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Abstract: The dynamics of a linear (or linearized) dynamic stochastic economic model can be expressed in terms of matrices (A, B, C, D) that define a state-space system. An associated state-space system (A, K, C, Σ) determines a vector autoregression (VAR) for observables available to an econometrician. We review circumstances in which the impulse response of the VAR resembles the impulse response associated with the economic model. We give four examples that illustrate a simple condition for checking whether the mapping from VAR shocks to economic shocks is invertible. The condition applies when there are equal numbers of VAR and economic shocks.

JEL classification: C32, E10

Key words: vector autoregression, economic shocks, innovations, invertibility

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A, B, C'S (AND D)'S FOR UNDERSTANDING VARS

"(Likelihood Principle) The information brought by an observation x about [a parameter] θ is entirely contained in the likelihood function." *The Bayesian Choice*, by Christian P. Robert, p. 15.[1]

"... with a specific parameterization of preferences the theory would place many restrictions on the behavior of endogenous variables. But these predictions do not take the form of locating blocks of zeros in a VAR description of these variables." *Money and Interest in a Cash-in-Advance Economy*, Robert E. Lucas, Jr., and Nancy L. Stokey, p. 512.[2]

I. INTRODUCTION

This paper is about inferring a set of meaningful economic shocks from the innovations to a VAR. Applied macroeconomists use unrestricted Vector Autoregressions (VARs) to answer questions about responses to economically interpretable shocks. For example: What is the effect of a technology shock on hours worked? How does output respond to monetary perturbations? What happens after a fiscal shock? VAR researchers hope that they can coax answers to such questions from unrestricted VARS and propose their estimated impulse responses functions as objects that subsequent quantitative theoretical models should aim to interpret in terms of structural parameters.

To get pertinent impulse responses, a researcher needs to transform the one-step ahead prediction errors in her VAR into shocks that impinge on an economic model, i.e., shocks to preferences, technologies, agents' information sets, and the economist's measurements. Unrestricted VAR researchers hope to accomplish this recovery job by imposing weak identification restrictions directly on the unrestricted VAR. The restrictions are called weak because the researcher wants them to hold for a class of models.

The preceding epigraphs frame our topic. At least up to a linear approximation, the theoretical vector autoregression implied by a model is a recursive expression of its conditional likelihood function, which according to the likelihood principle, contains all that the data have to say about the model's parameters. Lucas and Stokey construct a theoretical model whose equilibrium is a Markov process. They express doubts that Markov process implies

[1] See Robert, (2001).

[2] See Lucas and Stokey (1987).

zero restrictions on a vector autoregression. Lucas and Stokey indicate that it would be a good idea to deduce the restrictions that theoretical models like theirs put on VARs. This paper collects a set of convenient formulas that summarize such restrictions and describes the mapping from the economic shocks to the shocks in a VAR. We review conditions under which this mapping has an inverse that is one-sided in nonnegative powers of the lag operator, a prerequisite for having impulse response functions to VAR innovations that can potentially match impulse response functions to the economic shocks. We then focus on circumstances when the impulse response associated with a VAR mirrors the one associated with the economic theory. In an interesting special 'square case' in which the number of economic shocks equals the number of variables in a VAR, we provide an easy to check necessary and sufficient condition for the existence of an identification of VAR shocks that makes the impulse response associated with a VAR match the one associated with the economic theory.

Prominent macroeconomists have expressed skepticism about the value of incompletely theoretical VAR's as a research tool (see Chari, Kehoe, and McGrattan (2005)). By describing how VAR shocks recombine current and past realizations of the economic shocks hitting preferences, technologies, information sets, and measurements, formula (25) below helps us to express and evaluate diverse grounds for skepticism about VARs. Formula (25) imposes the following taxonomy of potential challenges in interpreting VAR shocks and the impulse responses of observables to them in terms of the economic shocks and their impulse responses.

First, for some theories, the number of economic shocks differs from the number of observables and therefore the number of shocks in the VAR. Second, even in the lucky situation in which the number of economic shocks equals the number of observables, the history of economic shocks can span a bigger space than the history of the observables, making it impossible to match up their impulse response functions; here there is said to be an invertibility problem because the economic shocks cannot be expressed as a linear combination of current and past VAR innovations. Third, even when the theory and measurements are such that there are equal numbers of economic and VAR shocks and there is no invertibility problem, there remains the challenge of partitioning the contemporaneous covariation among VAR shocks in a way that captures the contemporaneous covariance of economic shocks and measurement errors. Fourth, even when all of the first three problems can be resolved, because (25) is in general an infinite order VAR (technically, it is a finite order VARMA system), one must either include vector moving average terms or make sure to include a sufficient number of AR terms, perhaps guided by an information-theoretic (e.g., a Bayesian information criterion).

This hierarchy of problems has prompted many quantitative macroeconomists to forgo matching their theories to an unrestricted VAR. But because the VAR representation can be regarded as an expression of a conditional likelihood function, a researcher who believes fully in her theory cannot turn her back on the implications of her theory for a VAR.[3]

I.1. **Related literature.** The process of reverse engineering a subset of economic shocks from the innovations to a VAR is known to be fraught with hazards. Several authors have described the invertibility problem that we highlight in this paper. For examples, see Hansen and Sargent (1981, 1991c), Watson (1994), Reichlin and Lippi (1994), and Sims and Zha (2004). These papers present some examples in which the invertibility problem is 'fatal', but also indicate other examples in which it is not.

I.2. **Point of this paper.** This paper reviews what is known about the reverse engineering exercise and, for an interesting special case, describes an easy check for the presence of an invertibility problem. We present four examples that represent a variety of situations, some in which invertibility is a problem, and others in which it is not. The models are (1) a permanent income model in which lack of invertibility is endemic; (2) and (3) a model with two sources of technology change and a model with sticky prices, in both of which invertibility prevails, though in model (3) there is a benign eigenvalue of unity that prevents an infinite order VAR representation from existing ; and (4) a home production model in which, depending on the variables observed, invertibility may or may not be a problem.

For each example, we form a 4-tuple (A,B,C,D) for the economic model, then deduce the 4-tuple (A,K,C,Σ) for the associated VAR. Thus, these examples all involve 'engineering' a VAR from an economic theory rather than 'reverse engineering' features of the dynamics to economic shocks from a VAR. Our intention is that these 'engineering' examples will provide insights about the pitfalls and possibilities for successfully performing reverse engineering exercises. Nevertheless, we recognize that an analyst who is confident about his model and who knows the mapping from its deep parameters to the tuple (A,B,C,D) would not proceed by first estimating a VAR not restricted by his theory. Instead, he would use one of the likelihood based approaches – either maximum likelihood or a Bayesian procedure – and directly estimate the deep parameters in the manner recommended a quarter of a century ago by Hansen and Sargent (1981).[4] Practitioners who estimate relatively unrestricted VARs are doubtful about many details of the dynamics, and prefer to impose restrictions that they believe will be robust across a variety of specifications.[5] That is the

[3]A theorist who wants to analyze the data as if her theory *is not* true would not turn her back on the likelihood function either, but this robustness business is another story.

[4]Of course, as we remind the reader in section II.12, at each step in the iterative calculations in his estimations, he would form a restricted VAR in order to factor the conditional likelihood.

[5]This is explicitly the motivation of Jonas Fisher (2003).

audience that should be vitally interested in our reverse engineering exercise. A good way to shed light on that endeavor is to assemble some representative examples of environments ((A,B,C,D)'s) where reverse engineering can be done easily and others where it cannot.

I.3. **Organization.** Section II describes the mapping from the objects (A,B,C,D) that characterize (a linear approximation to) an economic model to objects (A,K,C,Σ) that define an infinite order vector autoregression. This section defines impulse response from economic shocks to observables and from VAR innovations to observables, reviews the connection between an infinite order VAR and a conditional likelihood, and describes the invertibility criterion in terms of the zeros of a particular matrix characteristic polynomial. Section III gives an easy to check condition for invertibility in terms of the eigenvalues of the matrix $A-BD^{-1}C$. Sections IV, V, VI, and VII apply this check to four models: a permanent income model, the two-shock model of Fisher (2003), the sticky price and wage model of Erceg, Henderson, and Levin (2000), and the household production model of Benhabib, Rogerson, and Wright (1991). We check the invertibility condition for both calibrated and estimated versions of these models and for alternative sets of observables, thereby illustrating an insight of Watson (1994). Section VIII briefly describes findings of two recent papers that address related issues. Section IX contains some concluding remarks. Three appendices describe the priors that we used to obtain posterior distributions of the parameters of several models; a fourth appendix gives formulas that map (A,B,C,D) into a finite order VAR.

II. Mapping from an economic model to a VAR

This section describes a class of economic model with shocks w_t to preferences, technologies, agents' information sets, and the economist's measurements. For a set of observables y_t, we let a_t's be innovations to a VAR. The innovations a_t can be expressed as a linear combination of the history of the w_s's up to t. We state conditions on the (linear approximation to) the economic model under which w_t can be recovered from the history of a_s's through t.

II.1. **Representation of an equilibrium.** We start with an equilibrium of an economic model or an approximation to it that has a representation in the state-space form

$$x_{t+1} = Ax_t + Bw_t \tag{1}$$

$$y_t = Cx_t + Dw_t \tag{2}$$

where w_t is a Gaussian vector white noise satisfying $Ew_t = 0$, $Ew_tw_t' = I$, $Ew_tw_{t-j} = 0$ for $j \neq 0$. Here x_t is an $n \times 1$ vector of possibly unobserved state variables, y_t is a $k \times 1$ vector of variables observed by an economist or econometrician, and w_t is an $m \times 1$ vector

of economic shocks and measurement errors impinging on the states and observables. The observation vector y_t typically includes some prices, quantities, and capital stocks. With m shocks in the economic model, n states, and k observables, A is $n \times n$, B is $n \times m$, C is $k \times n$, and D is $k \times m$. In general, $k \neq m$, although we shall soon devote some special attention to an interesting 'square case' in which $k = m$.

There are two popular ways to obtain equilibrium representations of the form (1)-(2). The first is to compute a linear or loglinear approximation of a nonlinear model about a nonstochastic steady state, as exposited for example, in Christiano (1990), Uhlig (1999), or the `dynare` manual.[6] It is straightforward to collect the linear or log linear approximations to the equilibrium decision rules and to arrange them into the state-space form (1)-(2). We provide an extended example in section V. A second way is to set (1)-(2) directly as a representation of a member of a class of dynamic stochastic general equilibrium models with linear transition laws and quadratic preferences. Ryoo and Rosen (2003), Topel and Rosen (1988), Rosen, Murphy, and Scheinkman (1994), and Hansen and Sargent (2005) provide many examples. We describe such an example in section IV.

The economic shocks, the w_t's, are comprised of two kinds of shocks, the first being the shocks to preferences, technologies, and information sets within an economic model, the second being errors in measuring y. To distinguish these two components, we can write

$$\begin{aligned} Bw_t &= \begin{bmatrix} B_1 & 0 \end{bmatrix} \begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix} \\ Dw_t &= \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} w_{1t} \\ w_{2t} \end{bmatrix}, \end{aligned}$$

where w_{1t} represents the economic shocks and w_{2t} represents pure y-measurement error.

II.2. **Argument in a nutshell.** The following simple argument isolates a main outcome and the major themes of this paper.[7] When D is square and D^{-1} exists, (2) implies $w_t = D^{-1}(y_t - Cx_t)$. Substituting this into (1) and rearranging gives $[I - (A - BD^{-1}C)L]x_{t+1} = BD^{-1}y_t$, where L is the lag operator. If the eigenvalues of $(A - BD^{-1}C)$ are strictly less than one in modulus, then the inverse of the operator on the left of this equation gives a square summable polynomial in L, and we can solve for $x_{t+1} = \sum_{j=0}^{\infty}[A - BD^{-1}C]^j BD^{-1}y_{t-j}$. Shifting back one period and substituting this equation into (1) gives

$$y_t = C \sum_{j=0}^{\infty} [A - BD^{-1}C]^j BD^{-1}y_{t-j-1} + Dw_t. \tag{3}$$

[6]Dynare is a suite of Matlab programs that computes linear approximations of a big class of dynamic stochastic general equilibrium models.

[7]The argument in this subsection is entirely due to our discussant Mark Watson.

Under the conditions used to derive it (i.e., D is invertible and the eigenvalues of $(A - BD^{-1}C)$ are strictly less than one in modulus), equation (3) defines a vector autoregression for y_t because Dw_t is orthogonal to y_{t-j} for all $j > 0$. The impulse response function associated with (1) describes both the VAR and the theoretical model. Thus, when $(A - BD^{-1}C)$ is a stable matrix,[8] the VAR matches up naturally with the theory. In the following sections we say more about this outcome. In addition, we explore why, when $(A - BD^{-1}C)$ is not a stable matrix, the impulse responses from a VAR cannot be made to match up with those from the economic model.

II.3. **Impulse response from economic shocks w to observables y.** VAR researchers are often interested in an impulse response function from the w_t's to the y_t's,

$$y_t = \mu_y + d(L)w_t \tag{4}$$

where L is the lag operator, $d(L) = \sum_{j=0}^{\infty} d_j L^j$, $\sum_{j=0}^{\infty} \text{trace}(d_j d_j') < +\infty$, and μ_y is the mean of y, which can be computed as follows. If all eigenvalues of A are less than unity in modulus, except for a single unit eigenvalue associated with a constant state variable, then the mean μ_x of the stationary distribution of x_t can be computed by appropriately scaling the eigenvector of A associated with the unit eigenvalue: $(I - A)\mu_x = 0$. After solving this equation for μ_x, the mean μ_y of the stationary distribution of y_t can be computed from $\mu_y = C\mu_x$.

Elementary calculations with system (1)-(2) deliver

$$y_t = \mu_y + [C(I - AL)^{-1}BL + D]w_t, \tag{5}$$

so that evidently

$$\begin{aligned} d_0 &= D \\ d_j &= CA^{j-1}B \quad j \geq 1. \end{aligned}$$

To economize on notation, from now on we shall assume that $\mu_y = 0$. Note that (4) transforms m shocks w_t into k observables y_{t+j}, $j \geq 0$. Formula (5) tells us how to compute the impulse response function directly from the state space representation (A,B,C,D) of the economic model.[9]

II.4. **Nonuniqueness of (A,B,C,D).** It is a sensible position to regard the basic theoretical object as being the impulse response function in (5). In general, there are multiple four-tuples of matrices (A,B,C,D) that can be used to represent an impulse response function in (5): different (A,B,C,D)'s can deliver the same $\mu_y, d(L)$. For convenience, one often selects a particular member of this class by choosing a minimum state realization of

[8] A square matrix is said to be stable if all its eigenvalues are strictly less than one in modulus.

[9] The Matlab control toolkit program `impulse.m` calculates $d(L)$ from (A,B,C,D).

$\mu_y, d(L)$.[10] We can sometimes exploit the freedom to switch among these representations in order to get a representation that satisfies our assumptions 1 and 2 in section III.

II.5. **The VAR and the associated impulse response.** An infinite order vector autoregression is defined by the projection equation

$$y_t = \alpha + \sum_{j=1}^{\infty} A_j y_{t-j} + a_t \tag{6}$$

where $\alpha = (I - \sum_{j=1}^{\infty} A_j)\mu_y$, $a_t = y_t - E[y_t|y^{t-1}]$, $\sum_{j=1}^{\infty} \text{trace}(A_j A_j') < +\infty$, and the A_js satisfy the least squares orthogonality conditions

$$E a_t y_{t-j}' = 0, \quad j \geq 1. \tag{7}$$

These least squares normal equations imply that $E a_t = 0$ and $E a_t a_{t-s}' = 0$ for $s \neq 0$. Letting $E a_t a_t' = \Omega = GG'$, we can represent $a_t = G\varepsilon_t$, where ε_t is a stochastic process that satisfies $E\varepsilon_t = 0$, $E\varepsilon_t\varepsilon_t' = I$, and $E\varepsilon_t\varepsilon_{t-j}' = 0$ for $j \neq 0$. Then write (6) as

$$y_t = \alpha + \sum_{j=1}^{\infty} A_j y_{t-j} + G\varepsilon_t \tag{8}$$

where $a_t = G\varepsilon_t$.

Compute the polynomial in the operator $c(L) = \sum_{j=0}^{\infty} c_j L^j = (I - \sum_{j=1}^{\infty} A_j L^j)^{-1} G$ and use it to form the moving average representation

$$y_t = \mu_y + c(L)\varepsilon_t. \tag{9}$$

This is said to be a Wold moving average representation.[11] The shock process ε_t is said to be 'fundamental for y_t' because it is by construction in the space spanned by square summable linear combinations of current and past values of the y_t process. The defining characteristic of a Wold representation is that the associated innovation is fundamental for y_t. An impulse response function associated with an infinite order VAR is by construction a Wold representation. In particular, representation (9) is a population version of the impulse response function reported by a typical VAR researcher.

[10] The Matlab control toolkit command `sys=ss(sys,'min')` replaces a four-tuple (A,B,C,D) with an equivalent minimal state realization.

[11] A VAR representation does not exist when $\det(c(z))$ has zeros on the unit circle. See Whittle (1983) and Hansen and Sargent (1991a).

II.6. **Main issue in unrestricted VAR identification.** We are interested in knowing the circumstances under which the impulse response function (9) associated with the VAR matches the theoretical impulse response function (4) from a theoretical model. Thus, we want a formula like (5) that is also cast in terms of (A,B,C,D), but that tells the response of y_t to current and past ε_t's. The key to constructing this representation is the *innovations representation,* to which we now turn.

II.7. **The innovations representation: the (A,K,C,Σ) System.** We seek a mapping from the matrices (A,B,C,D) for an equilibrium stochastic process for y_t to the autoregression coefficients $A_j, j = 1,\ldots$ and volatility matrix G in (8) and the associated moving average coefficient d_j in (9). The innovations representation is the recursive representation for y_t that corresponds to a Wold representation. Associated with any state space system (A,B,C,D) of the form (1)-(2) is another state-space system called the *innovations representation*:[12]

$$\hat{x}_{t+1} = A\hat{x}_t + KG\varepsilon_t \tag{10}$$

$$y_t = C\hat{x}_t + G\varepsilon_t, \tag{11}$$

where $\hat{x}_t = E[x_t|y^{t-1}], G\varepsilon_t \equiv a_t = y_t - E[y_t|y^{t-1}]$, K is the Kalman gain from the steady state Kalman filter equations:

$$\Sigma = A\Sigma A' + BB' - (A\Sigma C' + BD') \tag{12}$$
$$(C\Sigma C' + DD')^{-1}(A\Sigma C' + BD')'$$

$$K = (A\Sigma C' + BD')(C\Sigma C' + DD')^{-1} \tag{13}$$

where $\Sigma = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$. The covariance matrix of the innovations $a_t = G\varepsilon_t$ equals

$$Ea_t a_t' = GG' = C\Sigma C' + DD'. \tag{14}$$

With m shocks in the economic model, n states, and k observables, K is $n \times k$ and G is $k \times k$. The vector processes a_t and ε_t are each of dimension $k \times 1$, as is the y_t process, and the matrix G is $k \times k$.

We use the following

Definition II.1. *$H(z^t)$ is the Hilbert space consisting of all square summable linear combinations of the one-sided infinite history of random vectors z^t.*

[12] The conditions for the existence of this representation are stated carefully, among other places, in Anderson, Hansen, McGrattan, and Sargent (1996). The conditions are that that (A,B,C,D) be such that iterations on the Riccati equation for $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$ converge, which makes the associated Kalman gain K_t converge to K. Sufficient conditions are that (A',C') is stabilizable and that (A',B') is detectable. See Anderson, Hansen, McGrattan, and Sargent (1996, page 175) for definitions of stabilizable and detectable.

The Kalman filter applies a Gram-Schmidt procedure to the history y^t to construct a history a^t with orthogonal increments that spans $H(y^t)$, i.e., is such that $H(y^t) = H(a^t)$ and for which $Ea_t a_s' = 0$ for $t \neq s$.

The innovations representation (10)-(11) for the y_t process resembles the original representation (1)-(2). It differs from it in that (a) the $n \times k$ matrix KG replaces the $n \times m$ matrix B; (b) the $k \times k$ matrix G replaces the $k \times m$ matrix D; and (c) the $k \times 1$ process ε_t replaces the $m \times 1$ process w_t.

II.8. **Formula for the Wold moving average representation in terms of (A,B,C,D).** The innovations representation (10)–(11) can be rearranged to assume the form of a Wold moving average representation

$$y_t = [G + C(I - AL)^{-1}KGL]\varepsilon_t, \tag{15}$$

which is a version of (9) with

$$\begin{aligned} c_0 &= G \\ c_j &= CA^{j-1}KG, \quad j \geq 1. \end{aligned}$$

II.9. **Formula for the VAR representation in terms of (A,B,C,D).** By applying a partitioned inverse formula to invert the operator $[G + C(I - AL)^{-1}KGL]$ in (15), Hansen and Sargent (2005) show that when the eigenvalues of $A - KC$ are strictly less than unity in modulus, y_t has an autoregressive representation given by

$$y_t = C[I - (A - KC)L]^{-1}Ky_{t-1} + G\varepsilon_t \tag{16}$$

which is of the form (8) with

$$A_j = C(A - KC)^{j-1}K, \quad j \geq 0. \tag{17}$$

II.10. **Conditions for existence of an infinite order VAR.** Remember that K depends on (A,B,C,D) through formulas (12), (13). Equations (16) and (19) indicate that the dominant eigenvalue of $A - KC$ controls the rate at which the autoregressive coefficients A_j converge to zero. If all the eigenvalues of $A - KC$ are strictly less than unity in modulus, the A_j coefficients converge to zero and y_t is said to have an (infinite order) vector autoregressive representation.[13] If all of the eigenvalues of $A - KC$ are less than or equal to unity, but one or more is equal to unity in modulus, then y_t does not have an autoregressive representation. To explain what failure of an AR representation to exist in this case means, consider the nth order autoregressions

$$y_t = \mu_y + \sum_{j=0}^{n} A_j^n y_{t-j} + a_t^n, \tag{18}$$

[13] See page 112 of Anderson and Moore (1979).

where $\mu_y, A_1^n, \ldots, A_n^n$ satisfy the population orthogonality conditions $Ea_t^n = 0$ and $Ea_t a_{t-j}' = 0$, $j = 1, \ldots, n$. When the eigenvalue of $A - KC$ with maximum modulus is unity, the projections $\hat{y}_t^n = \mu_y + \sum_{j=0}^n A_j^n y_{t-j}$ still converge in mean square to $\hat{y}_t^\infty = \hat{E}[y_t | y_{t-1}, \ldots]$ where $\hat{E}$ is the linear least squares projection operator.[14] That an autoregressive representation fails to exist means that the AR coefficients A_j^n do not converge as $n \to \infty$. However, $\hat{y}_t^n \to \hat{y}_t^\infty$ in mean square as $n \to \infty$.

II.11. **Formula for the VARMA representation in terms of (A,B,C,D).** Representation (15) is an infinite order vector moving average, and (16) is an infinite order vector autoregression. In the special square case that $n = k$ and the $k \times n$ matrix is of rank n, it is easy to deduce a VARMA representation.

Premultiply both sides of (16) by C^{-1}, then premultiply both sides of the result by $[I - (A - KC)L]$ and rearrange to obtain:

$$C^{-1} y_t = [(A - KC)C^{-1} + K] y_{t-1} + C^{-1} G \varepsilon_t - (A - KC) C^{-1} G \varepsilon_{t-1}.$$

Premultiply both sides of this equation by C to obtain:

$$y_t = C[(A - KC)C^{-1} + K] y_{t-1} + G\varepsilon_t - C(A - KC)C^{-1} G \varepsilon_{t-1}$$

or

$$y_t = C[(A - KC)C^{-1} + K] y_{t-1} + C[I - (A - KC)L] C^{-1} G \varepsilon_t \tag{19}$$

Equation (19) is a first-order VARMA (vector autoregressive, moving average process) for y_t. The presence of the moving average component indicates that the pure VAR representation (16) is in general of infinite order. We have more to say about the square case $n = k$ in section III.

II.12. **The conditional Likelihood.** A theoretical infinite order VAR or an innovations representation implied by (A,B,C,D) contains all of the implications of the economic model for first and second moments of the process $\{y_t\}$. When w_t is Gaussian, it exhausts the implications of the model for the joint distribution of any sequence of y_t's. This claim follows from the fact that all of the information that a time series of observations $\{y_t\}_{t=1}^T$ contains about the economic parameters underlying (A,B,C,D) is contained in the model's likelihood function. The innovations representation (10)-(11) or the infinite order vector autoregression (8) contains all of the information needed to construct a Gaussian likelihood function conditional on an initial infinite history of observations.[15]

[14] See Hansen and Sargent (1991b), chapter 2.

[15] See Hansen and Sargent (2005), chapter 9, for how the Kalman filter can also be used to construct an unconditional likelihood function.

Denote the likelihood function of a sample of data $\{y_t\}_{t=1}^T$ conditional on the infinite history y^0 by $f(y_T,y_{T-1},\ldots,y_1|y^0)$. Factor this likelihood as

$$L = f(y_T,y_{T-1},\ldots,y_1|y^0) = f_T(y_T|y^{T-1})\,f_{T-1}(y_{T-1}|y^{T-2})\cdots f_1(y_1|y^0). \tag{20}$$

Under the assumption that w_t is a Gaussian process, the conditional density $f_t(y_t|y^{t-1})$ is $\mathcal{N}(C\hat{x}_t, GG')$. Recalling that $a_t = y_t - C\hat{x}_t$ from (11), it follows that $\log f(y_T,y_{T-1},\ldots,y_1|y^0)$, the log of the conditional likelihood (20), equals

$$\log L = -.5\sum_{t=1}^{T}\left\{k\log 2\pi + \ln|GG'| + a_t'(GG')^{-1}a_t\right\}. \tag{21}$$

II.13. **Comparison of impulse responses.** Comparing (1)-(2) with (10)-(11), notice that the representations are equivalent when $B = KG$, $G = D$, and $\varepsilon_t = w_t$. Note that B,D, and w_t are objects embedded in an economic theory, while K,G, and ε_t are objects that are functions of the economic-theory determined the four-tuple (A,B,C,D), functions pinned down by the Kalman filter equations given above.

II.14. **The mapping from economic to VAR innovations.** We can combine and rearrange the two representations (1)-(2) and (12)-(13) to obtain the following system that describes the mapping from the economic shocks w_t to the innovations $G\varepsilon_t$ in the innovations representation:

$$\begin{bmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ KC & A-KC \end{bmatrix}\begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} B \\ KD \end{bmatrix} w_t \tag{22}$$

$$G\varepsilon_t = \begin{bmatrix} C & -C \end{bmatrix}\begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + Dw_t. \tag{23}$$

Define

$$A^* \equiv \begin{bmatrix} A & 0 \\ KC & A-KC \end{bmatrix} \tag{24}$$

and write (22)-(23) as

$$G\varepsilon_t = \left\{D + \begin{bmatrix} C & -C \end{bmatrix}\left[I - A^*L\right]^{-1}\begin{bmatrix} B \\ KD \end{bmatrix}L\right\}w_t. \tag{25}$$

Equation (25) verifies that by construction $H(\varepsilon^t) = H(a^t) \subset H(w^t)$. We want to know whether $H(a^t) = H(w^t)$. If it is, we say that the mapping (25) is invertible. For the purpose of directly interpreting the shocks $G\varepsilon_t$ in a vector autoregression in terms of the economic shocks w_t, we would prefer that $\begin{bmatrix} C & -C \end{bmatrix}\left[I - A^*L\right]^{-1}\begin{bmatrix} B \\ KD \end{bmatrix} = 0$ so that (25) would collapse to

$$G\varepsilon_t = Dw_t.$$

In the following section, we give a neat condition for checking whether $H(a^t) = H(\varepsilon^t)$ in the 'square' case that there are as many observables as economic shocks.

III. THE SQUARE CASE

In this section, we focus on the square case with equal numbers of shocks and observables and assume that D^{-1} exists. This is the case that is 'least likely' to have an invertibility problem. We state a necessary and sufficient condition for invertibility directly in terms of (A,B,C,D).

We shall make the following assumptions:

ASSUMPTION 1. The state space system (1), (2) is stable: all eigenvalues of A are less than one in modulus, except possibly one associated with a constant.

ASSUMPTION 2: D is square and invertible.

III.1. **Simple check for invertibility.** Assumption 2 often applies to systems with equal numbers of economic shocks and observables (i.e., variables in the pertinent VAR). Under Assumptions 1 and 2, (25) can be represented as

$$G\varepsilon_t = \left\{ I + \begin{bmatrix} C & -C \end{bmatrix} \left[I - A^* L \right]^{-1} \begin{bmatrix} BD^{-1} \\ K \end{bmatrix} L \right\} D w_t \tag{26}$$

A sufficient condition for $H(\varepsilon^t) = H(w^t)$ is that the polynomial in L on the right side of (26) has a square-summable inverse in nonnegative powers of L. Such an inverse exists if and only if the zeros of $\det\left\{ I + \begin{bmatrix} C & -C \end{bmatrix} \left[zI - A^* \right]^{-1} \begin{bmatrix} BD^{-1} \\ K \end{bmatrix} \right\}$ are all less than unity in modulus. The following theorem gives an easy way to check this sufficient condition for $H(a^t) = H(w^t)$.

Theorem III.1. *Under assumption 1, when D^{-1} exists, the zeros of*

$$\det\left\{ I + \begin{bmatrix} C & -C \end{bmatrix} \left[zI - A^* \right]^{-1} \begin{bmatrix} BD^{-1} \\ K \end{bmatrix} \right\}$$

equal the eigenvalues of $A - BD^{-1}C$ and the eigenvalues of A.

Proof. Write

$$I + C^*(zI - A^*)^{-1}B^* = \left\{ I + \begin{bmatrix} C & -C \end{bmatrix} \left[zI - A^* \right]^{-1} \begin{bmatrix} BD^{-1} \\ K \end{bmatrix} \right\} \tag{27}$$

where

$$\begin{aligned} C^* &= \begin{bmatrix} C & -C \end{bmatrix} \\ B^* &= \begin{bmatrix} BD^{-1} \\ K \end{bmatrix}. \end{aligned}$$

Now set $a = I, b = C^*, c = B^*, d = (zI - A^*)$ in the partitioned inverse formula

$$\det(a)\det(d + ca^{-1}b) = \det(d)\det(a + bd^{-1}c) \tag{28}$$

to get

$$\det(I + C^*(zI - A^*)^{-1}B^*) = \frac{\det(zI - A^* + B^*C^*)}{\det(zI - A^*)}. \tag{29}$$

Compute

$$zI - A^* + B^*C^* = zI - \begin{bmatrix} A - BD^{-1}C & BD^{-1}C \\ 0 & A \end{bmatrix}, \tag{30}$$

an equation that shows that the zeros of $(\det zI - A^* + B^*C^*)$ equal the eigenvalues of $A - BD^{-1}C$ and the eigenvalues of A. Using this result in (29) shows that the zeros of $\det(I + C^*(zI - A^*)^{-1}B^*)$ equal the eigenvalues of $A - BD^{-1}C$ and the eigenvalues of A. □

Remark III.2. *If all zeros of* $\det\left\{ I + \begin{bmatrix} C & -C \end{bmatrix} \left[zI - A^* \right]^{-1} \begin{bmatrix} BD^{-1} \\ K \end{bmatrix} \right\}$ *are less than or equal to unity in modulus, but one or more zeros equal unity in modulus, then an autoregressive representation fails to exist. Nevertheless, it is true that* $H(a^t) = H(w^t)$. *See Whittle (1983) and Hansen and Sargent (2005), chapter 2.*

Remark III.3. *Under assumptions 1 and 2, to check whether* $H(\varepsilon^t) = H(w^t)$, *we can simply inspect the eigenvalues of* $A - BD^{-1}C$. *Thus, we can check whether* $H(\varepsilon^t) = H(w^t)$ *by knowing only the fundamental objects* A, B, C, D *and without actually computing the innovation representation and* K, Σ *via the Kalman filter.*

Another way to express this point is to note that we can compute K and Σ directly without having to solve the Riccati equation (13), as we show in the following theorem.

Theorem III.4. *Suppose that* D^{-1} *exists and* $A - BD^{-1}C$ *is a stable matrix. Then in the steady state Kalman filter,* $K = BD^{-1}$ *and* $\Sigma = 0$.

Proof. Notice that $\Sigma = 0$ solves the steady state Riccati equation (13). Notice also that with $\Sigma = 0$, equation (13) implies that $K = BD^{-1}$. Furthermore, the Riccati difference equation corresponding to the steady state equation (13) can be represented as

$$\begin{aligned} \Sigma_{t+1} &= (A - K_tC)\Sigma_t(A - K_tC)' + BB' \\ &+ KDD'K' - BD'K' - KDB' \end{aligned} \tag{31}$$

where

$$K_t = (A\Sigma_t C' + BD')(C\Sigma_t C' + DD')^{-1}.$$

Under the conditions of the theorem, $A - K_t C$ converges to a stable matrix $A - BD^{-1}C$ and successive iterates Σ_t converge to zero starting from any positive semidefinite initial Σ_0. □

Remark III.5. *When D^{-1} exists and $A - BD^{-1}C$ is a stable matrix, the implication $\Sigma = 0$ means that there are no hidden state variables. It follows from $\Sigma = 0$ that x_{t+1} belongs to the Hilbert space generated by y^t.*

Remark III.6. *Under assumption 1, when $A - BD^{-1}C$ is a stable matrix, all of the zeros of $det(I + C^*(zI - A^*)^{-1}B^*)$ are cancelled by poles.*[16] *This follows from (29) and the definition of A^*. This result reflects a situation in which $G\varepsilon_t$ equals Dw_t; in particular, the correlation between the w_t process and the ε_t process is entirely contemporaneous.*

Remark III.7. *The one step ahead errors covariance matrix from the economic model $E[y_t - E[(y_t|w^{t-1})])][y_t - E[(y_t|w^{t-1})])]' = DD'$; while the one step ahead errors covariance matrix from the VAR is $E[y_t - E[(y_t|y^{t-1})][y_t - E[(y_t|y^{t-1})]' = DD' + C\Sigma C'$. When the invertibility condition fails, the prediction error variance matrix for the VAR is larger.*

We have the following

Corollary III.8. *Under the conditions of theorem III.1, $Dw_t = G\varepsilon_t$ and the innovation covariance matrix $GG' = DD'$. Thus, we are free to set $G = D$. Of course, the choice of G is unique only up to postmultiplication by an orthogonal matrix.*

Proof. It can be verified directly from (25) that when the conditions of theorem III.1 hold and, therefore, $K = BD$, it follows that $G\varepsilon_t = Dw_t$. □

Remark III.9. *Under the conditions of theorem III.1, corollary III.8 gives a way to find the correct identification scheme for the VAR. If an eigenvalue of $A - BD^{-1}C$ equals 1 in modulus, the model remains invertible (see remark III.2) but it lacks an infinite order VAR representation (see subsection II.10).*

The assertions in theorems III.1 and III.4 can be viewed as extensions to a vector process of the following well-known example:

Example III.10. *Take the scalar pure m.a. process*

$$y_t = w_t + \alpha w_{t-1}.$$

[16] From (29), the zeros of $\det(I + C^*(zI - A^*)^{-1}B^*)$ are the zeros of $\det(zI - A^* + B^*C^*)$ and the poles are the zeros of $\det(zI - A^*)$.

Let the state be $x_t = w_{t-1}$ *so that we have a state space representation with* $A = 0$, $B = 1$, $C = \alpha$, *and* $D = 1$. *Evidently,*

$$A - BD^{-1}C = -\alpha,$$

which is a stable matrix if and only if $|\alpha| < 1$, *in which case* $K = B$.

III.2. **A quartet of examples.** In the following four sections, we present four models designed to illustrate the theoretical results of sections II and III. We select our four examples to document when invertibility is a problem, when it is not, and when we face benign borderline cases in which an eigenvalue of unity in modulus is a symptom that an infinite order VAR does not exist. Each model will teach us something of interest in a context we feel is representative of some typical applications in macroeconomics.

First, we use a permanent income model to express the point about invertibility made by Hansen, Roberds, and Sargent (1991) in terms of the objects in theorem III.1. If we observe total income and consumption, this model is always non-invertible because one eigenvalue of $A - BD^{-1}C$ would be equal to the inverse of the discount factor, and, consequently, bigger than one. The origin of the non-invertibility of this model is the presence of two income shocks that cannot be disentangled from observing total income and consumption.

Second, we discuss the model with investment-specific technological shocks as described in Fisher (2003). We show that for our choice of observables, the model is invertible both for a sensible calibration and for parameters estimated using the Bayesian approach.

Third, we use the model with sticky prices and sticky wages of Erceg, Henderson, and Levin (2000). For a particular set of observables, this model is invertible but does not have a VAR representation for a reasonable choice of parameter values. This result teaches us about benign borderline cases that sometimes occur in applications in macroeconomics. We also estimate the model using the Bayesian approach and show that, for our choice of observables, the posterior probability of the model being non-invertible is zero.

Finally, we study a model of household production described by Benhabib, Rogerson, and Wright (1991). We show that, for a sensible choice of parameter values, the model is invertible for one set of observables but non-invertible for another set. This model illustrates how the presence or absence of invertibility depends crucially on our choice of observables in ways that have been discussed by Hansen and Sargent (1981, 1991c), Watson (1994), and Reichlin and Lippi (1994). Finally, we estimate this model using the Bayesian approach and show how the posterior probability of the model being non-invertible is zero for our first set of observables, but one for the second set.[17]

[17] Note that our discussion of invertibility holds for linear or linearized models. If we work with non-linear economies, different issues appear as illustrated in Caballero and Engel, (2004).

IV. A BADLY BEHAVED EXAMPLE: A PERMANENT INCOME MODEL

This section briefly reviews how the structure of the permanent income models of Hansen, Sargent, and Roberds (1991) manifests itself in ways described by our theorem III.1. Assume that there is a representative household whose preferences over stochastic sequences of consumption c_t and capital accumulation $k_t - k_{t-1}$ are representable by the following utility function:

$$-.5\sum_{t=0}^{\infty}\beta^t[(c_t-b)^2+\varepsilon(k_t-k_{t-1})^2] \tag{32}$$

where $\varepsilon > 0$ is a very small number, making $\varepsilon(k_t - k_{t-1})^2$ a small adjustment cost that we include to select an interesting solution.[18]

The representative household maximizes utility function (32) subject to the asset accumulation equation:

$$k_t + c_t \leq Rk_{t-1} + d_t$$

with k_{-1} as an initial condition, and where the endowment d_t follows the two-component process described by:

$$d_t = \mu_d + \frac{1}{1-\rho_1 L}\sigma_1 w_{1t} + \frac{1}{1-\lambda_1 L}\sigma_2 w_{2t}$$

where $|\rho_1| < 1$, $|\lambda_1| < 1$, $w_{1t} \sim \mathcal{N}(0,1)$, and $w_{2t} \sim \mathcal{N}(0,1)$.

We follow Hall (1978) and set $R\beta = 1$ in order to deliver the outcome that k_t and c_t are cointegrated.[19] Our choice of parameter values is as follows: $R = 1.05$, $\rho_1 = 0.9$, $\lambda_1 = 0.6$, $\mu_d = 5$, and $b = 30$.

IV.1. **The A, B, C, and D matrices.** Let $d_{1t} = \frac{1}{1-\rho_1 L}\sigma_1 w_{1t}$, $d_{2t} = \frac{1}{1-\lambda_1 L}\sigma_2 w_{2t}$, and $d_t = \mu_d + d_{1t} + d_{2t}$. Define the state vector as $x_t = \begin{bmatrix} k_{t-1} & 1 & d_{1t} & d_{2t} \end{bmatrix}'$ and let the observable variables be $y_t = \begin{bmatrix} c_t & d_t \end{bmatrix}'$. We can write our *A*, *B*, *C*, and *D* matrices as follows:[20]

$$\begin{aligned} x_{t+1} &= Ax_t + Bw_t \\ y_t &= Cx_t + Dw_t \end{aligned}$$

[18] If we set $\varepsilon = 0$, the solution of the problem is $c_t = b$.

[19] This outcome occurs in the limit as $\varepsilon \searrow 0$.

[20] We used Hansen and Sargent's (2005) Matlab program `solvea.m` to compute (A,B,C,D) by setting `S=[sc;sd(1,:)], A=ao, B=c, C=S*ao, D=S*c`.

where $w_t = \begin{bmatrix} w_{1t} & w_{2t} \end{bmatrix}'$ is a vector of white noise with mean zero and identity contemporaneous covariance matrix and:

$$\begin{aligned} A &= \begin{bmatrix} 1.0000 & 0.0000 & 0.6667 & 0.8889 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.9000 & 0 \\ 0 & 0 & 0 & 0.6000 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.5000 & 0 \\ 0 & 0.8000 \end{bmatrix} \\ C &= \begin{bmatrix} 0.0500 & 5.0000 & 0.3333 & 0.1111 \\ 0 & 5.0000 & 0.9000 & 0.6000 \end{bmatrix} \\ D &= \begin{bmatrix} 0.1667 & 0.0889 \\ 0.5000 & 0.8000 \end{bmatrix}. \end{aligned}$$

It follows that:

$$A - BD^{-1}C = \begin{bmatrix} 1.0000 & 0.0000 & 0.6667 & 0.8889 \\ 0 & 1.0000 & 0 & 0 \\ -0.2250 & -20.0000 & -0.1500 & -0.2000 \\ 0.2250 & 15.0000 & 0.1500 & 0.2000 \end{bmatrix}.$$

This matrix has an eigenvalue of 1.05, which equals R. Therefore, the mapping (26) is not invertible. It follows that the Hilbert space $H(a^t)$ spanned by the history of VAR shocks is smaller than the space $H(w^t)$ spanned by the space of economic shocks. Furthermore, in general the shapes of the impulse responses to ε_t and w_t differ.[21]

Motivated by remark III.7, as a measure of the information lost in the history y^t conditioning the VAR compared to the history w^t of shocks in the economic model, we computed $C\Sigma C' = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0227 \end{bmatrix}$ and $DD' = \begin{bmatrix} 0.0357 & 0.1544 \\ 0.1544 & 0.8900 \end{bmatrix}$, where $y_t = \begin{bmatrix} c_t & d_t \end{bmatrix}'$. These matrices reveal that while the VAR correctly estimates the one-step ahead prediction error variance in consumption (this is after all the content of Hall's (1978) characterization of the linear-quadratic permanent income model), it overestimates the volatility of the aggregate endowment shock from the consumer's point of view. The overestimation of this volatility comes hand in hand with failing to match the impulse response function.

[21] Invertibility of the mapping (26) for the permanent income model is obtained if the observation vector is either $\begin{bmatrix} c_t & k_t \end{bmatrix}'$ or $\begin{bmatrix} c_t & k_t - k_{t-1} \end{bmatrix}'$. With either of these observation vectors, the offending zero at R flips to become a zero at R^{-1}.

IV.2. **Historical note.** Sargent (1987, chapter XIII), Hansen, Roberds, and Sargent (1991), and Roberds (1991) studied a version of this example in response to a question asked by Robert E. Lucas, Jr., at a 1985 Minneapolis Fed conference: with a constant interest rate, what restrictions the hypothesis of present value budget balance place on a vector autoregression for government expenditures and tax receipts? The permanent income model is isomorphic to a stochastic version of a tax smoothing model in the style of Barro (1979) with total tax collections τ_t replacing consumption c_t and government expenditures g_t replacing the endowment d_t. This model imposes two restrictions on the c_t and d_t process: (1) present value budget balance, and (2) c_t must be a martingale. Because it implies equal present values of the moving average coefficients of d_t and c_t to either economic shock w_{it}, present value budget balance puts a zero of R into the operator on the right side of (26) and is therefore the source of non-invertibility.

Hansen, Roberds, and Sargent (1991) went on to answer Lucas's question by showing that present value budget balance by itself puts no testable restrictions on the infinite order VAR of $\begin{bmatrix} c_t & d_t \end{bmatrix}'$.

The permanent income example with c_t and d_t as the observables is one in which the invertibility condition is bound to fail. That stands as a counterexample to a presumption that VAR shocks always readily match up with the economic shocks w_t. It is thus one important example of things that can go wrong. However, there are other examples in which things can go right. In the next sections, we turn to examples that are invertible.

V. A BETTER BEHAVED EXAMPLE: JONAS FISHER'S TWO TECHNOLOGY SHOCK MODEL

The model of Fisher (2003) is a good laboratory for us because (1) Fisher explicitly remarks that invertibility is a prerequisite for his interpretations to hold water; (2) at least with Fisher's observables, invertiblity can be established by a direct argument; and (3) Fisher's model directly confronts some of the issues about matching innovations from VARs to productivity shocks that have preoccupied critics of VARs (see Chari, Kehoe, and McGrattan (2005)).

Fisher (2003) assesses the impact of technology shocks on business cycles by imposing long-run restrictions on an estimated non-structural VAR. Fisher explicitly acknowledges that a necessary condition for his procedure to be compelling is that the mapping (26) be invertible, and he assumes but does not verify invertibility. He imposes a long-run restriction on G that is suggested by an analysis of his exogenous growth model with two orthogonal unit-root technology processes. In this section, we use our theorem III.1 to verify that that invertibility assumption is indeed valid at calibrated values for the parameters in Fisher's model.

Recovering the theoretical impulse responses from an unrestricted VAR requires assigning the correct identifying matrix G. In this section we also show that Fisher's choice is the right one. In the last part of the section we extend the results by reporting the posterior probability of the model being non-invertible using the Bayesian approach.

Fisher's model features a representative household whose preferences over stochastic sequences of consumption C_t and leisure $1-L_t$ are representable by the utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t + \psi \log(1-L_t)\right)$$

where $\beta \in (0,1)$ is the discount factor and E_0 is the conditional expectation operator. The resource constraint is:

$$C_t + X_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

and the law of motion for capital is:

$$K_{t+1} = (1-\delta) K_t + V_t X_t,$$

and:

$$\begin{aligned} A_t &= e^{\gamma + C_a(L)\sigma_a w_{at}} A_{t-1},\ \gamma \geq 0 \\ V_t &= e^{\upsilon + C_\upsilon(L)\sigma_\upsilon w_{\upsilon t}} V_{t-1},\ \upsilon \geq 0 \\ [w_{at}, w_{\upsilon t}]' &\sim \mathcal{N}(0, I) \end{aligned}$$

where $C_a(L)$ and $C_\upsilon(L)$ are square summable polynomials in the lag operator L. We assume that C_a and C_υ are both the identity operator.

V.1. **The A, B, C, and D matrices.** Since the model is non-stationary, we define the scaling variable $Z_t = A_{t-1}^{\frac{1}{1-\alpha}} V_{t-1}^{\frac{\alpha}{1-\alpha}} = \left(A_{t-1} V_{t-1}^{\alpha}\right)^{\frac{1}{1-\alpha}}$ and the transformations $\widetilde{C}_t = \frac{C_t}{Z_t}$ and $\widetilde{K}_t = \frac{K_t}{Z_t V_{t-1}}$. Using loglinearization, we compute policy functions for the transformed capital stock around the steady state value of the variables:

$$\log \widetilde{K}_{t+1} - \log \widetilde{K}_{ss} = a_1 \left(\log \widetilde{K}_t - \log \widetilde{K}_{ss}\right) + a_2 \sigma_a w_{a,t} + a_3 \sigma_\upsilon w_{\upsilon,t},$$

for hours worked:

$$\log L_t - \log L_{ss} = b_1 \left(\log \widetilde{K}_t - \log \widetilde{K}_{ss}\right) + b_2 \sigma_a w_{a,t} + b_3 \sigma_\upsilon w_{\upsilon,t},$$

and for consumption:

$$\log \widetilde{C}_t - \log \widetilde{C}_{ss} = c_1 \left(\log \widetilde{K}_t - \log \widetilde{K}_{ss}\right) + c_2 \sigma_a w_{a,t} + c_3 \sigma_\upsilon w_{\upsilon,t},$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3 are constants that depend on the structural parameters of the economy. For this model, it turns out that $a_2 = a_3$, $a_2 = -\frac{a_1}{1-\alpha}$, and $b_2 = -\frac{b_1}{1-\alpha}$. We use these loglinear decision rules and the definitions of the transformed variables to

obtain the following state-space system in logarithms of our original (untransformed) variables:[22]

$$\begin{bmatrix} 1 \\ \Delta k_{t+1} \\ l_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1-a_1}{1-\alpha}(\gamma+\upsilon) & a_1 & 0 \\ -b_1\frac{\gamma+\upsilon}{1-\alpha} & b_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \Delta k_t \\ l_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{\sigma_a}{1-\alpha}+a_2\sigma_a & \frac{\sigma_\upsilon}{1-\alpha}+a_3\sigma_\upsilon \\ b_2\sigma_a & b_3\sigma_\upsilon \end{bmatrix} \begin{bmatrix} w_{a,t} \\ w_{\upsilon,t} \end{bmatrix} \quad (33)$$

$$\begin{bmatrix} \Delta(y_t - l_t) \\ l_t \end{bmatrix} = \begin{bmatrix} \gamma+\alpha b_1\frac{\gamma+\upsilon}{1-\alpha} & \alpha(1-b_1) & 0 \\ -b_1\frac{\gamma+\upsilon}{1-\alpha} & b_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Delta k_t \\ l_{t-1} \end{bmatrix} + \begin{bmatrix} (1-\alpha b_2)\sigma_a & -\alpha b_3\sigma_\upsilon \\ b_2\sigma_a & b_3\sigma_\upsilon \end{bmatrix} \begin{bmatrix} w_{a,t} \\ w_{\upsilon,t} \end{bmatrix} \quad (34)$$

where $k_t = \log K_t$, $l_t = \log L_t$, and $y_t = \log Y_t$.

Equations (33) and (34) form a state space system of the form *A*, *B*, *C*, and *D*. We set parameter values to be: $\beta = 0.99$, $\psi = 2.2$, $\alpha = 0.33$, $\delta = 0.025$, $\gamma = 0.01$, $\upsilon = 0.001$, $\sigma_a = 0.5$, and $\sigma_\upsilon = 0.2$.

The system formed by (33) and (34) is a 'square system' with two shocks and two observables. The eigenvalues of $A - BD^{-1}C$ are all strictly less than one in absolute value, which means that (26) is invertible. It also means that by setting $G = D$, the impulse response function to ε_t associated with an identified VAR perfectly matches the impulse response function to the theoretical shocks w_t. This impulse response function is reported in the bottom two rows of panels of Figure 1.

Therefore, we can conclude that for this particular model and this particular set of observables, invertibility prevails so that we are assured that there exists some *G* satisfying $GG' = DD'$ that makes the impulse response for the identified VAR match the theoretical impulse response to the *w*'s. However, the example also confirms the doubt expressed in the epigraph from Lucas and Stokey at the beginning of this paper. The required *G* must be equal to

$$D = \begin{bmatrix} 0.4370 & -0.0252 \\ 0.1908 & 0.0763 \end{bmatrix} \quad (35)$$

which lacks zeros, as Lucas and Stokey feared.

As mentioned before, Fisher explicitly acknowledges that in order to recover the theoretical impulse response to the *w*'s using an unrestricted VAR, we need the mapping (26) to be invertible. But it is important to note that we also need a way of discovering *G* while initially being ignorant of *D*. Is Fisher able to do that? We analyze this question below.

[22] This is not a minimum state space representation. With some work, $\log L_{t-1}$ can be eliminated as a state variable.

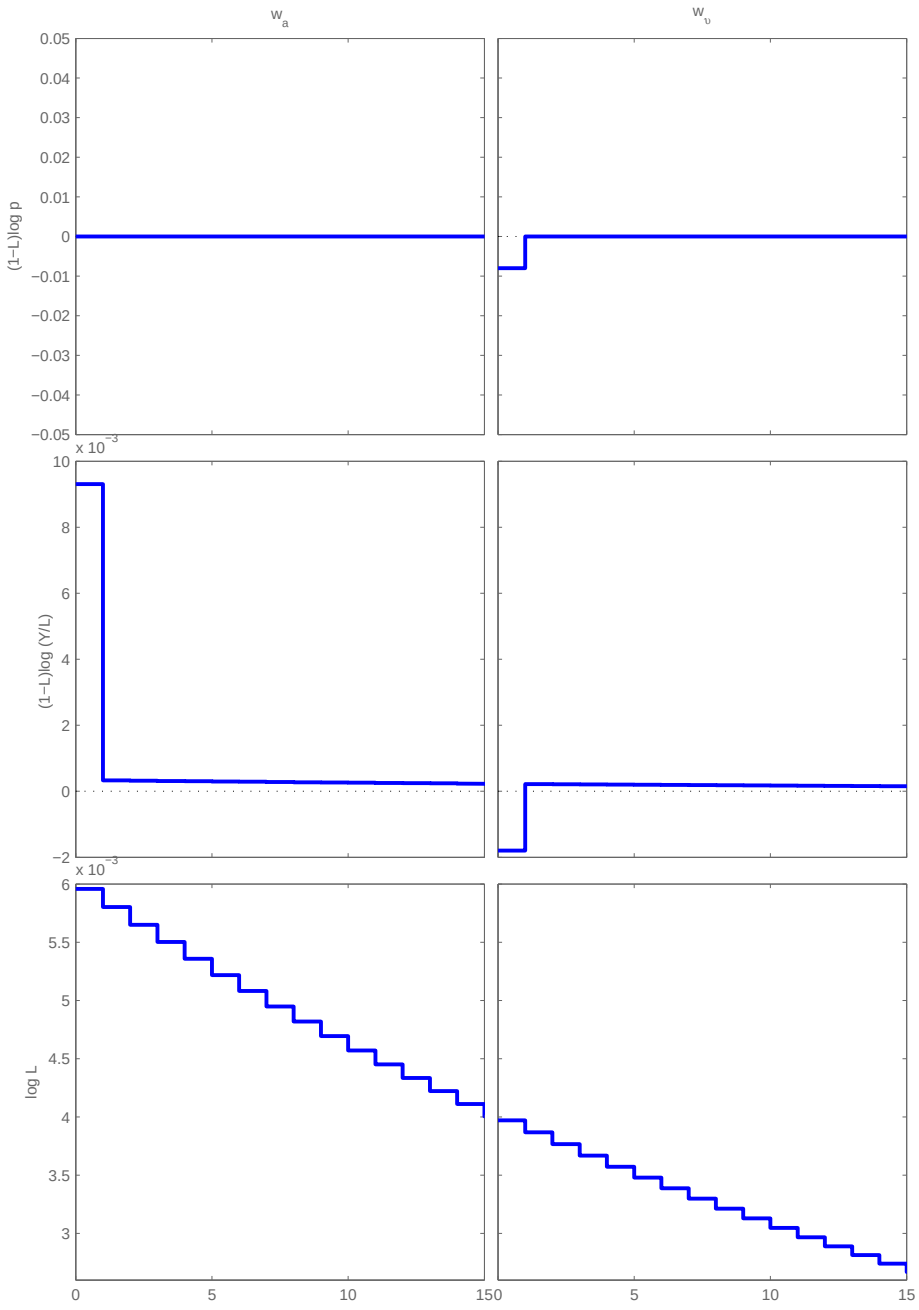

FIGURE 1. Common impulse response functions for VAR and economic structure for Fisher's model. For the two-observed-variable model, only the bottom two panels are pertinent.

V.2. **Fisher's identification procedure.** Fisher fits an unrestricted VAR with $\Delta \log p_t$ as an observable. Therefore, in order to explain his procedure, we need to define the state space system formed by (33) and the following observer equation:[23]

$$\begin{bmatrix} \Delta \log p_t \\ \Delta(y_t - l_t) \\ l_t \end{bmatrix} = \begin{bmatrix} -\nu & 0 & 0 \\ \gamma + \alpha b_1 \frac{\gamma+\upsilon}{1-\alpha} & \alpha(1-b_1) & 0 \\ -b_1 \frac{\gamma+\upsilon}{1-\alpha} & b_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Delta k_t \\ l_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & -\sigma_\upsilon \\ (1-\alpha b_2)\sigma_a & -\alpha b_3 \sigma_\upsilon \\ b_2 \sigma_a & b_3 \sigma_\upsilon \end{bmatrix} \begin{bmatrix} w_{a,t} \\ w_{\upsilon,t} \end{bmatrix} \tag{36}$$

Before we describe Fisher's bit of magic, we have to work around a technical difficulty. When using the three variable observation vector (36), we have to confront the fact that now we have a stochastically singular system. Two shocks are driving three observables (i.e., the system formed by (33) and (36) is not square). To eliminate the stochastic singularity problem, while staying as close as possible to Fisher's model, we add a very small normally distributed measurement error to $\log L_t$ with mean zero and standard deviation σ_{μ_1}.

To identify G from a three variable system, Fisher notes that $\Delta \log p_t = -\Delta \log V_t$ is an exogenous white noise that equals $w_{\upsilon,t}$. Therefore, any scheme for factoring GG' that identifies the row of $G\varepsilon_t$ associated with $\Delta \log p_t$ with $w_{\upsilon,t}$ should work. Fisher uses the following scheme that satisfies this condition.

Let $\Omega = C\Sigma C' + DD'$ be the covariance matrix of $G\varepsilon_t$ from the infinite order VAR (see equation (14)). Fisher (2003), footnote 5, applies a procedure of Blanchard and Quah to identify G. First, he forms $\hat{c}(1) = (I - \sum_{j=1}^{\infty} A_j)^{-1}$. Second, he computes a lower triangular Cholesky factor x of $\hat{c}(1)\Omega\hat{c}(1)'$, so that $xx' = \hat{c}(1)\Omega\hat{c}(1)'$. Third, after noting that $\hat{c}(1)G$ is a factor of $\hat{c}(1)\Omega\hat{c}(1)'$, he computes $G = \hat{c}(1)^{-1}x$.

This scheme succeeds in recovering a $G = D$. The impulse response associated with the infinite order VAR when $G = D$ conforms with the impulse response to the economic shocks. The impulse responses functions are reported in Figure 1. The only change from the VAR computed for our two variable system is the addition of the top panel in Figure 1.[24]

While these calculations confirm the validity of Fisher's identification procedure for his theoretical model, they do not really contradict the skepticism about zero restrictions on G or $\sum_{j=1}^{\infty} A_j L^j$ expressed in Lucas and Stokey's epigraph. The phrase '*endogenous* variables' in the epigraph bears remembering. Fisher's zero restriction that $\Delta \log p_t$ is never

[23]Notice that system (36) is by construction triangular, with the shock w_{at} being revealed by $\Delta \log p_t$. By an easy argument, it can then be shown that w_{vt} can be revealed from w_{at} and either of the remaining two observables. Therefore, the model with these observables is invertible.

[24]A simple alternative to Fisher's scheme would also work, namely, choosing G as a triangular Cholesky factor of the innovation covariance matrix Ω that sets $G_{12} w_{\upsilon,t} = \Delta \log p_t$.

influenced by $w_{a,t}$ comes from having specified the model so that $\Delta \log p_t$ is econometrically exogenous.[25]

V.2.1. *Finite order VARs for Fisher's model.* Using the projection formulas in Appendix D, we computed population versions of finite order vector autoregressions for both the two and three variable VARs implied by the Fisher's model. We computed VARs with 1 and 4 lags. Both gave such close approximations to the impulse response functions reported in Figure 1 that it was impossible to detect any difference when we plotted them on along side those in Figure 1. Therefore, for Fisher's model, a VAR with one lag that includes $\Delta \log p_t$, $\Delta(y_t - l_t)$, and l_t as regressors would do a fabulous job in matching the theoretical impulse responses if correctly identified.

V.3. **Posterior distribution for parameters of Fisher's Model.** We have argued that, when we observe $\Delta \log p_t$, $\Delta(y_t - l_t)$, and l_t, Fisher's model is invertible for a sensible choice of parameter values. In this subsection, we investigate whether the result also holds when we estimate the model using U.S. data.[26]

In order to do that, we employ the formulas reported in section II.12 to compute the likelihood function of Fisher's Model. Then, using the priors for the structural parameters reported in Appendix A, we draw from the posterior distribution of the parameters using McMc techniques. For each draw of the posterior, we evaluate our *A*, *B*, *C*, and *D* matrices. We find the eigenvalues associated with each draw of the matrices and compute the posterior probability of the model being non-invertible. The posterior mean, standard deviation, and a plot of the posterior distribution of the structural parameters are reported in Appendix A.

We observe $\Delta \log p_t$, $\Delta(y_t - l_t)$, and l_t, where $\log p_t$ is the log of real price of investment, $y_t - l_t$ is the log of labor productivity in consumption units, and l_t are logs of worked hours. We use quarterly data, with sample period 1955:01 to 2000:04. We follow Fisher and measure the real price of investment as the ratio of an investment deflator and a deflator for consumption derived from the National Income and Product Accounts (NIPA). In general, investment deflators are poorly measured, so we use Fisher's constructed investment deflator. Our consumption deflator corresponds to nondurable, services, the service flow from durables, and government consumption. Labor productivity is the non-farm business labor productivity series published by the Bureau of Labor Statistics (BLS). Per capita hours

[25]Fisher presents an informative discussion of this point in his paper and describes how the particular zero restriction that we have imposed would not prevail with a modified technology for producing investment goods.

[26]As noticed before, the system (33) and (36) is not square. In order to square the system we add a normally distributed measurement error to the observed worked hours, μ_1, with mean zero and standard deviation σ_{μ_1}.

are the BLS hours worked divided by population 16 and 65 years. In order to express labor productivity in consumption units per hour, we use the consumption deflator reported above.[27]

The results are that, given our priors, the posterior probability of the model being non-invertible is zero. This is not only true for the set of observables we report above, but it is also true for the case where, instead of log hours, we observe the difference of log hours, Δl_t.

VI. ANOTHER WELL BEHAVED EXAMPLE: THE EHL MODEL

Many sticky price models imply a reduction in hours worked after a positive productivity shock hits the economy (see Gali, (1999)). This theoretical finding has motivated some empirical work trying to identify a productivity shock and its consequences for hours worked.

We analyze Erceg, Henderson, and Levin's (2000) model with sticky prices and sticky wages. We use our theorem III.1 to verify that invertibility assumption is indeed valid at calibrated values. Since this model is well known in the literature, we will only present the equations describing the log deviation from steady-state values of the variables.

First, we have the Euler equation that relates output growth with the real rate of interest:

$$y_t = E_t y_{t+1} - \sigma(r_t - E_t \Delta p_{t+1} + E_t g_{t+1} - g_t)$$

where y_t denotes output, r_t is the nominal interest rate, g_t is the preference shifter shock, p_t is the price level, and σ is the elasticity of intertemporal substitution.

The production function and the real marginal cost of production are:

$$y_t = a_t + (1-\delta)n_t$$

$$mc_t = w_t - p_t + n_t - y_t$$

where a_t is a technology shock, n_t is the amount of hours worked, mc_t is the real marginal cost, w_t is the nominal wage, and δ is the capital share of output.

The marginal rate of substitution, mrs_t, between consumption and hours is:

$$mrs_t = g_t + \frac{1}{\sigma} y_t + \gamma n_t$$

where γ is the inverse elasticity of labor supply with respect to real wages. Hence, the preference shifter shock affects both the consumption Euler equation and the marginal rate of substitution.

[27] We thank Jonas Fisher for these data.

The pricing decision of the firm under a Calvo timing restriction delivers the following forward looking equation for price inflation, Δp_t:

$$\Delta p_t = \beta E_t \Delta p_{t+1} + \kappa_p (mc_t + \lambda_t)$$

where $\kappa_p = \frac{(1-\delta)(1-\theta_p\beta)(1-\theta_p)}{\theta_p(1+\delta(\varepsilon-1))}$ and $\bar{\varepsilon} = \frac{\bar{\lambda}}{\bar{\lambda}-1}$ is the steady state value of ε, the elasticity of substitution between types of goods. λ_t is the price markup shock, θ_p is the probability of keeping prices fixed during the period, and β is the discount factor.

Staggered wage setting delivers the following forward looking equation for wage inflation Δw_t:

$$\Delta w_t = \beta E_t \Delta w_{t+1} + \kappa_w (mrs_t - (w_t - p_t))$$

where $\kappa_w = \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\phi\gamma)}$, θ_w is the probability of keeping wages fixed in a given period, and ϕ is the elasticity of substitution between different varieties of labor in the production function. With staggered wage setting, it is no longer true that workers remain on their labor supply schedule. Hence, the driving force of current nominal wage growth is expected nominal wage growth, as well as the distance between the marginal rate of substitution and the real wage.

We use the following specification for the Taylor rule:

$$r_t = \rho_r r_{t-1} + (1-\rho_r)\left[\gamma_\pi \Delta p_t + \gamma_y y_t\right] + ms_t$$

where γ_π and γ_y are the long run responses of the monetary authority to deviations of inflation and output from their steady state values, and ms_t is the monetary shock. We include an interest rate smoothing parameter, ρ_r, following recent empirical work (see Clarida, Galí, and Gertler, (2000)).

To complete the model, we need the identity that links real wage growth, nominal wage growth and price inflation:

$$w_t - p_t = w_{t-1} - p_{t-1} + \Delta w_t - \Delta p_t.$$

Finally, we specify the shocks to follow:

$$\begin{aligned} a_t &= \rho_a a_{t-1} + w_t^a \\ g_t &= \rho_g g_{t-1} + w_t^g \\ ms_t &= w_t^{ms} \\ \lambda_t &= w_t^\lambda \end{aligned}$$

where each innovation w_t^i is distributed as $\mathcal{N}(0, \sigma_i^2)$ distribution, for $i = a, g, m, \lambda$. The innovations are uncorrelated with each other.

VI.1. **The A, B, C, and D matrices.** With the model in this loglinear form, we find that the coefficients of the policy function of the form:

$$k_t = Pk_{t-1} + Qz_t, \tag{37}$$

and

$$L_t = Rk_{t-1} + Sz_t, \tag{38}$$

where $k_t = [\ w_t - p_t \quad r_t \quad \Delta p_t \quad \Delta w_t \quad y_t\ ]'$, $L_t = [\ n_t \quad mc_t \quad mrs_t \quad c_t\ ]'$, and $z_t = [\ a_t \quad g_t \quad ms_t \quad \lambda_t\ ]'$.

A more convenient way of writing (37) and (38) is

$$\left[\ k_t' \quad z_t'\ \right]' = \begin{bmatrix} P & QN \\ N & 0 \end{bmatrix} \left[\ k_{t-1}' \quad z_{t-1}'\ \right]' + \begin{bmatrix} Q \\ I \end{bmatrix} w_t,$$

and

$$L_t = \left[\ R \quad SN\ \right] \left[\ k_{t-1}' \quad z_{t-1}'\ \right]' + Sw_t,$$

where $w_t = [\ w_t^a \quad w_t^{ms} \quad w_t^{\lambda} \quad w_t^{g}\ ]'$.

Let us consider the observables $Y_t = [\ \Delta p_t \quad \Delta n_t \quad y_t \quad w_t - p_t\ ]'$. Then, we obtain the following state-space system in log deviations from steady state:

$$\left[\ k_t' \quad z_t' \quad n_t\ \right]' = A \left[\ k_{t-1}' \quad z_{t-1}' \quad n_{t-1}\ \right]' + Bw_t, \tag{39}$$

$$Y_t = C \left[\ k_{t-1}' \quad z_{t-1}' \quad n_{t-1}\ \right]' + Dw_t, \tag{40}$$

where

$$A = \begin{bmatrix} P & QN & 0 \\ N & 0 & 0 \\ R_{1,\cdot} & (SN)_{1,\cdot} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} Q \\ R_{1,\cdot} \end{bmatrix}$$

$$C = \begin{bmatrix} & A_{3,\cdot} & \\ A_{6,\cdot} - [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\ ]' & & \\ & A_{5,\cdot} & \\ & A_{1,\cdot} & \end{bmatrix},$$

and

$$D = \begin{bmatrix} B_{3,\cdot} \\ B_{6,\cdot} \\ B_{5,\cdot} \\ B_{1,\cdot} \end{bmatrix}$$

where $A_{j,\cdot}$ stands for the jth row of matrix A and $x_t = \left[\ k_{t-1}' \quad z_{t-1}' \quad n_{t-1}\ \right]'$.

VI.2. **An empirical analysis of EHL's model.** Equations (39) and (40) form a state space system with matrices A, B, C, and D. Since the system is 'square', with four shocks and four observables, we can check its non-invertibility empirically. We do so from two empirical strategies.

First, we follow the literature and chose our parameter values to be: $\beta = 0.9$, $\bar{\varepsilon} = 6$, $\delta = 0.4$, $\sigma = 0.5$, $\theta_p = \theta_w = 0.9$, $\gamma = 2$, $\phi = 6$, $\gamma_y = 0.125$, $\gamma_\pi = 1.5$, $\rho_r = \rho_a = \rho_g = 0.9$, and $\sigma_i = 0.05$ for $i = a$, m, λ, and g.

Second, we estimate the model and compute the posterior probability of the system (39) and (40) being non-invertible as we did for the Fisher's model. The prior distributions, the posterior mean, standard deviation, and a plot of the posterior distribution of the structural parameters are reported in Appendix B. [28]

For both empirical strategies, the eigenvalues of $A - BD^{-1}C$ are all strictly less than one in absolute value except one that is exactly equal to one in absolute value. Therefore (26) is invertible. The unit eigenvalue means that the model does not have an infinite order VAR representation, but the fact that invertibility prevails means that to an arbitrarily good approximation the economic shocks can be expressed as linear combinations innovations in a sufficiently long finite order VARs.

VII. An Intermediate Example: the household production model

Benhabib, Rogerson, and Wright, (1991) and Greenwood and Hercowitz, (1991)introduced a household sector into a standard business cycle model. Their motivation was that the household sector is large both in terms of inputs (time worked and capital used) and in terms of output and that including this section improves the quantitative performance of the model along several dimensions. We use a model of household production to show how the invertibility or lack of invertibility of a model depends on the choice of observables.

We describe the Benhabib, Rogerson, and Wright (BRW) model of the business cycle as postulated in their 1991 paper. To show that the issues we are concerned with appear in the work of applied researchers, we calibrate the model with exactly the same parameter values than BRW did. Then we propose two simple VARs and we discuss its invertibility conditions.

[28] We observe Δp_t, Δn_t, y_t, and $w_t - p_t$, where Δp_t is the log of inflation, n_t is log of the share of per capita worked hours, y_t is the log of per capita output, and $w_t - p_t$ is the real wage. We use quarterly data for the sample period 1960:01 to 2001:04. Our measure of inflation is the nonfarm business sector deflator. Per capita hours are the BLS hours worked divided by the working age population. As the per capita output measure, we use the nonfarm business sector divided by the working age population. Finally, we take hourly compensation for the nonfarm business sector as nominal wages. We demean inflation and linearly detrend hours, output, and real wage.

The economy is populated by a representative household whose preferences are given by:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(a c_{mt}^{\theta} + (1-a) c_{ht}^{\theta} \right)^{\frac{1}{\theta}} + \psi \log\left(1 - l_{mt} - l_{ht}\right) \right\}$$

where c_{mt} is the consumption of the market good at time t, c_{ht} is the consumption of the household good, l_{mt} is labor in the market sector, l_{ht} is labor in the household sector, E_0 is the expectation operator, and $\beta \in (0,1)$ is the discount factor.

The technology to produce the market good y_{mt} and the household good y_{ht} is a Cobb-Douglas function of the form $y_{mt} = e^{z_{mt}} b k_{mt}^{\alpha} l_{mt}^{1-\alpha}$ and $y_{ht} = e^{z_{ht}} b k_{ht}^{\eta} l_{ht}^{1-\eta}$ where k_{mt} is the capital used in the market sector, k_{ht} is the capital used in the household sector, b measures the productivity level, and $\{z_{mt}, z_{ht}\}$ are shocks to the productivity levels, which follow an AR(1) process:

$$\begin{aligned} z_{mt} &= \rho_m z_{mt-1} + \varepsilon_{mt} \\ z_{ht} &= \rho_h z_{ht-1} + \varepsilon_{ht} \end{aligned}$$

where

$$\begin{pmatrix} \varepsilon_{mt} \\ \varepsilon_{ht} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma \right)$$

and $\Sigma = \begin{pmatrix} \sigma_m & \gamma \\ \gamma & \sigma_h \end{pmatrix}$. Finally, define $\begin{pmatrix} w_{mt} \\ w_{ht} \end{pmatrix} = \Sigma^{-1/2} \begin{pmatrix} \varepsilon_{mt} \\ \varepsilon_{ht} \end{pmatrix}$ such that:

$$\begin{pmatrix} w_{mt} \\ w_{ht} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, I \right).$$

The output of the market sector can be used for consumption or for investment while the output of the household sector can only be used for consumption.

Capital evolves according to $k_{t+1} = (1-\delta) k_t + i_t$ where δ is the depreciation factor and i_t is investment. Since capital can be moved across sectors without cost, the aggregate resource constraints of the economy are:

$$\begin{aligned} c_{mt} + i_t &= e^{z_{mt}} b k_{mt}^{\alpha} l_{mt}^{1-\alpha} \\ c_{ht} &= e^{z_{ht}} b k_{ht}^{\eta} l_{ht}^{1-\eta} \\ k_t &= k_{mt} + k_{ht} \end{aligned}$$

A competitive equilibrium for this economy can be defined in the standard way. Following BRW, we set the discount factor to $\beta = 0.99$ to match the interest rate, the participation of capital in each sector $\alpha = 0.33$ and $\eta = 0.08$, the depreciation $\delta = 0.025$, and the preference parameter $\theta = 0.8$ as in the related literature. We normalize the production function setting $b = 1$. The utility function parameters $a = 0.34$ and $\psi = 0.59$ are set to generate

a fraction of market work of 0.33 and of household work of 0.28. The stochastic process parameters $\rho_m = \rho_h = 0.95$, $\sigma_m = \sigma_h = 0.07$, and $\gamma = 2/3$. Note, however, that since we solve the model by linearization, the results of our discussion below are independent of the values of σ_m, σ_h, and γ because the coefficients of the policy functions are independent of them, the covariances do not affect the eigenvalues of $A - BD^{-1}C$.

We solve the model by loglinearizing its equilibrium conditions around the steady state. Then, we get a policy function for capital (where we use $\widehat{x_t} = \log x_t - \log x_{ss}$ to denote a variable value as a percentage deviations with respect to the steady state):

$$\widehat{k}_{t+1} = \gamma_k \widehat{k}_t + \gamma_m \rho_m z_{mt-1} + \gamma_h \rho_h z_{ht-1} + \gamma_m \varepsilon_{mt} + \gamma_h \varepsilon_{ht}$$

that, together with the law of motion for the technological shocks generate the transition equation:

$$\begin{bmatrix} \widehat{k}_{t+1} \\ z_{mt} \\ z_{ht} \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma_k & \gamma_m \rho_m & \gamma_h \rho_h & 0 \\ 0 & \rho_m & 0 & 0 \\ 0 & 0 & \rho_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ z_{mt-1} \\ z_{ht-1} \\ 1 \end{bmatrix} + \begin{bmatrix} \gamma_m & \gamma_h \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Sigma^{1/2} \begin{bmatrix} w_{mt} \\ w_{ht} \end{bmatrix} \tag{41}$$

VII.1. **Case I: things go well.** Now we illustrate how the concrete choice of observed variables renders the model invertible or not. First we assume that we observe market output and market hours. The policy functions for market output is given by:

$$\widehat{y}_{mt} = \eta_k \widehat{k}_t + \eta_m \rho_m z_{mt-1} + \eta_h \rho_h z_{ht-1} + \eta_m \varepsilon_{mt} + \eta_h \varepsilon_{ht}$$

or, in observed logs:

$$\log y_{mt} = \log y_{mss} + \eta_k \widehat{k}_t + \eta_m \rho_m z_{mt-1} + \eta_h \rho_h z_{ht-1} + \eta_m \varepsilon_{mt} + \eta_h \varepsilon_{ht}$$

The policy functions for hours:

$$\widehat{l}_t = \phi_k \widehat{k}_t + \phi_m \rho_m z_{mt-1} + \phi_h \rho_h z_{ht-1} + \phi_m \varepsilon_{mt} + \phi_h \varepsilon_{ht}$$

or in observed logs:

$$\log l_{mt} = \log l_{mss} + \phi_k \widehat{k}_t + \phi_m \rho_m z_{mt-1} + \phi_h \rho_h z_{ht-1} + \phi_m \varepsilon_{mt} + \phi_h \varepsilon_{ht}$$

Then, the measurement equation is:

$$\begin{bmatrix} \log y_{mt} \\ \log l_{mt} \end{bmatrix} = \begin{bmatrix} \eta_k & \eta_m \rho_m & \eta_h \rho_h & \log y_{mss} \\ \phi_k & \phi_m \rho_m & \phi_h \rho_h & \log l_{mss} \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ z_{mt-1} \\ z_{ht-1} \\ 1 \end{bmatrix} + \begin{bmatrix} \eta_m & \eta_h \\ \phi_m & \phi_h \end{bmatrix} \Sigma^{1/2} \begin{bmatrix} w_{mt} \\ w_{ht} \end{bmatrix} \tag{42}$$

For our calibration, the biggest eigenvalue of $A - BD^{-1}C$ is 0.910 and the model is invertible.

VII.2. **Case II: things go badly.** Now let us suppose that we change our observables and that we build a measurement equation with market consumption, whose policy function is:

$$\log c_{mt} = \log c_{mss} + \psi_k \widehat{k}_t + \psi_m \rho_m z_{mt-1} + \psi_h \rho_h z_{ht-1} + \psi_m \varepsilon_{mt} + \psi_h \varepsilon_{ht}$$

and labor. Then, we will have:

$$\begin{bmatrix} \log c_{mt} \\ \log l_{mt} \end{bmatrix} = \begin{bmatrix} \psi_k & \psi_m\rho_m & \psi_h\rho_h & \log c_{mss} \\ \phi_k & \phi_m\rho_m & \phi_h\rho_h & \log l_{mss} \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ z_{mt-1} \\ z_{ht-1} \\ 1 \end{bmatrix} + \begin{bmatrix} \psi_m & \psi_h \\ \phi_m & \phi_h \end{bmatrix} \Sigma^{1/2} \begin{bmatrix} w_{mt} \\ w_{ht} \end{bmatrix} \tag{43}$$

Now the biggest eigenvalue of $A - BD^{-1}C$ is 1.096 and the model is non-invertible.

VII.3. **Posterior for parameters of household production model.** We have seen that, for a sensible calibration, the household production model is invertible if we observe y_m and l_m, while non-invertible if we observe c_m and l_m. In this subsection we go further and compute the posterior probability of the systems (41) and (42) and (41) and (43) being non-invertible.

Our priors for the structural parameters are reported in Appendix C, as are the posterior mean, standard deviation, and a plot of the posterior distribution of the structural parameters.

In the first system, we observe $\log y_{mt}$ and $\log l_{mt}$, where $\log y_{mt}$ is the log of the per capita market output and $\log l_{mt}$ is log of the share of market worked hours. We define market output as the sum of real consumption, real private investment, and real government expenditures (all from BEA). To obtain output per capita, we divide output by civilian noninstitutional population between 16 and 65 years (BLS). The share of market worked hours is calculated as follows. We calculate per capita worked hours dividing hours worked in the nonfarm sector (BLS) by civilian noninstitutional population between 16 and 65 years (BLS). Then we divide per capita worked hours by 4000.[29] In the second system, we observe $\log c_{mt}$ and $\log l_{mt}$, where $\log c_{mt}$ is the log of the per capita market real consumption. We define market real consumption as the sum of real consumption of nondurables, real consumption of services, and real government expenditures (all from BEA). In order to obtain per capita market real consumption, we divide real consumption by civilian noninstitutional population between 16 and 65 years (from BLS). Finally, since our model does not have a balance growth path, we linearly detrend both per capita market output and per capita market real consumption. We use quarterly data and the sample period is 1964:01 to 2004:04.

[29] Implicitly we are assuming that the maximum number of hours that a person can work is 4000.

We find the following results. If we observe y_m and l_m, the posterior probability of the model being non-invertible is zero. If we observe c_m and l_m, the posterior probability of the model being non-invertible is one. These results confirm our calibration results.

VII.4. **Discussion.** Why do things go wrong in the second case and not in the first? Watson (1994) suggests that a researcher is most vulnerable to non-invertibility when her VAR excludes measures of important endogenous variables that depend on streams of expected future values of other variables. This is precisely the situation in our example. Models with household production limit the econometrician in terms of which activities of the household she observes. In general, the researcher can only measure market prices and quantities.

Furthermore the set of observables that generate the non-invertibilities is not obvious ex-ante. A researcher interested in the study of the interaction between consumption and hours can reasonably think about estimating a VAR with these two variables. How could she know, before computing the model, that this specification is non-invertible but one with market output and hours is?

VIII. Relation with the literature

There is a substantial critical literature evaluating the ability of VARs to document empirical phenomena. We do not attempt here to review this literature except two recent papers: Chari, Kehoe, and McGrattan (2005), or CKM, and Erceg, Guerrieri, and Gust (2004), or EGG. These two papers discuss issues related to our points and have received wide attention. Consequently, it is important to compare their findings with ours.

VIII.1. **Do technology shocks lead to a fall in hours?** CKM and EGG are motivated by the observation that Structural Vector Autoregressions (SVAR) have become popular as a procedure to isolate economic shocks. One of the most relevant examples is the discussion concerning the relation between hours and productivity shocks. A SVAR with the first differences of labor productivity[30] and first differences of hours, DSVAR from now on, and identified as proposed by Galí (1999), delivers that hours fall after a productivity shock. If, instead of the first difference of hours, we estimate a SVAR with hours in levels, LSVAR from now on, the evidence is ambiguous.

Both CKM and EGG specify simple business cycle models with shocks to technology, taxes, and, in the case of EGG, preferences and government consumption. They select parameters for their models (by ML estimation in CKM and by calibration in EGG) and use them as data generation processes for which they compute a DSVAR an LSVAR.

The results in CKM and EGG are similar. For example, if we compare figures 4, 6A, and 11A in CKM and figure 5A, panel 2 in EGG, we see that in both papers:

[30] All variables are expressed in logs

- The DSVAR gets the impulse response function wrong: the researcher that uses a DSVAR will find that hours respond negatively to a technology shock even when the true impulse response function is positive.
- The LSVAR estimates a impulse response function with the right sing but the wrong size. In addition, confidence bands are so big that the researcher cannot distinguish among competing models.

Moreover both papers document that the presence of capital is a probable cause of the bad behavior of SVARs. CKM also show that the eigenvalue of one induced by the use of hours in first differences in the DSVAR is empirically of little relevance for their finding. As the number of lags grow to a number too large for empirical applications, they can recover the right impulse response function.

However, CKM and EGG diverge dramatically in their reading of these findings. CKM conclude that SVARs are not a reliable technique to learn about the data. EGG are more sanguine. They recognize the limitations of SVARs, but they also emphasize that several remedies are available to avoid most the pitfalls of the tool and that, with the help of models serving as guideposts, SVARs are a fruitful approach to learn from the data.

VIII.2. **Comparison of CKM and EGG with our paper.** From the previous discussion, we can see how the focus of our paper is different from both CKM and EGG. Our paper is center on the ability of the researcher to recover economic shocks to the economy from the innovations of an unrestricted VAR of infinite order. CKM and EGG concentrate on the study of finite order SVARs.

CKM and EGG claim that non-invertibility is not a problem in their models. We find that, for the parameters they use, this is indeed the case when you consider a model without measurement errors. We also checked that when you use measurement errors, as CKM suggest when they estimate the model using a state space form, the model might be non-invertibility. CKM also argue that the presence of an eigenvalue equal to one induced by the first difference of hours is not important empirically. We corroborate their finding since in our theorems we document how eigenvalues of one do not cause lack of invertibility, for which we need eigenvalues strictly bigger than one. On the other hand, we show that eigenvalues of one imply that the model lacks of VAR resprentation (see subsection II.10), and that we can not be sure that $G = D$ is the correct identification scheme (see remark III.9). [31]

[31] We also agree with CKM disregard of the criticisms that since hours are bounded they cannot literally have a unit root.

IX. CONCLUDING REMARKS

We hesitate to draw sweeping conclusions about VARs from these exercises that apply our simple check for invertibility. For some (A,B,C,D)'s invertibility is a problem, and for others it is not. Some applications of VARs are informative about the shapes of impulse responses to some economic shocks that theories should attempt to match, others are not.

It is easy to reiterate the recommendation[32] to estimate the deep parameters of a complete and fully trusted model likelihood based methods. If you fully trust your model, that recommendation is incontrovertible. However, the enterprise of identifying shocks and responses to them by identifying SVARs aims to coax interesting patterns from the data that will prevail across a *set* of incompletely specified and not fully trusted models. If one is not dogmatic in favor of a particular fully specified model, it is easy to be sympathetic with the SVAR enterprise, despite its potential pitfalls.

[32]Offered for example by Hansen and Sargent (1981, 1991c).

	Mean	St. Deviation
β	0.9996	0.0002
ψ	5.3455	0.0156
α	0.4457	0.0164
δ	0.0005	0.0005
γ	0.0002	0.0002
υ	0.0108	0.0005
σ_a	0.0124	0.0007
σ_υ	0.0083	0.0804
σ_{μ_1}	0.0261	0.0014

TABLE 1. Posterior mean and standard deviation for the structural parameters of Fisher's model

APPENDIX A. PRIORS AND POSTERIOR ESTIMATES FOR FISHER'S MODEL

In this section we describe the priors of the structural parameters of Fisher's model used in section and the posterior distributions that we obtain.

Let us first describe the priors. Since we are mostly interested on how the likelihood function characterizes the posterior probability of the model being non-invertible, we use uniform priors for all the structural parameters. Therefore, we use the following prior distributions $\beta \sim U(0.90, 1.01)$, $\psi \sim U(0, 10)$, $\alpha \sim U(0, 1)$, $\delta \sim U(0.0, 0.1)$, $\gamma \sim U(0.0, 0.1)$, $\upsilon \sim U(0.0, 0.1)$, $\sigma_a \sim U(0.0, 0.2)$, $\sigma_\upsilon \sim U(0.0, 0.2)$, and $\sigma_{\mu_1} \sim U(0.0, 0.2)$, where $U(a, b)$ stands for the uniform distribution between a and b.

The above described prior distribution, the likelihood function of the model, and the Metropolis-Hastings algorithm are used to get 500.000 draws from the posterior distribution of the structural parameters. We obtain an acceptance ratio between 25 and 30 percent. The posterior mean and standard deviations of the parameters are reported in table 1, while the posterior distributions are drawn in figure 2.

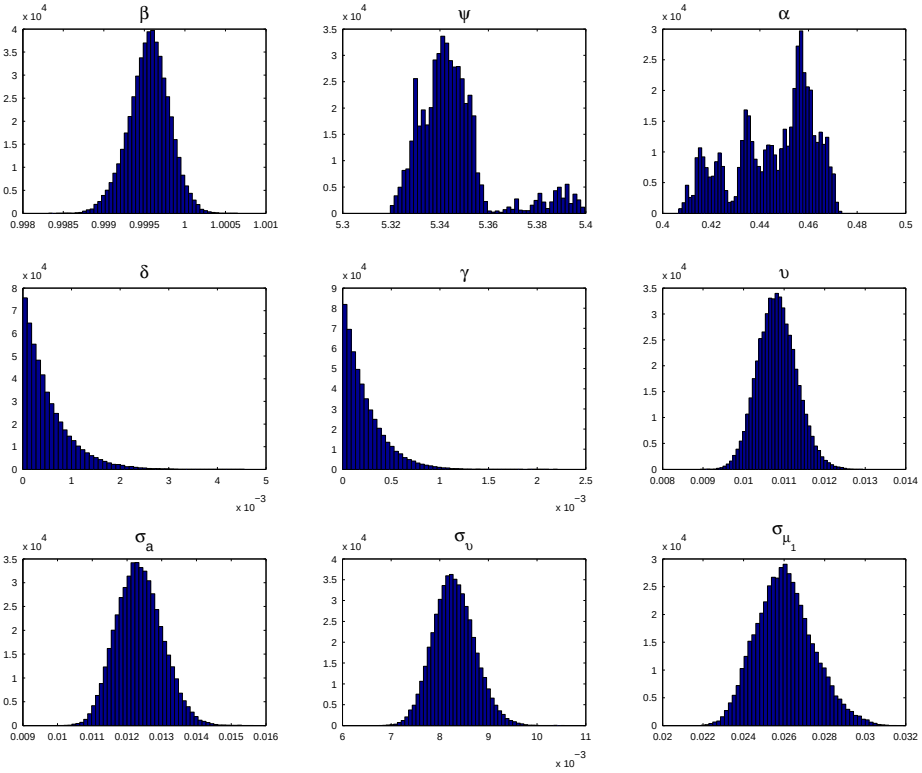

FIGURE 2. Posterior distribution for the structural parameters of Fisher's model.

APPENDIX B. PRIORS AND POSTERIOR ESTIMATES FOR EHL'S MODEL

In this section we describe the priors of the structural parameters of EHL's model used in section and the posterior distributions that we obtain.

We use the following prior distributions. The inverse of the elasticity of intertemporal substitution follows a gamma distribution, $\sigma^{-1} \sim Gamma(2, 1.25)$. This assumption implies a positive support for σ. We assume a gamma distribution for the average duration of prices, $1/(1-\theta_p) - 1 \sim Gamma(4,1)$. Thus, the average duration of prices has a prior mean of 3 and a prior standard deviation of 2. We also assume a gamma distribution for the average duration of wages, $1/(1-\theta_w) - 1 \sim Gamma(3,1)$. Hence, the average duration of prices has a prior mean of 2 and a prior standard deviation of 1.74. We assume a normal distribution for the inverse of the elasticity of the labor supply, $\gamma \sim \mathcal{N}(1, 0.5)$. We choose prior uniform distributions between 0 and 1 for the all the autorregresive parameters $\rho_r \sim U(0,1)$, $\rho_a \sim U(0,1)$, and $\rho_g \sim U(0,1)$. Regarding the Taylor rule coefficients, because we do not impose nonnegativity restrictions, we assume normal distributions, $\gamma_y \sim \mathcal{N}(1/8, 1/50)$ and $\gamma_\pi \sim \mathcal{N}(1.5, 1/10)$. Therefore, the prior means match Taylor's original guest. Finally, we choose prior uniform distributions between 0 and 1 for all standard deviations, $\sigma_a \sim U(0,1)$, $\sigma_{ms} \sim U(0,1)$, $\sigma_\lambda \sim U(0,1)$, and $\sigma_g \sim U(0,1)$. We impose dogmatic priors over the parameters β, δ, ϕ, and $\overline{\varepsilon}$. The reasons are as follows: First, because we do not consider capital, we have difficulty estimating β and δ. Second, there is an identification problem between the probability of the Calvo lottery, θ_p, and the mean of the price markup, $\overline{\varepsilon}$.[33] Therefore, it is impossible to identify θ_p and $\overline{\varepsilon}$ at the same time. Similarly, this problem emerges between θ_w and ϕ. The values we use ($\beta = 0.99$, $\delta = 0,36$, $\overline{\varepsilon} = 6$, and $\phi = 6$) are quite conventional in the literature.

The above described prior distributions, the likelihood function of the model, and the Metropolis-Hastings algorithm are used to get 500.000 draws from the posterior distribution of the structural parameters. We obtain an acceptance ratio between 25 and 30 percent. The posterior mean and standard deviations of the parameters are reported in table 2, while the posterior distributions are drawn in figure 3.

[33] The slope of the Phillips curve, κ_p, is the only equation containing θ_p and $\overline{\varepsilon}$.

	Mean	St. Deviation
σ	0.5502	0.0755
θ_p	0.9203	0.0023
θ_w	0.7479	0.0149
γ	1.7096	0.3268
ρ_r	0.9700	0.0089
γ_y	0.0887	0.0184
γ_π	1.5498	0.1005
ρ_a	0.6710	0.0156
ρ_g	0.1024	0.1428
σ_a	0.3855	0.0201
σ_{ms}	0.0094	0.0027
σ_λ	0.6841	0.0343
σ_g	0.4599	0.0477

TABLE 2. Posterior mean and standard Deviation for the structural parameters of EHL's model

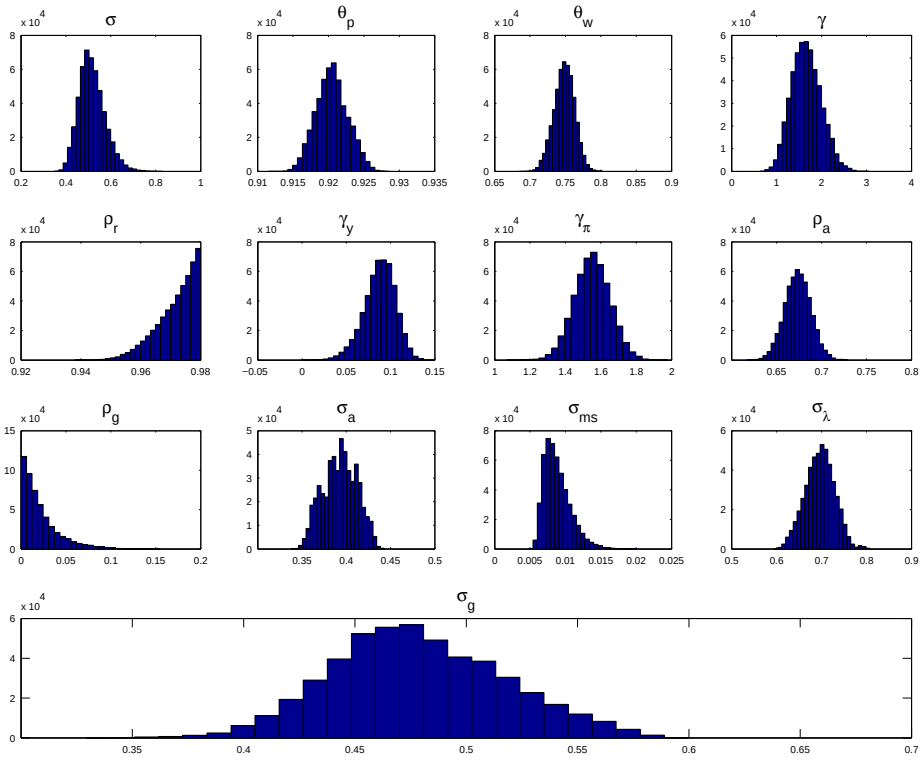

FIGURE 3. Posterior distribution for the structural parameters of EHL's model.

APPENDIX C. PRIORS AND POSTERIOR ESTIMATES FOR HOUSEHOLD PRODUCTION MODEL

In this section we describe the priors of the structural parameters of household production model used in section and the posterior distributions that we obtain.

We want to minimize the effects of the priors on the results, therefore we use uniform priors for all the structural parameters. Hence, we set that $\beta \sim U(0.9050, 0.9950)$, $\alpha \sim U(0,1)$, $\theta \sim U(0,1)$, $\eta \sim U(0,1)$, $\delta \sim U(0.0, 0.1)$, $\rho_m \sim U(0,1)$, $\rho_h \sim U(0,1)$, $\sigma_m \sim U(0.0, 0.01)$, $\sigma_h \sim U(0.0, 0.01)$, $b \sim U(-2.4641, 4.4641)$, and $\gamma \sim U(0,1)$. We have to impose dogmatic priors over the parameters $\psi = 0.58756$ and $a = 0.33707$. This two parameters fix the amount of leisure time allocated into market and household production. We find that there is not enough information in the data to estimate them (i.e., the likelihood function was almost flat in those dimensions). Hence, as suggested by Benhabib, Rogerson, and Wright (1991), we calibrate them to get 33 percent of time devoted to market production activities and 28 percent of time devoted to household production activities.

These prior distributions, the likelihood function of the model, and the Metropolis-Hastings algorithm are used to get 500.000 draws from the posterior distribution of the structural parameters. We obtain an acceptance ratio between 25 and 30 percent. Note that we estimate two models. In the first model, we observe y_m and l_m. In the second model, we observe c_m and l_m. For the first of the models, the posterior mean and standard deviations of the parameters are reported in table 3, while the posterior distributions are drawn in figure 4. For the second of the models, the posterior mean and standard deviations of the parameters are reported in table 4, while the posterior distributions are drawn in figure 5.

	Mean	St. Deviation
β	0.9390	2.2310^{-7}
α	0.4815	8.0410^{-7}
θ	0.9391	1.1310^{-6}
η	0.0132	4.5910^{-8}
δ	0.0318	8.9310^{-9}
ρ_m	0.6886	0.1899
ρ_h	0.8940	0.0076
σ_m	0.0056	0.0029
σ_h	0.0050	0.0029
γ	0.4457	0.2899
b	0.8907	2.8010^{-8}

TABLE 3. Posterior mean and standard Deviation for the structural parameters of household production model when we observe y_m and l_m.

	Mean	St. Deviation
β	0.9390	2.0110^{-7}
α	0.4815	4.4510^{-7}
θ	0.9391	6.3910^{-7}
η	0.0132	2.9010^{-8}
δ	0.0318	1.1310^{-8}
ρ_m	0.6601	0.1755
ρ_h	0.9220	0.0030
σ_m	0.0050	0.0029
σ_h	0.0050	0.0029
γ	0.4668	0.3141
b	0.8907	3.8810^{-8}

TABLE 4. Posterior mean and standard Deviation for the structural parameters of household production model when we observe c_m and l_m.

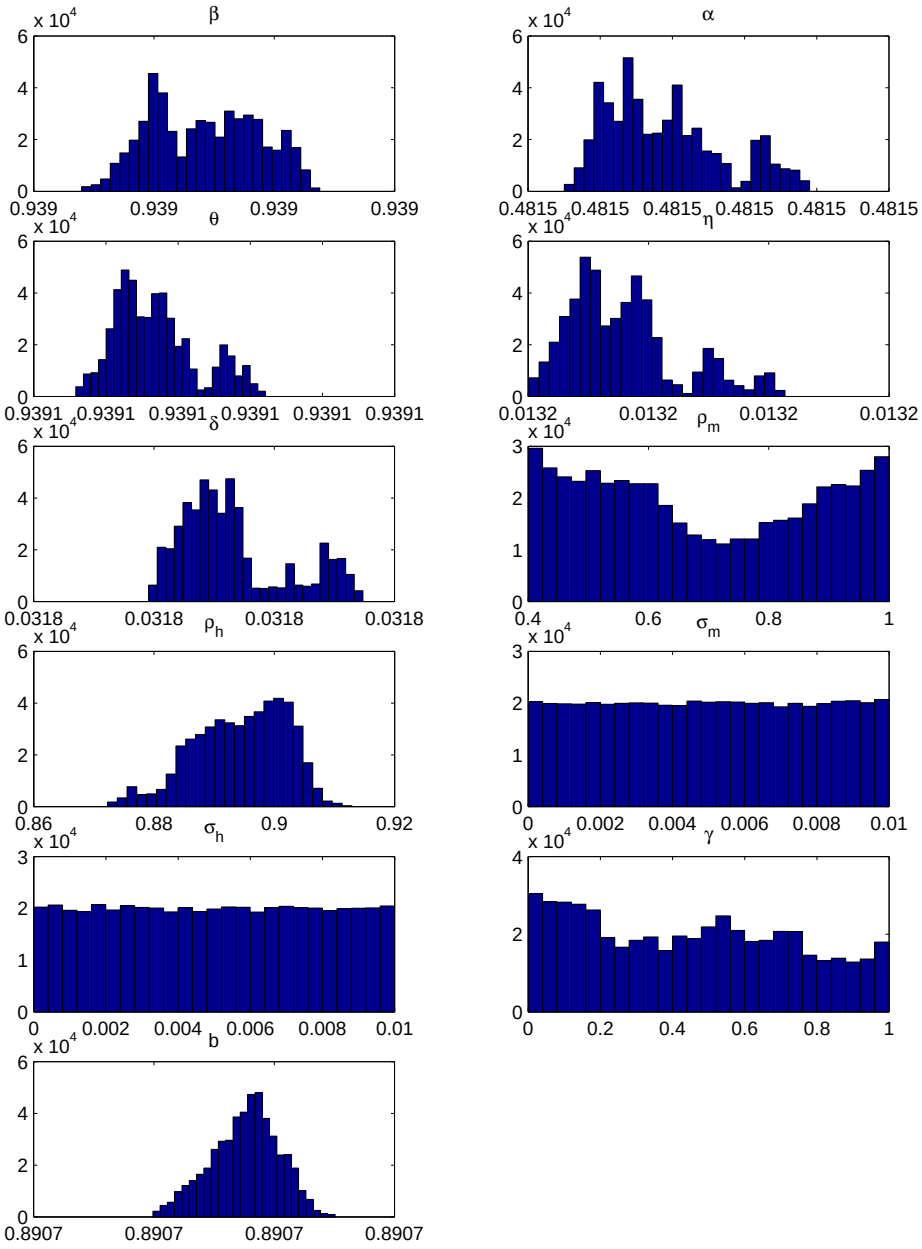

FIGURE 4. Posterior distribution for the structural parameters of household production model when we observe y_m and l_m.

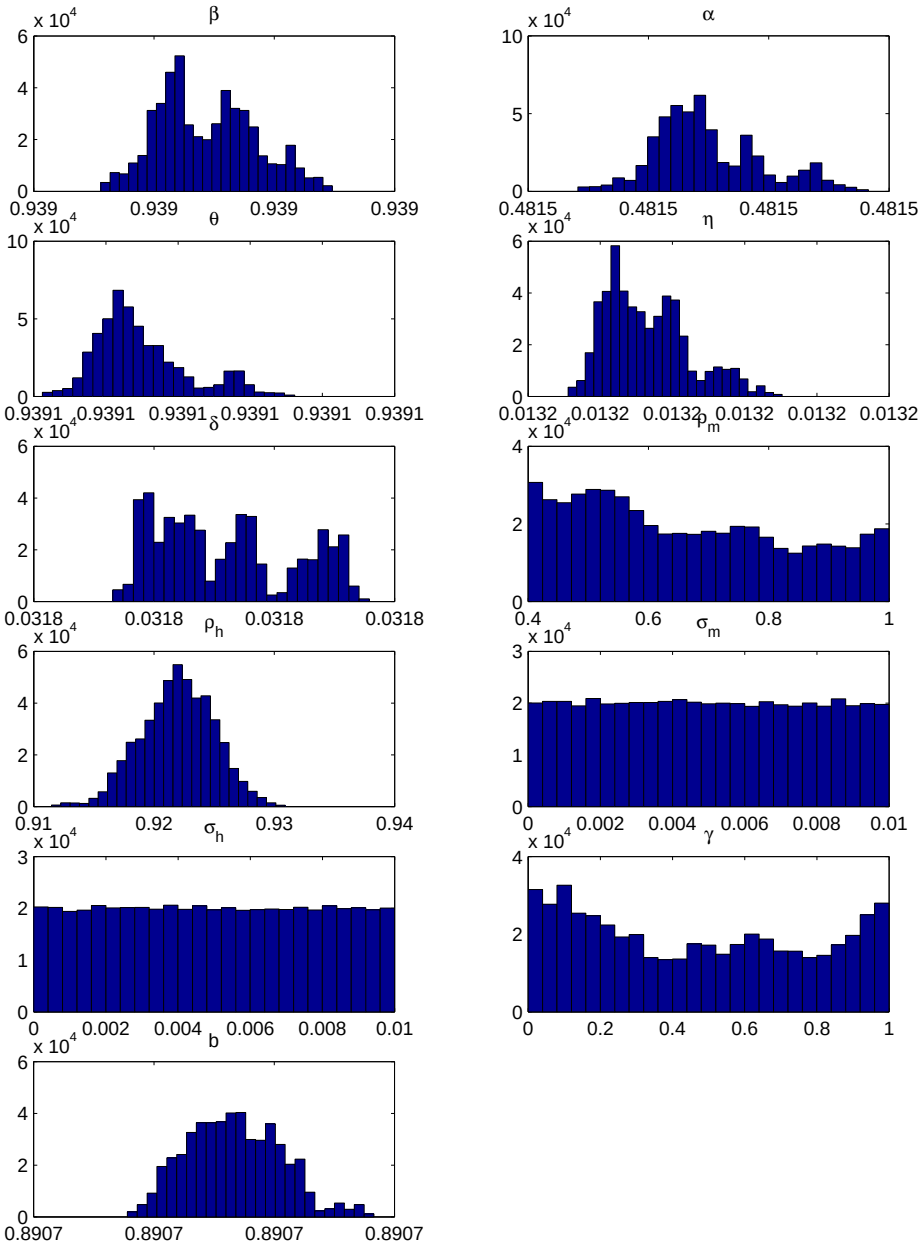

FIGURE 5. Posterior distribution for the structural parameters of household production model when we observe c_m and l_m.

APPENDIX D. FINITE ORDER AUTOREGRESSIONS

This appendix describes formulas for taking an A,B,C,D and forming the associated nth order vector autoregression.[34]

D.1. **Moment formulas.** Take an economic model in the state-space form (1)-(2). Assume that all of the eigenvalues of A are less than unity in modulus, except possibly for a unit eigenvalue that is affiliated with the constant. If present, the unit eigenvalue determines the unconditional mean vector μ_x of x via

$$(I-A)\mu_x = 0. \tag{A1}$$

The stationary covariance matrix of x is $c_x(0) = E(x-\mu_x)(x-\mu_x)'$ and can be computed by solving the discrete Sylvester equation

$$c_x(0) = Ac_x(0)A' + BB', \tag{A2}$$

which can be solved by Hansen and Sargent's matlab program `doublej`. (The indigenous matlab program `dlyap.m` works only when there are no unit eigenvalues of A.) The autocovariance of x is $c_x(j) = E(x_t-\mu_x)(x_{t-j}-\mu_x)'$ and can be computed from

$$c_x(j) = A^j c_x(0), \quad j \geq 1. \tag{A3}$$

Let $\mu_y = C\mu_x$ be the mean of y and $c_y(j) = E(y_t-\mu_y)(y_{t-j}-\mu_y)'$. Elementary calculations establish:

$$\begin{aligned} c_y(0) &= Cc_x(0)C' + DD' && \text{(A4)} \\ c_y(j) &= CA^j c_x(0)C' + CA^{j-1}BD', \quad j \geq 1 && \text{(A5)} \\ c_y(-j) &= c_y(j)', \quad j \geq 1. && \text{(A6)} \end{aligned}$$

D.2. **Projection formulas.** We want to calculate the nth order vector autoregressions

$$y_t - \mu_y = \sum_{j=1}^{n} A_j^{(n)}(y_{t-j}-\mu_y) + \varepsilon_t^{(n)} \tag{A7}$$

where $\varepsilon_t^{(n)}$ satisfies the orthogonality conditions

$$E[\varepsilon_t^{(n)}(y_{t-j}-\mu_y)'] = 0, \quad j = 1,\ldots,n. \tag{A8}$$

[34] Riccardo Colacito has written a Matlab programs `ssvar.m` that by implementing these formulas accepts an (A,B,C,D) and a positive integer n and yields all of the objects defining an nth order VAR. His program `varss.m` takes an nth order VAR and forms a state space system A,B,C,D, a useful tool for using Matlab to compute impulse response functions for estimated VAR's.

The orthogonality conditions, also known as the normal equations, can be written

$$c_y(k) = \sum_{j=1}^{n} A_j^{(n)} c_y(k-j)', \quad k = 1, \ldots, n. \tag{A9}$$

Writing out (A9) and solving for $\begin{bmatrix} A_1^{(n)} & A_2^{(n)} & \ldots & A_n^{(n)} \end{bmatrix}$ gives:

$$\begin{bmatrix} c_y(1)' \\ c_y(2)' \\ \vdots \\ c_y(n)' \end{bmatrix}' \begin{bmatrix} c_y(0) & c_y(1) & \cdots & c_y(n-1) \\ c_y(-1) & c_y(0) & \cdots & c_y(n-2) \\ \vdots & \vdots & & \vdots \\ c_y(1-n) & c_y(2-n) & \cdots & c_y(0) \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{(n)} & A_2^{(n)} & \ldots & A_n^{(n)} \end{bmatrix}. \tag{A10}$$

The covariance matrix $\Sigma^{(n)} = E\varepsilon_t^{(n)}\varepsilon_t^{(n)\prime}$ of the innovations is

$$\Sigma^{(n)} = c_y(0) - \begin{bmatrix} A_1^{(n)\prime} \\ A_2^{(n)\prime} \\ \vdots \\ A_n^{(n)\prime} \end{bmatrix}' \begin{bmatrix} c_y(0) & c_y(1) & \cdots & c_y(n-1) \\ c_y(-1) & c_y(0) & \cdots & c_y(n-2) \\ \vdots & \vdots & & \vdots \\ c_y(1-n) & c_y(2-n) & \cdots & c_y(0) \end{bmatrix} \begin{bmatrix} A_1^{(n)\prime} \\ A_2^{(n)\prime} \\ \vdots \\ A_n^{(n)\prime} \end{bmatrix}. \tag{A11}$$

Please note that the *n*th order autoregression can also be expressed as

$$y_t = \alpha^{(n)} + \sum_{j=1}^{n} A_j^{(n)} y_{t-j} + \varepsilon_t^{(n)} \tag{A12}$$

where

$$\alpha^{(n)} = (I - \sum_{j=1}^{n} A_j^{(n)})\mu_y. \tag{A13}$$

Here μ_y is a properly scaled eigenvector of $\sum_{j=1}^{(n)}$ associated with the unit eigenvalue, where the proper scaling assures that the mean of the constant 1 is 1. Our Matlab program `ssvar` takes an (A,B,C,D), with the understanding that the constant 1 is the first state variable, and computes an *n*th order VAR. Our program `varss` takes an *n*th order VAR and forms the pertinent (A,B,C,D).

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