ-compact Hausdorff spaces, a closed-valued correspondence is a random set if and only if $\{s : F(s) \cap K \neq \emptyset\} \in \Sigma$ for all compact sets K (cf. Himmelberg (1975) Theorem 3.5). In particular, this is true in separable locally compact Hausdorff spaces.

Clearly, all random correspondences are random sets. Though the converse is in general false, the next result, due to Debreu (1967), provides an important case where it holds (cf. Himmelberg (1975) pp. 57-58). Σ_* denotes the completion of Σ under P, i.e., the collection of all sets of the form $A \cup N$, where $A \in \Sigma$ and N is P-null.

Theorem 3 *Let* $F : S \to 2^X$ *be a closed-valued random set and suppose* X *is a Polish space. Then* $F^{-1}(A) \in \Sigma_*$ *for all* $A \in \mathcal{B}$*, and so* F *is a random correspondence provided* $\Sigma = \Sigma_*$*, i.e., provided* (S, Σ, P) *is a complete measure space.*

Theorem 3 suggests a simple way to extend our results to random sets even when (S, Σ, P) is not a complete measure space. For, let $F : S \to 2^X$ be a closed-valued random set. Its distribution ν is defined only on open sets and we have $\nu(G) = P(F^{-1}(G))$ for all open sets $G \subseteq X$. However, let P_*

be the unique extension of P to Σ_* and define a set function $\nu_* : \mathcal{B} \to [0,1]$ as follows:

$$\nu_*(A) = P_*\left(F^{-1}(A)\right)$$

for all $A \in \mathcal{B}$. If X is Polish, the set function ν_* is well defined by Theorem 3 and it coincides with ν on the open sets. Actually, more is true since it is easy to see that $F^{-1}(C) \in \Sigma$ for all closed sets $C \subseteq X$ when F is a random set. Hence, $\nu_*(C) = \nu(C)$ also for all closed sets $C \subseteq X$.

We call ν_* the *extended distribution* of F. It satisfies the same properties as the standard distributions induced by random correspondences.

Proposition 4 *Suppose X is Polish, and let $\nu_* : \mathcal{B} \to [0,1]$ the extended distribution induced by a compact-valued random set. Then:*

(i) $\lim_{n\to\infty} \nu_*(A_n) = \nu_*\left(\bigcap_{n\geq 1} A_n\right)$ *for all non-increasing sequences of Borel sets.*

(ii) $\lim_{n\to\infty} \nu_*(A_n) = \nu_*\left(\bigcup_{n\geq 1} A_n\right)$ *for all non-decreasing strictly monotone sequences of Borel sets.*

(iii) ν_* *is regular and tight.*

We can now extend Theorem 1 to random sets.

Proposition 5 *Let X be a Polish space and let ν_* be the extended distribution induced by a compact-valued random set $F : S \to 2^X$. Then*

$$\int u d\nu_* = \inf_{h\in S(F)} \int u dP_h \tag{4}$$

for all bounded and measurable functions $u : X \to \mathbb{R}$. If, in addition, u is lower Weierstrass, then in (4) we have a min *instead of an* inf*.*

Remark. Since ν and ν_* coincide on all open sets and on all closed sets, $\int u d\nu_* = \int u d\nu$ for all upper semicontinuous and all lower semicontinuous functions $u : X \to \mathbb{R}$.

5 Proofs and related material

5.1 Proof of Proposition 1

We only prove (ii), point (i) being well-known. Let $A_n^* = \{s : F(s) \subseteq A_n\}$ and $A^* = \{s : F(s) \subseteq A\}$. Clearly, $\bigcup_{n\geq 1} A_n^* \subseteq A^*$. We want to show that $\bigcup_{n\geq 1} A_n^* \supseteq A^*$. Let $\overline{s} \in A^*$. By strict monotonicity, $\bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} int(A_n)$, and so $F(\overline{s}) \subseteq \bigcup_{n\geq 1} int(A_n)$. Since $F(\overline{s})$ is compact, there exists a finite index set $I = \{n_1, ..., n_k\}$ such that $n_1 < n_2 < \cdots < n_k$ and $F(\overline{s}) \subseteq \bigcup_{j=1}^{k} int\left(A_{n_j}\right)$. Hence, $F(\overline{s}) \subseteq \bigcup_{i=1}^{n_k} A_i = A_{n_k}$, and so $\overline{s} \in A_{n_k}^*$. This completes the proof that $\bigcup_{n\geq 1} A_n^* = A^*$. Then,

$$\nu(A) = P(A^*) = P\left(\bigcup_{n\geq 1} A_n^*\right) = \lim_n P(A_n^*) = \lim_n \nu(A_n),$$

as desired. ■

5.2 Proof of Proposition 2

We prove a more general statement, where we use the following definition.

Definition 4 *A Choquet capacity $\nu : 2^X \to [0,1]$ is a capacity satisfying the following two properties*

(i) $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcap_{n\geq 1} A_n\right)$ *for all non-increasing sequences $\{A_n\}_{n\geq 1} \subseteq X$.*

(ii) $\lim_{n\to\infty} \nu(G_n) = \nu\left(\bigcup_{n\geq 1} G_n\right)$ *for all non-decreasing sequences $\{G_n\}_{n\geq 1} \subseteq X$ of open sets.*

A Dempster capacity is a Choquet capacity such that:

(iii) $\lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n\geq 1} A_n\right)$ *for all non-decreasing strictly monotone sequences $\{A_n\}_{n\geq 1} \subseteq X$ of Borel sets.*

Remarks. (i) We use this definition of Choquet capacities, which is a symmetric version of the more usual one, because we are mostly interested in lower envelopes. (ii) We named after Dempster the Choquet capacities that are continuous along non-decreasing strictly monotone sequences of Borel sets because, by Proposition 1 and Lemma 5, the distributions induced by compact-valued random correspondences can be extended on 2^X to a Dempster capacity.

We can now state the result.

Proposition 6 *Let ν be a convex Dempster capacity. If X is metric, then ν is regular. If, in addition, X is Polish, then ν is tight.*

The rest of the subsection is devoted to proving Proposition 6. We start by establishing a couple of lemmas. The first one is a variation of Choquet's capacitability results proved in Meyer (1966).

Lemma 2 *Let X be a metric space and ν a capacity on 2^X such that:*

(i) $\nu(A_n) \downarrow \nu(A)$ *if* $\{A_n\}_{n\geq 1}$ *is a sequence of Borel sets of* X *such that* $A_n \downarrow A$ *and* $A_n \supseteq cl(A_{n+1})$ *for all* $n \geq 1$;

(ii) $\nu(A_n) \uparrow \nu(A)$ *if* $\{A_n\}_{n\geq 1}$ *is a sequence of subsets of* X *such that* $A_n \uparrow A$;

(iii) $\nu(A_1 \cup A_2) + \nu(A_1 \cap A_2) \leq \nu(A_1) + \nu(A_2)$ *for any two Borel sets* A_1 *and* A_2.

Then, for all Borel sets A we have:

$$\begin{aligned} \nu(A) &= \inf\{\nu(G) : A \subseteq G \text{ and } G \text{ open}\} \\ &= \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\}. \end{aligned}$$

Proof. Let $\mathcal{F}$ be the set of closed sets. Since X is metric, all open sets belong to $\mathcal{F}_\sigma$, and so they are $\mathcal{F}$-analytic by Theorem 8 p. 34 of Meyer (1966). Hence, by Theorem 12 p. 35 of Meyer (1966), all Borel sets are $\mathcal{F}$-analytic. By Choquet Capacitability Theorem, since $\mathcal{F} = \mathcal{F}_\delta$ we have

$$\nu(A) = \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\} \tag{5}$$

for all Borel sets A (see Theorem 19 p. 39 of Meyer, 1966). Define ν_e on 2^X as follows:

1. $\nu(C) = \nu_e(C)$ for all closed sets C;
2. $\nu_e(T) = \sup\{\nu(C) : C \subseteq T \text{ and } C \text{ closed}\}$ for all $T \in \mathcal{F}_\sigma$;

3. $\nu_e(A) = \inf\{\nu_e(T) : A \subseteq T \text{ and } T \in \mathcal{F}_\sigma\}$ for all $A \subseteq X$.

Since (i) holds, by Theorem 23 of Meyer (1966),

$$\nu_e(A) = \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\}$$

for all Borel sets A. By (5), $\nu_e(A) = \nu(A)$ for all Borel sets A. Hence,

$$\nu(A) = \inf\{\nu(T) : A \subseteq T \text{ and } T \in \mathcal{F}_\sigma\} \tag{6}$$

for all Borel sets A. On the other hand, let C be a closed set. Since X is a metric space, there exists a sequence $\{G_n\}_{n\geq 1}$ of open sets such that $G_n \downarrow C$ and $G_n \supseteq cl(G_{n+1})$ for all $n \geq 1$. By property (i), $\lim_n \nu(G_n) = \nu(C)$, and so

$$\nu(C) = \inf\{\nu(G) : C \subseteq G \text{ and } G \text{ open}\}. \tag{7}$$

To prove the result we have to show that (7) holds for any Borel set A of X. Let $\varepsilon > 0$. By (6), there exists $T \in \mathcal{F}_\sigma$ such that $A \subseteq T$ and $\nu(T) < \nu(A) + \varepsilon$. Let $\{C_n\}_n$ be a non-decreasing sequence of closed sets such that $C_n \uparrow T$. By (ii), it follows that $\lim_n \nu(C_n) = \nu(T)$. On the other hand, by (7), for each n there exists an open set G_n such that $C_n \subseteq G_n$ and $\nu(G_n) < \nu(C_n) + \varepsilon 2^{-n}$. As ν is submodular by (iii), we have:

$$\nu\left(\bigcup_{i=1}^{n} G_i\right) + \sum_{i=1}^{n} \nu(C_i) \leq \nu\left(\bigcup_{i=1}^{n} C_i\right) + \sum_{i=1}^{n} \nu(G_i)$$

for all n (see, e.g., Meyer (1966) p. 41). This implies:

$$\nu\left(\bigcup_{i=1}^{n} G_i\right) \leq \nu(C_n) + \sum_{i=1}^{n} [\nu(G_i) - \nu(C_i)] < \nu(C_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \leq \nu(C_n) + \varepsilon. \tag{8}$$

Set $G = \bigcup_{n\geq 1} G_n$. By (ii), $\lim_n \nu\left(\bigcup_{i=1}^{n} G_i\right) = \nu(G)$, and so, being $\lim_n \nu(C_n) = \nu(T)$,

$$\nu(G) \leq \lim_n \nu(C_n) + \varepsilon = \nu(T) + \varepsilon < \nu(A) + 2\varepsilon.$$

Since ε was arbitrary, we conclude that $\nu(A) = \inf\{\nu(G) : A \subseteq G \text{ and } G \text{ open}\}$, as desired. ■

The next lemma shows an interesting compactness property of cores.

Lemma 3 *Let X be a metric space and $\nu : \mathcal{B} \to [0,1]$ a regular capacity. Then $core(\nu)$ is weak* compact. If, in addition, X is Polish, then for each $\varepsilon > 0$ there exists a compact set K_ε such that $P(K_\varepsilon) \geq 1 - \varepsilon$ for all $P \in core(\nu)$ (i.e., $core(\nu)$ is tight).*

Proof. By the Alaoglu Theorem it suffices to prove that $core(\nu)$ is weak*-closed. Let $\{p_\alpha\}_\alpha$ be a net in $core(\nu)$ that weak*-converges to $p \in ca(\mathcal{B})$. We want to show that $p \in core(\nu)$. Let C be a closed set. By the Portmanteau Theorem, $\limsup_\alpha p_\alpha(C) \leq p(C)$. Hence, $\nu(C) \leq p(C)$. By the regularity of ν,

$$\nu(A) = \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\}$$

for all Borel sets A. Hence, all probabilities in $ca(\mathcal{B})$ being regular,

$$\begin{aligned} \nu(A) &= \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\} \\ &\leq \sup\{p(C) : C \subseteq A \text{ and } C \text{ closed}\} = p(A). \end{aligned}$$

This proves that $p \in core(\nu)$, and so $core(\nu)$ is weak* compact. Next, suppose that X is Polish. Then, all measures in $core(\nu)$ are separable and by Theorem 7 p. 240 of Billingsley (1968), $core(\nu)$ is tight. ∎

The final lemma is Lemma 2.5 p. 254 of Huber and Strassen (1973).

Lemma 4 *Let ν be a convex Choquet capacity. If X is a Polish space, then for every Borel set $A \subseteq X$ there exists $p \in core(\nu)$ such that $\nu(A) = p(A)$.*

Proof of Proposition 6. Let ν be a convex Dempster capacity. Let $\overline{\nu}$ be the dual capacity of ν. It satisfies the hypotheses of Lemma 2. In particular, point (i) holds because $\left(\overline{A}\right)^c = int(A^c)$ and $\overline{(A^c)} = (int(A))^c$ for all subsets A of X. By Lemma 2,

$$\begin{aligned} \overline{\nu}(A) &= \sup\{\overline{\nu}(C) : C \subseteq A \text{ and } C \text{ closed}\} \\ &= \inf\{\overline{\nu}(G) : A \subseteq G \text{ and } G \text{ open}\}, \end{aligned}$$

for all Borel sets A. On the other hand,

$$\begin{aligned} \nu(A) &= 1 - \overline{\nu}(A^c) = 1 - \sup\{\overline{\nu}(C) : C \subseteq A^c\} \\ &= 1 - \sup\{1 - \nu(C^c) : A \subseteq C^c\} = \inf\{\nu(C^c) : A \subseteq C^c\} \\ &= \inf\{\nu(G) : A \subseteq G\}. \end{aligned}$$

Similarly, $\overline{\nu}(A) = \inf\{\overline{\nu}(G) : A \subseteq G \text{ and } G \text{ open}\}$ implies

$$\nu(A) = \sup\{\nu(C) : C \subseteq A \text{ and } C \text{ closed}\}.$$

We conclude that ν is regular.

Assume that X is Polish. By Lemma 3, *core* (ν) is tight. By Definition 4, ν satisfies the hypotheses of Lemma 4. Along with the tightness of *core* (ν), this implies that for each $\varepsilon > 0$ there exists a compact set K_ε such that $\nu(K_\varepsilon) \geq 1 - \varepsilon$. Let A be a Borel set. Since ν is regular, there exists a closed set $C_\varepsilon \subseteq A$ such that $\nu(A) - \nu(C_\varepsilon) \leq \varepsilon$. Set $K'_\varepsilon = K_\varepsilon \cap C_\varepsilon$. By the convexity of ν, we have

$$\begin{aligned} \nu(K'_\varepsilon) &\geq \nu(K_\varepsilon) + \nu(C_\varepsilon) - \nu(K_\varepsilon \cup C_\varepsilon) \\ &\geq (1-\varepsilon) + (\nu(A) - \varepsilon) - 1 = \nu(A) - 2\varepsilon, \end{aligned}$$

and so $\nu(A) - \nu(K'_\varepsilon) \leq 2\varepsilon$. Since this holds for any $\varepsilon > 0$, we conclude that $\nu(A) = \sup\{\nu(K) : K \subseteq A\}$. ■

Lemma 5 *Let $\nu : \mathcal{B} \to [0,1]$ be a convex capacity satisfying properties (i) and (ii) of Proposition 1. Then there exists a convex Dempster capacity that extends ν on 2^X.*

Proof. Define $\nu' : 2^X \to [0,1]$ by

$$\nu'(A) = \sup\{\nu(B) : B \subseteq A \text{ and } B \in \mathcal{B}\}$$

for all $A \subseteq X$. We now prove that ν' is a Dempster capacity that extends ν on 2^X. Property (iii) of Definition 4 holds because ν and ν' coincide on Borel sets. As to property (i) of Definition 4, consider the dual capacity $\overline{\nu}$ of ν. Clearly, it is a submodular capacity such that $\overline{\nu}\left(\bigcup_{n\geq 1} A_n\right) = \lim_n \overline{\nu}(A_n)$ for all non-decreasing sequences of Borel sets $\{A_n\}_n$. Define $\overline{\nu}^* : 2^X \to [0,1]$ by $\overline{\nu}^*(A) = \inf\{\overline{\nu}(B) : B \supseteq A \text{ and } B \in \mathcal{B}\}$. By Theorem 23 of Meyer (1966), applied to $\mathcal{B}$, the set function $\overline{\nu}^*$ is submodular and such that $\overline{\nu}^*\left(\bigcup_{n\geq 1} A_n\right) = \lim_n \overline{\nu}^*(A_n)$ for all non-decreasing sequences of subsets $\{A_n\}_n$ of X. It is easy to check that ν' is the dual capacity of $\overline{\nu}^*$, and so ν' satisfies also property (i) of Definition 4. It is therefore a convex Dempster capacity. ■

Proof of Proposition 2. It is now an immediate consequence of Proposition 6 and Lemma 5. ■

5.3 Proof of Theorem 1

The Theorem is based on some lemmas. The first one is a special case of results on p. 59 and p. 93 of Srivastava (1998).

Lemma 6 *Suppose* (X, τ) *is a Polish space and* $u : X \to \mathbb{R}$ *a Borel function. Then there is a finer Polish topology* τ_u *on* X *generating the same Borel* σ*-algebra and such that* u *is continuous with respect to* τ_u*. Moreover, if* K *is compact with respect to* τ_u*, the two topologies coincide on* K*.*

In the next lemma, which is of independent interest, we extend the Lusin Theorem to Dempster convex capacities.

Lemma 7 *Let* ν *be a convex Dempster capacity and* $u : X \to \mathbb{R}$ *a Borel measurable function. Suppose* X *is a Polish space. Given* $\varepsilon > 0$ *there exists a compact set* K_ε *such that* $\nu(K_\varepsilon) \geq 1 - \varepsilon$ *and* $u_{|K_\varepsilon}$ *is continuous.*

Proof. By Proposition 6, ν is regular. Moreover, by Lemma 4,

$$\nu(A) = \min \{p(A) : p \in core(\nu)\} \tag{9}$$

for all sets $A \in \mathcal{B}$. Let τ be the Polish topology on X and τ_u the finer Polish topology that, by Lemma 6, makes u continuous. The Borel σ-algebra generated by τ_u coincides with $\mathcal{B}$, the Borel σ-algebra generated by τ. The capacity ν is also regular with respect to τ_u. In fact, being ν regular with respect to τ, for all Borel sets A we have

$$\nu(A) = \sup \{\nu(C) : C \subseteq A \text{ and } C^c \in \tau\} = \inf \{\nu(G) : A \subseteq G \text{ and } G \in \tau\},$$

and so, being τ_u finer,

$$\begin{aligned} \nu(A) &= \inf \{\nu(G) : A \subseteq G \text{ and } G \in \tau\} \\ &\geq \inf \{\nu(G) : A \subseteq G \text{ and } G \in \tau_u\} \geq \nu(A), \end{aligned}$$

and

$$\begin{aligned} \nu(A) &= \sup \{\nu(C) : C \subseteq A \text{ and } C^c \in \tau\} \\ &\leq \sup \{\nu(C) : C \subseteq A \text{ and } C^c \in \tau_u\} \leq \nu(A). \end{aligned}$$

We conclude that ν is regular with respect to τ_u as well, i.e.,

$$\nu(A) = \sup \{\nu(C) : C \subseteq A \text{ and } C^c \in \tau_u\} = \inf \{\nu(G) : A \subseteq G \text{ and } G \in \tau_u\}.$$

Since ν is convex on $\mathcal{B}$ and regular on τ_u, by Lemma 3 *core* (ν) is tight with respect to τ_u. Along with (9), this implies that for each $\varepsilon > 0$ there exists a set K_ε, compact for τ_u (and so for τ as well), such that $\nu(K_\varepsilon) \geq 1-\varepsilon$. Clearly, $u_{|K_\varepsilon}$ is continuous with respect to τ_u. On the other hand, by the last part of Lemma 6, τ and τ_u coincide on K_ε. Hence, $u_{|K_\varepsilon}$ is also continuous with respect to τ. ■

The next lemma shows that for continuous functions a stronger version of Theorem 1 holds.

Lemma 8 *Let X be a separable metric space and ν be the distribution induced by a compact-valued random correspondence $F : S \to 2^X$. If X is σ-compact or complete, then for each continuous function $u : X \to \mathbb{R}$ there is $h \in S(F)$ such that, for all $t \in \mathbb{R}$,*

$$P_h(\{x : u(x) \geq t\}) = \nu(\{x : u(x) \geq t\}) \tag{10}$$

and

$$P_h(\{x : u(x) > t\}) = \nu(\{x : u(x) > t\}). \tag{11}$$

Proof. If X is complete, by Proposition 2 there exists a σ-compact set $K_\sigma \subseteq X$ such that $\nu(K_\sigma) = \nu(X)$. Hence, to prove the result we can assume that X is a σ-compact and separable metric space.

Since F is measurable and has closed values, and X is a σ-compact separable metric space, by Theorem 5.6 p. 62 of Himmelberg (1975), F admits a Castaing representation $\{f_i\}_{i=1}^\infty$. Define $g : S \to \mathbb{R}$ as follows: $g(s) = \inf_{i\geq 1} u(f_i(s))$ for each $s \in S$. Since u is continuous and $F(s) = \overline{\{f_i(s)\}_{i\geq 1}}$, it is easy to see that $\inf_{i\geq 1} u(f_i(s)) = \inf_{x\in F(s)} u(x)$. Hence, $g(s) \in \mathbb{R}$ for all $s \in S$ because $F(s)$ is compact. Moreover, being the infimum of a countable collection of measurable functions, g as well is measurable. Define the multifunction $\Gamma : S \to X$ as follows: $\Gamma(s) = u^{-1}(g(s)) \cap F(s)$ for all $s \in S$. By what we have just said, $g(s) = \inf_{x\in F(s)} u(x)$. Therefore, by the Weierstrass Theorem, $\Gamma(s) \neq \emptyset$ for all $s \in S$. Moreover, Γ has compact values. In fact, $u^{-1}(g(s))$ is a closed set because u is continuous, and so $\Gamma(s)$ as well is a closed set which is included in the compact set $F(s)$.

We now prove that Γ is measurable. We begin by proving that the multifunction $u^{-1} : \mathbb{R} \to X$ is measurable, i.e., that

$$\left(u^{-1}\right)^w(C) \equiv \left\{r \in \mathbb{R} : u^{-1}(r) \cap C \neq \emptyset\right\} \in \mathcal{B}(\mathbb{R})$$

for all closed sets C. Since X is σ-compact, it is enough to look at compact sets K. We prove that $(u^{-1})^w(K)$ is closed for each compact set K. Let $\{r_n\}_{n\geq 1} \subseteq (u^{-1})^w(K)$ and suppose that r_n converges to some $r \in \mathbb{R}$. We want to show that $r \in (u^{-1})^w(K)$. Since $\{x : u(x) = r_n\} \cap K \neq \emptyset$ for each $n \geq 1$, there is a sequence $\{x_n\}_{n\geq 1}$ such that $x_n \in \{x : u(x) = r_n\} \cap K$ for each $n \geq 1$. Since K is compact, there is a subsequence $\{x_{n_k}\}_{k\geq 1}$ that converges to some $x_0 \in K$. Since u is continuous,

$$r = \lim_k r_{n_k} = \lim_k u(x_{n_k}) = u(x_0),$$

and so $x_0 \in \{x : u(x) = r\} \cap K$. In turn this implies that $r \in (u^{-1})^w(K)$, and so $(u^{-1})^w(K)$ is a closed set.

Having established the measurability of u^{-1}, we can now prove that the multifunction Γ is measurable. Since F is measurable, again by Theorem 4.1 of Himmelberg (1975) it suffices to prove that the multifunction $u^{-1} \circ g : S \to 2^X$ is measurable. But this is indeed the case because

$$\left\{s : u^{-1}(g(s)) \cap K \neq \emptyset\right\} = g^{-1}\left(\left\{r : u^{-1}(r) \cap K \neq \emptyset\right\}\right)$$

and we just proved that the set $\{r : u^{-1}(r) \cap K \neq \emptyset\}$ is measurable.

Summing up, $\Gamma : S \to 2^X$ is a measurable multifunction with compact values. By the classic selection theorem of Ryll-Nardzewski and Kuratowski (see, e.g., Himmelberg (1975) p. 60), there exists a measurable function $\overline{h} : S \to X$ with $\overline{h}(s) \in \Gamma(s)$ for all $s \in S$. We then have:

$$\begin{aligned}\left\{s : u\left(\overline{h}(s)\right) \geq t\right\} &= \left\{s : \inf_{i\geq 1} u(f_i(s)) = g(s) \geq t\right\} \\ &= \{s \in S : u(f_i(s)) \geq t \text{ for all } i \geq 1\}.\end{aligned}$$

On the other hand,

$$\{s \in S : u(f_i(s)) \geq t \text{ for all } i \geq 1\} = \{s \in S : F(s) \subseteq \{x : u(x) \geq t\}\}.$$

For, $F(s) = \overline{\{f_i(s)\}_{i=1}^\infty}$ and, u being continuous, the sets $\{x : u(x) \geq t\}$ are closed. This proves (10). Equality (11) follows from (10) and the continuity property (i) in Proposition 1. ■

The last lemma is a continuity result.

Lemma 9 *Let* ν *be a convex capacity and suppose* $\{A_n\}_{n\geq 1}$ *is a monotone increasing sequence of Borel sets such that* $\lim_n \nu(A_n) = 1$. *For any bounded measurable function* $u : X \to \mathbb{R}$ *and* $C \subseteq core(\nu)$,

$$\lim_{n\to\infty}\left[\inf_{p\in C}\int_{A_n} udp\right] = \inf_{p\in C}\int udp.$$

Proof. Let $core_{ba}(\nu) = \{p \in ba : p \geq \nu \text{ and } p(X) = 1\}$. It is easy to check that $core_{ba}(\nu)$ is $\sigma(ba, B)$-compact. Let $u : X \to \mathbb{R}$ be a bounded measurable function. Without loss of generality, assume $u \geq 0$. For each $p \in ca(\mathcal{B})$ set $\phi_n(p) = \int_{A_n} udp$ and $\phi(p) = \int_A udp$. It is easy to check that ϕ and each ϕ_n are $\sigma(ba, B)$-continuous. Moreover, for each Borel set B we have

$$\lim_n p(B \cap A_n) = \lim_n [p(B) - p(B \cup A_n) + p(A_n)] = p(B)$$

since

$$1 = \lim_n \nu(A_n) \leq \lim_n p(A_n) \leq \lim_n p(B \cup A_n) \leq 1.$$

Hence,

$$\begin{aligned}\lim_n \phi_n(p) &= \lim_n \int_{A_n} udp = \lim_n \int (1_{A_n} u)\, dp \\ &= \lim_n \int p(A_n \cap (u \geq t))\, dt = \int p(u \geq t)\, dt = \phi(p).\end{aligned}$$

Therefore, $\{\phi_n\}_{n\geq 1}$ is an increasing sequence of $\sigma(ba, B)$-continuous functions that pointwise converges to the $\sigma(ba, B)$-continuous function ϕ on the $\sigma(ba, B)$-compact set $core_{ba}(\nu)$. By the Dini Theorem, the convergence is uniform, and so on each subset $C \subseteq core_{ba}(\nu)$ – in particular on each $C \subseteq core(\nu)$ – we have $\lim_{n\to\infty}\left[\inf_{p\in C}\int_{A_n} udp\right] = \inf_{p\in C}\int udp$. ■

Proof of Theorem 1. We divide the proof in two parts, part (i) establishes Eq. (1) and shows that it holds with a min for continuous functions $u : X \to \mathbb{R}$. Part (ii) shows that Eq. (1) holds with a min for all lower Weierstrass functions.

Part (i). Suppose u is continuous and bounded. Without loss of generality, assume that $u \geq 0$. It is easy to check that $P_h \in core(\nu)$ for all $h \in S(F)$. Hence, $\int u(x)\, d\nu \leq \inf_{h\in S(F)} \int u(x)\, dP_h$, and so to prove (1) we have to find a selection $\overline{h} \in S(F)$ such that $\int u(x)\,d\nu = \int u(x)\,dP_{\overline{h}}$. Because of Lemma 8, there is $\overline{h} \in S(F)$ such that:

$$\int_0^\infty \nu\left(\{x : u(x) \geq t\}\right) dt = \int_0^\infty P_{\overline{h}}\left(\{x : u(x) \geq t\}\right) dt,$$

and so $\int u(x)\,d\nu = \int u(x)\,dP_{\overline{h}}$, as wanted. This completes the proof of the theorem for continuous functions.

Next suppose u is a Borel and bounded function. Again without loss of generality, assume that $u \geq 0$. By Lemma 7, for each $n > 0$ there exists a compact set K_n such that $\nu(K_n) \geq 1 - 1/n$ and u is continuous on K_n. Since u is continuous on any finite union of these compact sets (see, e.g., Kuratowski (1966) p. 106), we can assume that $\{K_n\}_{n\geq 1}$ is an increasing sequence with $\lim_n \nu(K_n) = 1$. If we set $X_u = \bigcup_n K_n$, we have $\nu(X_u) = 1$.

Let $S_n = \{s : F(s) \subseteq K_n\}$. Since K_n is compact and F is measurable, it holds that $S_n = F^{-1}(K_n) \in \Sigma$ (cf. Theorem 3.5 of Himmelberg, 1975). Consider the multifunction $F_n : S_n \to 2^{K_n}$ defined by $F_n(s) = F(s)$ for all $s \in S_n$. For all Borel sets $A \subseteq K_n$, set $\nu_n(A) = P(\{s : F_n(s) \subseteq A\})$. The multifunction F_n is compact-valued on S_n. Moreover, F_n is measurable. In fact, for all Borel sets $A \subseteq K_n$

$$(F_n)^{-1}(A) = \{s \in S_n : F(s) \subseteq A\} = S_n \cap \{s \in S : F(s) \subseteq A\} = S_n \cap F^{-1}(A)$$

and, since F is measurable, we have $(F_n)^{-1}(A) \in \Sigma_{|S_n}$.

We conclude that $F_n : S_n \to 2^{K_n}$ is a measurable and compact-valued multifunction with values in the compact metric space K_n. Hence, by what we have just proved for continuous functions (which holds for any finite positive measure P not necessarily normalized to 1 on S),

$$\int_{K_n} u\,d\nu_n = \min_{h \in S(F_n)} \int_{K_n} u\,dP_h.$$

Since $S_n \in \Sigma$, it is easy to check that if $h \in S(F)$, its restriction $h_{|S_n}$ to S_n belongs to $S(F_n)$. The converse is true: each $h \in S(F_n)$ admits an extension to a $h^* \in S(F)$. For, let $h' \in S(F)$ and define $h^* : S \to X$ as follows: $h^*(s) = h(s)$ for all $s \in S_n$ and $h^*(s) = h'(s)$ for all $s \notin S_n$. Hence, h^* extends h on S and it is easy to check that $h^* \in S(F)$.

We conclude that $S(F_n) = \left\{h_{|S_n} : h \in S(F)\right\}$, and so

$$\int_{K_n} u\,d\nu_n = \min_{h \in S(F_n)} \int_{K_n} u\,dP_h = \min_{h \in S(F)} \int_{K_n} u\,dP_{h_{|S_n}}. \tag{12}$$

Since $P(S_n) = \nu(K_n) \geq 1 - 1/n$, for all $A \in \mathcal{B}$ and all $h \in S(F)$ we have:

$$\begin{aligned} P_h(A) &= P(\{h \in A\}) = P((\{h \in A\}) \cap S_n) + P((\{h \in A\}) \cap S_n^c) \\ &\leq P(\{s \in S_n : h \in A\}) + \frac{1}{n} = P_{h_{|S_n}}(A) + \frac{1}{n}. \end{aligned}$$

Hence, for all $h \in S(F)$ we have:

$$\begin{aligned} \int_{K_n} udP_h &= \int_0^\infty P_h((u \geq t) \cap K_n)\, dt \\ &\leq \frac{1}{n} + \int_0^\infty P_{h_{|S_n}}((u \geq t) \cap K_n)\, dt = \frac{1}{n} + \int_{K_n} udP_{h_{|S_n}}, \end{aligned}$$

and so

$$\inf_{h \in S(F)} \int_{K_n} udP_h \leq \inf_{h \in S(F)} \left(\frac{1}{n} + \int_{K_n} udP_{h_{|S_n}} \right) = \frac{1}{n} + \inf_{h \in S(F)} \int_{K_n} udP_{h_{|S_n}}.$$

Together with (12), this implies:

$$\inf_{h \in S(F)} \int_{K_n} udP_h - \frac{1}{n} \leq \int_{K_n} ud\nu_n. \tag{13}$$

On the other hand, for all Borel sets $A \subseteq K_n$ we have:

$$\begin{aligned} \nu_n(A) &= P(\{s : F_n(s) \subseteq A\}) = P(\{s \in S_n : F(s) \subseteq A\}) \\ &= P(\{s \in S : F(s) \subseteq A\}) = \nu(A). \end{aligned}$$

Hence, $\int_{K_n} ud\nu_n = \int_{K_n} ud\nu$, and together with (13) it implies:

$$\inf_{h \in S(F)} \int_{K_n} udP_h - \frac{1}{n} \leq \int_{K_n} ud\nu.$$

By Lemma 9,

$$\lim_{n\to\infty} \left[\inf_{h \in S(F)} \int_{K_n} udP_h \right] = \inf_{h \in S(F)} \int_{X_u} udP_h.$$

On the other hand, since the Choquet integral is monotone, $\int_{X_u} ud\nu \geq \int_{K_n} ud\nu$ for each $n \geq 1$, and so

$$\begin{aligned} \int_{X_u} ud\nu &\geq \lim_{n\to\infty} \int_{K_n} ud\nu \geq \lim_{n\to\infty} \left(\left[\inf_{h \in S(F)} \int_{K_n} udP_h \right] - \frac{1}{n} \right) \\ &= \inf_{h \in S(F)} \int_{X_u} udP_h \geq \int_{X_u} ud\nu. \end{aligned}$$

Hence,

$$\int_{X_u} ud\nu = \inf_{h \in S(F)} \int_{X_u} udP_h.$$

As $\nu(X_u) = P_h(X_u) = 1$ for all $h \in S(F)$ and ν is convex, we conclude that:

$$\int_X ud\nu = \int_{X_u} ud\nu = \inf_{h \in S(F)} \int_{X_u} udP_h = \inf_{h \in S(F)} \int_X udP_h,$$

as desired.

Part (ii). We prove that Lemma 8 holds for lower Weierstrass measurable functions. First notice that by Lemma 4, $\nu(A) = \min\{p(A) : p \in core(\nu)\}$ for all $A \in \mathcal{B}$. Next, given a lower Weierstrass measurable function $u : X \to \mathbb{R}$, let τ_u be the finer Polish topology that, by Lemma 6, makes u continuous. The correspondence F has still closed values with respect to τ_u. Moreover, since the Borel σ-algebra generated by τ_u coincides with $\mathcal{B}$, F is still $\mathcal{B}$-measurable. Finally, ν is also regular with respect to τ_u as shown in the proof of Lemma 7. By Lemma 3, $core(\nu)$ is tight with respect to τ_u. Then, being $\nu(A) = \min\{p(A) : p \in core(\nu)\}$ for all $A \in \mathcal{B}$, there is a set K_σ which is σ-compact with respect to τ_u and such that $\nu(K_\sigma) = 1$. Hence, to prove the result we can assume that the space X is a σ-compact and separable metric space.

By Theorem 5.6 of Himmelberg (1975), F has a Castaing representation. Define the correspondence $\Gamma : S \to 2^X$ as in the proof of Lemma 8. We have $\Gamma(s) \neq \emptyset$. In fact, though $F(s)$ may be not compact with respect to τ_u and so we cannot invoke the Weierstrass Theorem, we know that $\min_{x \in F(s)} u(x)$ exists because u is lower Weierstrass with respect to the original topology τ.

We can then proceed as in the proof of Lemma 8 to show that Γ is measurable and closed-valued. Again by Theorem 5.6 of Himmelberg (1975), Γ has a measurable selection $\overline{h}$. The rest of the proof is now as in Lemma 8. This completes the proof of part (ii). ■

5.4 Proof of Corollary 1

Given any weak*-continuous and linear functional $\phi : ca(\mathcal{B}) \to \mathbb{R}$, there exists $u \in C_b(X)$ such that $\phi(P) = \int udP$ for all $P \in ca(\mathcal{B})$ (see, e.g., Megginson (1998) p. 224). Hence, by Theorem 1,

$$\inf\{\phi(p) : p \in core(\nu)\} = \min\{\phi(p) : p \in A\},$$

where $A \equiv \{P_h : h \in S(F)\}$. In fact, as ν is convex, $\int u d\nu = \min_{p \in core(\nu)} \int u dp$ for all $u \in C_b(X)$. The result then follows from Lemma 3 and Theorem 13.B p. 74 of Holmes (1975). ■

5.5 Proof of Lemma 1

We start with a lemma.

Lemma 10 *If $F : S \to 2^X$ is a correspondence such that $S(F) \neq \emptyset$, then*

$$\int F dP = \left\{ \int h dP : h \text{ integrable and } h \in S(F) \right\}.$$

Proof. Clearly, it suffices to show that

$$\left\{ \int h dP : h \text{ integrable and } h \in \widetilde{S}(F) \right\} \tag{14}$$

$$\subseteq \left\{ \int h dP : h \text{ integrable and } h \in S(F) \right\}. \tag{15}$$

Let $\widetilde{h} \in \widetilde{S}(F)$ be integrable. Let $A \in \Sigma$ be such that $P(A) = 1$ and $\widetilde{h}(s) \in F(s)$ for all $s \in A$. As $S(F) \neq \emptyset$, let $h' \in S(F)$. Let $h''(s) = \widetilde{h}(s)$ for all $s \in A$ and $h''(s) = h'(s)$ for all $s \in A^c$. Then $h'' \in S(F)$. Moreover, since $\widetilde{h}$ is integrable and $h'' = \widetilde{h}$ P-a.e., h'' is integrable and $\int \widetilde{h} dP = \int h'' dP$. Hence, $\int \widetilde{h} dP \in \left\{ \int h dP : h \text{ integrable and } h \in S(F) \right\}$, which proves the inclusion (14). ■

Proof of Lemma 1. Let $u : X \to \mathbb{R}$ be a measurable function. By the argument used in Part (ii) of the proof of Theorem 1, we can assume that u is a continuous function, X is a σ-compact and separable metric space (not necessarily complete), and F closed-valued and $\mathcal{B}$-measurable (not necessarily compact-valued).

Define the correspondence $u \circ F : S \to 2^{\mathbb{R}}$ by $(u \circ F)(s) = \{u(x) : x \in F(s)\}$. By Theorem 5.6 of Himmelberg (1975), $S(F) \neq \emptyset$. Hence $S(u \circ F) \neq \emptyset$ since $u \circ h \in S(u \circ F)$ if $h \in S(F)$. Therefore, by Lemma 10,

$$\int (u \circ F) \, dP = \left\{ \int g dP : g \text{ integrable and } g \in S(u \circ F) \right\}.$$

Let $g \in S(u \circ F)$. Consider the correspondence $u^{-1} \circ g : S \to 2^X$. Since g is measurable, by proceeding as in the proof of Lemma 8 we can prove that the correspondence $u^{-1} \circ g : S \to 2^X$ is measurable. Since F is closed-valued and X is a σ-compact and separable metric space, by Corollary 4.2 of Himmelberg (1975) the closed-valued correspondence $(u^{-1} \circ g) \cap F : S \to 2^X$ is measurable. Then, by Theorem 5.6 of Himmelberg (1975), $S((u^{-1} \circ g) \cap F) \neq \emptyset$. Let $\gamma \in S((u^{-1} \circ g) \cap F)$. We have $u(\gamma(s)) = g(s)$ for all $s \in S$ and $\gamma \in S(F)$. All this implies that $S(u \circ F) = \{u \circ h : h \in S(F)\}$. Since u is bounded, all measurable selections of $u \circ F$ are integrable. We then have:

$$
\begin{aligned}
\int (u \circ F)\, dP &= \left\{ \int g dP : g \in S(u \circ F) \right\} \\
&= \left\{ \int (u \circ h) dP : h \in S(F) \right\} = \left\{ \int u dP_h : h \in S(F) \right\},
\end{aligned}
$$

as desired. ■

5.6 Proofs of Corollary 2 and Propositions 4 and 5

Corollary 2. As to point (ii), it is easy to define a lower semicontinuous function $g : X \to [0, 1]$ such that $G_i = \{x : g(x) > (1/i)\}$ for all $i \in (0, 1]$ and $G_0 = \emptyset$. Hence, by what has been proved in Part (ii) of the proof of Theorem 1, there is $h \in S(F)$ such that $P_h(G_i) = \nu(G_i)$. A similar argument proves point (i), where we take finite-valued functions in place of lower semicontinuous ones. Finally, as to point (iii), by Kuratowski (1966) p. 267, there exists a continuous function $g : X \to [0, 1]$ such that $C_i = \{x : g(x) \geq i\}$ for all $i \in [0, 1]$. The result then follows form Lemma 8. ■

Proposition 4. It is an immediate consequence of Theorem 3 and Propositions 1 and 2, when applied to (S, Σ_*, P_*). ■

Proposition 5. By Theorem 3, the random set F is a compact-valued random correspondence with respect to Σ_*. Let $S_*(F)$ be the set of all selections of F that are Σ_*-measurable. For $h \in S_*(F)$, let P_h^* be the distribution induced by h, i.e., $P_h^*(A) = P_*(h \in A)$ for all $A \in \mathcal{B}$. Apply Theorem 1 to (S, Σ_*, P_*). We have that

$$
\int u d\nu_* = \inf_{h \in S_*(F)} \int u dP_h^*
$$

for all bounded and measurable functions $u : X \to \mathbb{R}$.

Given $h \in S_*(F)$, it is easy to check that there is a Σ-measurable function h' such that $h'(s) = h(s)$ for all $s \in S$ except those in some set $A \in \Sigma$ with $P(A) = 0$. On the other hand, as $S(F) \neq \emptyset$, let $h'' \in S(F)$ and define a function $h''' : S \to X$ as follows: $h'''(s) = h'(s)$ for all $s \notin A$ and $h'''(s) = h''(s)$ for all $s \in A$. Then $h''' \in S(F)$ and $h''' = h$ P-a.e. on S. This implies that $\int udP_h^* = \int udP_{h'''}$ for all measurable functions $u : X \to \mathbb{R}$, and so $\inf_{h \in S(F)} \int udP_h \leq \inf_{h \in S_*(F)} \int udP_h^*$. The rest of the proof is now trivial. ■

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