The Implied Volatility of U.S. Interest Rates: Evidence from Callable U.S. Treasuries

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Abstract: The prices for callable U.S. Treasury securities provide the sole source of evidence concerning the implied volatility of interest rates over the extended 1926-1994 period. This paper uses the prices of callable as well as non-callable Treasury instruments to estimate implied interest rate volatilities for the past sixty years, and, for the more recent 1989-1994 period, the cross-sectional term structures of implied interest rate volatility. We utilize these estimates to perform cross-sectional richness/cheapness analysis across callable Treasuries. *Inter alia*, we develop the optimal call policy for deferred "Bermuda"-style options for which prior notification of intent to call is required—by introducing the concept of "threshold volatility" to measure the point when the time value of the embedded call option has been eroded to zero. Using this concept facilitates callable-bond valuation and documents the optimality of the Treasury's past call policy for U.S. government obligations.

JEL classification: G13, H63

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1 Introduction

Interest rate contingent claims in general, and those that depend on volatility in particular, have grown substantially in type, open interest, and trading volume. The volatility-dependent interest rate securities traded on and off exchanges now include caps and floors, options on bonds, options on interest rates and interest rate spreads, options on futures contracts, call and put swaptions, interest-only and principal-only securities and so-called indexed amortizing rate swaps. In addition, embedded options are found in callable and puttable bonds, mortgage-backed securities, and sinking-fund bonds.

Concomitant with these developments have come a number of interest rate valuation models. The literature originated with the Cox, Ingersoll, and Ross (CIR) (1985) model, with other one-factor models provided by Vasicek (1977), Dothan (1978), Courtadon (1982) and Brennan and Schwartz (1979). Brennan-Schwartz (1982) also provided the first two-factor interest rate model, which explicitly allowed for a non-perfect correlation between the long- and short-term rates of interest. Ho and Lee (1986) were the first to offer a stochastic interest rate model that exactly matched the currently observable term structure of interest rates. Black, Derman, and Toy (1990), Black and Karasinski (1991), and Hull and White (1990) extended the Ho-Lee model to match a term structure of volatility curve (or, equivalently, cap prices) in addition to the term structure. Longstaff and Schwartz (1992) and Fong and Vasicek (1991) considered two-factor models in which interest rates and interest rate volatility are both stochastically changing through time. Finally, Heath, Jarrow, and Morton (HJM) (1990, 1992) provided a rigorous theoretical framework for one- and two-factor no-arbitrage interest rate models.

Another important strand of the literature deals with the testing of the stochastic distribution of (the changes in) interest rates. Chan, Karolyi, Longstaff, and Sanders (CKLS) (1992) used historical time series to fit single-factor models of the spot rate of interest,

including CIR, Vasicek, Dothan, and Brennan and Schwartz. Flesaker (1992) tested the Ho-Lee model. Amin and Morton (1994) used short-dated options on Eurodollar futures contracts to estimate and test alternate formulations of the HJM model, while Bliss and Ritchken (1995) test a single-factor, two state-variable version of the HJM model against a single state-variable generalized Vasicek model.

The objective of this paper is to utilize the longest available time series of volatility-sensitive interest rate contingent securities to characterize the term structure of implied U.S. interest rate volatilities and to extract the cross-sectional richness/cheapness implications of these volatilities. We do this by utilizing the market prices of callable U.S. Treasury securities, available on the CRSP data tapes for the period 1926–1994.[1] In so doing, we examine implied interest-rate term structures representing significantly longer-dated maturities than those analyzed by Amin and Morton (1994).

The use of callable Treasury securities requires that we confront the issue of possibly irrational pricing of these securities. Longstaff (1992) and Edleson, Fehr, and Mason (1993) have found negative option values implicit in the market prices of callable Treasuries, while Jordan, Jordan, and Jorgensen (1995) believe they have resolved this issue by explicit consideration of tax effects. We show that the negative option values noted by Longstaff and others exist predominantly when the option is deep out-of-the-money, and may be attributed to pricing errors well within the normal range found in non-callable Treasury securities.

In addition to characterizing the historical behavior of the term structure of interest rate volatility, we derive the optimal call policy for callable Treasury securities and then examine whether the Treasury has in the past implemented such an optimal call policy for these government obligations.[2] Ingersoll (1977) was among the first to examine optimal call policies by examining corporate convertible callable securities. Vu (1986) focused his attention on callable, non-convertible corporate securities. Longstaff's (1992) analysis of callable Treasuries, based on Vu (1986), overlooks certain critical features of Treasury call provisions.

[1]Over this period, the U.S. Treasury also issued two puttable bonds and one callable perpetuity.

[2]Bühler and Schultze (1992) investigated the German government's call policy and concluded that rational call opportunities were frequently missed. Not surprisingly, they also document the presence of implied negative option values in the German bond market.

First, Longstaff bases his call/no-call decision on the market price of the callable bond. In contrast, we argue that the callable bond's price cannot provide sufficient information to make an unambiguous decision and that an option valuation model—such as the one we posit—is required in order to make the tradeoff between the coupon rate and interest-rate volatility. Further, Longstaff ignores the 120-day call notification period, which the Treasury must provide to holders of callable bonds.[3] Longstaff notes instances of bonds trading in excess of par but is unable to explain this "puzzling" anomaly. By recognizing the notification period, we demonstrate that callable bonds could rationally trade at premiums to par as observed.

The paper is organized as follows. Section 2 introduces the stochastic interest rate model we utilize. Given this interest rate model, we use numerical techniques to value callable Treasury securities. Section 3 discusses the data. Specifically, we address the oft-raised issue of whether callable Treasuries are rationally priced relative to their non-callable counterparts. We do so by estimating the term structure of spot interest rates and computing the values of the bonds' embedded options. Section 4 presents empirical estimates of the implied volatilities observed in the marketplace, as well as a comparison of callable-bonds' implied volatilities with those calculated from options on Treasury bond futures contracts. Subsequently, Section 5 addresses the Treasury's past observed call policy to determine its optimality. Section 6 presents estimates of the term structure of implied volatilities in interest rates. Section 7 summarizes and discusses future research objectives.

2 Calculation of Implied Volatility

2.1 Stochastic Models of Interest Rate Movements

As reviewed above, the finance literature has provided an abundance of stochastic interest rate models. We propose to use the following risk-neutral stochastic model for spot interest

[3]The predecessor paper, Vu (1986), also ignores the less-onerous requirement for corporate bonds, which typically have a prior notification period of between 30 and 90 days.

rate movements:[4]

$$dr = \mu_t r\, dt + \sigma r\, dz, \tag{1}$$

where

$dr \equiv$ the change in the short-term rate of interest r;

$\mu_t \equiv$ the expected change in dr/r; μ_t is chosen to precisely match the observable term structure of interest rates;

$\sigma \equiv$ a measure of the standard deviation; thus, $\text{Var}(dr) = \sigma^2 r^2 dt$;

$dz \equiv$ is the stochastic Browning process, with $E(dz) = 0$ and $\text{Var}(dz) = dt$.

In implementing the above model, we precisely match the term structure of interest rates.[5] As is well-known, models such as equation (1) are in wide usage both in academia and industry. These models are not, however, without their academic critics. Specifically, in discrete time, the use of the vector $\boldsymbol{\mu}_0 \equiv [\mu_1, \mu_2, \ldots, \mu_T]'$ (designed to match the term structure exactly) typically gives rise to intertemporal inconsistency. Intertemporal consistency would require that the vector of the time drift parameters at time 1, $\boldsymbol{\mu}_1$, equal that at time 0 less μ_1.[6] This property is typically not satisfied with arbitrary term structures for times 0 and 1. Nevertheless, we adopt the usage of equation (1) due to its important ability of exactly matching the observable term structure of interest rates. In so doing, we make maximal use of the information available in the term structure of interest rates and are able to match exactly the prices of underlying non-callable assets.

[4]Any reference hereafter to expectations $E(\cdot)$ or variance $\text{Var}(\cdot)$ constitutes a reference to the variable's risk-neutral distribution.

[5]This means that under the risk-neutral expectation generated by eq. (1), for given σ, μ_t is chosen so that

$$E\left(\exp\left\{-\int_0^T r_t\, dt\right\}\right) = \text{PV}_T,$$

where PV_T is the price of a zero-coupon bond of maturity T.

[6]Formally, intertemporal consistency implies that, for given T, $\boldsymbol{\mu}_0$, $\boldsymbol{\mu}_1$ should satisfy the property that

$$\boldsymbol{\mu}_0 = [\mu_1, \boldsymbol{\mu}_1]'.$$

2.2 The Term Structure of Volatility

As has been recognized by Black, Derman and Toy (1990), Black and Karasinski (1991), and Hull and White (1990), the volatility σ in equation (1) is not in fact constant. Ideally, one would seek to estimate a term structure of volatility, $\sigma(t)$, using, say, a set of cap prices.[7]

The sparse set of callable Treasury securities available over the past sixty years limits our ability to estimate the entire term structure of local volatility each period. In principle, the volatility implied by bond i, σ_i, is given by the entire term structure of volatility up to its maturity date:

$$\sigma_i = f\left(\{\sigma(t)\},\, 0 \leq t \leq T_{mi}\right),$$

where $f(\cdot)$ is an as-yet-unspecified function and T_{mi} is bond i's maturity date.

Note that this procedure is quite analogous to the one we could implement for Black-Scholes equity options. Specifically, if we valued equity options under a term structure of implied volatilities $\sigma(t)$, we would have $\sigma_i^2 = (1/T)\int_0^T \sigma^2(s)\, ds \equiv g(T)$ for the appropriately defined g.[8]

Given the evidence, from the prices of exchange-traded options, that implied volatilities depend on whether the option is at- or away-from-the-money, we allow implied volatilities of bonds that are out-of- or in-the-money to differ from those at-the-money. Thus, we augment $f(\cdot)$ with the argument $|\, \text{FP}_i - 100\,|$, where FP_i is the forward price on the first call date of a non-callable bond with the same coupon rate and maturity date as bond i and $|\cdot|$ denotes "absolute value":

$$\sigma_i = f\left(\{\sigma(t)\},\, 0 \leq t \leq T_{mi};\, |\, \text{FP}_i - 100\,|\right).$$

[7]Alternatively, Black and Karasinski (1991) and Hull and White (1990) have suggested modeling a term structure of volatility by incorporating mean-reversion in eq. (1). For example, with σ set to a constant, and a term structure of volatility achieved through the selection of a time-dependent rate of mean reversion, β_t :

$$dr = (\mu_t - \beta_t r)\, dt + \sigma\, dz.$$

[8]Of course, for the constant volatility case $\sigma(s) = \sigma$ for all s, $\sigma_i = \sigma$ and so the appropriate g is $g(T) \equiv \sigma$.

For any callable bond, there are two key dates: time to first call T_{ci} (which can be zero, for currently callable bonds) and time to maturity T_{mi}. Letting $\{\sigma(t)\}$ be approximated by a functional form to be estimated, we model the maturity-dependent change in σ by attributing it to the (callable) bond-specific indicatives: time to first call date, time to maturity, and measures of away-from-the-money. Thus, let

$$\sigma_{it} = f\left(T_{ci}, T_{mi}, \mid \mathrm{FP}_i - 100 \mid\right), \tag{2}$$

where σ_{it} is the implied volatility for bond i at time t.

2.3 Numerical Implementation Techniques

To begin, we estimate the term structure of interest rates using three alternative procedures:

1. Yields on stripped Treasury coupons, C-STRIPS
2. Implementation of the Fama-Bliss (1987) method of extracting the pure discount rate curve from a sample of (non-callable) bonds.[9]
3. Implementation of the Nelson-Siegel-Bliss method of extracting the pure discount rate curve from a sample of (non-callable) bonds. See Bliss (1994) for details.

See Appendix A for a description of the latter two estimation methods.

This produces three alternate estimates of the current risk-free present value function, PV_t for $0 \leq t \leq T$; the value of T will be dictated by the maturity date of the longest-maturity non-callable bond in each monthly sample. Given the term structure, Jamshidian (1991) provides the forward-induction procedures used for estimating the vector of drifts μ_0 in an efficient manner in the context of a binomial tree for the short-term rate of interest.

The binomial tree is built through appropriate calculation of the centrality parameters of the tree for all future periods. Thus, for a period of length Δ, calculating the centrality

[9] The CRSP Fama-Bliss files extend only to a maturity of five years. The same underlying procedure was utilized to extend our calculations to the maturity date of the longest non-callable bond.

parameter for the second time interval requires solving the non-linear equation for u_Δ as given by

$$\mathrm{PV}_{2\Delta} = \mathrm{PV}_{\Delta}\, E\left(\exp\left\{-u_{2\Delta} e^{\sigma \tilde{z}}\right\}\right),$$

where the discretized Brownian motion $\tilde{z}$ has $E(\tilde{z}) = 0$ and $\mathrm{Var}(\tilde{z}) = \Delta$.

A recursive procedure is then used to construct the tree for successive time intervals. Appendix B provides a numerical example of the implementation of this procedure for the lognormal interest rate distribution in eq. (1). In the implementation of the interest rate tree, we utilize a six-month time step for Δ. However, we need to account for the time interval from the quote date to the first coupon date, which will generally be different from six months. Appendix B also demonstrates the numerical procedure implemented to account for the distinct length of the first time interval from subsequent ones.

Once we have completed the calculation of $u_{i\Delta}$ for all periods i, for a given σ we can calculate the value of the callable bond through backward induction. The implied volatility is chosen to equate the model's bond value with the market price.

3 Data

3.1 Data Period

In order to perform our tests, we require an estimate of the term structure of interest rates, which estimate we obtain from the prices of non-callable Treasury securities. To avoid using extrapolated data, the current study will cover only those subperiods wherein the maturities of non-callable bonds permit us to estimate an accurate measure of the term structure. Figure 1 presents the availability and maturities of callable U.S. government bonds for the period 1926–1994, after excluding those callable bonds whose maturities are not spanned by their non-callable counterparts. Also excluded are flower bonds, and bonds with unusual provisions.

3.2 Arbitrage Bounds on the Pricing of Callable Bonds

There are two well-known arbitrage bounds for callable bonds that should be satisfied by any no-arbitrage valuation model. These pertain to the relationship between the value of the callable (at par) bond B, on the one hand, and the values of two hypothetical non-callable bonds (with coupons identical to B) maturing at the first call date and the maturity date. We denote the values of these bonds as S and L, respectively. It can be shown that, in the absence of arbitrage,

$$B \leq \min\{S, L\}. \tag{3}$$

Intuitively, eq. (3) is motivated by the fact that the option not to precommit on the call/no-call decision is valuable.[10] More formally, if $B > S$, sell short the callable bond and buy the short bond, pocketting the difference. On the first call date, S will pay \$100 which is then used to call the shorted bond. Similarly, if $B > L$, a long position in the long bond coupled with a short position in the callable bond will earn arbitrage profits equal to at least B-L. If rate go up, so that both bonds trade at discounts do nothing. Since their coupon and principal amounts exactly offset each other net future cash flows are zero. However, if at any call date interest rates have declined sufficiently so that the long bond is selling at a premium, then selling the long bond and calling the shorted bond will earn an additional profit equal to the premium.

Any violation of relation (3) gives rise to an arbitrage opportunity and hence cannot be rationally explained by an arbitrage-free valuation model. Yet it is precisely this violation that has been detected in the data by Longstaff (1992) and Edleson, Fehr, and Mason (1993) and addressed by Jordan, Jordan, and Jorgensen (1995).

3.3 "Irrationalities" in the Pricing of Callable Treasuries

The primary interest of this paper lies in eliciting the interest rate volatilities implicit

[10] Jordan, Jordan, and Jorgensen (1995) point out that this relation must hold for all call dates, not merely the first and last ones. Our optimality conditions, developed below, implicitly accounts for all the intermediate dates, not just the one at the extremes.

in callable U.S. Treasuries. Hence, we do not seek to explain "irrationalities" in the pricing of callable bonds, just as we do not seek to explain why the prices of non-callable U.S. government bonds cannot all be determined by a unique term structure of interest rates.[11] Rather, our objective is to extract the information implicit in callable bonds regarding the future volatility of U.S. risk-free rates of interest.

Nevertheless, it is incumbent upon us to obtain a better understanding of this "irrationality." Thus, consider the forward price of the long non-callable bond on the first call date:

$$\mathrm{FP} = \frac{\mathrm{DIB}}{\mathrm{PV}_S}, \tag{4}$$

where

$\mathrm{FP} \equiv$ forward price, inferred from today's term structure, of the long non-callable bond on the callable bond's first call date;

$\mathrm{DIB} \equiv$ value today of a deferred interest non-callable bond, which begins accruing interest on the first call date and matures on the same date as the callable bond; and

$\mathrm{PV}_S \equiv$ present value factor of \$1 payable on the first call date.

Now,

$$\mathrm{DIB} = L - S + 100\,\mathrm{PV}_S. \tag{5}$$

Define a measure of "away-from-the-money forward" as

$$\delta \equiv \mathrm{FP} - 100. \tag{6}$$

Combining eqs. (4) - (6) yields

$$\delta = \frac{L - S}{\mathrm{PV}_S}.$$

To characterize the problem of "irrationally"-priced callable bonds, consider a sample of bonds observed over a period of time, ranked in increasing order of δ. Of those bonds,

[11] Bliss (1994) attributes this inability to price all non-callable U.S. Treasuries via a unique term structure to "non-PV" effects, including: market segmentation, liquidity effects and tax-timing options.

select the 100 with the lowest δ (i.e., the observations that are most out-of-the-money) and calculate the fraction for which $\min\{L, S\} - B$ is positive. Then, drop the bond with the lowest δ, add the bond with the 101st lowest value of δ, and recalculate the fraction for which $\min\{L, S\} - B$ is positive. Continue until the last window consists of 100 bonds with the highest δ's. Using the three alternative estimates of the term structure of interest rates for the post-1985 data period—C-STRIPS prices, the Fama-Bliss method, and the Nelson-Siegel-Bliss method—Figures 2a–2c plot the moving average of this fraction against the measure of away-from-the-money forward, δ. We conclude that the violations of (3) occur most frequently when the bonds' call options are out-of-the-money, i.e., $\delta < 0$. Indeed, these results are consistent with Jordan, Jordan, and Jorgensen's (1995) finding of a small tax effect that is important only for out-of-the-money calls.

For the purpose of this study, this result is fortuitous. We are not interested in estimating implied volatilities for options far away-from-the-money. Indeed, it is most difficult to do so: the option's vega (defined as $\partial V/\partial\sigma$ where model value equals market price) will be extremely low, rendering the implied volatility estimate unreliable. Thus, we will not be concerned with having to discard those observations for which the implied volatility could not in any case have been reliably estimated.

3.4 Apparent Underpricing of Callable Bonds

In calculating implied volatilities for callable bonds, we imposed an upper bound of 100% on the permissible volatility. This additional constraint resulted in a classification of callable bonds into three types: those with "negative" option values alluded to in the previous section, those with ("good") implied volatilities in the range [0, 100%], and those whose market prices were less than the bonds' fair values at volatilities of 100%, which we classified as "huge" option values. Figure 3 depicts the annual classifications of available observations into these three categories. We can observe that in the pre-1960 period, "huge" implied volatilities constituted a more significant problem than the "negatives." Since the mid-1980s, the incidence of "huge" and "negative" problem observations has declined markedly. This may explain why Jordan, Jordan, and Jorgensen (1995), with their data sample taken from this later period, find a relatively small number of overpriced bonds relative to Longstaff (1992) and our current work.

Our analysis of the data underlying Figure 3 indicates that the "huge" volatilities may arise from (at least) three possible sources. First, short maturity bonds combined with a fixed-interval (six months) binomial lattice produce an insufficiently dense distribution of bond prices. Second, long-dated callables are priced under an interest rate distribution that has sparse density at reasonable interest rates; the remaining few nodes at the center of the tree have insufficient density to produce meaningful changes in the bonds' present values as σ changes. The consequence of both these phenomena is to produce extremely low vegas, with bond values that are insensitive to changes in volatility. Finally, we observe a number of intermediate-maturity, low-coupon bonds, primarily in the 1950s, whose prices appear unreasonably low and cannot be attributed to numerical problems. Their (reasonable) vegas indicate a price sensitivity to volatility changes, but even "large" volatilities are unable to explain the large option value.

4 Initial Empirical Results

4.1 Time Series of U.S. Interest Rates' Implied Volatilities

Multiple observations of rationally priced callable bond prices with vegas exceeding .05 are available only for the period 1926–1946 and 1987–1994. In the period 1947–1986, such usable data occurred only infrequently.

Figure 4a presents the vega-weighted implied volatilities generated each month using the three term structures of interest rates for the latter period, 1987–1994. Define an intermediate-term bond to be of a maturity between five and twenty years; define a long-term bond as having a maturity in excess of twenty years. We examined the behavior of the intermediate-term and long bonds separately in order to allow for a possible maturity effect on the time series of implied volatilities;[12] there was an insufficient number of short maturity

[12]Section 6 will explicitly model a term structure of implied volatilities.

bonds in this period to provide a reliable implied-volatility time series.

Over the period 1987–1994, the patterns of the implied volatilities are similar across the three term structure estimation methods. For the intermediate term, the implied volatilities were increasing monotonically from 2% to 14%. The long-maturity implied volatilities are generally more variable than the intermediate-term volatilities. Furthermore, these implied volatilities increase in the late '80s and early '90s but evidence a decline after 1991.

Figure 4b presents all callable bonds' vega-weighted implied volatilities for the period 1926–1955 using the Fama-Bliss method for the estimation of the term structures of interest rates. The time series pattern here is one of an increasing volatility in this period, leveling off towards the latter part of the period.

4.2 Comparison with T-Bond Futures Options

Table 1 provides summary statistics and correlations for the vega-weighted callable bonds' implied volatilities using the three estimation methods, as well as the correlation with the implied volatility of the short-maturity at-the-money option on the T-Bond futures contract. The latter uses the Black (1976) model to infer a price volatility from the closing prices of short-term call options and their associated futures contracts.

For both the long and intermediate term, the callable bonds' implied volatilities are significantly positively correlated across estimation methods. For the long-term maturities, the three methods' vega-weighted implied volatilities are negatively correlated with the futures options implied volatilities, and these correlations are all statistically significant. For the intermediate term, the correlations of callable bonds' implied volatilities with that of the futures option's are not statistically different from zero. These results suggest that there is information in the callable bonds' implied volatilities over and above that available from the futures option's implied volatilities. Further, Figure 4a indicates that the implied volatilities from the intermediate-term maturities provide incremental information regarding the future evolution of interest rate volatilities not available solely from the behavior of long-maturity implied volatilities.

4.3 Treasury 11.75s of 11/15/09–14

The inception of the Treasury's STRIPS program has permitted the efficient trading of stripped zero-coupon Treasury securities. The Treasury has permitted only one callable bond to be included in this program, the 11.75s of 11/15/09–14.[13] This particular bond may therefore have received more attention in pricing than other callable securities. It is also one of the longer time series of implied volatilities for any individual bond. Furthermore, the availability of C-STRIPS prices permits the efficient evaluation of its embedded option. It is thus of interest to inspect Figure 5, which plots these implied volatilities for the post-'85 period for this bond.

The one conclusion that can be gleaned from Figure 5 is that implied volatilities on such bonds typically range from 5% to 20% and lie for the most part in the 10% to 15% range. These numbers will be useful in examining the optimality of the Treasury's call policy, since they are an indication of the volatility the Treasury should consider when making the call/no-call decision. We can also conclude that implied volatilities in the post-'89 period are in broad agreement across the different term structure estimation methods.

5 Examining the Optimality of the Treasury's Call Policy

We seek to analyze the optimality of the Treasury's call policy by examining the policies pursued at every event date when the Treasury could have called a bond. The indenture provisions for (most) callable U.S. Treasury securities call for a 120-day notification period prior to a call. The call itself can take place only on a coupon payment date, i.e., twice per annum.

One alternative to examining the optimal call policy consists of an examination of the market price of the callable bond on call notification dates. This approach suffers from the

[13] Realizing the undesirability of the "tail" consisting of the principal plus the coupons, if any, after the first call date, the Treasury ceased issuing callable bonds. All Treasury securities issued since 1985 have been non-callable.

lack of informativeness of the callable bond's current market price, B. For example, if $B > S$, the callable bond is clearly mispriced rich. That does not, however, indicate that it ought necessarily to be called: if its fair value is less than S, the optimal strategy is to short the callable bond, and go long the replicating portfolio at the fair value. Now, assume that $B = S$ exactly. Then even if S is indeed the fair value, we have insufficient information upon which to determine whether or not we are *indifferent* between calling and not calling the bond. Finally assume that $B < S$. Clearly, we should not call the bond; but, if its fair value indicates it *should* be called, we should purchase as much as possible of that bond (with a limit price of S), and call the remainder. Thus, if the bond is marginally in- or out-of-the-money, the Treasury has relatively little, and imprecise, information on which to act. Further, a single bond may be mispriced, due to minor market frictions or the possible impact of taxation.[14]

In order to base the call/no-call decision on the wide array of information available in the prices of traded bonds—i.e., in the fair value of the bond—we estimate the term structure of interest rates from the prices of non-callable securities, and then use the callable bond's coupon rate to assess the threshold volatility, the volatility at which the call option time value would have eroded to zero, and thus the bond should be called (or purchased, if $B < S$). Setting the callable bond price to just below S, we solve for the implied volatility at that price. This threshold volatility is denoted σ_T. We show that when σ_T is "large" (in the sense defined below), the bond should be called.

Consider now a bond that is approaching the end of its call protection period. Since Treasury securities typically pay interest mid-month, and given our *end*-of-month price observations, we constrain the Treasury to decide whether or not to call no later than 4.5 months prior to a coupon payment date. Clearly, any decision at 5.5 or 6.5 months prior to the next coupon payment would be suboptimal. At 4.5-months prior to the next call date,

[14] An analogy from equity markets may be useful: European-style options, as they near their expiration dates, may sell at a slight discount to their intrinsic value due to the tax implications of exercising a call option. Tax effects have been documented in bond markets by Schaefer (1982), Ronn (1987), Bliss (1994) and Jordan, Jordan, and Jorgensen (1995) and hence constitute potential "non-PV" effects in bond prices.

the Treasury should give call notification if both of the following two conditions are satisfied:

1. On the notification date, the option is in-the-money forward:

   I.e., the forward price of the long bond L as of the next call date, observed on the notification date, is greater than par. Recall that L was previously defined to be an otherwise-equivalent non-callable bond that matures on the callable bond's final maturity date. Let $S_{4.5\,m}$ be an otherwise-equivalent bond that matures in 4.5 months. This first necessary condition specifies that if

$$\mathrm{FP}_{4.5\,m} \equiv \frac{L - S_{4.5\,m} + 100\,\mathrm{PV}_{4.5\,m}}{\mathrm{PV}_{4.5\,m}} > 100, \tag{7}$$

   then the option's intrinsic value is positive.

2. The time value of the option has eroded to zero.

   The value of the currently callable bond at a notification date 4.5 months prior to the next coupon date is:

$$B_0 = \min\left\{\frac{B^u_{4.5\,m} + B^d_{4.5\,m}}{2}\,\mathrm{PV}_{4.5\,m},\ S_{4.5\,m}\right\},$$

   where

   $B_0 \equiv$ is today's value of the callable bond;
   $B^s_{4.5\,m} \equiv$ the value of the callable bond 4.5 months hence, if not called today, in the up- ($s = u$) and down-states ($s = d$) next period.

   The first term in the min operator is the value of the callable bond if the option is not exercised; the second term is the value if it is. The difference $\frac{1}{2}\left(B^u_{4.5\,m} + B^d_{4.5\,m}\right)\mathrm{PV}_{4.5\,m} - S_{4.5\,m}$ represents the time value remaining in the option.

   Now, define the threshold volatility, σ_T, implicitly by the relation

$$B_0(\sigma_T) \equiv \min\left\{\frac{B^u_{4.5\,m}(\sigma_T) + B^d_{4.5\,m}(\sigma_T)}{2}\,\mathrm{PV}_{4.5\,m},\ S_{4.5\,m}\right\} = \lim_{\epsilon\to 0} S_{4.5\,m} - \epsilon, \tag{8}$$

for a positive but arbitrarily small ϵ (e.g., $\epsilon = 5$ cents).[15] Note that σ_T is defined only on notification dates, i.e. 4.5 months prior to the next call date/coupon payment date.

σ_T represents the maximum volatility consistent with an extinguished time value. If this maximum volatility is large relative to "normal" market volatilities, we can conclude that there is no time value remaining in the call option, and the Treasury should call the bond if it is in-the-money forward.[16] On the other hand, if the threshold volatility is small, there must be time value remaining in the option at normal levels of volatility, and the Treasury should refrain from calling the bond. Note that σ_T depends only on the (model and the) prices of non-callable bonds needed to generate the term structure—it does not depend on the observed prices of callable instruments.

When the option is, as is the case here, in-the-money (forward), then for a positive but "low" σ, $\min\left\{\left[B^u_{4.5m}(\sigma) + B^d_{4.5m}(\sigma)\right] \mathrm{PV}_{4.5m}/2,\, S_{4.5m}\right\} = S_{4.5m}$. At higher levels of σ, we find that threshold value σ_T for which equation (8) obtains. If, for a very large σ (we use $\sigma = 100\%$), it is still true that $\min\left\{\left[B^u_{4.5m}(\sigma) + B^d_{4.5m}(\sigma)\right] \mathrm{PV}_{4.5m}/2,\, S_{4.5m}\right\} = S_{4.5m}$, we conclude that σ_T is arbitrarily large and cease our search. Clearly, for such large σ_T's, the true σ cannot exceed σ_T, and the bond should be called.

Given observed information on or beliefs regarding volatility, we are now in a position to examine the Treasury's call policy. We deem it optimal when the Treasury calls a bond if and only if both (1) $\mathrm{FP}_{4.5m} > 100$ *and* (2) σ_T is "large" are satisfied and suboptimal whenever it clearly deviates from such a policy. Where the threshold volatility approximates normal market volatility between 8% and 15%, we designate these cases as "ambiguous."

Table 2 documents the threshold volatilities for U.S. Treasury callable bonds for the time period for which C-STRIPS prices are available: 1985–1994. During this time frame, there were three bonds that ended their call protection period: the 7.5s of Aug. 1988–1994, the 7s of May 1993–1998, and the 8.5s of May 1994–1999.

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[15]If $B(\sigma = 0) < S_{4.5m}$, the threshold volatility condition (8) cannot be satisfied (the threshold volatility is not defined), and the bond should not be called.

[16]Alternatively, the Treasury should purchase the bond in the marketplace if its current price is less than $S_{4.5m}$. For example, the 7s of May '93–98 could have been acquired at 101.76 rather than called at the (effective) price of $S = 102.26$.

Several important conclusions can be drawn from Table 2:

1. The term structure estimates provided by the C-STRIPS data, and the Fama-Bliss and Nelson-Siegel-Bliss methods, are in broad agreement with respect to the present values of cash flows: i.e., $\mathrm{FP}_{4.5m}$, $S_{4.5m}$ and L. Further, all term structure estimates are in agreement with respect to cases when $L < S_{4.5m}$. Therefore, the usage of Fama-Bliss and Nelson-Siegel-Bliss term structures is appropriate when C-STRIPS prices are not available.

2. The three term structure estimates yield essentially similar estimates of the threshold volatility σ_T.

3. Note from the values of both $S_{4.5m}$ as well as the callable-bond (full) market prices, that a premium of over $1 per $100 face value is not at all uncommon. Contrary to Longstaff (1992), the combination of prior notification and Bermuda-style option can easily explain a notification-date price- premium in excess of 61 cents.

4. We conclude from Table 2 that, at least in recent years, the Treasury has called the bonds optimally. They did not call the bond when the forward price was at a discount, and they did not call precipitously: the threshold volatility was at least in the 20% range when a call was triggered. Note, specifically, that the Treasury declined to call when, on 910328, the option was in-the-money forward, but the threshold volatility was a normal 7.5% to 10.3%. The $7\frac{1}{2}$s of Aug'88–93 provide the only clean test of the optimality of the call policy. In this case the bonds were in-the-money forward in March 1991, but with relatively low threshold volatilities. When the Treasury did call the following September, both conditions were clearly met.

Table 3 presents evidence from the Fama-Bliss term structure estimates for the earlier part of the century, the years 1932 through 1971.[17] The table displays the call decisions for the 41 callable bonds that had moved beyond their call protection period in the four decades beginning in the 1930s. Several conclusions can be drawn from Table 3:

[17]There were no currently callable bonds, with maturities spanned by non-callable bonds, between Sep. '72 and Aug. '88.

1. In the earlier part of the century, the Treasury issued callable bonds with other than a five-year terminal call period.

2. A clearly suboptimal decision is calling a bond when the forward price is at a discount because that implies that the option is being exercised when it is out-of-the-money. There are only four instances when the Treasury has called a bond when the option was out-of-the-money. In each case the spot price was at a premium, but the forward price was at a discount, suggesting that the Treasury ignored the forward nature of the call.

3. On the other hand, for the most part the Treasury also did not wait unduly to call. Thus, we see only four examples of high threshold volatilities and forward prices in excess of par unaccompanied by a call.

4. In comparing the Price column with $S_{4.5m}$, we observe that the market was reasonably efficient in anticipating the Treasury's call: in virtually all cases, Price = $S_{4.5m}$ when a subsequent call notice was given.[18]

5. There were a total of 17 instances of apparently suboptimal Treasury behavior. For the remaining 27 cases, the call decision was optimal, giving the Treasury the benefit of the doubt when the threshold volatility was in the normal range of 8–15%.

While we cannot justify each Treasury call decision, the overall Treasury call policy appears consistent with financial principles. This result may be a fortuitous consequence of a naive strategy, as only the single case noted above provides an opportunity to observe a more sophisticated decision process.

[18] A notable exception is the 3.5s of 6/15/47. When the call notice was given in 1935, $S_{4.5m}$ = 101.70, but the market price was quoted at an inflated 104.70.

6 Estimating the Term Structure of Volatilities

Recall that eq. (2) provided a general theoretical framework for estimating a term structure of volatilities. Now, taking a first-order Taylor's series expansion of $\ln \sigma_{it}$ in eq. (2) yields

$$\ln \sigma_{it} = c_{0t} + c_{1t} T_{ci} + c_{2t}\left(T_{mi} - T_{ci}\right) + c_{3t} \mid \mathrm{FP}_i - 100 \mid + \epsilon_{it}. \tag{9}$$

The parameter vector $\boldsymbol{c}_t \equiv [c_{0t},\, c_{1t},\, c_{2t},\, c_{3t}]'$ is deliberately subscripted by t to allow for changes in the implied volatility relationship between one month and the next.[19] Eq. (9) fits a log-linear approximation to the term structure of implied volatility for a given month. This guarantees that $\widehat{\sigma}_{it}$ will not be negative. Further, in order to place greater emphasis on observations that contain the most information on implied volatility, the observations are weighted by each bond's vega . Eq. (9) thus specifies the functional form of eq. (2).

Our priors on the coefficient vector $\boldsymbol{c}_t$ are that

$$c_{1t} < 0, \quad c_{2t} < 0, \quad c_{3t} > 0.$$

If interest rates display mean-reversion, we would expect a declining term structure of volatility and therefore a negative set of c_{1t}'s and c_{2t}'s. Further, the "smile" effect should result in positive values for the c_{3t}'s.

Table 4 reports the empirical results of implementing eq. (9) to the prices of callable U.S. Treasury securities over the period Jan. 1989–Dec. 1994. The present value factors for these regressions were obtained from the prices of C-STRIPS.

Note, from Table 4, that the coefficients in the 9106–9312 period display remarkable stability. Further, at least over this period, the coefficients are in line with prior expectations: The signs of the constants c_{1t} and c_{2t} are consistent with a declining term structure of volatility $\sigma(t)$, or, alternatively, with an interest rate process exhibiting (possibly time-dependent) mean-reversion.

[19] Thus, the procedure allows for the eliciting of regime changes or turning points in financial markets: e.g., the change in Fed monetary policy, October 1979; the stock market crash, October 1987.

Figure 6 considers the time series pattern of the fitted price errors by maturity, including those of the bonds for which the option value was "negative" or the implied volatility "huge." Thus, consider

$$e_{it} \equiv P_{it} - V_{it}\left[\hat{\sigma}\left(T_{ci}, T_{mi} - T_{ci}, \mid \mathrm{FP}_i - 100 \mid\right)\right],$$

where $V_{it}\left[\hat{\sigma}\left(T_{ci}, T_{mi} - T_{ci}, \mid \mathrm{FP}_i - 100 \mid\right)\right]$ is bond i's fitted price at time t given its parameters $\{T_{ci}, T_{mi} - T_{ci}, \mid \mathrm{FP}_i - 100 \mid\}$. Note that the pricing errors are comparable in magnitude to those observed for non-callable bonds—see Bliss (1994). This is particularly noteworthy as we have included those bonds with "negative" or irrationally large option values and have used a log-linear approximation to the term structure of volatilities.

Further, in order to consider the richness/cheapness implications of our model, we regressed e_{it} on a constant and $e_{i,t-1}$. If the model is able to identify measures of richness/cheapness, then such a regression should report a contracting of those pricing errors e_{it}. Table 5 presents the result of this analysis:

Table 5 demonstrates the existence of mean-reversion in the fitted price errors. Since the slope coefficient is greater than zero and significantly less than unity, the price errors display a reversion towards zero; since the β's are greater than zero, this reversion is not immediate. Hence we conclude that this model provides a measure of callable bonds' richness or cheapness. However, operationalizing a trading strategy to take advantage of such mispricing would require shorting (or repo-ing) the overvalued issues. In practice, doing so is likely to be costly.

7 Summary

Using a lognormally distributed interest rate model, and a binomial lattice for a numerical implementation thereof, we have examined the valuation of callable U.S. Treasury securities from the late 1920s to the present and calculated the volatilities implicit in the pricing of these instruments.

We find that the prices of such securities exhibit "irrationalities"— in the sense that they appear mispriced relative to similar non-callable securities—predominantly in cases where the option is away-from-the-money forward, precisely those cases wherein the estimate of implied volatility will be unreliable due to the bond's low vega. From 1987 forward, these implied volatilities are typically in the range of 8% to 15%.

We formally derive an optimal bond call policy, taking into account the required prior notification period of intent to call a Treasury bond. We do this by developing the concept of the "threshold volatility," which we use to determine when the option's time value has been driven to zero. Applying this technique, we then examine the optimality of the Treasury's observed call policy and conclude on balance that this policy has been reasonably optimal.

Finally, we show that the cross-sectional term structure of implied volatilities for these instruments declines with increased time to call and maturity and is increasing the greater the option is away-from-the-money forward. Further, we show that these term structures of implied volatility have power to perform cross-sectional richness/cheapness analysis across the set of callable bonds.

The paper's analyses are suggestive of the following future work. First, it may be of interest to use this model to test for the appropriate interest rate process. Specifically, such a test would attempt to determine the value of γ in

$$dr = \mu_t r\, dt + \sigma r^{\gamma}\, dz$$

that best fits the time series process of implied volatilities over time. This would provide an econometric discrimination in the ability of these models to explain market prices. Second, for those months for which both types of bonds with embedded options were available, a contrast of the implied volatility in callable, vs. puttable, U.S. Treasury securities, would lend insight into the volatility exhibited in the prices of these two types of bonds with embedded options. Given the discrepancies between the implied volatilities of callable and puttable bonds observed by market participants in the corporate sector, it is of interest to see what can be gained from an examination of these bonds in the Treasury market.

Appendix A Descriptions of Term Structure Estimation

A.1 The Unsmoothed Fama-Bliss Method

The Unsmoothed Fama-Bliss method is a minor variant on the term structure estimation method employed in Fama and Bliss (1987). The underlying idea of the Fama-Bliss method is as follows:

- Assume the shortest maturity issue is a bill (as is always the case). We can then compute the value of the term structure at that maturity which would exactly price the bill.

- We assume that the term structure from maturity 0 to the maturity of the first issue is constant (flat).

- We then examine the next longest maturity issue. Taking the previously determined term structure as given, we compute the constant forward rate over the interval from the previously included maturity to this issue's maturity, which would result in the current issue being accurately priced.

- We continue to extended the term structure by adding longer and longer maturity issues, at each point computing the (constant) forward rate necessary to exactly price the new issue, holding fixed the previously determined term structure.

- The Unsmoothed Fama-Bliss method achieves a continuous term structure by linearly interpolating the discount rates between maturities.

A number of filter rules are iteratively applied during the process to exclude issues that produced large jumps or reversals of the fitted term structure at adjacent maturities—the logic being that, while most issues are correctly priced, data collection and transcription procedures occasionally introduce erroneous quotes that do not reflect useful information. This method necessarily prices the remaining included issues exactly. The reader is referred to the Fama and Bliss (1987) paper for the full details of the original method, including filter rules, parameters used, and the iterative selection procedure.

The Fama-Bliss method reduces quotations to a single value, the mean of the bid and asked quotes, as is the practice with most methods.

A.1.1 Differences from Previous Work

The application of the Fama-Bliss method in this paper differs from the original in several minor ways.

- The term structure beyond five years was computed and used.
- Rather than picking off selected values at fixed maturities, as was done in Fama and Bliss (1987), we use here the full information set produced by the Fama-Bliss estimation method with discount rates determined at each horizon corresponding to the maturity of a bond in the sample. This preserves all the information in the prices of the included issues. When pricing bonds with principal and coupon payments, which may occur at any point along the term structure, this additional detail is necessary.
- The original Fama-Bliss method forced in the longest available issue. This did not matter since the term structure was subsequently truncated at 5 years. In this paper, where we utilize the full term structure, we have chosen to let the longest maturity bond be included or excluded on the basis of the same filters used for other issues.

A.2 The Nelson-Siegel-Bliss Method

The Nelson-Siegel-Bliss method developed in Bliss (1994) introduces a new estimation method to fit a modified version of the approximating function developed by Nelson and Siegel (1987). The Nelson-Siegel-Bliss method brings together several desirable characteristics—accounting for bid/asked spreads, fitting the discount rate function directly to bond prices, and producing an asymptotically flat term structure[20]—using the following approximating function:[21]

$$r(m) = \beta_0 + \beta_1 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} \right) + \beta_2 \left(\frac{1 - e^{-m/\tau_2}}{m/\tau_2} - e^{-m/\tau_2} \right). \qquad (10)$$

The parameter vector $\phi \equiv \{\beta_0, \beta_1, \beta_2, \tau_1, \tau_2\}$ is then estimated using the following non-linear, constrained optimization, estimation procedure:

$$\min_{\phi} \sum_{i=1}^{N_t} (w_i \epsilon_i)^2,$$

where

$$\epsilon_i = \begin{cases} P_i^A - \hat{P}_i & \text{if } \hat{P}_i > P_i^A \\ P_i^B - \hat{P}_i & \text{if } \hat{P}_i < P_i^B \\ 0 & \text{otherwise} \end{cases}$$

and the weights, w_i, are defined in terms of Macaulay duration, d_i, measured in days

$$w_i = \frac{1/d_i}{\sum_{j=1}^{N_t} 1/d_j}$$

subject to:

$$0 \leq r(m_{\min})$$

[20]Livingston and Jain (1982) and Siegel and Nelson (1988) demonstrate that this property is appropriate if forward rates are finite.

[21]Nelson and Siegel (1987) state that

> "...if the instantaneous forward rate at maturity m is given by the solution to a second-order differential equation with real and unequal roots, we would have
>
> $$f(m) = \beta_0 + \beta_1 \cdot \exp(-m/\tau_1) + \beta_2 \cdot \exp(-m/\tau_2),$$
>
> where τ_1 and τ_2 are time constants associated with the equation ..."

Integrating this forward rate function produces the discount rate function used here. Nelson and Siegel found that for their sample of Treasury bills this equation is over-parameterized and therefore set $\tau_1 = \tau_2$. With the longer maturities used in this study over-parameterization is not a problem and tests of the 4-parameter versus 5-parameter versions of the approximating function found that the 5-parameter version used here produces better results based on criteria similar to those developed in this paper.

$$0 \leq r(\infty)$$

and

$$\exp\left[-r(m_k)\, m_k\right] \geq \exp\left[-r(m_{k+1})\, m_{k+1}\right] \quad \forall m_k < m_{\max}.$$

The constraints ensure that the discount function is non-increasing (non-negative forward rates) and that the short and long ends of the discount rate function are positive. Only the long-rate constraint proved binding, and then only rarely.

Appendix B Computing Implied Variances from Callable Bond Prices

B.1 Iterative Search Procedure

The term structure model used in this paper is the Black and Karasinski (1991) lognormal process with constant proportional variance, zero mean reversion, and time-varying drift, μ_t, calibrated to the current term structure:

$$dr = \mu_t \, r \, dt + \sigma \, r \, dz.$$

The process for finding the implied interest rate volatility for a particular callable bond has three steps:

1. The term structure of interest rates is estimated from the prices of non-callable bonds using any of several estimation methods or from the prices of Treasury C-STRIPS.

2. A value for the short-interest rate volatility is assumed. This volatility is then used to build a binomial tree of the future evolution of the short rate. This tree is constructed to match the term structure estimated in the first step. To enhance speed of execution the tree is constructed using the forward induction method developed by Jamshidian (1991). The nodes of the tree are placed at horizons corresponding to the cash flows of the particular bond being priced.

3. The fitted price for the callable bond is next determined by backward induction beginning with the tree nodes at the maturity of the bond. This fitted value is conditional upon the assumed σ, as well as the estimated term structure, bond cash flows, and call timing, which remain invariant.

A search is conducted over feasible values of σ, repeating (2) and (3) at each iteration, until a σ is found for which the fitted price is equal to the observed price for that bond.

B.2 Matching the Term Structure in a No-Arbitrage Recombining Interest-Rate Lattice

In Step (2) above the short-rate tree, r_{tj},[22] is constructed for a given value of σ to guarantee that three conditions are met:

At each node (t, j),

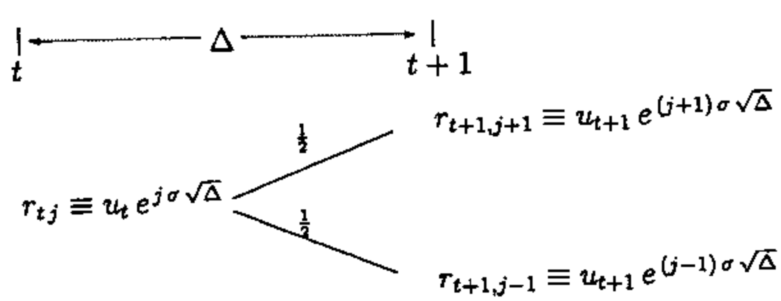
- The expected change in the natural logarithm of the short rate is the same regardless of the actual state j at time t (though it will vary for different time horizons, t).
- The variance of the change in the logarithm of the short rate is equal to σ regardless of the actual state j or time t.

And lastly:

- At each horizon t the "median" interest rate u_t is selected so the price of a zero-coupon bond maturing at the horizon will conform to the pre-specified discount function.

The construction of the tree guarantees that the first two conditions are met. These conditions also ensure that the process underlying the short rate is Markov.

A generic node of the binomial tree looks like:



[22] The tree is constructed for $\{(t, j) : t = 0, 1, \ldots, T; j = t, t-2, \ldots, -t\}$, where t indexes the time horizons, j indexes the possible states at that horizon, and T corresponds to the maturity of the callable bond being priced.

The expected change in the short rate and its variance are:[23]

$$\begin{aligned} E(\triangle \ln r_{t+1} \mid r_t) &= \ln\left(\frac{u_{t+1}}{u_t}\right); \\ \mathrm{Var}(\triangle \ln r_{t+1} \mid r_t) &= \sigma^2 \Delta. \end{aligned}$$

Notice that $E(\triangle \ln r_{t+1} \mid r_t)$ is independent of j and $\mathrm{Var}(\triangle \ln r_{t+1} \mid r_t)$ is independent of both j and t.

For valuing coupon bonds the normal interval of interest Δ corresponds to the semi-annual coupon payment dates.[24]

B.3 A Numerical Example

A simple numerical example will demonstrate the lattice, or "tree," approach to the valuation of callable bonds. Suppose $\Delta = 1$ year, and the present value factors have been calculated to be

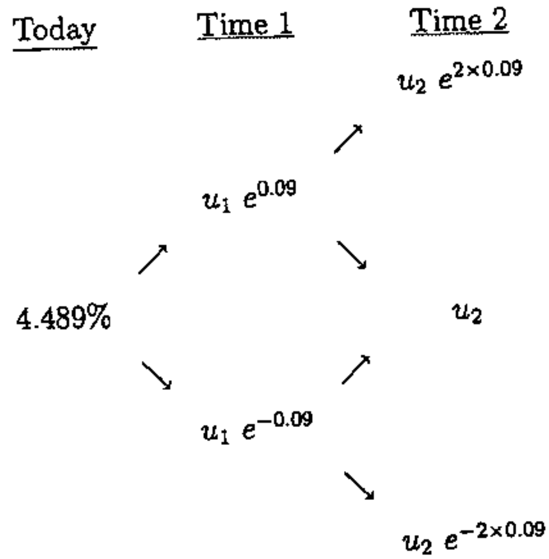
$$\begin{aligned} \mathrm{PV}_1 &= 0.9561, \\ \mathrm{PV}_2 &= 0.9028. \end{aligned}$$

Thus, $r_1 = \ln(1/0.9561) = 4.489\%$. Note that the forward rate here is $-\ln(.9561/.9028) = 5.736\%$. Assume further that $\sigma = 9\%$.

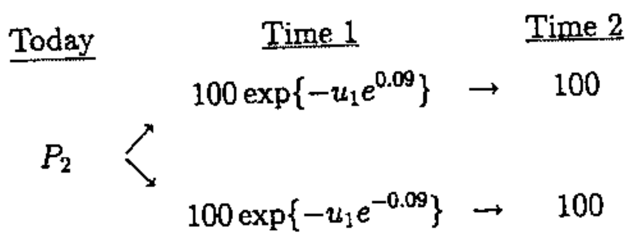
Using a volatility of 9%, the interest rate tree specifies that the short-term rate moves in a binomial fashion:

[23]We are cognizant of the fact that the time index is more properly incremented by Δ than by unity. For example, the expectation in the text should be written $\ln(u_{t+\Delta}/u_t)$. We maintain the integer increment for simplicity of notation and understanding in this exposition.

[24]We ignore slight differences in the length of time between coupons arising from variation in the number of days in a half year and the adjustment for coupon payment dates that fall on weekends.



We seek to find a parameter, designated u_1, such that the price of \$100 in two years' time conforms to the current yield curve's implied value of \$90.28. The price behavior of the two-year zero-coupon bond is:



The no-arbitrage properties of the model imply that the price of the zero-coupon bond must equal the discounted expected value, using probabilities of 0.5,[25] of the cash flows next period. In other words, P_2 must satisfy

$$P_2 = 100\ e^{-0.0459}\left(\frac{1}{2}\exp\{-u_1 e^{0.09}\} + \frac{1}{2}\exp\{-u_1 e^{-0.09}\}\right).$$

[25]The choice of risk neutral probability of $\{0.5, 0.5\}$ is innocuous in this application. Any other valid pair of probabilities could be used, and the values of u_i's would adjust. The resulting tree would price securities identically to the tree using $\{0.5, 0.5\}$.

Since we know $P_2 = 90.28$ from the current yield curve, we can solve for $u_1 = 5.714\%$. (Note that this u_1 is approximately equal to the forward rate, 5.36%.) This procedure can then be repeated to derive the interest rate tree out to 30 or more years.

Suppose we now seek to find the no-arbitrage value of a two-year, 6% bond callable at par in one year's time. The interest rate tree is

$$
4.59\% \begin{cases} \nearrow & 5.714\% \times e^{0.09} = 6.252\% \\ \searrow & 5.714\% \times e^{-0.09} = 5.222\% \end{cases}
$$

The price behavior of the two-year callable 6% coupon bond is:

Today	Year 1	Year 2
	$6 + \min\left\{106e^{-0.06252}, 100\right\} = 6 + 99.58 \rightarrow$	106
B $\begin{matrix}\nearrow \\ \searrow\end{matrix}$		
	$6 + \min\left\{106e^{-0.05222}, 100\right\} = 6 + 100$	

The bond is called at Year 1 if rates decline to 5.222%. Hence, the current bond value B is given by

$$
B = e^{-0.0459}\left(\frac{1}{2}105.58 + \frac{1}{2}106\right) = 101.04.
$$

With the current term structure, the value of a non-callable two-year 6% bond is given by the present value of the cash flows:

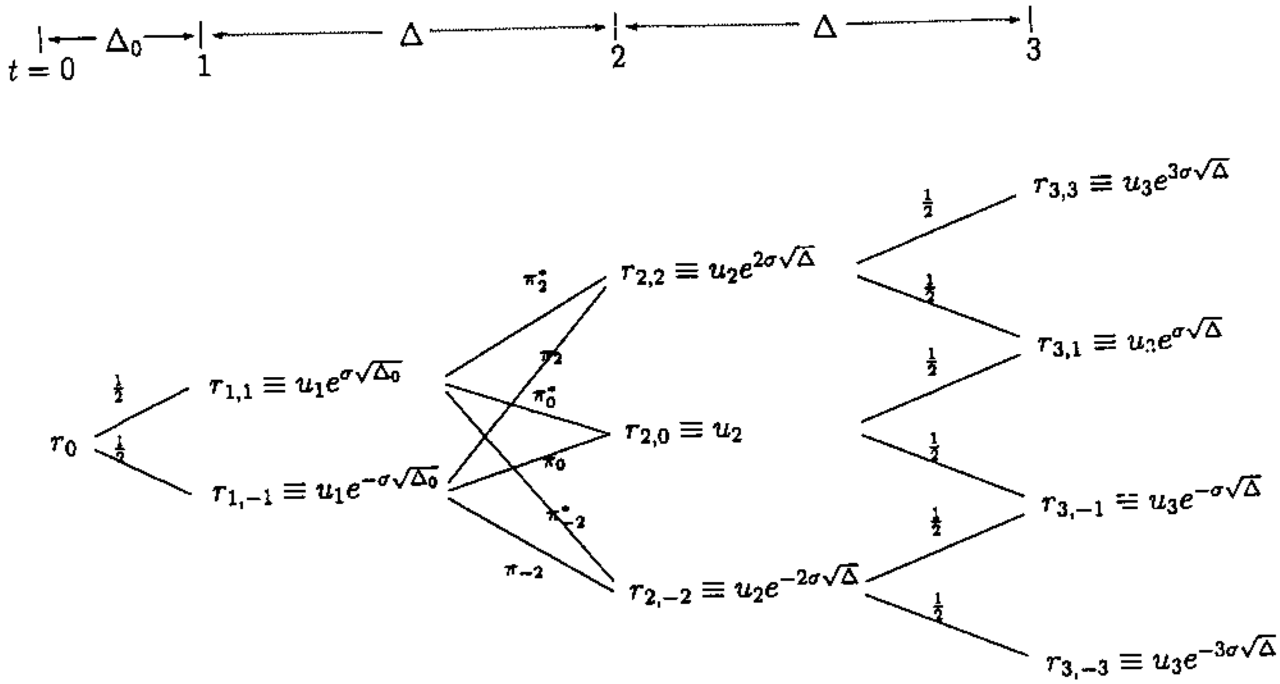
$$
6 \times P_1 + 106 \times P_2 = 6 \times 0.9561 + 106 \times 0.9028 = 101.43.
$$

Thus, the value of the one-year call is $101.43 - 101.04 = 0.39$.

B.4 Accounting for the Length of the First Time Interval in the Construction of the Interest Rate Tree

The time to the first remaining coupon will rarely equal six months,[26] so we must construct the tree with a short first time interval, Δ_0.

Because of the short first interval, the two possible $t = 1$ interest rates are "too close together" and the tree must be adjusted to ensure that the variance of changes from $t = 1$ to $t = 2$ equals σ. This is done by using a trinomial tree over that interval as follows:



The binomial sections of the tree have only one degree of freedom, u_t, and this is used to match the term structure. Over the second interval we have five degrees of freedom, $\{u_2, \pi_2^*, \pi_2, \pi_0^*, \pi_0\}$, and five conditions to meet: that the expected changes in log-interest rate are identical whether we begin at $r_{1,1}$ or $r_{1,-1}$, that the variance of changes in the log-interest rate is likewise invariant and equal to σ, and finally that the tree through $t = 2$ correctly prices a $(\Delta_0 + \Delta)$-period zero-coupon bond.

[26]This arises because most Treasury notes and bonds mature on the 15th of the month and our quotes are all month-end quotes. For this reason $\frac{1}{2} \leq \Delta_0 \leq 5\frac{1}{2}$ months while $\Delta = 6$ months.

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Figure 1: Maturities of Usable Callable Bonds

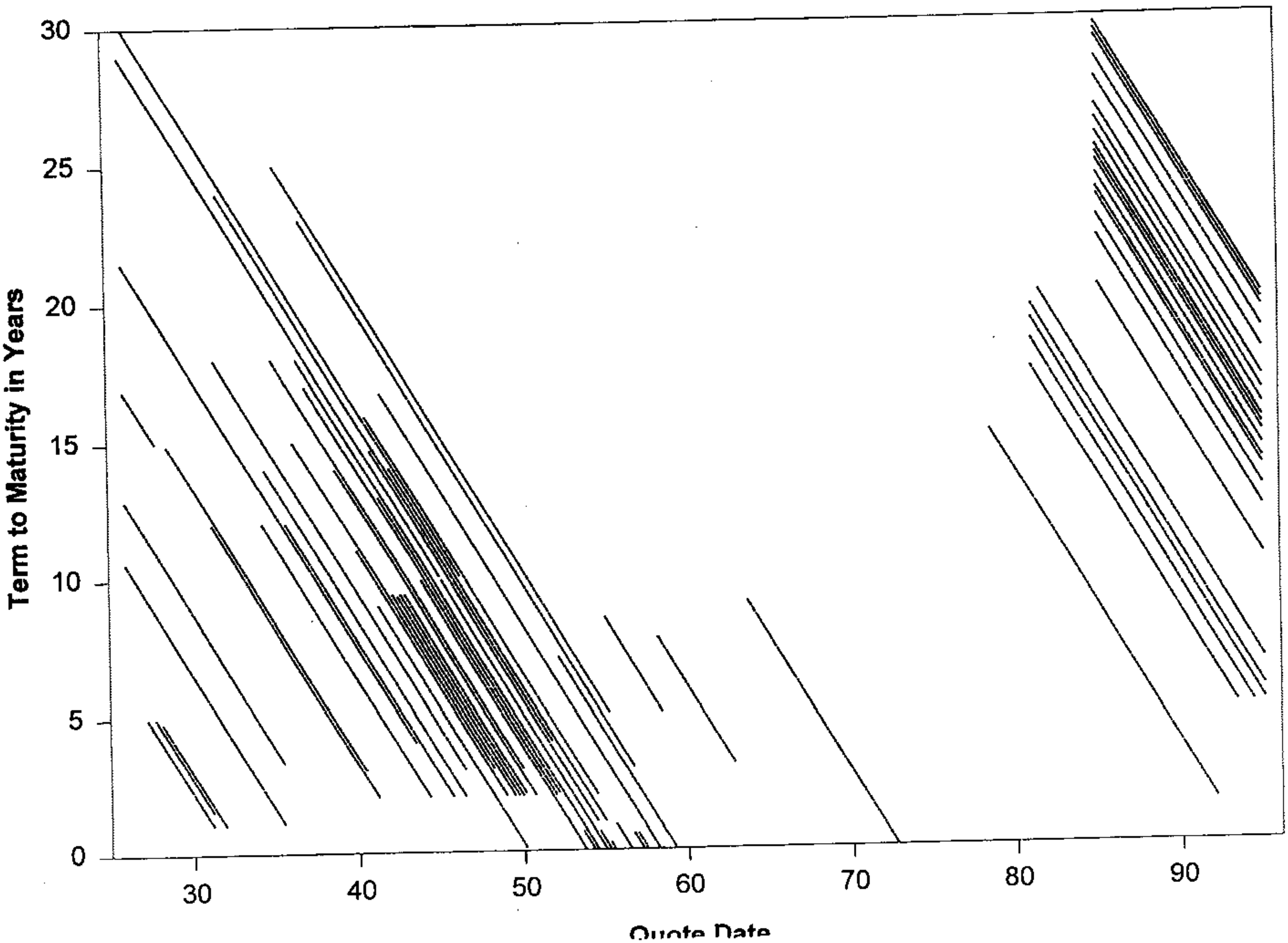

Figure 2a: Proportion of Rationally Priced Callable Bonds
(Using Treasury Strips Term Structures)

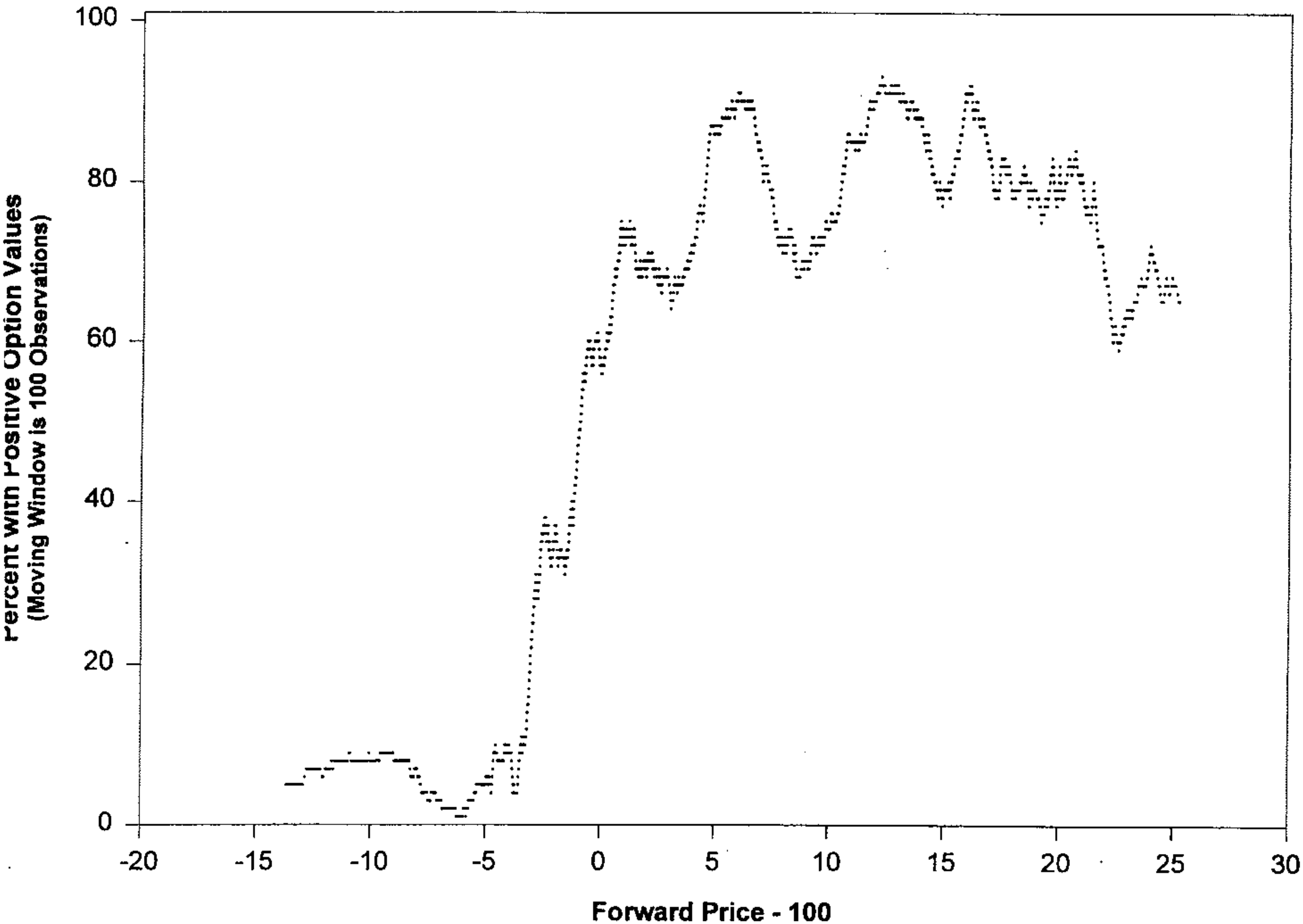

Figure2b: Proportion of Rationally Priced Callable Bonds
(Using Fama-Bliss Term Structures)

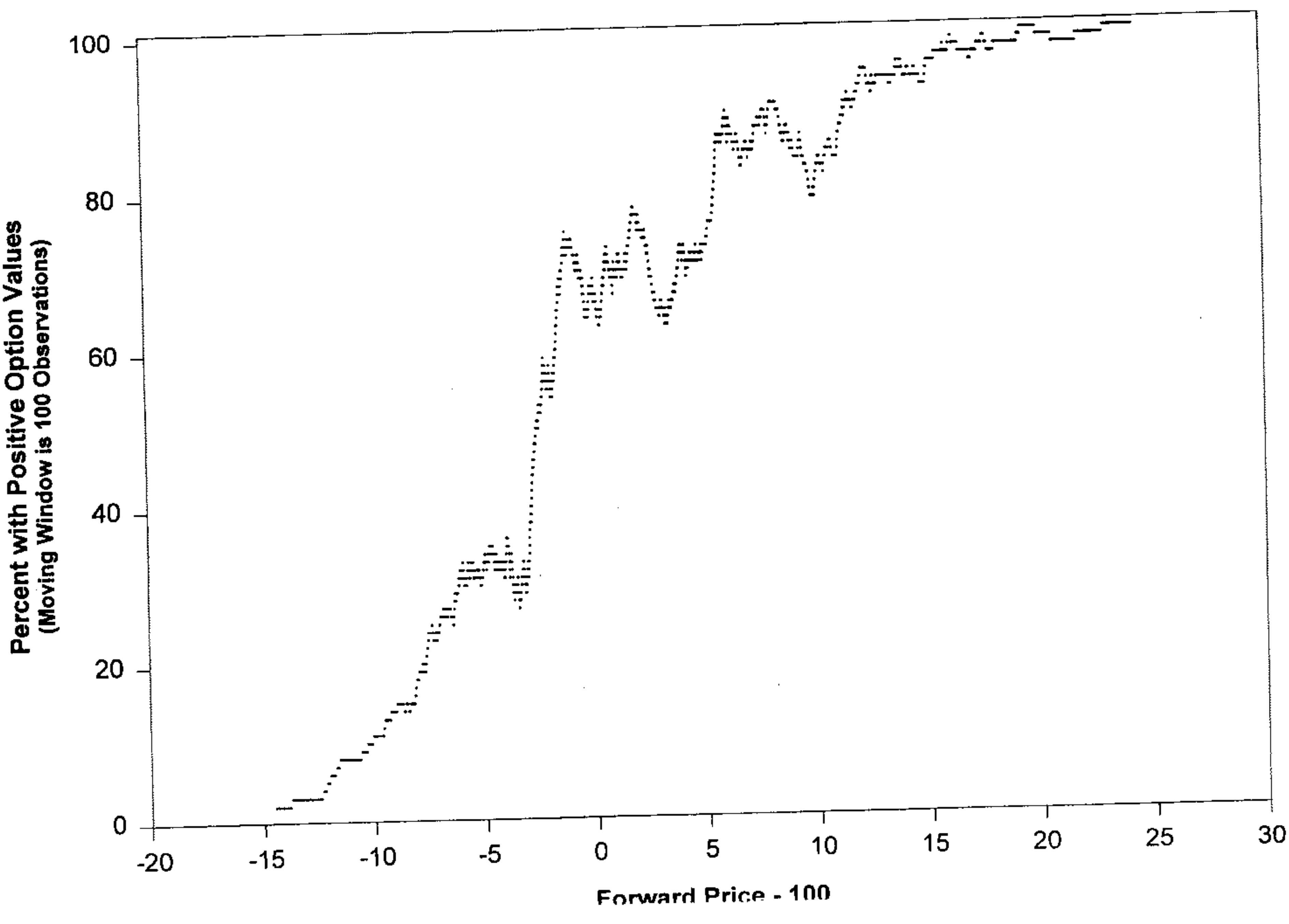

Figure 2c: Proportion of Rationally Priced Callable Bonds
(Using Nelson-Siegel-Bliss Term Structures)

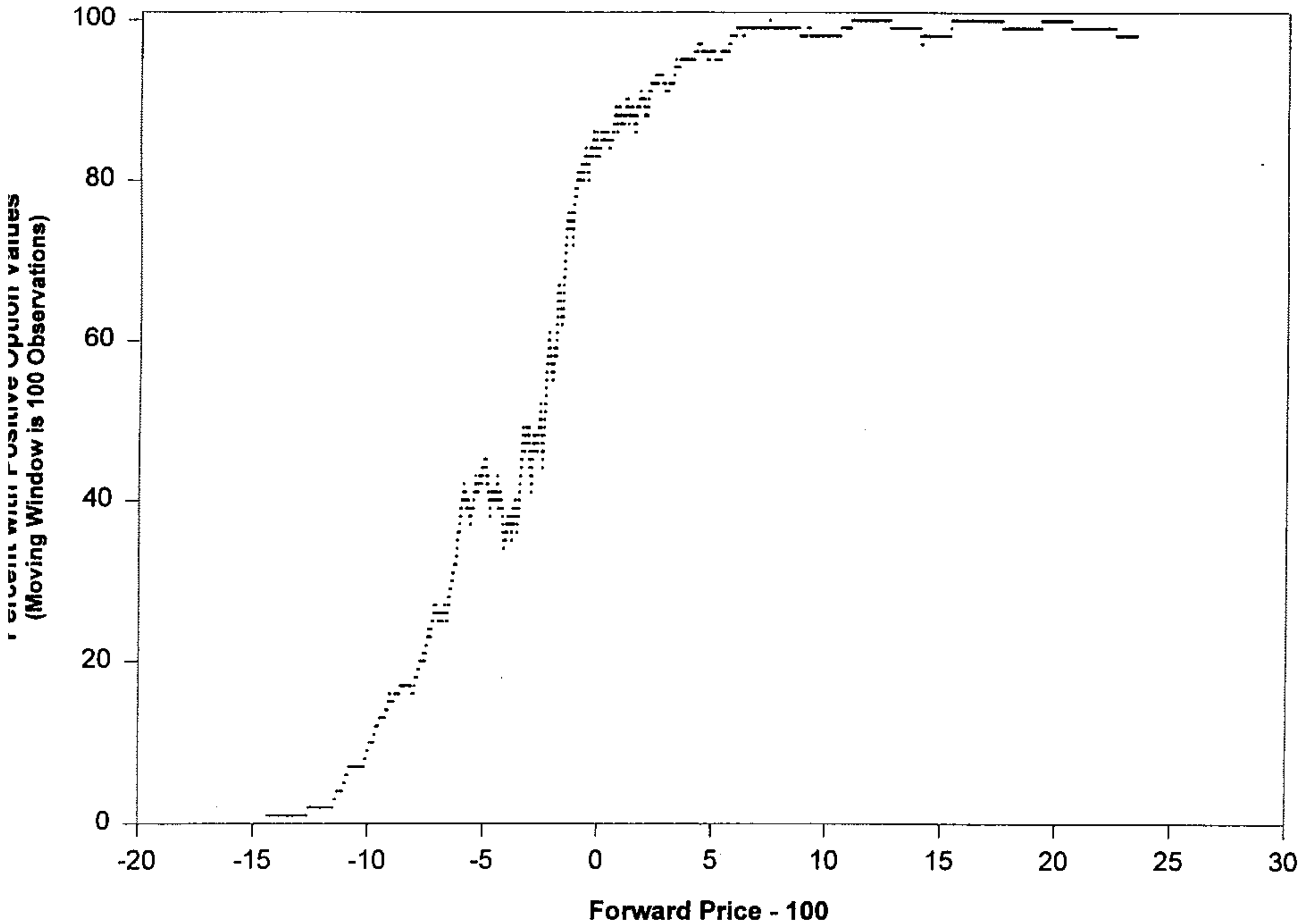

Figure 3: Classification of Callable Bonds by Implied Volatilities
(Using Fama-Bliss Term Structures)

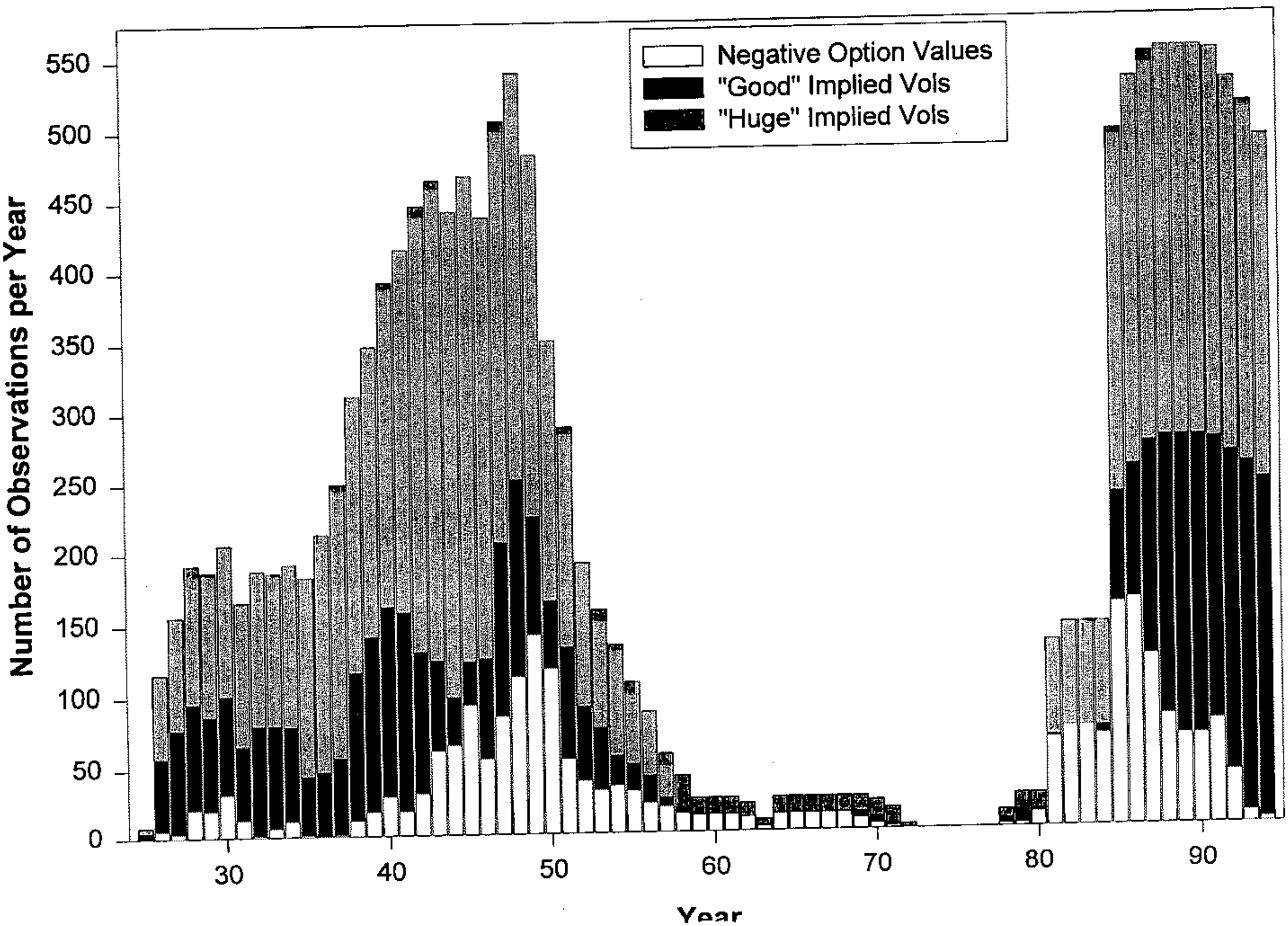

Figure 4a: Vega Weighted Implied Volatilities

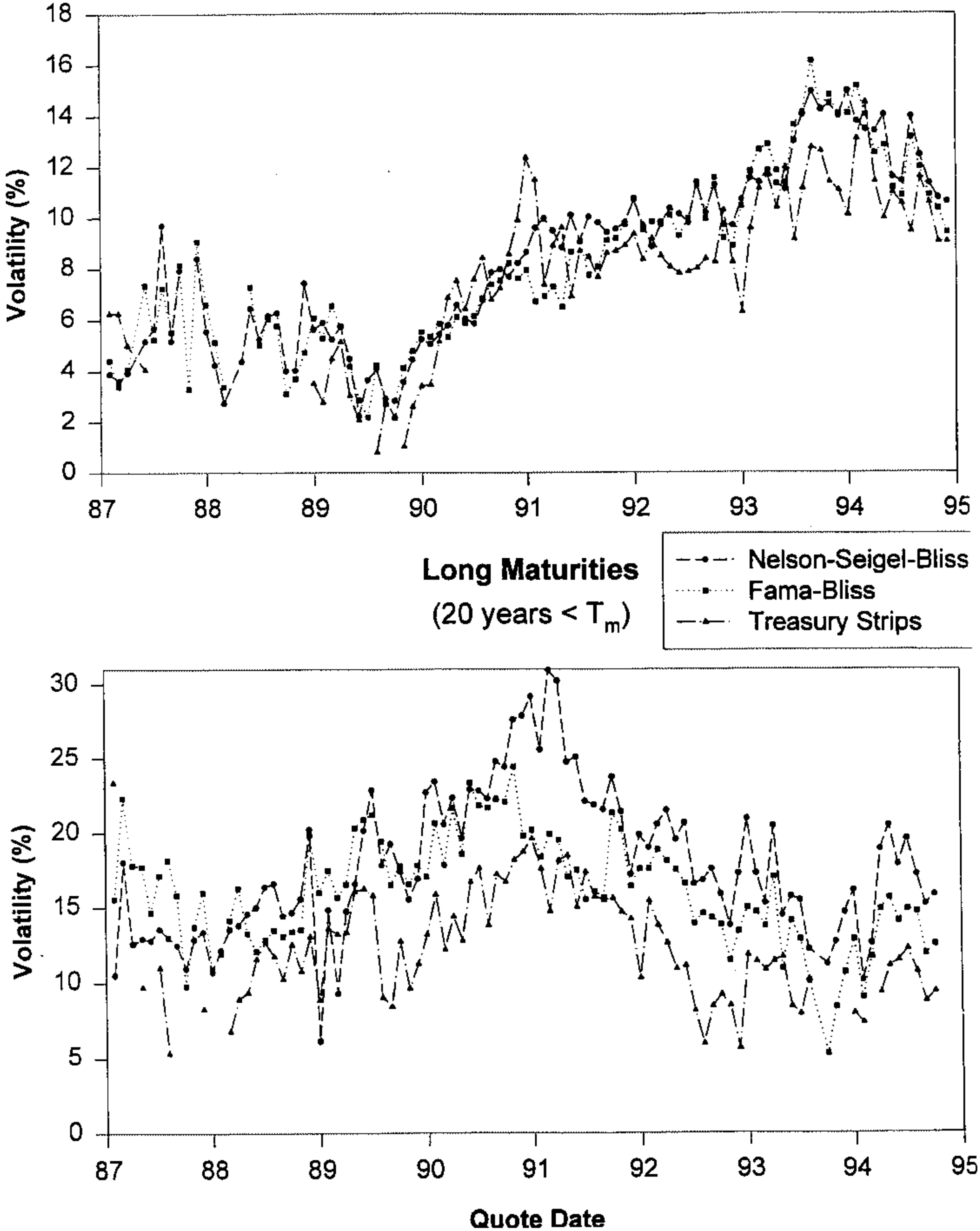

Figure 4b: Vega Weighted Implied Volatilities
(All maturities w/ minimum vega of 2%)

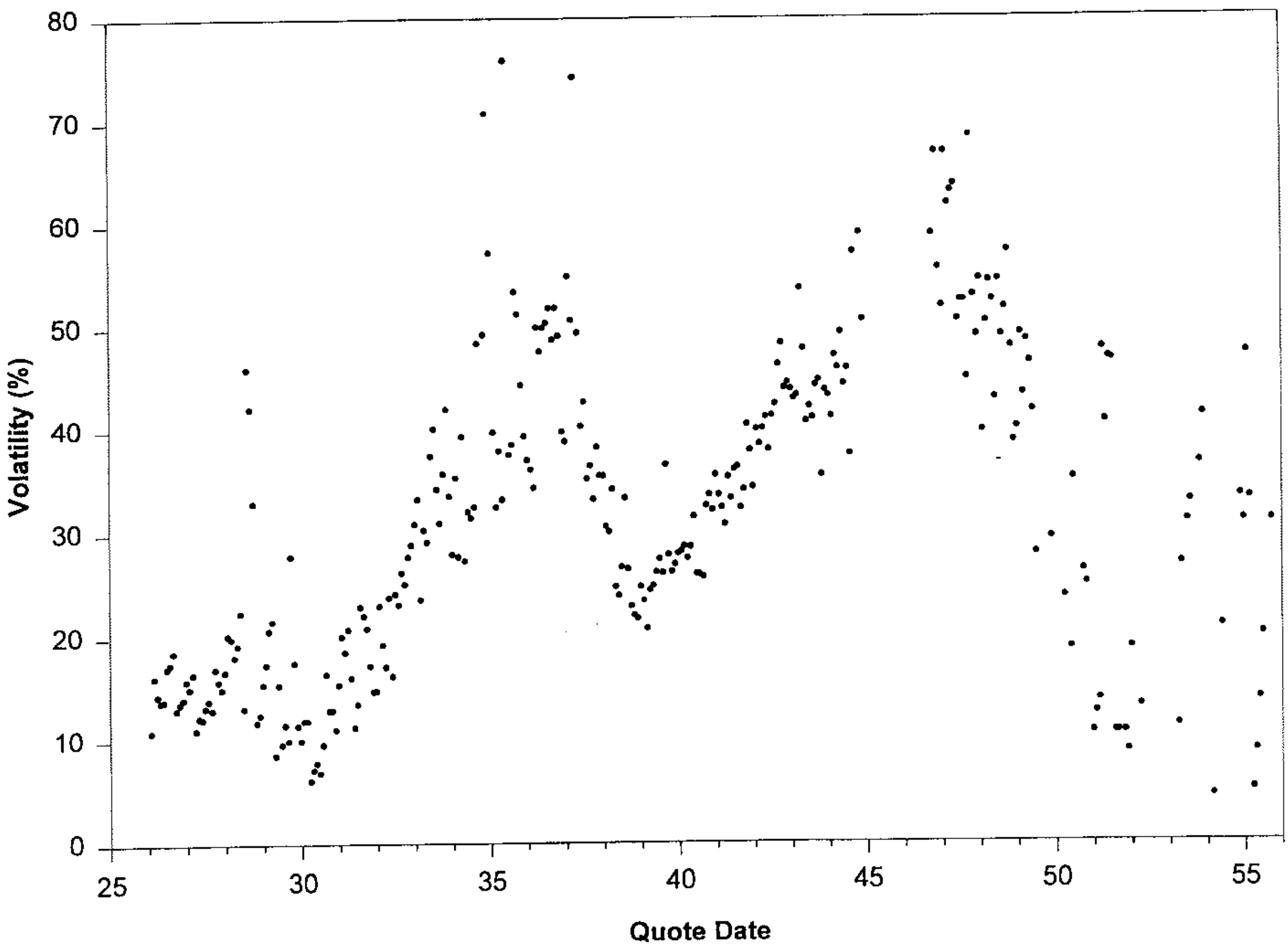

Figure 5: Implied Volatilities of $11^3/_4$'s of 2009 - 2014 Treasury Bond

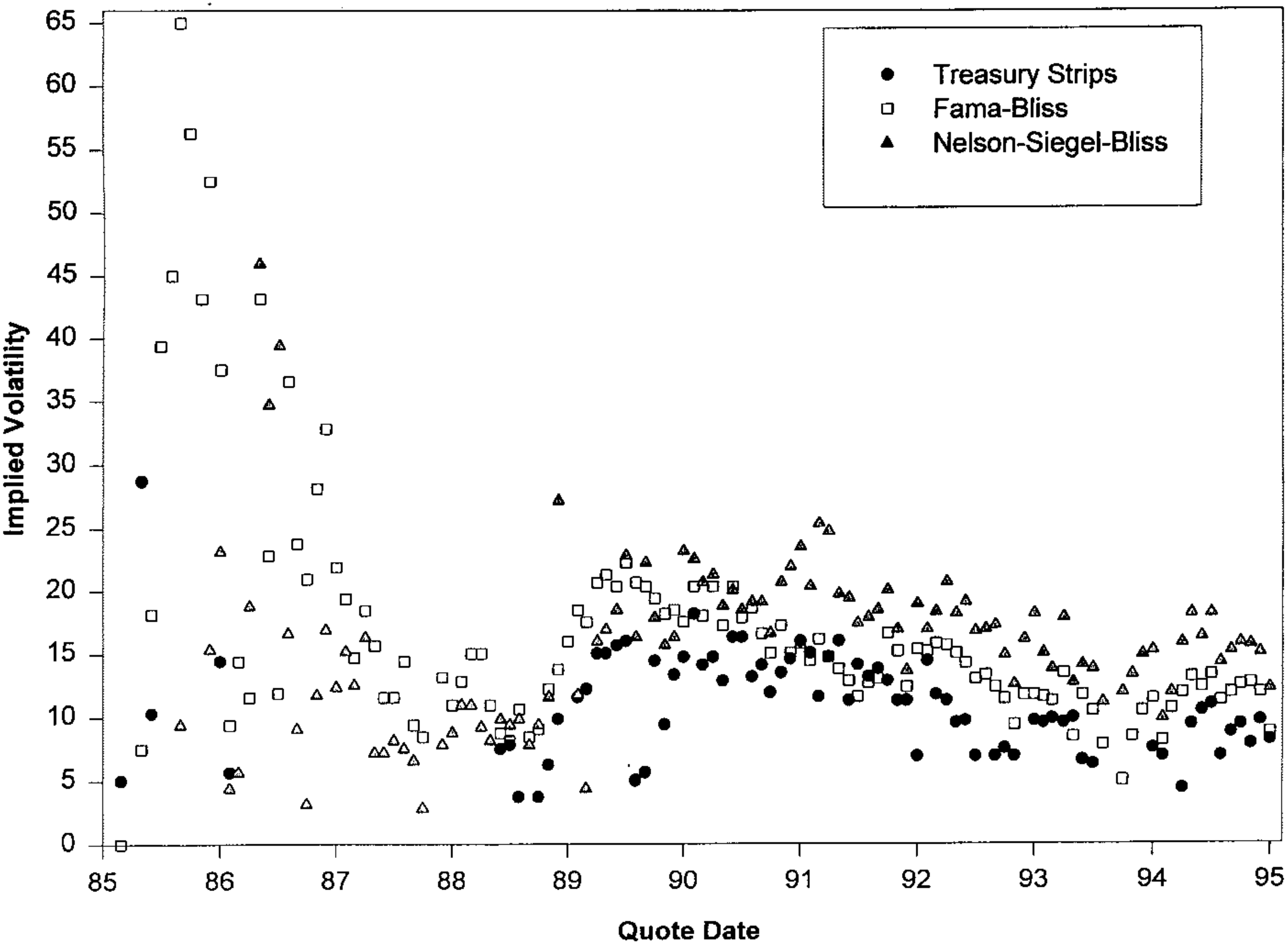

Figure 6: Callable Bond Fitted Price Errors
Using Strips Term Structure and Fitted Volatilities

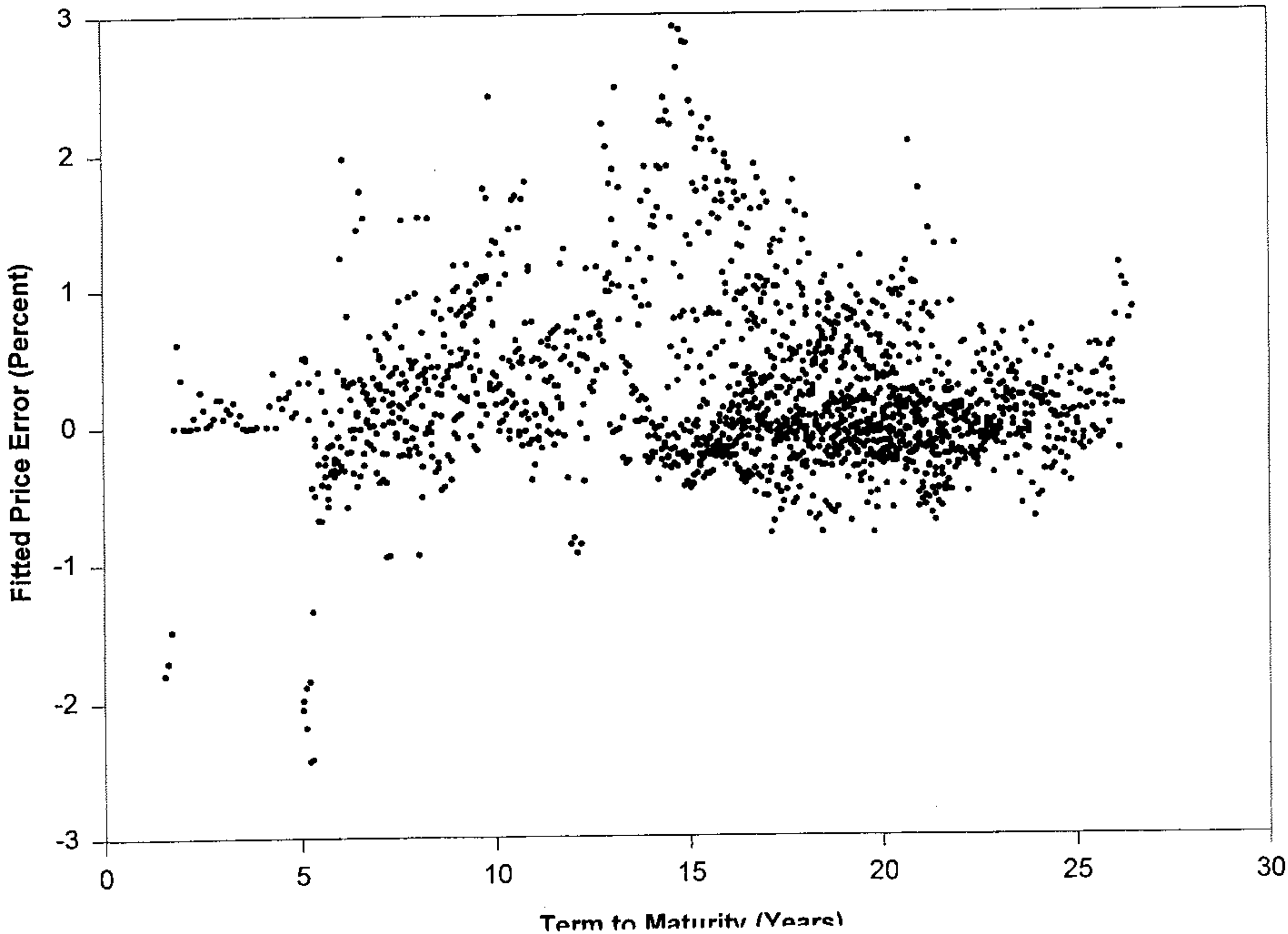

Table 1a — Summary Statistics for Implied Volatilities
Data Period: 1987 – 1994

Estimation Method	Intermediate Maturities (5 – 20 Years)			Long Maturities (> 20 Years)		
	Mean Implied Vol	Standard Deviation	N (months)	Mean Implied Vol	Standard Deviation	N (months)
Fama-Bliss	7.60%	3.12%	80	15.61%	3.45%	82
Nelson-Siegel-Bliss	7.72%	3.07%	80	16.52%	4.26%	82
C-STRIPS	7.55%	2.86%	62	12.56%	3.54%	71
Futures Options*	10.28%	2.27%	83	10.28%	2.27%	83

Table 1b — Correlation Matrices for Implied Volatilities
Data Period: 1987 – 1994

Estimation Method	Intermediate Maturities (5 – 20 Years)				Long Maturities (> 20 Years)			
	Fama-Bliss	Nelson-Siegel-Bliss	C-STRIPS	Futures Options	Fama-Bliss	Nelson-Siegel-Bliss	C-STRIPS	Futures Options
Fama-Bliss	1.000				1.000			
Nelson-Siegel-Bliss	0.954	1.000			0.741	1.000		
C-STRIPS	0.819	0.816	1.000		0.600	0.562	1.000	
Futures Options*	−0.324	−0.345	−0.051	1.000	−0.205	−0.448	−0.156	1.000

* The implied *price*-volatilities estimated from options on futures contracts are not maturity-dependent.

Table 2 — Analysis of Threshold Volatilities
Dates: March 1988 – December 1994

			Term Structure Estimation Technique Used											
			C-STRIPS				Fama-Bliss				Nelson-Siegel-Bliss			
Quote Date	Full Price	T_m	$FP_{4.5m}$	$S_{4.5m}$	L	σ_T	$FP_{4.5m}$	$S_{4.5m}$	L	σ_T	$FP_{4.5m}$	$S_{4.5m}$	L	σ_T
$7\frac{1}{2}$'s of Aug'88–93														
880331	98.08	5.37	96.40	101.31	97.80	Neg.	96.60	101.42	98.09	Neg.	96.68	101.30	98.06	Neg.
880930	96.31	4.88	95.25	100.81	96.20	Neg.	95.53	100.83	96.48	Neg.	95.63	100.77	96.52	Neg.
890331	93.29	4.37	93.08	100.17	93.49	Neg.	93.29	100.25	93.76	Neg.	93.15	100.19	93.58	Neg.
890929	97.67	3.88	97.55	100.61	98.24	Neg.	97.29	100.58	97.96	Neg.	97.21	100.55	97.84	Neg.
900330	97.30	3.38	96.51	100.65	97.27	Neg.	96.78	100.66	97.54	Neg.	96.73	100.63	97.45	Neg.
900928	99.15	2.88	98.04	100.84	98.94	Neg.	98.17	100.87	99.09	Neg.	98.32	100.84	99.21	Neg.
910328	101.35	2.38	100.02	101.45	101.47	7.5%	100.18	101.41	101.58	10.3%	100.18	101.41	101.59	9.7%
910930	102.34	1.88	101.85	101.73	103.55	66.2%	101.90	101.73	103.60	67.5%	101.86	101.71	103.53	66.2%
Called Feb'92				Call was optimal				Call was optimal				Call was optimal		
7's of May'93–98														
921231	101.76	5.37	102.37	102.26	104.61	20.3%	102.37	102.27	104.61	19.8%	102.15	102.20	104.33	19.4%
Called May'93				Call was optimal				Call was optimal				Call was optimal		
$8\frac{1}{2}$'s of May'94-99														
931231	102.96	5.37	113.25	103.04	116.14	62.8%	113.34	103.02	116.20	63.1%	113.26	102.96	116.06	63.1%
Called May'94				Call was optimal				Call was optimal				Call was optimal		
$7\frac{7}{8}$'s of Feb'1995-2000														
940930	101.83	5.38	101.67	101.96	103.59	11.0%	101.58	101.90	103.45	10.1%	101.58	101.93	103.48	10.1%
Called Feb'95				Call was optimal				Call was optimal				Call was optimal		

Notes:

1. $T_m =$ Time to maturity, in years
2. $L =$ Present value of otherwise equivalent non-callable bond to T_m
3. $S_{4.5m} =$ Present value of otherwise equivalent non-callable bond which matures in 4.5 months
4. Full Price = Market price, including accrued interest
5. $\sigma_T =$ Threshold volatility; "Neg" means *negative* option value
6. Appendix B describes the term structure estimation techniques
7. $FP_{4.5m} =$ Forward price of L on next call date

Table 3 — Analysis of Threshold Volatilities
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5\,m}$	$S_{4.5\,m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$3\frac{1}{2}$'s of Mar'30 32	290830	99.10	2.54	99.93	101.01	100.94	Negative	No		
	300228	101.17	2.05	99.90	101.58	101.48	Negative	No		
	300829	102.31	1.55	100.53	101.64	102.17	70.0	Yes		
	310228	101.67	1.05	101.51	101.68	103.20	> 100%	Yes	Mar'30	Suboptimal (Late)
$3\frac{1}{2}$'s of Sep'30 32	300228	101.17	2.55	99.90	101.55	101.46	Negative	No		
	300829	102.31	2.05	100.70	101.64	102.34	48.7	Yes		
	310228	101.67	1.55	101.68	101.68	103.36	> 100%	Yes	Mar'30	Suboptimal (Late)
$3\frac{1}{2}$'s of Dec'30 32	300529	101.96	2.55	100.40	102.14	102.53	15.0	No		
	301129	102.63	2.05	100.87	101.61	102.48	58.1	Yes		
	310529	103.12	1.55	102.13	101.71	103.84	> 100%	Yes		
	311130	101.64	1.04	100.89	101.67	102.56	> 100%	Yes	Dec'30	Suboptimal (Late)
4's of Nov'27 42	270430	101.87	15.54	109.77	102.08	111.67	11.2	?		
	271031	101.65	15.04	110.74	101.88	112.60	17.3	Yes	May'28	Optimal

- "Optimal to Call?"
  - **"Yes"** indicates (1) the threshold volatility, σ_T, is higher than normal levels of volatility observed in the market, indicating that the time value of the option has eroded to zero, and (2) the forward price is in the money, i.e. $FP_{4.5\,m} > 100$.
  - **"No"** indicates either (1) σ_T is low relative to normal and therefore that the option time has value remaining or that (2) $FP_{4.5\,m} < 100$.
  - **"?"** indicates σ_T is comparable to market levels and that calling or not is a matter of indifference.
- "Evaluation"
  - **"Optimal"** indicates either
    - the call was made at the first call date at which it was clearly indicated by the forward price and high threshold volatility, or
    - the bond was clearly not optimal call at each notification date and the call never occurred.
  - **"Suboptimal (Late)"** indicates that the call was rational but should have occurred at another, earlier date.
  - **"Irrational"** indicates that the bond was called either when the forward price was not in-the-money or when the time value remaining in the option was high, as indicated by a low threshold volatility.

Table 3 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5m}$	$S_{4.5m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$3\frac{3}{8}$'s of Mar'41–43	401031	102.80	2.37	106.42	101.63	108.05	> 100%	Yes	Mar'41	Optimal
$3\frac{3}{8}$'s of Jun'40–43	400131	102.43	3.37	109.02	101.64	110.65	> 100%	Yes	Jun'40	Optimal
$3\frac{1}{4}$'s of Apr'44–46	431130	101.46	2.38	104.34	101.38	105.72	> 100%	Yes	Apr'44	Optimal
$3\frac{3}{8}$'s of Jun'43–47	430130	101.49	4.38	108.53	101.57	110.09	> 100%	Yes	Jun'43	Optimal
$3\frac{1}{2}$'s of Jun'32–47	320130	94.91	15.38	98.54	100.72	99.27	Negative	No		
	320730	101.56	14.88	103.42	101.62	105.04	7.7%	?		
	330131	103.76	14.37	106.80	101.74	108.54	17.3%	Yes		
	330731	103.06	13.87	107.35	101.68	109.03	16.4%	Yes		
	340131	101.76	13.37	104.56	101.58	106.14	8.4%	?		
	340731	104.38	12.87	108.78	101.73	110.51	26.4%	Yes		
	350131	104.70	12.37	111.81	101.70	113.50	34.4%	Yes	Jun'35	Suboptimal (Late)
4's of Jun'32–47	320130	95.50	15.38	104.26	100.96	105.18	4.8%	No		
	320730	100.71	14.88	109.20	101.87	111.05	16.4%	Yes		
	330131	101.99	14.37	112.56	101.99	114.55	26.4%	Yes		
	330731	101.50	13.87	112.93	101.93	114.86	25.3%	Yes	Jun'35	Suboptimal (Late)*
$2\frac{3}{4}$'s of Sep'45–47	450430	101.28	2.38	103.28	101.11	104.39	> 100%	Yes	Sep'45	Optimal
3's of Jun'46–48	460131	101.39	2.37	104.29	101.22	105.49	> 100%	Yes	Jun'46	Optimal
$3\frac{1}{8}$'s of Jun'46–49	460131	101.43	3.37	106.93	101.28	108.20	> 100%	Yes	Jun'46	Optimal
2's of Mar'48–50	471031	100.61	2.37	101.87	100.68	102.55	> 100%	Yes		
	480430	101.50	1.88	100.89	100.64	101.53	> 100%	Yes		
	481029	101.02	1.38	100.55	100.59	101.14	> 100%	Yes		
	490429	100.93	0.88	100.39	100.56	100.95	> 100%	Yes	Mar'48	Suboptimal (Late)

* The 4's of Jun'32-47 were called two years after the last date for which we have price quotations.

Table 3 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5m}$	$S_{4.5m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
2's of Dec'48–50	480730	100.68	2.38	101.17	100.62	101.79	92.5%	Yes	Dec'48	Optimal
$2\frac{3}{4}$'s of Mar'48–51	471031	101.04	3.37	105.00	101.06	106.04	> 100%	Yes	Mar'48	Optimal
2's of Jun'49–51	490131	100.57	2.37	101.30	100.56	101.86	97.5%	Yes	Jun'49	Optimal
2's of Sep'49–51	490429	100.54	2.38	101.47	100.56	102.03	> 100%	Yes	Sep'49	Optimal
2's of Dec'49–51	490729	100.68	2.38	101.52	100.64	102.15	> 100%	Yes	Dec'49	Optimal
2's of Mar'50–52	491031	100.61	2.37	101.64	100.61	102.24	> 100%	Yes	Mar'50	Optimal
2's of Sep'50–52	500428	100.60	2.38	101.17	100.58	101.74	90%	Yes	Sep'50	Optimal
$2\frac{1}{2}$'s of Sep'50–52	500428	100.89	2.38	102.15	100.83	102.97	> 100%	Yes	Sep'50	Optimal
$3\frac{1}{8}$'s of Dec'49–52	490729	101.30	3.38	105.48	101.20	106.65	> 100%	Yes	Dec'49	Optimal
2's of Sep'51–53	510430	100.25	2.38	101.51	100.46	101.97	> 100%	Yes		
	511031	100.35	1.87	100.10	100.42	100.52	25.0%	Yes		
	520430	100.44	1.38	100.18	100.42	100.60	75.0%	Yes		
	521031	100.27	0.87	100.03	100.34	100.37	> 100%	Yes	Never	Suboptimal (Late)
$2\frac{1}{4}$'s of Dec'51-53	490729	103.46	4.38	103.79	100.76	104.54	77.5%	Yes		
	500131	102.79	3.87	103.00	100.71	103.70	76.2%	Yes		
	500731	102.06	3.37	102.40	100.69	103.08	77.5%	Yes		
	510131	101.29	2.87	101.40	100.59	101.98	58.1%	Yes		
	510731	100.72	2.37	100.64	100.59	101.22	39.4%	Yes	Dec'51	Suboptimal (Late)
$2\frac{1}{2}$'s of Dec'49–53	490729	100.99	4.38	104.76	100.89	105.63	93.7%	Yes	Dec'49	Optimal
$2\frac{1}{2}$'s of Mar'52–54	511031	100.60	2.37	101.07	100.67	101.73	62.5%	Yes	Mar'52	Optimal

Table 3 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5m}$	$S_{4.5m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
2's of Jun'52-54	510131	100.79	3.37	100.90	100.47	101.37	30.9%	Yes		
	510731	100.41	2.87	100.16	100.47	100.63	11.2%	?		
	520131	100.26	2.37	100.01	100.39	100.40	11.2%	?		
	520731	100.10	1.87	99.88	100.32	100.20	Negative	No		
	530130	100.03	1.38	99.84	100.34	100.18	Negative	No		
	530731	100.00	0.87	99.76	100.23	99.99	Negative	No	Never	Suboptimal (Late)
$2\frac{3}{4}$'s of Jun'51 54	510131	100.95	3.37	103.09	100.84	103.91	82.5%	Yes	Jun'51	Optimal
2's of Dec'44 54	450131	101.62	9.87	110.36	100.75	111.09	59.1%	Yes		
	450731	103.05	9.37	110.38	100.70	111.05	66.2%	Yes		
	460131	104.93	8.87	110.74	100.72	111.43	89.4%	Yes		
	460731	104.03	8.37	109.34	100.67	109.99	81.2%	Yes		
	470131	103.26	7.87	106.77	100.70	107.45	53.0%	Yes		
	470731	103.10	7.37	106.85	100.72	107.55	63.1%	Yes		
	480130	101.38	6.88	103.79	100.63	104.40	35.2%	Yes		
	480730	101.43	6.38	104.10	100.62	104.71	41.2%	Yes		
	490131	101.82	5.87	103.94	100.56	104.49	45.9%	Yes		
	490729	102.57	5.38	103.45	100.64	104.08	51.6%	Yes		
	500131	102.10	4.87	103.01	100.59	103.58	51.6%	Yes		
	500731	101.74	4.37	101.96	100.57	102.52	43.1%	Yes		
	510131	100.88	3.87	100.99	100.47	101.45	26.2%	Yes		
	510731	100.35	3.37	100.16	100.47	100.63	8.1%	?		
	520131	100.23	2.87	99.96	100.39	100.35	0.0%	No		
	520731	100.05	2.37	99.80	100.32	100.12	Negative	No		
	530130	99.91	1.88	99.65	100.34	100.00	Negative	No		
	530731	99.72	1.37	99.60	100.23	99.84	Negative	No		
	540129	100.97	0.88	100.28	100.79	101.07	> 100%	Yes	Never	Suboptimal (Late)
4's of Dec'44 54	440731	101.86	10.37	129.98	101.76	131.67	> 100%	Yes	Dec'44	Optimal

Table 3 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5m}$	$S_{4.5m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
2's of Jun'53 55	520131	101.44	3.37	99.85	100.39	100.25	Negative	No		
	520731	100.91	2.87	99.68	100.32	100.00	Negative	No		
	530130	100.38	2.38	99.52	100.34	99.86	Negative	No	Jun'53	Irrational
$2\frac{1}{4}$'s of Jun'52 55	520131	100.46	3.37	100.58	100.51	101.09	21.2%	Yes		
	520731	100.38	2.87	100.28	100.45	100.73	16.2%	Yes		
	530130	100.25	2.38	100.01	100.46	100.47	6.2%	?		
	530731	99.97	1.87	99.77	100.36	100.12	Negative	No		
	540129	100.72	1.38	100.63	100.92	101.54	> 100%	Yes	Jun'54	Suboptimal (Late)
3's of Sep'51 55	510430	101.05	4.38	106.90	100.96	107.82	> 100%	No	Sep'51	Optimal
2's of Dec'51 55	510731	100.28	4.37	100.19	100.47	100.65	5.6%	?		
	520131	100.16	3.87	99.70	100.39	100.09	Negative	No		
	520731	100.03	3.37	99.52	100.32	99.85	Negative	No		
	530130	99.63	2.88	99.45	100.34	99.79	Negative	No		
	530731	99.31	2.37	99.16	100.23	99.39	Negative	No		
	540129	100.93	1.88	100.52	100.79	101.32	72.5%	Yes		
	540730	100.93	1.38	101.05	100.92	101.97	> 100%	Yes	Dec'54	Suboptimal (Late)
$3\frac{3}{4}$'s of Mar'46 56	451031	101.60	10.37	128.94	101.57	130.43	> 100%	Yes	Mar'46	Optimal
$2\frac{1}{4}$'s of Jun'54 56	540129	100.75	2.38	101.15	100.92	102.07	77.5%	Yes	Jun'54	Optimal
$2\frac{1}{2}$'s of Mar'56 58	551031	100.25	2.37	100.23	100.51	100.74	0.0%	No		
	560430	99.00	1.88	98.93	100.20	99.14	Negative	No		
	561031	99.29	1.37	99.29	100.14	99.44	Negative	No		
	570430	99.59	0.88	99.32	100.12	99.45	Negative	No	Never	Optimal
$2\frac{3}{8}$'s of Mar'57 59	561031	98.11	2.37	98.17	100.08	98.26	Negative	No		
	570430	98.20	1.88	98.50	100.06	98.57	Negative	No		
	571031	98.30	1.37	98.78	99.76	98.56	Negative	No		
	580430	100.67	0.88	100.47	100.77	101.24	> 100%	Yes	Sep'58	Optimal

Table 3 — Analysis of Threshold Volatilities (Cont'd)
Dates: January 1932 – October 1971

Bond ID	Quote Date	Full Price	T_m	Fama-Bliss Term Structure Estimates $FP_{4.5m}$	$S_{4.5m}$	L	σ_T	Optimal to Call?	When Called	Evaluation
$2\frac{3}{4}$'s of Sep'56-59	560430	100.53	3.38	99.22	100.32	99.55	Negative	No	Sep'56	Irrational
$2\frac{7}{8}$'s of Mar'55-60	541029	101.19	5.38	103.95	101.11	105.05	47.3%	Yes	Mar'55	Optimal
$2\frac{3}{4}$'s of Jun'58-63	580131	100.73	5.37	99.26	100.72	99.98	Negative	No	Jun'58	Irrational
$2\frac{3}{4}$'s of Dec'60-65	600729	100.52	5.38	96.81	100.38	97.22	Negative	No		
	610131	100.82	4.87	95.86	100.45	96.35	Negative	No		
	610731	100.85	4.37	96.34	100.50	96.87	Negative	No		
	620131	100.70	3.87	95.91	100.33	96.28	Negative	No		
	620731	100.66	3.37	97.13	100.31	97.48	Negative	No	Dec'62	Irrational
$2\frac{1}{2}$'s of Sep'67-72	670428	91.30	5.38	90.19	99.76	90.09	Negative	No		
	671031	88.44	4.87	87.46	99.46	87.14	Negative	No		
	680430	89.19	4.38	88.46	99.15	87.85	Negative	No		
	681031	91.94	3.87	91.26	99.19	90.62	Negative	No		
	690430	90.88	3.38	90.73	98.95	89.89	Negative	No		
	691031	87.88	2.87	89.28	98.45	88.03	Negative	No		
	700430	89.03	2.38	90.17	98.60	89.03	Negative	No		
	701030	93.19	1.88	93.98	99.05	93.16	Negative	No		
	710430	97.13	1.38	97.46	99.74	97.23	Negative	No		
	711029	98.61	0.88	98.95	99.67	98.64	Negative	No	Never	Optimal

Table 4 — Regression Analysis

The regression equation, using vega-weighted least squares, is:

$$\ln \sigma_{it} = c_{0t} + c_{1t} T_{ci} + c_{2t}\,(T_{mi} - T_{ci}) + c_{3t}|\mathrm{FP}_i - 100| + \epsilon_{it}$$

Date	$\hat{c}_0$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	t_0	t_1	t_2	t_3	R^2	d.f.
8901	0.566	0.064		0.072	2.88	3.72		4.86	0.55	17
8902	0.699	0.063		0.073	4.91	5.46		7.00	0.79	17
8903	7.094	0.045	−1.327	0.111	1.74	2.35	1.60	7.22	0.70	16
8904	−1.460	0.164		0.103	1.59	2.45		4.17	0.72	12
8906	13.708	0.181	−3.037	0.061	3.19	4.43	3.43	2.31	0.90	10
8909	9.301	0.122	−1.963	0.058	11.15	7.86	11.02	5.10	0.95	9
8910	3.855	0.021	−0.615	0.060	1.69	0.44	1.29	1.82	0.56	10
8911	3.698	0.038	−0.617	0.062	2.36	1.50	1.90	3.60	0.62	10
8912	4.643	−0.053	−0.742	0.148	1.34	1.42	1.05	5.37	0.46	15
9001	7.898	0.099	−1.708	0.112	1.21	0.79	1.09	3.61	0.59	10
9002	0.628	0.032		0.085	0.51	0.37		3.78	0.54	9
9003	0.984	0.038		0.066	1.16	0.66		4.35	0.66	9
9004	0.228	0.064		0.090	0.23	0.94		4.79	0.51	9
9005	−0.121	0.090		0.087	0.12	1.27		4.35	0.58	9
9006	0.349	0.075		0.073	0.54	1.63		5.64	0.75	9
9007	0.920	0.020		0.081	1.16	0.35		5.05	0.64	9
9008	−0.635	0.107		0.132	0.27	0.66		2.81	0.42	9
9009	1.106	0.041		0.068	2.45	1.35		7.41	0.91	8
9010	1.674	0.013		0.068	4.47	0.50		8.35	0.95	8
9011	1.561	0.023		0.061	4.31	0.91		7.99	0.96	8
9012	−0.240	0.148		0.053	0.80	5.39		2.61	0.90	9
9101	1.216	−0.020		0.127	5.28	0.89		7.13	0.64	11
9102	0.787	0.012		0.108	0.58	0.13		3.85	0.59	9
9103	3.505	0.064	−0.646	0.101	1.68	0.78	1.21	4.05	0.75	9
9104	0.970	0.007		0.118	3.96	0.26		5.63	0.57	12
9105	3.227	0.020	−0.526	0.117	0.99	0.54	0.79	4.32	0.68	10
9106	0.787	0.010		0.130	2.18	0.30		4.65	0.57	11
9107	3.284	−0.023	−0.401	0.126	0.63	0.83	0.38	5.62	0.65	12
9108	2.011	−0.065		0.111	13.94	4.06		8.31	0.77	13
9109	2.065	−0.058		0.097	13.09	3.01		6.19	0.73	13
9110	2.114	−0.053		0.091	19.57	4.50		10.07	0.88	12
9111	2.067	−0.047		0.093	14.96	3.12		7.11	0.80	13
9112	2.575	−0.134		0.111	20.63	8.20		8.70	0.89	15

Table 4 — Regression Analysis (Cont'd)

Date	$\hat{c}_0$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	t_0	t_1	t_2	t_3	R^2	d.f.
9201	1.736	−0.016		0.086	8.92	0.68		4.30	0.50	15
9202	1.689	−0.032		0.096	9.10	1.40		5.26	0.49	14
9203	1.837	−0.047		0.093	10.95	2.48		6.37	0.67	13
9204	1.986	−0.072		0.102	11.15	3.78		7.33	0.70	12
9205	1.828	−0.040		0.080	15.42	2.96		7.80	0.79	13
9206	2.135	−0.081		0.086	15.23	4.24		5.31	0.79	13
9207	2.679	−0.171		0.123	11.51	5.79		4.87	0.79	12
9208	2.121	−0.087		0.105	8.72	3.09		4.69	0.68	16
9209	2.337	−0.085		0.103	14.83	4.93		6.94	0.67	16
9210	1.995	−0.065		0.092	12.60	3.54		5.91	0.45	16
9211	1.939	−0.069		0.088	9.08	2.44		1.93	0.68	11
9212	2.288	−0.061		0.071	18.28	4.08		6.47	0.68	17
9301	2.655	−0.091		0.074	18.26	5.79		7.07	0.80	16
9302	2.688	−0.088		0.066	14.65	4.49		5.39	0.75	18
9303	2.514	−0.086		0.073	10.68	3.51		5.12	0.74	17
9304	2.564	−0.063		0.054	15.15	3.57		4.83	0.78	18
9305	2.451	−0.071		0.054	12.64	3.51		4.49	0.81	18
9306	2.773	−0.077		0.038	23.13	6.11		5.53	0.96	17
9307	2.868	−0.067		0.033	23.96	5.19		7.08	0.99	14
9308	2.883	−0.065		0.023	18.56	3.73		3.56	0.97	14
9309	2.859	−0.063		0.021	20.96	4.23		3.76	0.98	14
9310	2.862	−0.066		0.019	19.66	4.08		3.14	0.98	14
9311	2.599	−0.057		0.029	19.96	3.81		5.04	0.98	14
9312	2.913	−0.078		0.032	35.80	8.61		8.84	0.99	15
9401	3.200	−0.101		0.033	22.73	7.56		3.71	0.77	17
9402	2.879	−0.099		0.043	17.79	5.65		4.10	0.71	15
9403	2.707	−0.110		0.059	16.73	5.71		5.06	0.67	18
9404	2.629	−0.089		0.061	18.94	5.18		5.86	0.68	18
9405	2.520	−0.068		0.057	20.38	4.38		6.36	0.70	17
9406	2.200	−0.048		0.060	13.94	2.38		5.04	0.62	17
9407	2.621	−0.062		0.046	25.74	4.76		6.40	0.71	17
9408	2.429	−0.068		0.063	15.00	3.36		5.23	0.62	17
9409	2.024	−0.029		0.055	13.80	1.54		4.56	0.62	16
9410	1.845	−0.009		0.051	12.50	0.48		4.23	0.66	16
9411	0.891	0.078		0.040	3.76	2.39		2.12	0.71	16
9412	1.763	−0.010		0.050	10.29	0.44		4.36	0.70	15

Table 5—Regression Analyses of the Pricing Errors e_{it} : $e_{it} = \alpha + \beta e_{i,t-1}$

$e_{it} \equiv P_{it} - V_{it}\left[\hat{\sigma}\left(T_{ci}, T_{mi} - T_{ci}, \mid \mathrm{FP}_i - 100 \mid\right)\right]$

Data Period: 1985–1994

	Estimation Method		
	STRIPS	Fama-Bliss	Nelson-Siegel-Bliss
N	943	1127	1175
R^2	0.609	0.587	0.713
α	−0.193 (−2.90)	−0.023 (−2.98)	−0.020 (−2.80)
β	0.771 (34.7)	0.758 (33.1)	0.826 (43.8)

Note: The numbers in parentheses are t-statistics.