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DUAL REPRESENTATIONS OF STRONGLY MONOTONIC UTILITY FUNCTIONS

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Dual representations of strongly monotonic utility functions

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Abstract

We present theorems that establish dualities, i.e., bijections, between specified sets of direct utility functions, indirect utility functions and expenditure functions. The substantive properties characterizing the specified set of direct utility functions are strong monotonicity, upper semicontinuity and quasi-concavity. Our results are strictly intermediate between two classes of analogous results in the literature. We also provide applications that use all the three classes of duality results.

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1 Introduction

Consider a consumer who faces positive prices and has consumption set $\Re^n_+$. Krishna and Sonnenschein [5] (henceforth, abbreviated to K-S) study the problem of classifying equivalent representations of such a consumer's preference.[1] They define a set of direct utility functions $\mathcal{U}$, a set of expenditure functions $\mathcal{E}$, a set of indirect utility functions $\mathcal{V}$, and mappings $\phi : \mathcal{U} \to \mathcal{E}$, $\psi : \mathcal{V} \to \mathcal{E}$ and $\chi : \mathcal{U} \to \mathcal{V}$. The substantive restrictions imposed on $u \in \mathcal{U}$ are that u should be unbounded above, quasi-concave, upper semicontinuous and weakly monotonic, i.e., $x \geq y$ implies $u(x) \geq u(y)$. K-S show that ϕ, ψ and χ are bijections and provide explicit characterizations of ϕ^{-1}, ψ^{-1} and χ^{-1}. We shall henceforth refer to triples like $(\mathcal{U}, \mathcal{E}; \phi)$, $(\mathcal{V}, \mathcal{E}; \psi)$ and $(\mathcal{U}, \mathcal{V}; \chi)$ as dualities if the first two elements in the triple are sets and the third element is a bijection between these sets.

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[1]This and related problems have generated a rich literature that is summarized in Diewert [3].

If $u \in \mathcal{U}$, then routine arguments show that (a) the consumer's Marshallian demand has nonempty, compact and convex values, and (b) some element of the demand exhausts the budget. An important motivation for duality theory is to enable precise and tractable specification of consumer preferences in applications, where (a) and (b) might be the minimum required of demands. As the properties defining $\mathcal{U}$ are close to being the "minimal" general conditions that ensure (a) and (b), the class of preferences characterized in K-S seems the largest that might be relevant for applications. While the K-S results may be seen as the most inclusive dualities, it is also useful to study their finer structure by deriving analogous dualities for subsets of the classes of functions considered in K-S.

Jackson [4] contains dualities $(\mathcal{U}^*, \mathcal{E}^*; \phi)$, $(\mathcal{V}^*, \mathcal{E}^*; \psi)$ and $(\mathcal{U}^*, \mathcal{V}^*; \chi)$, where $\mathcal{U}^*$, $\mathcal{E}^*$ and $\mathcal{V}^*$ are proper subsets of $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{V}$ respectively. $\mathcal{U}^*$ is the subset of $\mathcal{U}$ whose elements are continuous and strongly monotonic, i.e., $x \gg y$ implies $u(x) > u(y)$.

In Theorems 2.4 and 3.2 of this paper we derive dualities $(\mathcal{U}^{**}, \mathcal{E}^{**}; \phi)$, $(\mathcal{V}^{**}, \mathcal{E}^{**}; \psi)$ and $(\mathcal{U}^{**}, \mathcal{V}^{**}; \chi)$, where $\mathcal{U}^* \subset \mathcal{U}^{**} \subset \mathcal{U}$, $\mathcal{E}^* \subset \mathcal{E}^{**} \subset \mathcal{E}$ and $\mathcal{V}^* \subset \mathcal{V}^{**} \subset \mathcal{V}$, with all inclusions strict. These dualities are strictly intermediate between those in K-S and Jackson [4] because $\mathcal{U}^{**}$ is the subset of $\mathcal{U}$ whose elements are strongly monotonic but not necessarily continuous.

If $u \in \mathcal{U}^{**}$, then strong monotonicity rules out thick indifference curves which are permitted in the K-S theory, and consequently, property (b) of demands is strengthened to (c) every element of the demand exhausts the budget. If $u \in \mathcal{U}^*$, then we have the additional property (d) the demand is upper hemicontinuous. Naturally, the strong monotonicity condition satisfied by elements of $\mathcal{U}^{**}$ entails stronger properties of the elements in $\mathcal{E}^{**}$ and $\mathcal{V}^{**}$. While an expenditure function $e \in \mathcal{E}$ is lower semicontinuous in the utility level, our duality results strengthen this property to full continuity of $e \in \mathcal{E}^{**}$ in the utility level. While an indirect utility function $v \in \mathcal{V}$ is non-decreasing in wealth, our results make $v \in \mathcal{V}^{**}$ increasing in wealth.

A fuller analysis of the implications of our results is available in Sections 2 and 3. Applications of this class of duality results are considered in Section 4. We show in Theorems 4.1 and 4.2 that $v \in \mathcal{V}$ is upper semicontinuous and $e \in \mathcal{E}$ is lower semicontinuous. Another application studied in Theorems 4.3 and 4.4 is a sharpening of the classical result that specific taxes and subsidies are weakly dominated by lump-sum transfers.

The three above-mentioned classes of dualities allow greater flexibility in modelling a consumer's preference as they justify the specification of the preference equivalently as either $u \in \mathcal{U}$, or $\phi(u) \in \mathcal{E}$, or $\chi(u) \in \mathcal{V}$. This is useful as it enables the modeller to specify a preference precisely to satisfy a particular need, thereby sharpening Occam's razor. For instance, assuming the relevant functions are differentiable, Shephard's lemma implies that the price derivative of $e \in \mathcal{E}$ is the Hicksian demand generated by $\phi^{-1}(e) \in \mathcal{U}$ and Roy's identity implies that the negative of the price derivative of $v \in \mathcal{V}$, divided by the wealth derivative of v, is the Marshallian demand generated by $\chi^{-1}(v) \in \mathcal{U}$.[2] This ability to legitimately derive these demands directly from $e \in \mathcal{E}$ and $v \in \mathcal{V}$, rather than indirectly *via* $\phi^{-1}(e) \in \mathcal{U}$ and $\chi^{-1}(v) \in \mathcal{U}$, increases the precision with which a preference may be specified to satisfy a particular property of these demands.

We conclude this section by describing our formalism. $\mathcal{N}$, $\mathcal{Z}$ and $\Re$ denote the sets of natural numbers, integers and real numbers respectively. Given $x, y \in \Re^n$, we say $x \geq y$ if $x_i \geq y_i$ for $i = 1, \ldots, n$; $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ if $x_i > y_i$ for $i = 1, \ldots, n$. Moreover, $\Re^n_+ = \{x \in \Re^n \mid x \geq 0\}$ and $\Re^n_{++} = \{x \in \Re^n \mid x \gg 0\}$. Define the budget mapping $B : \Re^n_{++} \times \Re_+ \Rightarrow \Re^n_+$ by $B(p, w) = \{x \in \Re^n_+ \mid p.x \leq w\}$; we use $\Rightarrow$ in place of $\rightarrow$ to denote set-valued mappings. Consider a utility function $u : \Re^n_+ \rightarrow \Re$. Define $F_u : \Re_+ \Rightarrow \Re^n_+$ by $F_u(v) = u^{-1}([v, \infty))$; $F_u(v)$ is the upper contour set of utility function u for utility level v. Provided F_u has nonempty values, the expenditure function generated by u is the mapping $\phi(u) : \Re_+ \times \Re^n_{++} \rightarrow \bar{\Re}$ defined by $\phi(u)(v, p) = \inf\{p.x \mid x \in F_u(v)\}$. Our assumptions about u will ensure that F_u has nonempty values and replace "inf" with "min", thereby making $\phi(u)$ real-valued. The indirect utility function generated from u is the mapping $\chi(u) : \Re^n_{++} \times \Re_+ \rightarrow \bar{\Re}$ defined by $\chi(u)(p, w) = \sup u \circ B(p, w)$. Our assumptions will ensure that B has nonempty values and replace "sup" with "max", thereby making $\chi(u)$ real-valued. Define the Marshallian demand mapping $m : \Re^n_{++} \times \Re_+ \Rightarrow \Re^n_+$ by $m(p, w) = \cap_{y \in B(p,w)} \{x \in B(p, w) \mid u(x) \geq u(y)\}$, and assuming the values of $\phi(u)$ are in $\Re_+$, define the Hicksian demand mapping $h : \Re_+ \times \Re^n_{++} \Rightarrow \Re^n_+$ by $h(v, p) = m(p, \phi(u)(v, p))$.

The plan of the rest of this paper is as follows. In Sections 2 and 3, we establish the dualities $(\mathcal{U}^{**}, \mathcal{E}^{**}; \phi)$ and $(\mathcal{U}^{**}, \mathcal{V}^{**}; \chi)$ respectively. We illustrate the uses of the above-described dualities in Section 4; the proofs of the results in this section are collected in the Appendix. We conclude in Section 5.

2 Duality between direct utility and expenditure functions

We begin by defining a set $\mathcal{U}^{**}$ of direct utility functions.

Definition 2.1 $\mathcal{U}^{**}$ *is the set of functions* $u : \Re^n_+ \rightarrow \Re$ *satisfying the following properties:*

(a) $u(0) = 0$,

(b) u *is unbounded above,*

[2] Of course, differentiability of the expenditure function in prices does not have to be assumed as it can be shown to follow from general properties of support functions (Rockafellar [6], Section 23).

(c) u is strongly monotonic, i.e., if $x, y \in \Re^n_+$ and $x \gg y$, then $u(x) > u(y)$,

(d) u is upper semicontinuous, i.e., $F_u(v)$ is closed in $\Re^n_+$ for every $v \in \Re_+$, and

(e) u is quasi-concave, i.e., $F_u(v)$ is convex for every $v \in \Re_+$.

The only departure from the definition of $\mathcal{U}$ is that (c) replaces property (U1) in K-S, which requires that u be weakly monotonic. It is easily checked that (c) and (d) combine to imply (U1) in K-S, i.e., $\mathcal{U}^{**} \subset \mathcal{U}$. The set $\mathcal{U}^*$ in Jackson [4] differs from $\mathcal{U}^{**}$ by strengthening (d) to full continuity. Thus, $\mathcal{U}^* \subset \mathcal{U}^{**}$. Define $u : \Re^n_+ \to \Re$ by $u(x) = \max\{k \in \mathcal{Z} \mid \sum_{i=1}^n x_i \geq k\}$ for every $x \in \Re^n_+$; as $u \in \mathcal{U} - \mathcal{U}^{**}$, we have $\mathcal{U}^{**} \neq \mathcal{U}$. Now define $u : \Re^n_+ \to \Re$ by $u(x) = \sum_{i=1}^n x_i$ for every $x \in \Delta \equiv \{y \in \Re^n_+ \mid \sum_{i=1}^n y_i < 1\}$ and $u(x) = 1 + \sum_{i=1}^n x_i$ for $x \in \Re^n_+ - \Delta$; as $u \in \mathcal{U}^{**} - \mathcal{U}^*$, we have $\mathcal{U}^{**} \neq \mathcal{U}^*$.

An alternative characterization of $\mathcal{U}^{**}$ that is easy to confirm is: $u \in \mathcal{U}^{**}$ if and only if $u \in \mathcal{U}$ and u is locally non-satiated. We work with Definition 2.1 in order to maintain easy comparability with K-S and Jackson [4].

(a) is not a substantive restriction on $\mathcal{U}$ as it does not restrict the class of preferences whose representations satisfy the other defining conditions. (b) is a substantive restriction on $\mathcal{U}$ as there exist preferences whose utility representations do not satisfy (b) but do satisfy all the other properties of $\mathcal{U}$. For instance, $u : \Re^n_+ \to \Re$, defined by $u(x) = 0$ for every $x \in \Re^n_+$, satisfies (a), (U1) in K-S, (d) and (e), but not (b), and therefore $u \notin \mathcal{U}$. Moreover, there is no function ordinally equivalent to u, i.e., an increasing transformation of u, that belongs to $\mathcal{U}$. On the other hand, as we show in Remark 2.2, neither (a) nor (b) are substantive restrictions on $\mathcal{U}^{**}$ as they do not restrict the set of preferences whose representations satisfy the other defining conditions of $\mathcal{U}^{**}$. In the context of $\mathcal{U}$, (b) is a weaker non-satiation assumption than local non-satiation. In the context of $\mathcal{U}^{**}$, (c) is not only stronger than local non-satiation, it also implies (b).

Remark 2.2 *Let $u : \Re^n_+ \to \Re$ satisfy (c), (d) and (e). Define $U : \Re^n_+ \to \Re$ by $U(x) = u(x) - u(0)$. U satisfies (a), (c), (d) and (e), and is equivalent to u. If U does not satisfy (b), then let $\alpha = \sup U(\Re^n_+)$, and define $v : \Re^n_+ \to \Re$ by $v(x) = U(x)/[\alpha - U(x)]$. (a) and (c) imply that $\alpha - U(x) > 0$ for every $x \in \Re^n_+$, $\alpha > 0$ and $U(\Re^n_+) \subset [0, \alpha)$. Note that $v(x) = f \circ U(x)$ where $f : [0, \alpha) \to \Re$ is given by $f(r) = r/(\alpha - r)$. As f is increasing, v is equivalent to U, and therefore, to u. Clearly, $v(0) = 0$. (c), (d) and (e) follow from the fact that v and U are equivalent. To check (b), let $r \in \Re$. For some $k \in \mathcal{N}$, $f(\alpha - 1/k) = k\alpha - 1 > r$. As $\alpha = \sup U(\Re^n_+)$, there exists $x \in \Re^n_+$ such that $U(x) > \alpha - 1/k$. Consequently, $v(x) = f \circ U(x) > f(\alpha - 1/k) > r$.*

Thus, (c), (d) and (e) are the substantive restrictions on $\mathcal{U}^{**}$. As is well-known and evident from the proofs of Theorems 1 and 2 in K-S, (e) does not imply any restrictions on the properties of $\phi(u)$ and $\chi(u)$ for $u \in \mathcal{U}$; the role of property (e) is to ensure that ϕ and χ are injective. Thus, the properties of u that determine the substantive properties of $\phi(u)$ and $\chi(u)$ are the monotonicity properties such as (c) or (U1) in K-S, the non-satiation properties such as (b) or (c), and the continuity properties such as (d). Consequently, the results in K-S, Jackson [4] and this paper essentially serve to classify the trade-offs implied by various combinations of these three classes of properties.

(a) and (c) imply that u is non-negative. (b) ensures that $F_u(v) \neq \emptyset$ for every $v \in \Re_+$. (d) and (e) imply that $F_u(v)$ is convex and closed in $\Re^n_+$ for every $v \in \Re_+$. Thus, for every $u \in \mathcal{U}^{**}$ and $(v,p) \in \Re_+ \times \Re^n_{++}$, $\phi(u)(v,p) = \min\{p.x \mid x \in F_u(v)\} = p.y \in \Re_+$ for some $y \in F_u(v)$. Next, we define the set of expenditure functions.

Definition 2.3 *$\mathcal{E}^{**}$ is the set of functions $e : \Re_+ \times \Re^n_{++} \to \Re_+$ that satisfy the following properties. For every $v \in \Re_+$ and $p \in \Re^n_{++}$,*
(a) $e(v,p) = 0$ if and only if $v = 0$,
(b) $e(.,p)$ is unbounded above,
(c) if $v' \geq v$, then $e(v',p) \geq e(v,p)$,
(d) $e(.,p)$ is continuous,
(e) if $p' \geq p$, then $e(v,p') \geq e(v,p)$,
(f) $e(v,tp) = te(v,p)$, for every $t > 0$, and
(g) $e(v,.)$ is concave.

The only departure from the set $\mathcal{E}$ in K-S is property (d), which is stronger than property (e2) in K-S that requires $e(.,p)$ to be lower semicontinuous. The set $\mathcal{E}^*$ in Jackson [4] replaces the weak monotonicity conditions (c) and (e) by a condition that, among other things, implies stronger monotonicity properties than (c) and (e). We now have all the ingredients for our first duality theorem.

Theorem 2.4 *$(\mathcal{U}^{**}, \mathcal{E}^{**}; \phi)$ is a duality with $\phi^{-1} : \mathcal{E}^{**} \to \mathcal{U}^{**}$ given by $\phi^{-1}(e)(x) = \sup \cap_{p \in \Re^n_{++}} \{v \in \Re_+ \mid p.x \geq e(v,p)\}$.*

Proof. Consider $u \in \mathcal{U}^{**}$. We first show that $\phi(u) \in \mathcal{E}^{**}$. As $\mathcal{U}^{**} \subset \mathcal{U}$, we have $u \in \mathcal{U}$, and therefore by Theorem 1 in K-S, $e \equiv \phi(u) \in \mathcal{E}$. Consequently, e satisfies properties 2.3(a)-(c), (e)-(g), and $e(.,p)$ is lower semicontinuous for $p \in \Re^n_{++}$.

We show that $e(.,p)$ is upper semicontinuous, and therefore, e satisfies property 2.3(d) and $e \in \mathcal{E}^{**}$. Fix $p \in \Re^n_{++}$, and let $\alpha \in \Re$ and $U = \{v \in \Re_+ \mid e(v,p) \geq \alpha\}$. It suffices to show that U is closed in $\Re_+$. Let (v_k) be a sequence in U converging to $v \in \Re_+$. By properties 2.1(b) and (d), $F_u(v)$ is nonempty and closed in $\Re^n_+$. Thus, there exists $x \in F_u(v)$ such that $e(v,p) = p.x$. We show that $v \in U$, i.e., $p.x \geq \alpha$. Property 2.1(c) implies that there exists a sequence $(x_k) \subset \Re^n_+$ converging to x such that $u(x_k) > u(x)$ for every k. Then, $r_k \equiv u(x_k) - u(x) > 0$. As (v_k) converges to v, for every k, there exists i_k such that $v_{i_k} < v + r_k$. By definition, $u(x_k) = u(x) + r_k \geq v + r_k > v_{i_k}$. Hence, $p.x_k \geq e(v_{i_k}, p) \geq \alpha$ for every k. Letting $k \uparrow \infty$, we have $p.x \geq \alpha$.

By Theorem 3 in K-S, $\phi : \mathcal{U} \to \mathcal{E}$ is a bijection. As $\mathcal{U}^{**} \subset \mathcal{U}$, $\phi : \mathcal{U}^{**} \to \mathcal{E}^{**}$ is injective. In order to show that $\phi : \mathcal{U}^{**} \to \mathcal{E}^{**}$ is surjective, consider $e \in \mathcal{E}^{**}$. As $\mathcal{E}^{**} \subset \mathcal{E}$, we have $e \in \mathcal{E}$, $\phi^{-1}(e) \in \mathcal{U}$ and $\phi \circ \phi^{-1}(e) = e$. It only remains to show that $\phi^{-1}(e) \in \mathcal{U}^{**}$. As $\phi^{-1}(e) \in \mathcal{U}$, $\phi^{-1}(e)$ is weakly monotonic and satisfies properties 2.1(a), (b), (d) and (e). We show that $\phi^{-1}(e)$ satisfies property 2.1(c), and therefore, $\phi^{-1}(e) \in \mathcal{U}^{**}$.

Consider $p \in \Re^n_{++}$ and $x \in \Re^n_+$. Property 2.3(a) implies that $e(0, p) = 0 \leq p.x$. Properties 2.3(b), (c) and the lower semicontinuity part of (d) combine to imply that $\{v \in \Re_+ \mid p.x \geq e(v, p)\}$ is closed in $\Re_+$ and bounded above. Thus, given $x \in \Re^n_+$, $\cap_{p \in \Re^n_{++}} \{v \in \Re_+ \mid p.x \geq e(v, p)\}$ is nonempty, closed in $\Re_+$ and bounded above. Consequently, $\phi^{-1}(e)(x) = \max \cap_{p \in \Re^n_{++}} \{v \in \Re_+ \mid p.x \geq e(v, p)\}$.

Let $x \in \Re^n_+$, $y \gg x$ and $v \equiv \phi^{-1}(e)(x) \geq \phi^{-1}(e)(y)$. As $\phi^{-1}(e) \in \mathcal{U}$, $v \in \Re_+$. By the definition of $\phi^{-1}(e)$, we have $p.x \geq e(v, p)$ for every $p \in \Re^n_{++}$. We derive a contradiction.

As $v \geq \phi^{-1}(e)(y)$, we have $y \notin F \equiv \cup_{k \in \mathcal{N}} F_{\phi^{-1}(e)}(v + 1/k)$. By property 2.1(b), $F_{\phi^{-1}(e)}(v + 1/k) \neq \emptyset$ for every $k \in \mathcal{N}$; thus, $F \neq \emptyset$. By property 2.1(e), $F_{\phi^{-1}(e)}(v + 1/k)$ is convex for every $k \in \mathcal{N}$. As the constituent sets are nested, F is convex. By the separating hyperplane theorem (Berge [2], First separation theorem, page 163), there exists $q \in \Re^n - \{0\}$ such that $q.y \leq q.z$ for every $z \in F$. The definition of F and the weak monotonicity of $\phi^{-1}(e)$ imply that $q \geq 0$. As $q \neq 0$, we have $q > 0$. Therefore, $q.x < q.y \leq q.z$ for every $z \in F$. The continuity of $p \mapsto p.x$ implies that there exists $p \gg q$ such that $p.x < q.y$. Since $F \subset \Re^n_+$, $p.z \geq q.z \geq q.y$ for every $z \in F$. Therefore, $p.x < q.y \leq \inf\{p.z \mid z \in F\} \leq \inf\{p.z \mid z \in F_{\phi^{-1}(e)}(v + 1/k)\} = \phi \circ \phi^{-1}(e)(v + 1/k, p) = e(v + 1/k, p)$ for every $k \in \mathcal{N}$. Letting $k \uparrow \infty$ and using the upper semicontinuity part of property 2.3(d), we have $p.x < q.y \leq e(v, p)$, a contradiction. ∎

3 Duality between direct and indirect utility functions

We first define the set of indirect utility functions $\mathcal{V}^{**}$.

Definition 3.1 *$\mathcal{V}^{**}$ is the class of functions $v : \Re^n_{++} \times \Re_+ \to \Re$ that satisfy the following properties. For every $p \in \Re^n_{++}$ and $w \in \Re_+$,*

(a) $v(p, 0) = 0$,

(b) $v(p, .)$ is unbounded,

(c) if $w' > w$, then $v(p, w') > v(p, w)$,
(d) if $p' \geq p$, then $v(p', w) \leq v(p, w)$,
(e) if $t > 0$, then $v(tp, tw) = v(p, w)$,
(f) $v(p, .)$ is upper semicontinuous, and
(g) v is quasi-convex.

Property (c) is the only departure from the properties defining $\mathcal{V}$ in K-S. (c) is stronger than the corresponding condition (V1) in K-S that requires $v(p, .)$ to be non-decreasing. This has a useful implication for applications of these results. If $v \in \mathcal{V}$ is differentiable, then Roy's identity generates the Marshallian demand provided the wealth derivative of v is positive. However, the weak monotonicity of v with respect to wealth does not guarantee a positive wealth derivative. This problem is substantially alleviated if $v \in \mathcal{V}^{**}$. While this does not guarantee a positive derivative for all wealth levels, it does imply that, for any profile of positive prices, the set of wealth levels where the wealth derivative vanishes is negligible in the sense that it is of Lebesgue measure zero. The exceptional set consists of points of inflection, which cannot be eliminated without additional information. We are now ready to prove our second duality theorem.

Theorem 3.2 *$(\mathcal{U}^{**}, \mathcal{V}^{**}; \chi)$ is a duality with $\chi^{-1} : \mathcal{V}^{**} \to \mathcal{U}^{**}$ given by $\chi^{-1}(v)(x) = \sup \cap_{p \in \Re^n_{++}} \{u \in \Re \mid v(p, p.x) \geq u\}$.*

Proof. Consider $u \in \mathcal{U}^{**}$. We first show that $v \equiv \chi(u) \in \mathcal{V}^{**}$. As $\mathcal{U}^{**} \subset \mathcal{U}$, we have $u \in \mathcal{U}$, and therefore by Theorem 2 in K-S, $v \in \mathcal{V}$. Consequently, v satisfies properties 3.1(a), (b), (d)-(g), and $v(p, .)$ is non-decreasing.

We show that $v(p, .)$ is increasing. Thus, $v \in \mathcal{V}^{**}$. Fix $(p, w') \in \Re^n_{++} \times \Re_+$ and $w > w'$. As $B(p, w')$ is nonempty and compact, property 2.1(d) implies that $v(p, w') = \max u \circ B(p, w') = u(x')$ for some $x' \in B(p, w')$. Then $p.x' \leq w' < w$. Clearly, there exists $x \gg x'$ such that $x \in B(p, w)$. Property 2.1(c) implies that $v(p, w) \geq u(x) > u(x') = v(p, w')$.

By Theorem 4 in K-S, $\chi : \mathcal{U} \to \mathcal{V}$ is a bijection. As $\mathcal{U}^{**} \subset \mathcal{U}$, $\chi : \mathcal{U}^{**} \to \mathcal{V}^{**}$ is injective. In order to show that $\chi : \mathcal{U}^{**} \to \mathcal{V}^{**}$ is surjective, consider $v \in \mathcal{V}^{**}$. As $\mathcal{V}^{**} \subset \mathcal{V}$, we have $v \in \mathcal{V}$, $\chi^{-1}(v) \in \mathcal{U}$ and $\chi \circ \chi^{-1}(v) = v$. It only remains to show that $\chi^{-1}(v) \in \mathcal{U}^{**}$. As $\chi^{-1}(v) \in \mathcal{U}$, $\chi^{-1}(v)$ is weakly monotonic and satisfies properties 2.1(a), (b), (d) and (e). We show that $\chi^{-1}(v)$ satisfies property 2.1(c), and therefore, $\chi^{-1}(v) \in \mathcal{U}^{**}$.

Given $p \in \Re^n_{++}$ and $x \in \Re^n_+$, properties 3.1(a) and (c) imply $v(p, p.x) \geq v(p, 0) = 0$; moreover, $\{u \in \Re \mid v(p, p.x) \geq u\}$ is closed in $\Re$ and bounded above. Given $x \in \Re^n_+$, $\cap_{p \in \Re^n_{++}} \{u \in \Re \mid v(p, p.x) \geq u\}$ is nonempty, closed in $\Re$ and bounded above. Thus, $\chi^{-1}(v)(x) = \max \cap_{p \in \Re^n_{++}} \{u \in \Re \mid v(p, p.x) \geq u\}$.

Let $x, y \in \Re^n_+$ such that $y \gg x$. As $\chi^{-1}(v) \in \mathcal{U}$, we have $\chi^{-1}(v)(y) \geq \chi^{-1}(v)(x) \equiv \alpha$. Suppose $\chi^{-1}(v)(y) = \alpha$. We derive a contradiction.

$\chi^{-1}(v)(y) = \alpha$ implies that $y \notin F_{\chi^{-1}(v)}(\alpha + 1/k)$ for every $k \in \mathcal{N}$. Thus, $y \notin F \equiv \cup_{k\in\mathcal{N}} F_{\chi^{-1}(v)}(\alpha + 1/k)$. Properties 2.1(b) and (e) imply that $F_{\chi^{-1}(v)}(\alpha + 1/k)$ is nonempty and convex for every $k \in \mathcal{N}$. As the constituent sets are nested, F is nonempty and convex. By the separating hyperplane theorem (Berge [2], First separation theorem, page 163), there exists $q \in \Re^n - \{0\}$ such that $q.y \leq q.z$ for every $z \in F$. As $\chi^{-1}(v)$ is nondecreasing, $q \in \Re^n_+$. As $q \neq 0$, $q > 0$. Setting $\beta = q.y$, we have $q.x < \beta \leq q.z$ for every $z \in F$. As $q > 0$ and $y \gg x \geq 0$, we have $\beta > 0$. Thus, $q.z \geq \beta > 0$ for every $z \in F$. It follows that $z > 0$ for every $z \in F$. By the continuity of linear functionals, there exists $p \gg q$ such that $p.x < \beta \leq q.z < p.z$ for every $z \in F$. It follows from $\chi^{-1}(v)(x) = \alpha$ and property 3.1(c) that $\alpha \leq v(p, p.x) < v(p, \beta)$. So, $v(p, \beta) > \alpha + 1/k$ for some $k \in \mathcal{N}$. Consequently, $\chi \circ \chi^{-1}(v)(p, \beta) = v(p, \beta) > \alpha + 1/k$. This implies that there exists $z \in B(p, \beta) \cap F_{\chi^{-1}(v)}(\alpha + 1/k)$. Since $z \in B(p, \beta)$, we have $p.z \leq \beta$, and as $z \in F_{\chi^{-1}(v)}(\alpha + 1/k)$, we have $z \in F$, which implies $p.z > \beta$, which is a contradiction. ■

4 Some applications

Duality results have two competing aspects. On the one hand, when choosing a dual representation such as $e \in \mathcal{E}$ or $v \in \mathcal{V}$ to specify a preference, we want the properties defining $\mathcal{E}$ or $\mathcal{V}$ to be minimal so that the modeller has greater latitude in selecting an appropriate e or v. On the other hand, when using the chosen dual representation, the modeller is free to use not only the properties used to define $\mathcal{E}$ or $\mathcal{V}$, but also other stronger properties possessed by elements of $\mathcal{E}$ or $\mathcal{V}$. So, an important aspect of duality theory is to derive various non-definitional properties possessed by dual representations.

A useful implication of duality theory is that it justifies a variety of equivalent strategies for deriving the non-definitional properties of dual representations, so that one may freely choose among these strategies using simplicity as the criterion. For instance, consider the question: what properties of the elements of $\mathcal{V}$ are true in addition to those used to define the set? One way to derive additional properties is to use the defining properties of $\mathcal{V}$ as axioms and directly derive their implications. An equivalent way is to use the duality $(\mathcal{U}, \mathcal{V}; \chi)$ and derive the properties of the values of χ.

We illustrate this indirect strategy *via* two sets of results. Both results are consequences of one part of the Maximum Theorem (Berge [2], Theorem 2 in Section VI.3). However, the applicability of this result is not automatic in both cases and requires some care in our setting. The problem in both cases is to show that the "feasibility" mapping, the budget mapping $B : \Re^n_{++} \times \Re_+ \Rightarrow \Re^n_+$ in one case and the upper contour set mapping $F_u : \Re \Rightarrow \Re^n_+$ in the other, is upper hemicontinuous. As $\Re^n_+$ is not compact, it is not enough to show that the feasibility mapping has a closed graph. We overcome this problem by a localization argument. Given a point in the domain, we find a neighborhood of the point such that the values of the feasibility mapping over this neighborhood may be restricted to a fixed compact subset of $\Re^n_+$ without artificially restricting the optimization problem being studied. Given this set-up, if the feasibility mapping restricted to this neighborhood has a closed graph, then the feasibility mapping is upper hemicontinuous at the specified point in the domain. It is important for our argument that all prices in the domain are positive.

Theorem 4.1 *If $v \in \mathcal{V}$, then*

(A) v is upper semicontinuous, and

(B) if $v \in \mathcal{V}^{**}$, then for every $(p, w) \in \Re^{n+1}_{++}$, $p' \gg p$ implies $v(p', w) < v(p, w)$.

The first part of this result strengthens property (V2) in K-S, that $v(p, .)$ is upper semicontinuous, to joint upper semicontinuity in prices and wealth. The second part is also proved as part of Theorem 2 in Jackson [4] for $v \in \mathcal{V}^*$; we merely point out that it relies on strong monotonicity of u but not its lower semicontinuity. The next set of results concerns the duality between direct utility and expenditure functions.

Theorem 4.2 *If $e \in \mathcal{E}$, then*

(A) e is lower semicontinuous,

(B) for every $(v, p) \in \Re^{1+n}_{++}$, $p' \gg p$ implies $e(v, p') > e(v, p)$, and

(C) if $\phi^{-1}(e)$ is lower semicontinuous, then for every $(v, p) \in \Re_+ \times \Re^n_{++}$, $v' > v$ implies $e(v', p) > e(v, p)$.

As $e \in \mathcal{E}$ is concave in prices, $e(v, .)$ is continuous. K-S show that e is lower semicontinuous in utility levels. (A) strengthens this to joint lower semicontinuity of e in prices and utility levels. For $e \in \mathcal{E}^{**}$, we have continuity in prices and utility levels and joint lower semicontinuity. (B) notes that e is increasing in prices.

We conclude this section with a classical economic application whose analysis illustrates the roles played by the monotonicity and continuity properties that separate the sets $\mathcal{U}$, $\mathcal{U}^*$ and $\mathcal{U}^{**}$.

Theorem 4.3 *Let $(p', w) \in \Re^{n+1}_{++}$, $p \in \Re^n_{++}$, $u \in \mathcal{U}$, $v = \chi(u)$, $e = \phi(u)$ and $x \in m(p, w)$.*

(A) $\tau^* \equiv \max\{\tau \in \Re \mid v(p', w - \tau) \geq v(p, w)\} \geq w - e(v(p, w), p') \geq w - p'.x \geq (p - p').x$.

(B) If $u \in \mathcal{U}^{**}$, then $e(v(p, w), p) = w$.

(C) If $u \in \mathcal{U}^*$, then $v(p', e(v(p, w), p')) = v(p, w)$.

(D) If $u \in \mathcal{U}^*$ and $h(v(p, w), p') \cap h(v(p, w), p) = \emptyset$, then $e(v(p, w), p') < p'.x$.

We interpret $p - p'$ as a vector of specific taxes; p' is the price vector excluding the taxes and p is the price vector including the taxes. By convention, a positive component of $p - p'$ represents a tax and a negative component represents a subsidy on the relevant commodity. τ^* is the maximal lump-sum tax subject to the constraint that the utility after the lump-sum tax is not less than the utility after the specific taxes. (A) implies that τ^* weakly exceeds the government's revenue from the specific taxes $p - p'$. Thus, the consumer and the government weakly prefer the lump-sum tax τ^* to the specific taxes $p - p'$, i.e., the lump-sum tax weakly dominates the specific taxes. This classical result is remarkably robust as an examination of the proof reveals that just properties 2.1(b) and (d) of u are required in the argument. A sharper result than (A) requires either of the two parties to be strictly better-off, i.e., either (1) $\tau^* > (p-p').x$, or (2) $v(p', w - \tau^*) > v(p, w)$.

If $u \in \mathcal{U}^*$, then setting $\tau \equiv w - e(v(p, w), p')$ and using (C) implies $v(p', w - \tau) = v(p', e(v(p, w), p')) = v(p, w)$. Property 3.1(c) implies that $\tau^* = w - e(v(p, w), p')$, i.e., the first inequality in (A) is strengthened to an equality. Thus, if $u \in \mathcal{U}^*$, then (A) cannot be sharpened *via* (2) and the only way to ensure (1) is to sharpen the second or third inequalities in (A). If $u \in \mathcal{U}^{**}$, then $p.x = w$, thus strengthening the third inequality in (A) to an equality. Therefore, the only way to sharpen (A) is to provide sufficient conditions for $e(v(p, w), p') < p'.x$. (D) provides one such sufficient condition.

Given $u \in \mathcal{U}^{**}$, showing $e(v(p, w), p') < p'.x$ for all possible vectors of specific taxes is not possible, even locally, as every open neighborhood of p' has a price vector $p = (1+t)p'$, with $t > 0$, such that the consumer utility and revenue impacts of the implied vector of specific taxes tp' will be identical to the effect of the equivalent lump-sum tax $tw/(1+t)$. Although (D) seems to promise a general sharper result, the assumption underlying it is not generally satisfied by all p and p'. For instance, if the vector of specific taxes is tp' for some $t > 0$, then $h(v(p, w), p') = h(v(p, w), (1+t)p') = h(v(p, w), p)$ as Hicksian demands are homogeneous of degree 0 in prices. The required assumption cannot be satisfied generally, even locally, as t can be made arbitrarily small. However, this problem in the use of (D) can be avoided, at least locally, by imposing a mild restriction on the set of admissible specific tax vectors.

Consider $\bar{v} \in \Re_{++}$. Let $P = \{(q, 1) \mid q \in \Re^{n-1}_{++}\}$ be the set of price vectors. From the perspective of characterizing $h(\bar{v}, .)$, this restriction is without loss of generality because, for every $p \in \Re^n_{++}$, $p/p_n \in P$ and $h(\bar{v}, p) = h(\bar{v}, p/p_n)$. However, from the perspective of characterizing the effects of specific taxes, the restriction to P rules out a tax on the n-th commodity, which now serves as the *numéraire*.

Theorem 4.4 *Let $u \in \mathcal{U}$, $e = \phi(u)$, $\bar{v} \in \Re_{++}$ and $(q', 1) \in P$. If* *(a) u is strictly quasi-concave,*

(b) $e(\bar{v}, ., 1)$ is twice continuously differentiable on an open neighborhood $N \subset \Re^{n-1}_{++}$ of q', and

(c) $D_{qq}e(\bar{v}, q', 1)$ has rank $n-1$,

then there exists an open neighborhood $U \subset \Re^{n-1}_{++}$ of q' such that $h(\bar{v}, q, 1) \neq h(\bar{v}, q', 1)$ for every $q \in U - \{q'\}$.

Thus, the sufficient conditions in Theorem 4.3(D) are satisfied if $u \in \mathcal{U}^*$, the n-th commodity is not taxed and the vector of specific taxes on the other commodities is sufficiently small. Consequently, under these conditions, the lump-sum tax yields more revenue than the vector of specific taxes while leaving the consumer indifferent between the two tax regimes. A *numéraire* commodity is required for this result as the Slutsky matrix $D_{pp}e(\bar{v}, p)$ is singular. As $e \in \mathcal{E}$, property 2.3(a) implies $e(0, p) = 0$ for every $p \in \Re^n_{++}$. Therefore, $h(0, p) = m(p, e(0, p)) = m(p, 0) = 0$ for every $p \in \Re^n_{++}$. Thus, an injectivity result like Theorem 4.4 requires $\bar{v} > 0$.

5 Conclusions

We have established dualities $(\mathcal{U}^{**}, \mathcal{E}^{**}; \phi)$ and $(\mathcal{U}^{**}, \mathcal{V}^{**}; \chi)$, where $\mathcal{U}^{**}$, $\mathcal{E}^{**}$ and $\mathcal{V}^{**}$ are sets of direct utility functions, expenditure functions and indirect utility functions respectively. The substantive properties of $u \in \mathcal{U}^{**}$ are strong monotonicity, upper semicontinuity and quasi-concavity. The duality $(\mathcal{V}^{**}, \mathcal{E}^{**}; \phi \circ \chi^{-1})$ is an immediate consequence of our results. These dualities are strictly intermediate between the analogous results in K-S and Jackson [4]. We have also provided some applications of the class of duality characterizations to which our results belong. While two of the applications derive potentially useful continuity and monotonicity properties of expenditure and indirect utility functions, the third analyzes the classical relationship between lump-sum transfers and specific taxes and subsidies.

Appendix

Proof of Theorem 4.1 Fix $v \in \mathcal{V}$. By Theorem 4 in K-S, $u \equiv \chi^{-1}(v) \in \mathcal{U}$ and $\chi(u) = v$. Consequently, $v(p, w) = \sup u \circ B(p, w)$ for every $(p, w) \in \Re^n_{++} \times \Re_+$.

(A) By Theorem 2 in Section VI.3 in Berge [2], it suffices to show that B is upper hemicontinuous with nonempty and compact values. It is easily confirmed that B has nonempty and compact values. Fix $(p^*, w^*) \in \Re^n_{++} \times \Re_+$. It suffices to show that B is upper hemicontinuous at (p^*, w^*).

As $p^* \gg 0$, there exists $r > 0$ such that $B_r(p^*) = \{p \in \Re^n \mid \|p - p^*\| \leq r\} \subset \Re^n_{++}$. As $B_r(p^*)$ is compact, $R = B_r(p^*) \times [0, 2w^*]$ is compact. Clearly, $\{(p, w, x) \in R \times \Re^n_+ \mid p.x \leq w)\}$ is closed in $R \times \Re^n_+$, i.e., the restriction of B to R has a closed graph. As the projection mapping π_i is continuous,

$\alpha \equiv \max_{i=1,\ldots,n} \max\{w/\pi_i(p) \mid (p,w) \in R\}$ exists. Clearly, $[0,\alpha]^n$ is compact and $B(p,w) = \{x \in \Re^n_+ \mid p.x \leq w\} \subset \cap_{i=1}^n \{x \in \Re^n_+ \mid \pi_i(x) \leq w/\pi_i(p)\} \subset \cap_{i=1}^n \{x \in \Re^n_+ \mid \pi_i(x) \leq \alpha\} \subset [0,\alpha]^n$ for every $(p,w) \in R$. By Theorem 7 in Section VI.1 of Berge [2], the restriction of B to R is upper hemicontinuous. Thus, B is upper hemicontinuous at (p^*,w^*).

(B) Fix $v \in \mathcal{V}^{**}$. By Theorem 3.2, $u \equiv \chi^{-1}(v) \in \mathcal{U}^{**}$ and $\chi(u) = v$. Fix $(p,w) \in \Re^{n+1}_{++}$ and $p' \gg p$. As $B(p',w)$ is nonempty and compact, property 2.1(d) implies that $v(p',w) = u(x')$ for some $x' \in B(p',w)$. Property 2.1(c) implies that $p'.x' = w$. As $w > 0$, we have $x' > 0$. As $p' \gg p$, we have $p.x' < p'.x' < w$. Thus, there exists $x \gg x'$ such that $x \in B(p,w)$. By property 2.1(c), $v(p',w) = u(x') < u(x) \leq v(p,w)$. ■

Proof of Theorem 4.2 Fix $e \in \mathcal{E}$. By Theorem 4 in K-S, $u \equiv \phi^{-1}(e) \in \mathcal{U}$ and $\phi(u) = e$.

(A) As $e(v,p) = \inf\{p.x \mid x \in F_u(v)\} = -\sup\{-p.x \mid x \in F_u(v)\}$ for every $(v,p) \in \Re_+ \times \Re^n_{++}$, it suffices to show that the mapping $(v,p) \mapsto \sup\{-p.x \mid x \in F_u(v)\}$ is upper semicontinuous at every $(v,p) \in \Re_+ \times \Re^n_{++}$, say (v^*,p^*).

Property 2.1(b) implies that there exists $x^* \in \Re^n_+$ such that $u(x^*) > v^*$. Define $G : \Re^n_{++} \Rightarrow \Re^n_+$ by $G(p) = \{x \in \Re^n_+ \mid p.x \leq p.x^*\}$. It follows that $\sup\{-p.x \mid x \in F_u(v)\} = \sup\{-p.x \mid x \in F_u(v) \cap G(p)\}$ for every $(v,p) \in [0,u(x^*)] \times \Re^n_{++}$. As $(v^*,p^*) \in [0,u(x^*)) \times \Re^n_{++}$, Theorem 2 in Section VI.3 of Berge [2] implies that it is sufficient to show that the mapping $\Gamma : [0,u(x^*)] \times \Re^n_{++} \Rightarrow \Re^n_+$, given by $\Gamma(v,p) = F_u \circ \pi_1(v,p) \cap G \circ \pi_2(v,p)$, is upper hemicontinuous with nonempty compact values.

Consider $(v,p) \in [0,u(x^*)] \times \Re^n_{++}$. Clearly, $x^* \in \Gamma(v,p)$. $F_u(v)$ is closed in $\Re^n_+$ by property 2.1(d) and $G(p)$ is compact as $p \gg 0$. Therefore, $\Gamma(v,p)$ is compact. Note that $\mathrm{Gr}\, F_u \circ \pi_1 = g^{-1}(\mathrm{Gr}\, F_u)$, where $g : (v,p,x) \mapsto (v,x)$. As the mapping $(v,x) \mapsto u(x) - v$ is upper semicontinuous (Berge [2], Theorem 5 in Section IV.8), $\mathrm{Gr}\, F_u$ is closed in $\Re_+ \times \Re^n_+$. As the projection mapping g is continuous, $\mathrm{Gr}\, F_u \circ \pi_1 = g^{-1}(\mathrm{Gr}\, F_u)$ is closed in $\Re_+ \times \Re^n_{++} \times \Re^n_+$. Therefore, the restriction of $F_u \circ \pi_1$ to $[0,u(x^*)] \times \Re^n_{++}$ has a closed graph. By Theorem 7 in Section VI.1 of Berge [2], it is now sufficient to show that $G \circ \pi_2$ is upper hemicontinuous. As π_2 is continuous, it is sufficient to show that G is upper hemicontinuous.

Consider $p \in \Re^n_{++}$. As $p \gg 0$, there exists $r > 0$ such that $B_r(p) = \{p' \in \Re^n \mid \|p' - p\| \leq r\} \subset \Re^n_{++}$. By construction, $B_r(p)$ is compact. Clearly, $\{(p',x) \in B_r(p) \times \Re^n_+ \mid p'.x \leq p'.x^*\}$ is closed in $B_r(p) \times \Re^n_+$, i.e., the restriction of G to $B_r(p)$ has a closed graph. As the projection mapping π_i is continuous, $\alpha \equiv \max_{i=1,\ldots,n} \max\{p'.x^*/\pi_i(p') \mid p' \in B_r(p)\}$ is well-defined. Clearly, $[0,\alpha]^n$ is compact and $G(p') \subset \cap_{i=1}^n \{x \in \Re^n_+ \mid \pi_i(x) \leq p'.x^*/\pi_i(p')\} \subset \cap_{i=1}^n \{x \in \Re^n_+ \mid \pi_i(x) \leq \alpha\} = [0,\alpha]^n$ for every $p' \in B_r(p)$. By Theorem 7 in Section VI.1 of Berge [2], the restriction of G to $B_r(p)$ is upper hemicontinuous with compact values. Thus, G is upper hemicontinuous at

p.

(B) Fix $(v,p) \in \Re^{1+n}_{++}$ and $p' \gg p$. As u is upper semicontinuous, $e(v,p') = p'.x$ for some $x \in F_u(v)$. As $v > 0$, we have $x \in \Re^n_+ - \{0\}$. Therefore, $e(v,p') = p'.x > p.x \geq e(v,p)$.

(C) Fix $(v,p) \in \Re_+ \times \Re^n_{++}$ and $v' > v$. As u is upper semicontinuous, $e(v,p) = p.x$ and $e(v',p) = p.x'$ for some $x \in F_u(v)$ and $x' \in F_u(v')$. Suppose $p.x = p.x'$. As $u \in \mathcal{U}$, $u(x) = v < v' = u(x')$. As u is lower semicontinuous, there exists $r > 0$ such that $y \in B_r(x') \equiv \{y \in \Re^n_+ \mid \|y - x'\| < r\}$ implies $u(y) > v$. As $u(x') = v' > v \geq 0$, we have $x' > 0$ and there exists $y \in B_r(x')$ such that $y < x'$. Thus, $y \in F_u(v)$ and $p.y < p.x' = p.x = e(v,p)$, a contradiction. ■

Proof of Theorem 4.3 (A) As $u \in \mathcal{U}$, properties 2.1(b) and (d) imply that there exists $x' \in F_u(v(p,w))$ such that $p'.x' = e(v(p,w),p')$. By definition, $x' \in B(p', e(v(p,w),p'))$ and $u(x') \geq v(p,w)$. It follows that

$$v(p', e(v(p,w),p')) \geq u(x') \geq v(p,w) \qquad (A.1)$$

If $\tau \equiv w - e(v(p,w),p')$, then (A.1) implies $v(p', w-\tau) = v(p', e(v(p,w),p')) \geq v(p,w)$, which proves the first inequality. As $x \in m(p,w)$, we have $u(x) = v(p,w)$. So, $x \in F_u(v(p,w))$ and

$$e(v(p,w),p') \leq p'.x \qquad (A.2)$$

The second inequality follows. As $x \in m(p,w)$, we have $p.x \leq w$, hence the last inequality.

(B) As $u \in \mathcal{U}^{**}$, there exists $y \in B(p,w)$ such that $u(y) = v(p,w)$ and $p.y = w$. As $y \in F_u(u(y))$, $e(v(p,w),p) = e(u(y),p) \leq p.y = w$. Suppose $e(v(p,w),p) < w$. Then, there exists $z \in F_u(v(p,w))$ such that $p.z = e(v(p,w),p) < w$. As $z \in F_u(v(p,w))$, we have $u(z) \geq v(p,w)$. As $p.z < w$ and u is strongly monotonic, there exists $z' \gg z$ such that $p.z' \leq w$ and $u(z') > u(z) \geq v(p,w) = \max u \circ B(p,w) \geq u(z')$, a contradiction.

(C) Suppose $v(p', e(v(p,w),p')) > v(p,w)$. Consequently, there exists $y \in B(p', e(v(p,w),p'))$ such that $u(y) = v(p', e(v(p,w),p')) > v(p,w)$ and $p'.y \leq e(v(p,w),p')$. As $v(p,w) \geq u(0)$, we have $y > 0$. As $u \in \mathcal{U}^*$, u is lower semicontinuous. Therefore, $A = \{z \in \Re^n_+ \mid u(z) > v(p,w)\}$ is open in $\Re^n_+$. Since $y \in A$ and $y > 0$, there exists $z \in A$ such that $z < y$. It follows that $z \in \Re^n_+$ and $u(z) > v(p,w)$, i.e., $z \in F_u(v(p,w))$. As $p'.z < p'.y \leq e(v(p,w),p') \leq p'.z$, we have a contradiction. The result follows from (A.1).

(D) Using (B), $x \in m(p,w) = m(p, e(v(p,w),p)) = h(v(p,w),p)$. By (A.2), $p'.x \geq e(v(p,w),p')$. Suppose $p'.x = e(v(p,w),p')$. Then, $x \in B(p', e(v(p,w),p'))$. If $x \in m(p', e(v(p,w),p')) = h(v(p,w),p')$, then $x \in h(v(p,w),p) \cap h(v(p,w),p')$, a contradiction. So, $x \notin m(p', e(v(p,w),p'))$. Consequently, there exists $y \in B(p', e(v(p,w),p'))$ such that $u(y) > u(x)$. As

$u \in \mathcal{U}^*$, u is lower semicontinuous. Consequently, there exists $z \in \Re^n_+$ such that $z < y$ and $u(z) > u(x) = v(p, w)$. As $p'.z < p'.y \le e(v(p, w), p') \le p'.z$, we have a contradiction. ■

Proof of Theorem 4.4 By (a), m is a function, and so h is a function. Define $\hat{h} : N \to \Re^{n-1}_+$ by $\hat{h}(.) = (h_1(\bar{v}, ., 1), \ldots, h_{n-1}(\bar{v}, ., 1))$. By Shephard's lemma and (b), $\hat{h}(.) = D_q e(\bar{v}, ., 1)$ and $\hat{h}$ is continuously differentiable. By (c), $D\hat{h}(q') = D_{qq} e(\bar{v}, q', 1)$ has rank $n - 1$, and so is injective. By the injective mapping theorem (Bartle [1], Theorem 41.5), there exists an open neighborhood $U \subset \Re^{n-1}_{++}$ of q' such that $\hat{h}$ is injective on U. Therefore, if $q \in U - \{q'\}$, then $h(\bar{v}, q, 1) = (\hat{h}(q), h_n(\bar{v}, q, 1)) \neq (\hat{h}(q'), h_n(\bar{v}, q', 1)) = h(\bar{v}, q', 1)$. ■

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