Applying Economic Restrictions to Foreign Exchange Rate Dynamics: Spot Rates, Futures, and Options

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Abstract: Extant models of exchange rate behavior have typically relied on statistical rather than economic considerations. The approach has been to employ a variant of the generalized central limit theorem to develop tests for the models proposed.

We propose a minimal set of simple economic restrictions—symmetry, invariance, and non-negativity—that must be satisfied by an exchange rate process. By symmetry, we mean that both the direct and indirect exchange rate processes must belong to the same class of distributions. By invariance, we mean that the distribution for an exchange rate must be invariant to changes in the currency unit. By non-negativity, we mean that the exchange rate process must preclude negative values.

We identify various alternative specifications for exchange rate processes and show that some of them do not possess some or all of the above properties. Finally, we propose a new exchange rate process—the mean-reverting logarithmic process (MRL)—and develop valuation equations for several exchange rate instruments, from forward and futures contracts to straight options on the spot rates to options on the futures contracts.

JEL classification: F31, G13, G15

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APPLYING ECONOMIC RESTRICTIONS TO FOREIGN EXCHANGE RATE DYNAMICS: SPOT RATES, FUTURES, AND OPTIONS

Introduction

Previous examinations of foreign exchange (FX) rate dynamics have relied on *statistical* rather than *economic* considerations. Typically, academics have relied on the generalized central limit theorem (CLT) to fashion tests for proposed exchange rate processes. Some have argued that FX rates—which depend upon the combined influence of many unrelated factors—must be either general stable or normal stable.[1] Others have postulated pricing models for FX options or futures prices predicated upon lognormal distributions.[2] Unfortunately, empirical tests of these statistically motivated models have uncovered persistent pricing biases even when the models are adjusted for premature exercise.[3]

Rather than following the usual statistical path, this paper proposes a minimal set of three simple *economic* restrictions to FX rate dynamics. In addition to providing economic appeal, the restrictions greatly reduce the feasible set of potential distributions. We examine in detail two potential candidates from the admissible set and empirically test resulting option and futures pricing formulations.

Simple economic intuition requires that every exchange rate and its inverse should possess the following characteristics:

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[1]Empirical tests of stable or normal stable behavior have been inconclusive or contradictory. Westerfield [17] confirmed stable distributions, while Rogalski and Vinso [15] claimed that t-distributions fit the data better. However, McFarland, Petit, and Sung [13] rejected both the normal and the general stable distributions, at least for daily data.

[2]For example, Garman and Kohlhagen [9] wrote a modified Black and Scholes [3] formula accounting for the foreign interest rate as the yield on the underlying foreign currency. Grabbe [10] derived a modified Merton [14] formula with random domestic and foreign government bond prices.

[3]E.g., Bodurtha and Courtadon [4], who found that lognormal models consistently overprice out-of-the-money options and underpriced in-the-money options.

Property 1: Symmetry

The distribution of any exchange rate and its inverse must be symmetric, i.e., the distribution functions for the FX rate and its inverse will be identical, but the parameters may be different. In other words, it doesn't matter whether traders consider deutschemarks to dollars or vice versa: the distributional characteristics will be the same.

Property 2: Invariance

The distribution of any exchange rate must be invariant to changes in the unit of currency. In other words, changing the domestic currency's name from dollars to ducats and defining ten ducats per old dollar should not change the characteristics of the exchange rate process. Mathematically, given the present rate (but subject to an adjustment of parameters) the conditional distribution must be homogenous of degree zero in the present and future rates of exchange.

Property 3: Non-negativity

The distribution of any exchange rate must preclude negative values. The distribution function for any FX process is entirely determined by a pair of functionals $\{(\mu, \sigma^2)\}$ called the *drift* and *diffusion* coefficients along with specified *boundary* conditions for the process.[4] The economically appealing symmetry, invariance, and non-negativity properties restrict the admissible

[4]Note that in general the coefficients are functions, not constants. See Karlin and Taylor [12] for a detailed description. Interestingly, a single pair of coefficients can result in two different distribution functions depending upon boundary behavior.

set of drift and diffusion coefficients to a family of functional pairs $\{(\mu^*, \sigma^{*2})\}$ with identifiable structure. By examining the required structure of the functionals, it is relatively straightforward to test whether a proposed process is inadmissible.

As an example, the family of lognormal processes is generally *inadmissible* under the proposed economic restrictions. However, a special form of the process with the diffusion coefficient exceeding the drift by precisely twice the current exchange rate is admissible. Unfortunately, the conditional mean and variance of such a restricted lognormal (RLN) process grows without bound; the volatility is constant no matter what the level of the exchange rate, and forecasts for future exchange rates and their reciprocals move in the same direction through time.[5] Each of these characteristics conflicts with observed exchange rate behavior. Furthermore, previous empirical testing of exchange rate options based on underlying lognormal exchange rates has uncovered consistent pricing biases.

One potential admissible distribution with enormous appeal is the *mean-reverting logarithmic* (MRL) process. In addition to satisfying the minimal set of economic restrictions described in Properties 1-3, the mean-reverting properties of the MRL process insure that an exchange rate will be restricted to a bounded range set by policymakers, and its volatility will increase with the level of rates. Furthermore, the resulting characterization of equilibrium option prices promises to modify some of the biases encountered in empirical tests of lognormal processes. Simple simulations show that if MRL parameters are set to provide an identical at-the-

[5]For example, suppose we make forecasts for the exchange rate at increasing intervals like 1 week, 2 weeks, 3 weeks ahead. Each successive forecast will be larger than the prior forecast if the drift coefficient is positive. Intuitively, we would expect the corresponding forecasts for the reciprocal (other-country) exchange rate to be successively smaller. Instead, they too are strictly increasing through time.

money price to an RLN option, then the MRL process will generate lower out-of-the-money prices and higher in-the-money prices than the RLN process.

The paper is organized as follows. Section I formulates mathematical restrictions for Ito diffusion processes implied by the minimal set of economic Properties 1-3 and identifies a test for admissible processes. Section II presents examples of admissible FX dynamics, including RLN and MRL processes. Section III presents explicit futures and options valuation formulas for MRL processes. Compared with the RLN process, the new valuation formula produces higher prices for in-the-money call options and lower prices for out of-the-money call options, correcting pricing biases observed with lognormal models. Section IV briefly summarizes results.

I. Economic Restrictions and Admissible Distributions for Exchange Rates

By economic definition, the equilibrium foreign exchange rate is a strictly positive stochastic process, $x_t > 0$, measuring units of domestic currency per unit of foreign currency. By assumption, x_t is an Ito diffusión process whose evolution can be characterized by

$$dx = \mu(x,\bar{\pi})\,dt + \sigma(x,\bar{\pi})dW \tag{1}$$

where $\bar{\pi}$ is a *vector* of time-invariant, constant parameters in R^k, W is a standard Wiener process, and $\mu(x,\bar{\pi})$ and $\sigma(x,\bar{\pi})$ are functions $\mu: R_+ \times R^k \mapsto R$ and $\sigma: R_+ \times R^k \mapsto R$ such that x is non-negative and bounded on finite time intervals. Applying Ito's formula to (1) implies that the reciprocal (*other-country*) exchange rate must obey

$$\begin{aligned} d(1/x) &= [-(1/x)^2 \mu(x,\bar{\pi}) + (1/x)^3 \sigma^2(x,\bar{\pi})]dt - (1/x)^2 \sigma(x,\bar{\pi})dW \\ &\equiv \mu[(1/x),\bar{\pi}]dt + \sigma[(1/x),\bar{\pi}]dW \end{aligned} \tag{2}$$

In (1) and (2), the functions $\mu(x,\bar{n})$ and $\mu[(1/x),\bar{n}]$ are the **drift** coefficients for the exchange rate and reciprocal (*other country*[6]) exchange rate, respectively. Similarly, $\sigma^2(x,\bar{n})$ and $\sigma^2[(1/x),\bar{n})$ are the **diffusion** coefficients. It is well-known[7] that the drift and diffusion functions are a sufficient set of parameters to write the Kolmogorov backward equation. Since the distribution function for the exchange rate satisfies the backward equation written with an appropriate drift and diffusion coefficient, specification of an admissible set of functions $\{\mu(x,\bar{n}), \sigma^2(x,\bar{n})\}$ plus appropriate boundary conditions will be entirely equivalent to specifying the admissible set of distribution functions for the exchange rate. The remainder of this section will provide a complete characterization of the class of functions $\{\mu, \sigma^2\}$ for the exchange rate (and hence for the distribution functions for the exchange rate) admissible under the symmetry, invariance, and non-negativity properties.

Lemma 1 below provides a complete description of the set of stochastic differential equations that is consistent with the symmetry requirement for exchange-rate dynamics.[8]

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[6]The instantaneous covariance between the exchange rate and its reciprocal process can be readily calculated from (1) and (2) as $\text{cov}[dx, d(1/x)] = -[\sigma^2(x,n)/x^2]dt$. Hence, the exchange rate and its reciprocal process are perfectly negatively correlated as expected. Similarly, the instantaneous deviation of the reciprocal process may be written as $stddev[d(1/x)] = \left|-(1/x)^2\sigma(x,n)\right|$. Hence the negative sign prior to the diffusion coefficient in (2) merely reflects the fact that the reciprocal exchange rate is perfectly negatively correlated to the Wiener process dW.

[7]E.g., Karlin and Taylor [12].

[8]The class of admissible exchange rate distributions could be further reduced by extending the symmetry property to the **moments** of the exchange rate distribution over a finite time interval. Strengthening the symmetry property might be supported by the following intuition. When asked to produce subjective forecasts of a future exchange rate and its inverse, a consumer generally submits reciprocal numbers. If the forecast of the Swiss franc in U.S. dollars is 1/2, then the forecast of the U.S. dollar in Swiss francs is 2. If investors forecast using the mean of the exchange rate distribution, symmetry extended to the actual forecast value would require that $E[(1/x_t)|x_0] = 1/E[x_t|x_0]$. However, Jensen's inequality implies that this relationship can obtain only for a deterministic exchange rate. Thus, strengthening restrictions on the parameter space n to assure symmetry of moments is infeasible. Fortunately, the economic intuition that investors provide symmetric forecasts from symmetric distributions still obtains if investors use the median

Lemma 1: A Precise Formulation of the *Symmetry* Property

A precise formulation of the symmetry property for Ito processes is that, subject to an adjustment of parameters, whenever 1/x is substituted for x in equation (1), the resulting drift and diffusion coefficients must equal those of equation (2). Formally, there must exist a vector of parameters $\bar{n}'$ such that

$$\mu[1/x,\bar{n}'] = -(1/x)^2\mu(x,\bar{n}) + (1/x)^3\sigma^2(x,\bar{n}) \tag{3}$$

and

$$\sigma[(1/x),\bar{n}'] = -(1/x)^2\sigma(x,\bar{n})\,. \tag{4}$$

The formulation in (3) and (4) represents a system of *functional equations*, i.e., a system of equations in two unknown *functions*, $\mu(x,\bar{n})$ and $\sigma^2(x,\bar{n})$ (the drift and diffusion coefficients), rather than in two unknown numbers. Standard methods show that any autonomous (time-invariant) process that satisfies (3)-(4) must follow the stochastic differential equation

$$\begin{aligned} dx = [\tfrac{1}{2} n_1^2 x f^2(\log x,\log n_2) + n_3 x g(\log x,\log n_4)]dt \\ + n_1 x f(\log x,\log n_2)dW \end{aligned} \tag{5}$$

where $\bar{n}=\{n_1,n_2,n_3,n_4\}$, a set of four scalars, and the functions $f(\cdot)$ and $g(\cdot)$ must both be **odd**[9] in the logarithms of the exchange rate x and the parameters $\bar{n}$. In other words, the dynamics of all feasible exchange rate processes satisfying the symmetry restriction must be consistent with (5) with the additional proviso that the functions $f(\cdot)$ and $g(\cdot)$ must both be odd.

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for their forecasts. Specifically, if $m(x_t|x_0)$ is the median for the future exchange rate conditional upon the current level, then for any absolutely continuous distribution $m[(1/x_t)|x_0] = 1/m[x_t|x_0]$. However, this provides no additional restriction on the class of admissible distributions.

[9] $f(\cdot)$ is odd in x and $\bar{n}$ if $f[\log(1/x),\log(1/n_2)] = -f[\log x,\log n_2]$.

Proof: See Appendix A.

Lemma 2 below describes the complete set of drift and diffusion functional pairs, which is consistent with Property 2 (the invariance property) for FX rate dynamics. This lemma further restricts the class of admissible drift and diffusion functions (and hence distribution functions) for foreign exchange processes.

Lemma 2: A Precise Formulation of the *Invariance* Property

A precise formulation of the invariance property requires that the equilibrium exchange rate distribution be invariant to changes in the unit of currency. Mathematically, given the present rate (but subject to an adjustment of parameters) the conditional distribution must be homogenous of degree zero in the present and future rates of exchange. Let $P(\cdot)$ represent the distribution function[10] for the logarithm of the exchange rate and let A be an interval on the positive real line. Then the invariance property formally requires that for any $k \in R_+$

$$P(s, \log x, t, A | n') = P(s, k \log x, t, kA | n''). \tag{6}$$

In other words, subject to an adjustment of parameters, the transition density of the logarithmically transformed process must be spatially homogeneous, depending only on the difference between future and present rates. In terms of the diffusion and drift coefficients defining the transition density of the transformed process, the parameters of the functional equations (3) and (4) must enter **multiplicatively** in the log only:

$$\mu(x, \overline{n}, f, g) = \left\{ \tfrac{1}{2} n_1^2 f^2 [\log(x n_2)] + n_3 g[\log(x n_4)] \right\} x \tag{7}$$

[10]Technically, the distribution function P(s,x,t,A|n) provides the probability that, starting from the level x at time s, the exchange rate will lie somewhere in the interval A at future date t when the function is parameterized by the vector n.

and

$$\sigma(x,\bar{n},f)=\{n_1 f[\log(xn_2)]\}x. \tag{8}$$

Combining Lemmas 1 and 2 allows statement of the following main result.

Theorem 1

All equilibrium exchange rate processes that satisfy both the symmetry and invariance properties must follow the stochastic differential equation (SDE)

$$\begin{aligned} dx &= [\tfrac{1}{2} n_1^2 x f^2[\log(xn_2)] + n_3 x g[\log(xn_4)]]dt + n_1 x f[\log(xn_2)]dW \\ &\equiv \mu^*(x,\bar{n},f,g)dt + \sigma^*(x,\bar{n},f)dW \end{aligned} \tag{9}$$

subject to the proviso that the functions $f(\cdot)$ and $g(\cdot)$ must both be odd in the logarithm of the exchange rate x and parameters $\bar{n}$.

To summarize the results of this section, note that any (transition) distribution function for an Ito diffusion process is completely characterized by a pair of functions$\{(\mu, \sigma^2)\}$ representing the drift and diffusion coefficients as written in equation (1). Symmetry and invariance properties substantially restrict the admissible coefficient pairs. Specifically, any admissible pair $\{(\mu, \sigma^2)\}$ must belong to the set of pairs of functionals $\{(\mu^*, \sigma^*)\}$ defined by Theorem 1. In order to satisfy Property 3 (non-negativity), the potential SDE must satisfy boundary classifications ensuring that zero is an inaccessible barrier for the exchange rate.[11]. *These joint criteria completely characterize the set of admissible distributions for autonomous exchange rate processes.* A salient feature of admissible functionals μ^* and σ^* in Theorem 1 is their definition in terms of functions $f(\cdot)$ and $g(\cdot)$, which are odd in the logarithms of the exchange rate and parameters $\bar{n}$. Theorem 1 may be used to readily test whether any potential pair of drift and diffusion coefficients

[11]See Karlin and Taylor [12] for a discussion of boundary classifications.

(and hence distribution) is admissible for exchange rate processes since the functions $f(\cdot)$ and $g(\cdot)$ must be odd in log x and log n. For example, no functions $f(\cdot)$ and $g(\cdot)$ and parameters $\bar{n}$ can be found to make the general geometric Wiener process dx = axdt + bxdW a member of the admissible class. However, as shown by example in the next section, the RLN subset of the family of geometric Wiener processes is admissible.

II. Examples of Suitable Exchange Rate Processes

This section presents examples of two admissible distributions: a restricted lognormal process (RLN) and a mean-reverting logarithmic process (MRL). It shows that the RLN process suffers several economic shortcomings rectified by the MRL process. Section III will present futures and options prices corresponding to each process. Section IV will present empirical results for estimators of each process and corresponding tests of resulting option prices.

Example 1: Restricted Lognormal (RLN)

A restricted subset of lognormal processes satisfies Properties 1-3 (symmetry, invariance, and non-negativity); the SDEs describing dynamics of the RLN process and its reciprocal are:

$$dx = \tfrac{1}{2} n_1^2 x dt + n_1 x dW; \tag{10a}$$

$$d(1/x) = \tfrac{1}{2} n_1^2 (1/x) dt - n_1 (1/x) dW. \tag{10b}$$

In other words, all admissible geometric Wiener processes exhibit the mathematical property that their diffusion coefficients exceed their drift coefficients by precisely twice the current exchange rate.[12] No other geometric Wiener processes are admissible.

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[12]Once again, the minus sign preceding the random component in (10b) does not signify a negative standard deviation. It merely implies that the two exchange-rate processes that depend upon the same Wiener noise are perfectly negatively correlated. Review footnote (5).

Proof.

For the functions $f(\cdot)$ and $g(\cdot)$ in equation (9), use

$$f[\log(n_2 x)] = sgn(\log n_2); \tag{11a}$$

$$g[\log(n_4 x)] = 0. \tag{11b}$$

For positive n_2, this is the same as writing $f(\cdot) = 1$; however, the notation in equation (11) stresses the fact that $f(\cdot)$ is odd in $\log n_2$ and $\log x$.

The RLN process has one major advantage: it is simple and well understood. Unfortunately, RLN dynamics imply three counterintuitive properties relative to foreign exchange rates. First, the conditional mean and variance for both the exchange rate and its reciprocal grow unboundedly as shown by the following equations:

$$E[x_t | x_0] = x_0 e^{n_1^2 t/2}; \tag{12a}$$

$$E[(1/x_t) | x_0] = (1/x_0) e^{n_1^2 t/2}; \tag{12b}$$

$$\text{var}[x_t | x_0] = x_0^2 e^{n_1^2 t}[e^{n_1^2 t} - 1]; \tag{12c}$$

$$\text{var}[(1/x_t) | x_0] = (1/x_0^2) e^{n_1^2 t}[e^{n_1^2 t} - 1]. \tag{12d}$$

Second, the symmetry restriction as formulated in Property 1 does not force the drift coefficient of the exchange rate to move in the opposite direction from the drift of the reciprocal rate.[13] Hence, expected values of both the exchange rate and its reciprocal are strictly increasing functions of time, which is highly counterintuitive. Third, modeled exchange rate volatility will be constant at all FX levels, implying that a currency devaluation would have no impact on

[13]If the drift coefficient were modeled with mean-reversion, this problem would disappear.

volatility.[14] In addition, previous empirical testing of exchange rate options based on underlying lognormal exchange rates has uncovered consistent pricing biases.[15]

To rectify the serious economic shortcomings of the RLN process, this paper proposes an alternative mean-reverting logarithmic (MRL) process with volatility proportional to the level of the exchange rate. Although significantly more complicated than the RLN process, it satisfies Properties 1-3, causes the expected exchange rate and its reciprocal to move in opposite directions, exhibits increased volatility at higher rate levels, and admits empirical testing using nonlinear maximum-likelihood methods.

Example 2: Mean-Reverting Logarithmic (MRL) Process

Let $y \equiv \log(x/n_2)$ with $x \geq n_2$. In other words, n_2 may be interpreted as a critical floor for the exchange rate x and a critical ceiling for the reciprocal exchange rate. Then the following SDE represents the dynamics for both an exchange rate x and its reciprocal (1/x) that satisfy Properties 1-3.

$$dy = n_3[\log(n_4/n_2) - y]dt + n_1 y^{1/2} dW. \qquad (13)$$

Proof.

For the functions $f(\cdot)$ and $g(\cdot)$ in equation (9), use

$$f[\log(n_2 x)] = -sgn[\log(n_2/x)] |\log(n_2/x)|^{1/2}\,; \qquad (14a)$$

$$g[\log(n_4 x)] = \log(n_4/x). \qquad (14b)$$

Both are odd in the parameters and exchange rate. Then the reciprocal exchange rates evolve as

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[14]Lognormal stock prices driving the Black-Scholes option-pricing model exhibit both of these shortcomings. In the equity markets, practitioners have circumvented the first problems by arguing that economic expansion causes stock prices to grow through time. This argument clearly fails when applied to FX markets. Practitioners address the second problem by seeking volatility adjustment factors.

[15]See Bodurtha and Courtadon [4].

$$dx = \left(\tfrac{1}{2}n_1^2 x\left|\log(n_2/x)\right| + n_3 x \log(n_4/x)\right)dt \\ - \operatorname{sgn}[\log(n_2/x)]n_1\, x\left|\log(n_2/x)\right|^{\frac{1}{2}} dW \tag{14c}$$

$$d(1/x) = \left(\tfrac{1}{2}n_1^2(1/x)\,\left|\log[(1/n_2)/(1/x)]\right| + n_3(1/x)\log[(1/n_4)/(1/x)]\right)dt \\ - \operatorname{sgn}\left(\log[(1/n_2)/(1/x)]\right)n_1(1/x)\left|\log[(1/n_2)/(1/x)]\right|^{\frac{1}{2}} dW. \tag{14d}$$

Making the substitution $y \equiv \log(x/n_2)$ into (14c) or (14d) and using the requirement that $x \geq n_2$ gives (13). As long as $n_3 > 0$, both conditional mean and variance for the exchange rate and its reciprocal are bounded. For the exchange rate x,

$$E[x_t|x_0] = n_2\left(\frac{d}{d-1}\right)^{q+1} c^{\left(\frac{d}{d-1}\right)} \tag{15a}$$

$$\operatorname{var}[x_t|x_0] = n_1^2\left[\left(\frac{d}{d-2}\right)^{q+1} c^{\frac{2d}{d-2}} - \left(\frac{d}{d-1}\right)^{2(q+1)} c^{\frac{2d}{d-1}}\right] \tag{15b}$$

where
$$c \equiv \left(\frac{x_t}{n_2}\right)^{e^{-n_3 t}}, \quad d \equiv \frac{2n_3}{2n_3(1-e^{-n_3 t})}, \quad q \equiv \frac{2n_3\log(n_4/n_2)}{n_1^2} - 1 \tag{15c}$$

(13) shows that a logarithmic transform of the exchange rate results in a mean-reverting, square-root process whose properties are well known.[16] The seemingly complex system in (14) allows each parameter $\{n_1, n_2, n_3, n_4\}$ a realistic economic interpretation. The transform's diffusion coefficient n_1^2 measures the response of the exchange rate to random noise in the economy; n_2 may be interpreted as a *critical floor* where the home government intervenes to strengthen (weaken) its currency; $n_3 > 0$ is a traditional speed of adjustment; and n_4 represents

[16]MRS processes have been studied intensively by Feller [8] and Cox, Ingersoll, and Ross [6].

the (logarithmic) *long-run mean*. With these interpretations, both the long-run mean and exchange rates must exceed the critical floor: $n_4 > n_2$ and $x_t > n_2$. For the reciprocal process (1/x), parameters n_1 and n_3 maintain identical interpretations, while n_2 becomes a *critical ceiling* and $1/n_4$ a (logarithmic) *long-run mean*. Notice that if the reciprocal rate 1/x hits the critical ceiling $1/n_2$, then it must immediately fall since the positive contribution from $dy = n_3\{\log[(1/n_2)/(1/n_4)]\}dt$ coupled with the absence of noise from the diffusion coefficient will force $y = \log[(1/n_2)/(1/x)]$ to increase. Similarly, if the exchange rate x hits the critical floor n_2, it must instantly rise. Thus, n_2 is a reflecting barrier whenever $n_4 > n_2$. This reflection at n_2 and $1/n_2$ leads to the interpretation that either the home government has intervened to strengthen (weaken if the exchange rate is less than one) its currency at the critical floor or the foreign government has intervened to weaken (strengthen if the reciprocal exchange rate exceeds one) its currency at the critical ceiling.

There are, of course, many other admissible processes that could be specified and tested. For example, using the following as the pair of functions $f(\cdot)$ and $g(\cdot)$ in equation (9) will result in an admissible process:

$$\begin{aligned} &f[\log(n_2 x)] = sgn(\log n_2) \\ &g[\log(n_4 x) = \log(n_4 / x) \end{aligned} \tag{16}$$

However, we will concentrate on the RLN and MRL processes for the remainder of this paper.

III. Futures and Options Pricing Formulas

A. The General Case

Domestic and foreign interest rates together with the feasible equilibrium exchange rate processes specified by (13) determine equilibrium prices for futures contracts and call option prices on both foreign exchange and foreign exchange futures. It is well-known that both futures and option prices may be calculated as conditional expectations of the contract's terminal value after the exchange rate process has been suitably modified for risk. Alternatively, both contract prices may be calculated as solutions to appropriate partial differential equations satisfied by the correct conditional expectations. The terminal condition for each partial differential equation is then the terminal value of the contract.[17]

The appropriate modification for risk for both futures and options involves replacing the multiplicative component of the foreign exchange rate drift term (the term $\{\frac{1}{2}n_1^2 f^2[\log(xn_2)] + n_3 g[\log(xn_4)]\}$ from equation (7)) with the instantaneous interest premium of the domestic market over the foreign market. Denote the domestic interest rate at time t by i_t and the foreign rate by j_t. Then the premium at any instant $[i_t - j_t]$ may be either positive or negative, depending on whether $i_t > j_t$ or vice versa. In either case, the appropriate risk-adjusted exchange rate process that satisfies Properties 1-3 must follow the SDE

$$dx_t = (i_t - j_t)x_t dt + n_1 x_t f[\log(n_2 x_t)]dW_t. \tag{17}$$

In equilibrium, exchange rates and interest rates are probably correlated random processes. However, bond-pricing formulas driven by short-term interest rates closely approximate discount factors for constant interest rates when bond maturities are less than a year.[18] Similarly, the instantaneous riskless rate is treated as a constant in the Black-Scholes model even though stock prices are undoubtedly correlated to interest rates. Consequently, when

[17] These results have appeared in many places in the literature (e.g., see Cox, Ingersoll, and Ross [7] or Harrison and Kreps [11]). A specific application to futures contracts appears in Cox, Ingersoll, and Ross [5].

[18] E.g., Cox, Ingersoll, and Ross [6].

valuing short-maturity futures and options contracts, it appears reasonable to replace the stochastic premium $i_t - j_t$ in (17a) by a constant premium (i-j) to obtain the risk-adjusted SDE

$$dx_t = (i-j)x_t dt + n_1 x_t f[\log(n_2 x_t)] dW_t. \tag{18}$$

Let the value of a futures contract be designated as h(x,t). Then for any exchange rate process x whose value at future time T will be x_T, the futures price must satisfy

$$h(x,t) = E_t^*[x_T | x_t] \tag{19}$$

where the expectation will be calculated relative to a risk-adjusted transition density implied by the SDE (18). Alternatively, the futures contract must satisfy the valuation partial differential equation (PDE)

$$\tfrac{1}{2} n_1^2 x^2 f^2[\log(n_2 x)] h_{xx} + (i-j)xh_x + h_t = 0 \tag{20}$$

subject to the terminal condition h(x, T) = x_T.

Similarly, for a call option u(x,t) on the spot exchange rate x and an exercise price of k, the conditional expectation analogous to (19) is merely[19]

$$u(x,t) = E_t^*[\max(x_T - k, 0)]. \tag{21}$$

The analogous valuation PDE is

$$\tfrac{1}{2} n_1^2 x^2 f^2[\log(n_2 x)] u_{xx} + (i-j)xu_x - iu + u_t = 0 \tag{22}$$

subject to the terminal condition u(x, T) = max(x_T-k,0).

Finally, consider a call option v(x,t) on the futures contract h with an exercise price of k. The appropriate conditional expectation is

$$v(x,t) = E_t^*[\max(h_T - k, 0)]. \tag{23}$$

The corresponding valuation PDE for the option on the futures contract is

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[19]The necessary adjustment for a put option should be obvious.

$$\frac{1}{2} n_1^2 x^2 f^2[\log(n_2 x)] v_{xx} - iv + v_t = 0 \tag{24}$$

subject to the terminal condition v(x, T) = max(h_T-k,0).

Valuation equations (19)-(24) apply for all dynamics satisfying Properties 1-3. Heuristically, they follow from the familiar ***risk-neutral hedge*** obtained by selling short one unit of foreign currency and exchanging it for x units of domestic currency to purchase $/u_x$ futures or options on the foreign currency. Maintenance of the short position in foreign currency requires payment of interest at the rate jdt (translating to the rate xjdt in domestic units). The change in value of the hedge portfolio less short interest must equal the return on a portfolio with equivalent equity invested at the domestic rate.

In futures markets, contracts are effectively ***marked to market*** each day, whereas no additional cash flows occur in options markets except those required to adjust the hedge. Thus, solutions for futures and options contracts differ for two reasons: the terminal conditions contrast since the contracts have different final payouts, and the option PDE (22) contains an extra term iu because options markets do not require the instantaneous settlement of futures markets.

B. Prices with RLN dynamics

Garman and Kohlhagen [9] have previously obtained equilibrium futures and options pricing formulas appropriate for any lognormal dynamics including the RLN process.[20] The appropriate solution for the futures price is

$$h(x,t) = x_t e^{(i-j)(T-t)}. \tag{25}$$

Thus, the futures price depends solely upon the current exchange rate, the interest rate differential, and the contract's maturity.

[20]Note that drift terms are absent from (20) or (22). Under risk-neutral heuristics, investors price all processes as if the deterministic component of movement is irrelevant.

The corresponding valuation formula for a call option with exercise price h is a variant of the familiar Black-Scholes equation

$$
\begin{aligned}
u(x,t) &= e^{-j(T-t)}x_t N(d_1) - e^{-i(T-t)}hN(d_2) \\
d_1 &= \frac{\log(x_t/h) + (i-j)(T-t)}{\sigma\sqrt{(T-t)}} \\
d_2 &= d_1 - \sigma\sqrt{(T-t)} \quad .
\end{aligned}
\tag{26}
$$

C. Prices with MRL Dynamics

For MRL dynamics, the appropriate risk-adjusted SDE follows immediately from (14c) and (17b) as

$$dx = (i-j)xdt - sgn[\log(n_2/x)]n_1 x\left|\log(n_2/x)\right|^{1/2} dW. \tag{27}$$

Hence the specific valuation PDEs for the futures contract, the call option on spot exchange, and the call option on futures corresponding to the general cases (20), (22), and (24) are given by the following three equations.

For the futures contract,

$$\tfrac{1}{2} n_1^2 x^2 \left|\log(n_2/x)\right| h_{xx} + (i-j)xh_x + h_t = 0 \tag{28a}$$

, subject to $h(x, T) = x_T$;

for the option on spot exchange,

$$\tfrac{1}{2} n_1^2 x^2 \left|\log(n_2/x)\right| u_{xx} + (i-j)xu_x - iu + u_t = 0 \tag{28b}$$

subject to $u(x, T) = \max(x_T - k, 0)$;

and for the option on the futures contract,

$$\tfrac{1}{2} n_1^2 x^2 \left|\log(n_2/x)\right| v_{xx} - iv + v_t = 0 \tag{28c}$$

subject to $v(x, T) = \max(h_T - k, 0)$.

Each of the valuation PDEs in (28) may be solved by standard *numerical* methods. In fact, American[21] option prices will have analytical (closed-form) solutions only if the interest rate premium (i - j) is *sufficiently large* (operationally, this means numerical methods indicate no premature exercise would occur for the spread ([i - j]).

For the futures contract and for European options, it is possible to write analytic solutions for (28) subject to one very interesting proviso. In the SDE (27) make the variable substitution $y \equiv \log(x/n_2)$ and assume that $x \geq n_2$ providing the economically appealing interpretation that n_2 is a critical floor at which policymakers will provide all-out intervention to strengthen the currency. Then the logarithmically transformed exchange rate process follows the complementary SDE

$$dy = [(i-j) - \frac{n_1^2}{2} y]dt + n_1 y^{1/2} dW \tag{27'}$$

and the risk-adjusted conditional expectations describing the futures and options values become

$$\begin{aligned} h(x,t) &= E_t^*[n_2 e^{y_T} | y_t] \\ u(x,t) &= E_t^* \max[n_2 e^{y_T} - k, 0 | y_t] \\ v(x,t) &= E_t^* \max[h_T - k, 0 | y_t] \end{aligned} \tag{29}$$

The conditional expectations in (29) should be evaluated relative to a risk-adjusted transition density corresponding to the *mean-reverting, square-root* (MRS) process in (27'). MRS processes have been extensively studied by Feller [8] and utilized by Cox, Ingersoll, and Ross [6], and Cox, Ingersoll, and Ross [7] in the financial literature. Interestingly, the *single SDE* (27') supports *two transition densities* with entirely different properties. If the interest rate spread

[21] American options may be exercised at any time prior to their maturity dates; European options may be exercised only on their maturity dates.

(i-j) is strictly positive, the transition density on the logarithmically transformed exchange rate sums to 1 and is given by[22]

$$p[y_\tau; y_t] = d(e^{-d[y_\tau + y_0 e^{-a\tau}]})[\frac{y_\tau}{y_0 e^{-a\tau}}]^{\frac{q}{2}} I_q[2de^{-\frac{a\tau}{2}}(y_0 y_\tau)^{1/2}], \tag{30}$$

where

$a \equiv \frac{n_1^2}{2}$, $d \equiv \frac{1}{1-e^{-a\tau}}$, $q \equiv \frac{2(i-j)-n_1^2}{n_1^2}$, and $I_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\Gamma(k+q+1)} (\frac{z}{2})^{q+2k}$, $I_q(.)$ is the modified Bessel function of order q.

However, if $(i-j) \le 0$, the resulting *norm reducing* or *defective*[23] density sums to less than 1 and is given by:

$$p[y_\tau; y_t] = d(e^{-d[y_0 + y_\tau e^{-a\tau}]})[\frac{y_\tau e^{-a\tau}}{y_0}]^{\frac{q}{2}} I_q[2de^{-\frac{a\tau}{2}}(y_0 y_\tau)^{1/2}], \tag{31}$$

where $d \equiv \frac{1}{1-e^{-a\tau}}$, $q \equiv \frac{n_1^2 - 2(i-j)}{n_1^2}$, and $I_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\Gamma(k+q+1)} (\frac{z}{2})^{q+2k}$,

$I_q(.)$ is the modified Bessel function of order q, and $a \equiv \frac{n_1^2}{2}$.

[22]Feller calls this a *norm-preserving* solution. Cox, Ingersoll, and Ross [6] used a similar density to describe the evolution of equilibrium interest rates, which would be reflected back into the positive domain if they ever reached zero.

[23]Feller calls this a *norm-decreasing* solution. Cox, Ingersoll, and Ross [6] used a similar density to describe the behavior of options on stocks, which pay constant proportional dividend streams and whose volatilities are proportional to the stock price. They pointed out that the amount by which the cumulative distribution function falls below 1 represents the probability that the stock price hits 0 at some time prior to option maturity and is absorbed (i.e., remains at zero). Feller noted that in cases where the constant term in the drift is less than or equal to zero, it is not possible to arbitrarily prescribe boundary conditions (i.e., to arbitrarily erect a reflecting barrier at the critical floor).

Hence, the simplifying assumption of a constant interest rate spread leads to the rather disconcerting notion of two different closed-form solutions for futures and options depending upon the sign of the interest premium. However, this paradox has a ready economic explanation. For the normal or the *norm-preserving* density case, we use (23) to obtain the following price function for the futures contract:

$$h(x,t) = x_t e^{(i-j)(T-t)}. \qquad (32)$$

As can be seen above the expression is identical to the one obtained under the RLN process assumptions. The corresponding European call option price is obtained by using the standard valuation relationship as given below,

$$u(x_t, t, \tau, h, T) \equiv E_t^*[e^{-i\tau} Max[0, x_T - h]],$$

where

τ = T-t is the time to maturity,

T = the terminal date,

h = the strike price of the option,

i = the domestic risk-free interest rate, and

x_T = the terminal value of x_t .

With the *norm-preserving* transition density function as given per (30), we obtain the following pricing function for the option:

$$u(x_t, t, \tau, h, T) = [x_t e^{-j\tau}][1 - \mathrm{X}^2[(\frac{2Ln(\frac{h}{n_2})}{(e^{a\tau}-1)});\ (\frac{2c}{a});\ (\frac{2Ln(\frac{x_t}{n_2})}{(1-e^{-a\tau})})]] -$$

$$[he^{-i\tau}][1-X^2[(\frac{2Ln(\frac{h}{n_2})}{(1-e^{-a\tau})}); (\frac{2c}{a}); (\frac{2Ln(\frac{x_t}{n_2})}{(e^{a\tau}-1)})]] \tag{33}$$

where $a \equiv \frac{n_1^2}{2}$, $c \equiv (i-j)$, j is the foreign risk-free interest rate, and all the other terms are as defined before. In (33) above, X^2 represents the non-central chi-squared distribution function, defined as a weighted sum of central chi-squared functions P:

$$X^2(s,v,\lambda) = \sum_{n\geq 0} e^{-\frac{\lambda}{2}} \frac{(\frac{\lambda}{2})^n}{n!} P(s, v+2n). \tag{33'}$$

To price the call option on the futures contract notice that the hedge between a futures contract and a futures option is different from the hedge between a cash position in the foreign exchange and a cash (or spot) option. While the cash position requires an investment, the futures position requires only that there be an interest-bearing good faith margin. Consequently, the operating SDE is given per (24) (see Black [2]).

Thus a European call option on a futures contract with the same expiration date as the futures contract has the given price below:

$$v(x_t, \tau, h) = [x_t][1-X^2[(\frac{2Ln[\frac{he^{-(i-j)\tau}}{n_2}]}{(e^{a\tau}-1)}) ; 0 ; (\frac{2Ln[\frac{x_t e^{-(i-j)\tau}}{n_2}]}{(1-e^{-a\tau})})]] -$$

$$[he^{-i\tau}][1-X^2[(\frac{2Ln[\frac{he^{-(i-j)\tau}}{n_2}]}{(1-e^{-a\tau})}) ; 0 ; (\frac{2Ln[\frac{x_t e^{-(i-j)\tau}}{n_2}]}{(e^{a\tau}-1)})]], \tag{34}$$

where x is the futures contract price, h is the exercise price of the option contract on the futures contract, and all the other terms are as defined before.

This European option formula is particularly interesting because of the intention of the Chicago Board Options Exchange to trade options that cannot be exercised until the last day of trading. In the case of American options, we will need to employ (24) together with the "tight-fit" condition per (34') and solve for the corresponding option values numerically.

When the premium (i-j) is not sufficiently large, an early exercise of the call option (of the American type) may be optimal; then the exchange rate at which the option may be optimally exercised is an increasing function of the time to maturity. Let Γ_t={x: early option exercise occurs at t}, i.e., Γ_t is the set of all the exchange rate values at which it is optimal to exercise the option at time t. In this context note that t is an optimal stopping time. Let $\partial\Gamma_t$, be the boundary of the set Γ_t. In cases where Γ_t is non-null, both the option prices and the optimal exercise policy must be derived from the SDE for the option (22), with its corresponding terminal condition and the "tight-fit" or "smooth-pasting" condition (see Shiryayev [16] or Williams [18]):

$$\frac{\partial u}{\partial x}\Big|_{x \in \partial\Gamma_t} = 1. \tag{34'}$$

Equation (34') indicates that at the time of premature exercise, the option value is changing unit-for-unit with changes in the exchange rate.

Because of the "risk-neutral hedging argument," the option price on the exchange rate is independent of both drift parameters n_3(the speed of adjustment) and n_4 (the long-run [logarithmic]) mean. The comparative statics of the analytical formula per (33) and the related numerical solution when the potential for early exercise exists uphold the intuition: the call price is an increasing function of the current exchange rate, x; the time to maturity, τ; the domestic risk-free rate of interest, i; and the volatility parameter, n_1, of the exchange rate. It is a decreasing function of the exercise price, h; the foreign risk-free rate of interest, j; and the critical floor, n_2. This last result indicates that the options are more valuable the less often the government intervenes.

We now turn our attention to what in our minds is another important benefit of the MRL process over the standard geometric Wiener (GW) process, insofar as the pricing biases observed for options are in a band of exercise prices around the current exchange rate. We will show below that the pricing biases from the MRL process are lower than those obtained from the GW process.

We compare the option prices generated by the exchange rate process per (10a), called by convention the geometric Wiener (GW, or the RLN) process, versus prices generated per (14c), i.e., by the mean-reverting logarithmic (MRL) process. Consider first a situation where the foreign country pays a "low" interest rate so that the premium (i-j) of the domestic (read U.S.) over foreign interest rates is relatively large. Then premature exercise of the call option will not occur, so the "American feature" of the option provides no additional value, and therefore (33) provides the exact valuation when the exchange rate follows the MRL process. Exchange rates observed to be following this process include the British pound, the German mark, the Swiss franc, the French franc, and the Canadian dollar.

As a comparative example, consider an option with a three-month maturity ($\tau = 0.25$), the domestic interest rate ($i = 0.09$), the foreign interest rate ($j = 0.04$), and the exercise price ($h = 0.40$). If the underlying exchange rate follows the GW (or RLN) process with a volatility parameter, $n_1 = \sigma = 0.16646$, the resulting option price when the current exchange rate, x, equals the exercise price is $u(x; \tau, \sigma) = u(0.40; 0.25, 0.16646) = 0.015671$. However, if the underlying exchange rate follows an MRL process with a critical floor of $n_2 = 0.2$, then an MRL volatility parameter $n_1 = 0.20$ produces an identical option price $u(x; \tau, n_1, n_2) = u(0.40; 0.25, 0.20, 0.20) = 0.015671$, when all the other parameters are held the same. Maintaining these volatility parameters and varying only the levels of the current exchange rate, x, then produces contrasting option prices under the GW and the MRL processes. Column 2 of Table 1 contains the option prices under the MRL process with $n_1 = 0.20$ and $n_2 = 0.20$ at various current exchange rate levels; column 3 contains the corresponding values when the underlying process is GW with $\sigma = 0.16646$. Differences in prices between the option models appear in column 4; the differences as a proportion of the GW price occur in column 5.

It is clear that the MRL process produces lower option prices than the GW process for current exchange rates below the exercise price and higher option prices for the exchange rate levels above the strike price. Furthermore, the magnitudes of both the difference in option values and the difference relative to the GW price are greater for a given percentage decline of the exchange rate below the strike price of 0.40 than for the same percentage increase above 0.40 in the interval of exchange rates within 15 percent of the exercise price. Thus if at-the-money options are defined by the current exchange rate lying in some interval around the strike price, for example $x \in [\ 0.38, 0.42\ ]$, then the MRL process will produce lower at-the-money prices than the GW process in empirical tests. Finally, the relative difference in price is always greater in magnitude for exchange rates below the strike price than for those above the strike price.

With the above-cited example parameters, numerical methods indicate that premature exercise of the foreign exchange call option under the MRL process might occur if the foreign interest rate, j, rose to 0.0475. In this case, deeply in-the-money options with maturities of τ less than 0.1375 may warrant premature exercise because the probability of further increases in the exchange rate does not justify carrying the short position in the foreign currency at the higher interest rate. Appropriate current exchange rate levels and option maturities that justify early exercise when the foreign interest rate, j, is equal to 0.0475 appear in Table 2. In cases of early exercise, solution per (33) is no longer appropriate since the American feature of the option provides additional value; then numerical solutions are required.

Larger absolute and percentage differences in option prices generated by the MRL and the GW processes occur when the exchange rate volatility, option maturity, and the interest rate premium rise. Table 3 presents results of an analysis in the format of Table 1, when the selected parameters are increased to the following levels: $i = 0.15$, $j = 0.075$, $\tau = 0.375$, $h = 1.60$, $n_1 = 0.75$, $n_2 = 1.20$, and $\sigma = 0.392693$. With these altered parameter values for the MRL process, the option value will be positive (0.000906) even though the current exchange rate has hit the critical floor level, i.e., $x = n_2 = 1.20$.

We have hitherto focused our attention on the prices for the futures and the corresponding option contracts, when the underlying exchange rate process follows the MRL process with a *norm-preserving* transition density function.

We now turn our attention to the pricing functions for the futures and the option contracts, where the underlying transition density function is *defective or norm-reducing*. In the case of the *defective* density case, we have the following price functions for the futures contract and the call option.

The price of the futures contract *h(x, t)* is as given below:

$$h(x,t) = [x_t e^{c\tau}].[\frac{e^{-2a\tau}}{(2e^{-a\tau}-1)}].\left[\frac{n_2}{x_t}\right]^{\frac{2(e^{-a\tau}-1)}{(2e^{-a\tau}-1)}}.\left[2e^{-a\tau}-1\right]^{[\frac{c}{a}]}; \tag{35}$$

and the corresponding price for the call option contract, $u(x_t, t, \tau, h, T)$, is as given below:

$$u(x_t, t, \tau, h, T) = [n_2 e^{-(j+a)\tau}][2e^{-a\tau}-1]^{(2-\frac{c}{a})}].$$

$$.\ [\frac{x_t}{n_2}]^{(1/(2e^{-a\tau}-1))}[1 - X^2[(\frac{(1-e^{-a\tau})}{(2e^{-a\tau}-1)})(2Ln(\frac{h}{n_2});\ 2(2-\frac{c}{a});\ (\frac{2Ln(\frac{x_t}{n_2})}{(2-e^{a\tau})(1-e^{a\tau})})]]$$

$$-\ he^{-(i-a)\tau}][1 - X^2[(\frac{2Ln(\frac{h}{n_2})}{(e^{a\tau}-1)});\ 2(2-\frac{c}{a});\ (\frac{2Ln(\frac{x_t}{n_2})}{(1-e^{-a\tau})})]], \tag{36}$$

where $a \equiv \frac{n_1^2}{2}$ and $c \equiv (i-j)$.

As can be seen from above, the price functional forms for both the futures contract and the call option on the exchange rate are radically different in the *defective density* case from the corresponding ones in the *norm-preserving* density case. As a consequence, any tests of the pricing model must take this into consideration. At this stage however, we have not yet derived similar comparative statics using the defective density function for the MRL process and contrasted with the corresponding GW process values.

IV. Conclusion

This paper develops a restricted class of distribution functions appropriate for reciprocal foreign exchange rate pairs. The restrictions are based on three economic requirements:

I) symmetry of the distribution for the exchange rate and its inverse; II) invariance of the exchange rate distribution to changes in the unit of account; and III) non-negativity of the exchange rate pair.

Prior theoretical work on the futures and options prices for foreign exchange has assumed that the underlying exchange rate follows a geometric Wiener (GW) process. Subsequent empirical tests of the realized prices versus those predicted by the Black-Scholes model then require only the identification of the process's volatility parameter. However, only a subset of the family of the GW processes satisfies the invariance and symmetry properties required for the exchange rate. This subset consists of the class of the GW processes whose drift parameter is exactly equal to one-half its diffusion parameter. This result provides a second (or joint) mechanism to empirically test the validity of options and futures models based on an underlying geometric Wiener process.

As an alternative to the specialized subset of the GW process, a mean-reverting logarithmic (MRL) process is developed to potentially describe the behavior of the exchange rate. The MRL process satisfies the symmetry, invariance, and non-negativity properties; furthermore, each of its four parameters has a reasonable economic interpretation. Closed-form solutions for the futures and the options prices based on an underlying MRL process are developed. It is also pointed out that the same SDE for the exchange rate process can result in different transition density functions or, equivalently, different pricing equations for different specifications of the primary parameters of interest, namely, the domestic risk-free interest rate, i, and the foreign risk-free interest rate, j.

Employing the option price function derived using the MRL process with the norm-preserving density function and contrasting with the GW process prices, we prove the following claim: If the volatility parameters of the MRL and the standard GW process of the Black-Scholes model (GW process), are set to achieve identical option prices under both processes at the striking price, then the option price under the MRL process will always be less than the price under the GW process for exchange rate values below the exercise level and always be higher for exchange rate values above the exercise level. However, both absolute and proportional differences between the respective prices will be greater for exchange rates below the exercise price than for rates above the exercise level. Therefore, we claim that if empirical tests define at-the-money options as those where the current exchange rate lies in some relatively narrow interval around the strike price, then the MRL process will generate lower prices for these options than the GW process.

Empirical testing of the option and futures pricing models implied by the MRL process requires the identification of two parameters, the volatility parameter (n_1) and the critical floor (n_2). Potentially, the critical floor may be inferred from government policy, so empirical tests can be conducted to determine the volatility parameter.

Finally, there are, of course, many other feasible exchange rate processes satisfying the functional equations necessary for symmetry, invariance, and non-negativity. Other potential cases and the development of the pricing models based on them are topics for continuing research in this area.

Appendix A

Consider the *functional equations* (3) and (4), which represent a system of equations in two unknown *functions*, $\mu(x,\bar{n})$ and $\sigma^2(x,\bar{n})$. Standard methods exist to solve functional equations. For example, consider a slightly simpler (non-parameterized) version of (4):

$$\sigma[(1/x)] = -(1/x)^2\sigma(x). \tag{A1a}$$

Defining $h(x) \equiv (1/x)\sigma(x)$, (A1a) can be rewritten as

$$h(1/x) = -h(x). \tag{A1b}$$

Furthermore, defining

$$f(\log x) = h(x) \tag{A1c}$$

implies that the function f is odd, i.e.,

$$f(-y) = -f(y). \tag{A1d}$$

Without the assumption that h has continuous derivatives, the solution to (A1b) is the set of functions with the unique form

$$h(x) = f(\log x), \tag{A2a}$$

where f is odd in the logarithm of x, i.e.,

$$h(1/x) = f[\log(1/x)] = -h(x). \tag{A2b}$$

Similar methods show that the solution to the more complicated system of equations (3) and (4) (where no assumptions on the continuity of derivatives of μ and σ are made) is given by the set of functionals $\{\mu(x,\bar{n},f,g), \sigma(x,\bar{n},f)\}$ with the unique form

$$\mu(x,\bar{n},f,g) = \tfrac{1}{2}n_1^2 x f^2(\log x, \log n_2) + n_3 x g(\log x, \log n_4) \tag{A3}$$

and

$$\sigma(x,\bar{n},f) = n_1 x f(\log x, \log n_2). \tag{A4}$$

In (A3) and (A4) the vectors of parameters in the functions f and g are limited to single parameters n_2 and n_4, respectively. This simplifies the notation but does not restrict the generality of the subsequent results. Furthermore, the functions f and g must be odd in the logarithms of the exchange rate x and the parameters n, i.e.,

$$f[\log(1/x), \log(1/n_2)] = -f[\log x, \log n_2], \tag{A5}$$

and

$$g[\log(1/x), \log(1/n_4)] = -g[\log x, \log n_4]. \tag{A6}$$

Using equations (A3) through (A6), any equilibrium exchange rate process that satisfies the symmetry property follows the stochastic differential

$$\begin{aligned} dx = {} & [\tfrac{1}{2} n_1^2 x f^2(\log x, \log n_2) + n_3 x g(\log x, \log n_4)]dt \\ & + n_1 x f(\log x, \log n_2) dW \end{aligned} \tag{A7}$$

for all functions $f(\cdot)$ and $g(\cdot)$, which are odd in the logarithm of the exchange rate and the parameters n_i. Thus, the set of drift and diffusion coefficients implied by (A7) completely characterizes the set of admissible distributions for autonomous (time-invariant) exchange rate processes under the symmetry property.

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Table 1
Comparative Option Values Under the MRL and GW Processes

(1) Exchange Rate x	(2) MRL Process Option Value	(3) GW Process Option Value	(2) - (3) Difference in Option Value	[(2) - (3)]/(3) Difference in Value Relative to GW Price (in percent)
0.32	2.1	5.3	-3.2	-60.38
0.33	9.0	16.2	-7.2	-44.44
0.34	29.5	42.8	-13.3	-31.07
0.35	79.1	99.5	-20.4	-20.50
0.36	180.5	206.6	-26.1	-12.63
0.37	360.7	387.6	-26.9	-6.94
0.38	633.7	665.2	-21.5	-3.23
0.39	1,044.5	1,056.2	-11.7	-1.11
0.40	1,567.1	1,567.1	0.0	0.00
0.41	2,204.1	2,193.7	10.4	.47
0.42	2,940.1	2,922.5	17.6	.60
0.43	3,755.2	3,734.7	20.5	.55
0.44	4,629.9	4,610.0	19.9	.43
0.45	5,547.0	5,529.7	17.3	.31
0.46	6,492.3	6,478.8	13.5	.21
0.47	7,455.9	7,446.0	9.9	.13
0.48	7,420.7	8,424.0	6.7	.08

The assumed exchange rate processes are given by (26) for the GW process and by (33) for the MRL process. The parameter values employed are, respectively, the time to maturity of the option, $\tau = 0.25$; the domestic interest rate, $i = 0.09$; the foreign interest rate, $j = 0.04$; the exercise or strike price, $h = 0.40$; the volatility parameter, $n_1 = 0.20$; the critical floor, $n_2 = 0.20$ for the MRL process; and the volatility parameter, $\sigma = 0.16646$ for the GW process. The trading vehicle is assumed to cover 100,000 units of the foreign currency.

Table 2

Optimal Exercise Levels of the Exchange Rate at Different Maturities for the MRL Process

Time to Maturity (τ)	Exchange Rate Triggering Premature Exercise
0.25	↑
0.224	No
0.20	Premature
0.175	Exercise
0.15	↓
0.125	x = 0.598
0.10	x = 0.594
0.075	x = 0.589
0.05	x = 0.5835
0.025	x = 0.5765

This table lists the level that the current exchange rate must reach to create value for the premature exercise feature for the American call option, with the maturities listed keeping all the other parameters except the foreign interest rate, j, at the same levels as those listed in Table 1. The foreign interest rate is $j = 0.475$ for this exercise. With $j = 0.04$, it is never optimal to prematurely exercise the option. Note that the exchange rate triggering premature exercise is an increasing function of the time to maturity, τ.

Table 3

Comparing Option Values with Increased Volatility, Maturity, and Interest Rate Spreads

(1) Exchange Rate, x	(2) MRL Process Option Value	(3) GW Process Option Value	(2) - (3) Difference in Option Value	[(2) - (3)]/(3) Difference in Option Value Relative to the GW Price (in percent)
1.40	5,763.9	7,478.1	-1,714.2	-22.92
1.45	8,130.0	9,438.3	-1,309.3	-13.87
1.50	10,804.3	11,674.0	-869.7	-7.45
1.55	13,751.5	14,177.1	-425.6	-3.00
1.60	16,938.6	16,938.6	0.0	0.00
1.65	20,335.7	19,944.9	390.8	1.96
1.70	23,656.2	23179.6	736.5	3.18
1.75	27,656.2	26,624.7	1,031.5	3.87
1.80	31,535.2	30,261.2	1,274.0	4.21

The assumed exchange rate processes are given by (26) for the GW process and by (33) for the MRL process. The parameter values employed are, respectively, the time to maturity of the option, $\tau = 0.375$; the domestic interest rate, $i = 0.15$; the foreign interest rate, $j = 0.075$; the exercise or strike price, $h = 1.60$; the volatility parameter, $n_1 = 0.75$; the critical floor, $n_2 = 1.20$ for the MRL process; and the volatility parameter, $\sigma = 0.392693$, for the GW process. The trading vehicle is assumed to cover 100,000 units of the foreign currency.