The Canonical Extensive Form of a Game Form: Part II - Representation*

Peter Sudhölter Joachim Rosenmüller Bezalel Peleg†

Institut für Mathematische Wirtschaftsforschung

Universität Bielefeld

D-33615 Bielefeld

Germany

April 14, 1999

Abstract

This paper exhibits to any noncooperative game in strategic or normal form a 'canonical' game in extensive form that preserves all symmetries of the former one. The operation defined this way respects the restriction of games to subgames and yields a minimal total rank of the tree involved. Moreover, by the above requirements the 'canonical extensive game form' is uniquely defined.

Key words: Games, Extensive Form, Normal Form, Strategic Form.
AMS(MOS) Subject Classification: 90D10, 90D35, 05C05.

---

*The authors acknowledge the useful comments of an anonymous referee.

†also at the Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Israel.

0 Introduction

This paper represents the second part of a study devoted to the relationship between a strategic game (game form) and its possible representations by extensive games (game forms). To be explicit, a representation of a strategic game G is an extensive game Γ the 'normal', 'strategic' or 'von Neumann-Morgenstern form' of which is G.

The aim of this paper is to exhibit the choice of a method to represent strategic games by extensive games in a way that essentially preserves all symmetries of the strategic game but in addition satisfies additional axioms. These axioms concern robustness under restriction and minimality of the tree involved. Eventually, we want these axioms to characterize the representation and this is the main theorem of this paper (i.e., Theorem 6.6).

We should emphasize that our starting point is a game in strategic form. The transition from the extensive form to the strategic form as defined by von Neumann and Morgenstern [5] has already been investigated extensively (see Kohlberg and Mertens [3] for a recent treatment of this topic). The transition in the opposite direction is considered 'trivial' and conceptually straightforward. It is the purpose of our work to show that this is not true: The choice of a representation of strategic games by extensive games which respects symmetries of strategic games leads to difficult conceptual problems and deep mathematical results.

The previous results obtained (see [6]) and the drive of the present continuation can at best be explained by pointing to the 'Battle of the Sexes'. Some of the problems motivating our treatment are explained at best within the context of this example.

A strategic version of this game is represented by the following 'bimatrix':

$$\begin{array}{cc} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

Here C is interpreted as 'attending a concert' and S is 'mingling with the crowds at a soccer game'. There are two 'focal' equilibria in this game: (C,C) and (S,S). Myerson discusses these games nicely in [4], Section 3.5.

There is a standard convention in Game Theory according to which two simple versions of a representation for this game by an extensive game are exhibited (see Figure 0.1).

The above convention which leads to multiple representations has the following two problematic aspects.

(1) The transition to the extensive form might influence focality. Consider the extensive game Γ_1. It is common knowledge in this game that player 1 moves first. Therefore

Figure 0.1: Two representations

she has the option to choose C before player two makes his choice. Thus, it seems to us that in Γ_1 the pair (C, C) of strategies is more likely to be played than (S, S). Our feeling is supported by the experimental work of RAPOPORT [7]. Clearly, in Γ_2 the pair (S, S) may be the dominant focal equilibrium rather than (C, C).

(2) The transition to the extensive form may destroy symmetry. The game G is 'symmetric' in the following sense: It has an automorphism which permutes players 1 and 2 (for a definition of automorphism, i.e. an isomorphism of G to itself, see HARSANYI AND SELTEN [2], Section 3.4). This automorphism is given explicitly in Example 4.6 (1) of [6]. However, Γ_1 and Γ_2 are totally asymmetric; more precisely, if $\Gamma = \Gamma_1$ or $\Gamma = \Gamma_2$, then there is no non-trivial automorphism of Γ that respects the temporal ordering of moves in Γ.

The discussion in the last paragraph leads naturally to the following basic question:

Let G be a game in strategic form. When is G 'symmetric'? (In particular, is the Battle of Sexes a symmetric game?)

Quite surprisingly there was no satisfactory answer available to this question. If we follow our mathematical intuition and define a strategic game G to be symmetric if all possible joint renamings of players and strategies are automorphisms of G (see HARSANYI AND SELTEN [2], p. 71, for the precise definition of renaming), then the class of symmetric games reduces to the trivial class of all games whose payoff functions are constant and equal. Also, this definition is incompatible with the definition of symmetric bimatrix games (see VAN DAMME [1], p.211).

In [6] we presented an answer to this basic question. A symmetry of G, according to the definition given in [6], is a *permutation* π *of the players* for which there exists an automorphism $\alpha = (\pi, \varphi)$ of G (here φ is a renaming of strategies which is compatible with π). Thus, our definition of symmetries (of strategic games) is different from that of HARSANYI AND SELTEN [2], p. 73. The game G is *symmetric* if every permutation of the players is a symmetry of G. Thus, in particular, the Battle of Sexes is symmetric according to our definition. Our definition has the following desirable properties.

(1) The class of symmetric games is a nontrivial interesting class.

(2) It is possible to use similar ideas to define symmetries of extensive games (see Definition 1.14).

(3) It is possible to compare the symmetry groups of a strategic game and its coalitional form (see Theorem 4.11 of [6]).

As far as we could check, symmetries of games in extensive form which preserve the partial ordering on the nodes that is induced by the game tree were not considered previously. Thus, the treatment of symmetry groups of extensive games, as offered in [6], was entirely new.

We now present the solution to the problem of representing the Battle of Sexes by an extensive game.

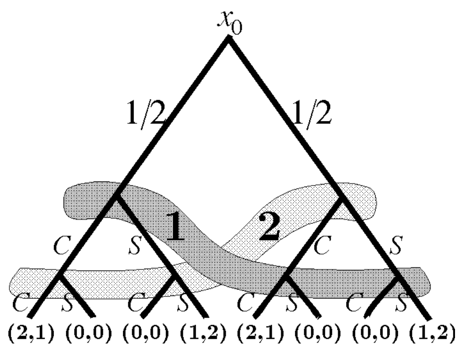

Figure 0.2: The 'canonical' Battle of Sexes

A symmetry of a (strategic) game G, according to our definition, as presented in [6] (see Definition 1.13), is a *permutation* π *of the players* for which there exists a *renaming* φ of strategies which is compatible with π such that $\alpha = (\pi, \varphi)$ is an automorphism of G. The game G is *symmetric* if every permutation of the players is a symmetry of G. Similar definitions have been offered for extensive games and this way it is seen that Figure 0.2 indeed shows a symmetric representation of a symmetric game.

However, it is by no means true that it is the only such symmetric representation.

As a further property it can be seen easily that Figure 0.2 indicates a graph 'minimal' in the class of all representations (of the Battle of Sexes) which preserve the symmetries.

It is our aim within this paper to generalize the foregoing construction to all finite strategic games and to show that that there is a 'canonical' way of representing a strategic game by a symmetry preserving and minimal extensive game. By various reasons we speak of games and game forms simultaneously within this context.

Clearly some further difficulties appear at once. Let us list two of them which appear to be quite diametral.

(1) For example, a 2×3 two-person game has no symmetries. Therefore, a 'canonical' representation for such a game should have suitable properties of a more general

character. If we add the requirement that our representations of 2×3 games are consistent with our representations of 2×2 games, then we have an additional tool (and an additional obstacle) to be concerned with.

(2) On the other hand, given a 'square' n-person strategic game form (i.e. all players having the same number of strategies), it is not clear how to find for it a minimal and symmetry preserving representation since square game forms in principle allow for complete symmetry between the players.

Other than in the example above, when the number of players is greater than two, then there is no obvious solution to the problem of representing square game forms. For this purpose it is necessary to first define the simplest ('atomic') representations of square game forms and then build 'symmetrizations' of such 'atoms' in order to obtain symmetry-preserving representations.

Based on these procedures we eventually come up with a general canonical method of representing strategic games (game forms) by extensive games (game forms): We axiomatically justify a certain symmetrization of atoms (the one which in every branch exhibits a clear 'time structure' or 'order of play') to be the unique representation which preserves symmetries, respects restrictions and shows a minimal tree.

We now shortly review the contents of the paper. The versions of strategic and extensive game forms and games we are dealing with have been introduced in [6]. We will, however, shortly repeat the definitions in order to make this work self contained. The same is true for the basic notions of isomorphisms between games and game forms.

One of the major aspects of [6] is an alternative definition of symmetry deviating from the one given by previous authors. Actually a symmetry is a residual class of automorphisms of game forms (games), the elements of which differ by an impersonal motion (automorphism) only. Hence the content of Section 1 is a short review of the necessary definitions.

Section 2 is devoted to the definition of faithful representations. An extensive game form γ is a *representation* of a game form g if for every choice of a vector u of payoff functions the normal form of the extensive game $\Gamma(\gamma, u)$ is (impersonally) isomorphic to the strategic game $G(g, u)$. A representation γ of a game form g is *faithful* if, for every choice of a vector of payoff functions u, the games $\Gamma(\gamma, u)$ and $G(g, u)$ have the same symmetry group. Theorem 2.5 shows that the outcome function of a representation does not depend on chance moves.

Within Section 3 we set out to construct and characterize faithful representations of general and square game forms. The main result of this section is presented in Corollary 3.6. The group of automorphisms of a square game form is the surjective image of the corresponding group of any extensive game form representing it, if and only if the representation is faithful.

In Section 4 we consider the minimal representations of square and general game forms which are called *atoms*.

In Section 5 it is first of all shown (Theorem 5.4) that symmetrizations of atoms are faithful representations of strategic game forms that are square and general. The main result of this section, Theorem 5.7, proves a converse result: A minimal and faithful representation of a general and square strategic game form must be a symmetrization of an atom.

An atom is *time structured* if the order of play is the same for all n-tupels of strategies. Section 6 is devoted to the proof that symmetrizations of time structured atoms yield canonical representations of finite strategic games. First, Theorem 6.5 shows that minimal and faithful representations of square and general strategic game forms must be time-structured (provided the number of strategies is at least three).

A representation (in the sense of Theorem 6.6) is a mapping which assigns to every game form (in the domain of the mapping), an extensive game form that represents it. A representation is *canonical* if it is faithful, respects isomorphisms, is consistent with restriction, and has a minimality property.

Theorem 6.6, the main result of this work, proves that there exists a unique canonical representation of strategic games which is given by (generalized) time structured representations.

1 Prerequisits

The structure presenting the fundamental strata of our considerations is the one of a game tree. Since it is not unfamiliar in Game Theory we describe it mainly verbally, however the details are exactly the ones presented in [6].

A ***game tree*** is a set of data $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ the elements of which are described as follows.

E is a finite set (the ***nodes***) and $\prec$ is a binary relation on E such that $(E, \prec)$ is a tree (the ***root*** is x_0, the generic element (node) is ξ, and the set of ***endpoints*** is ∂E). The ***rank function*** $\mathbf{r}$ defined on the nodes counts the number of steps from the nodes to the root, e.g. $\mathbf{r}(x_0) = 0$. The set $\{\xi \mid \mathbf{r}(\xi) = t\} =: \mathcal{L}(E, \prec, t)$ constitutes the ***level*** t and $R(E, \prec) := \sum_{\xi \in \partial E} \mathbf{r}(\xi)$ denotes the ***total rank*** of the tree. A ***path*** is a sequence of consecutive nodes. A ***play*** is any path that connects the root with an endpoint.

$\mathbf{P}$ is a partition of $E - \partial E$, the ***player partition***. There is a distinguished element $P_0 \in \mathbf{P}$ (which may be the empty set) representing the ***chance moves.*** All other player sets are assumed to be nonvoid. The elements of $\mathbf{P}$ represent sets of nodes at which a certain player is in command of the next move.

$\mathbf{Q}$ represents the ***information partition.*** $\mathbf{Q}$ is a refinement of $\mathbf{P}$; thus an element $Q \in \mathbf{Q}$, $Q \subseteq P$, is an information set of the player who commands the elements of

P. In particular it is required that $\mathbf{Q}$ refines P_0 to singletons.

$\mathbf{C}$ is a family of partitions representing ***choices***. To explain this object, let

$$C(\xi) \ := \ \{\xi' \mid \xi \prec \xi'\}$$

denote the successors of $\xi \in E$ and define for $Q \in \mathbf{Q}$

$$C(Q) \ := \ \bigcup_{\xi \in Q} C(\xi).$$

Now we assume that for any $Q \in \mathbf{Q}$ we are given a partition $\mathbf{C}(Q)$ of $C(Q)$ such that

$$S \ \in \ \mathbf{C}(Q) \implies |S \cap C(\xi)| = 1 \quad (\xi \in Q)$$

holds true. then the system of choices is described by $\mathbf{C} \ := \ (\mathbf{C}(Q))_{Q \in \mathbf{Q}}$.

$p \ = \ (p^\xi)_{\xi \in P_0}$ is a family of probability distributions (of chance moves), i.e., p^ξ is a probability on the successors of ξ for every $\xi \in P_0$. We assume that $p^\xi(\xi')$ is positive for every successor ξ' of ξ. Intuitively, chance determines the successor of $\xi \in P_0$ according to the random mechanism described by p^ξ.

We are now in the position to discuss *preforms.* Extensive preforms are obtained by assigning the names of the players to the corresponding elements of the player partition of a game tree, more precisely:

Definition 1.1. *An* ***extensive preform*** *is a tuple*

$$\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota) \tag{1.1}$$

where N is a finite set (the set of **players***) of at least two elements, the next five data constitute a game tree, and $\iota : \ \mathbf{P} - \{P_0\} \to N$ is a bijective mapping.*

Intuitively, ι assings the nodes of $P \in \mathbf{P} - \{P_0\}$ to the player $\iota(P) \in N$, who is thought to be in charge of choosing a successor when this node occurs during a play. We use the notation $\iota^{-1}(i) = P_i$ to refer to 'the nodes of player i'.

In order to describe a *strategic* preform we need much less preliminary framework.

Definition 1.2. *A* ***strategic preform*** *is a pair $e = (N, S)$, where N is a finite set (the set of players, $|N| \ \geq 2$) and $S = \prod_{i \in N} S_i$ is the product of finite sets $S_i \neq \emptyset \ \ (i \in N)$ (the strategy sets).*

Game forms are now obtained by assigning *outcomes* to the choice of a play or to a strategy profile respectively. In fact, a strategy profile in the strategic context can frequently be identified with an outcome (the 'general' case). However, it is more appropriate to choose an abstract set of outcomes and assign its elements to plays or profiles accordingly.

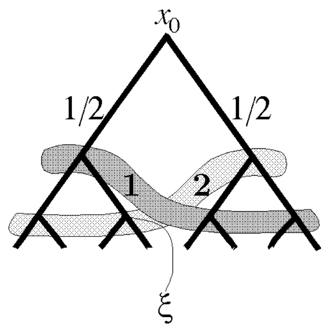

Figure 1.3: An extensive preform

Definition 1.3. *(1) An* ***extensive game form*** *is a tuple*

$$\gamma = (\epsilon; A, \eta) = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota; A, \eta). \tag{1.2}$$

Here, ϵ is an extensive preform and A is a finite set (the ***outcomes****) while $\eta : \partial E \to A$ is a surjective mapping called* ***outcome function****, which assigns outcomes to endpoints of the graph $(E, \prec)$.*

(2) A ***strategic game form*** *is a quadruple $g = (e; A, h) = (N, S; A, h)$. Here e is a preform, A a finite set and $h : S \to A$ is a surjective mapping again called the* ***outcome function.*** *If h is a bijection, then g is called* ***general****.*

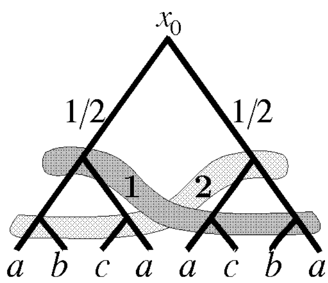

Figure 1.4: An extensive game form

The term 'general' is not suitable for extensive game forms as (unless in a special case) it is frequently to be expected that different plays influenced by a chance move result in the same outcome.

We now turn to the notion of (extensive and strategic) *games.* In a game we have payoffs in terms of utilities, i.e., real valued functions. Eventually, this includes also the possibility of computing expectations when the chance influence is taken into account.

Definition 1.4. *(1) An* ***extensive game*** *is a tuple*

$$\Gamma = (\epsilon; \upsilon) = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota; \upsilon) \tag{1.3}$$

such that

$$\upsilon = (\upsilon_i)_{i \in N} : \partial E \rightarrow I\!\!R^N \tag{1.4}$$

represents the system of player's **utility functions** *depending on endpoints of the graph* $(E, \prec)$.

(2) *A* ***strategic game*** *is a tuple*

$$G = (e; u) = (N, S; u) \tag{1.5}$$

Again,

$$u = (u_i)_{i \in N} : S \rightarrow I\!\!R^N \tag{1.6}$$

constitutes the player's **utility functions** *defined on strategy profiles.*

Remark 1.5. *Obviously, games can be generated from game forms by means of a utility defined on the alphabet. Formally, if* γ *and* g *are (extensive and strategic) game forms and*

$$U : A \rightarrow I\!\!R^N$$

is a ('utility') function defined on outcomes, then

$$\upsilon := U \circ \eta, \quad u := U \circ h \tag{1.7}$$

induce games

$$\Gamma := U * \gamma := U * (\epsilon; A, \eta) = (\epsilon; U \circ \eta) \tag{1.8}$$

and

$$G := U * g := U * (e; A, h) = (e; U \circ h). \tag{1.9}$$

Next we shortly review the basic ideas of isomorphisms, motions, and symmetries as presented in [6].

Isomorphisms for strategic objects are comparably easy to understand, therefore we start with this notion.

Given strategic preforms $e = (N, S)$ and $e' = (N, S')$ we consider bijective mappings $\pi : N \rightarrow N$ and $\varphi_i : S_i \rightarrow S'_{\pi(i)}$ $(i \in N)$. π renames the players and φ_i maps strategies of player i into strategies of his double.

The pair (π, φ) of course induces a simultaneous reshuffling of strategies, i.e., a mapping

$$\varphi^\pi : S \rightarrow S', \ (\varphi^\pi(s))_{\pi(i)} \ := \ \varphi_i(s_i) \quad (i \in N). \tag{1.10}$$

Definition 1.6. *An* ***isomorphism*** *between strategic preforms* e *and* e' *is a family* (π, φ) *of bijective mappings*

$$\pi : N \rightarrow N, \ \varphi_i : S_i \rightarrow S'_{\pi(i)} \ \ (i \in N)$$

such that

$$(\pi, \varphi)e = (\pi, \varphi)(N, S) := (\pi N, \varphi^\pi(S)) = (N, S') \tag{1.11}$$

holds true.

As to extensive objects, the definition of isomorphisms requires more effort. We start out with game trees. Consider a bijective mapping ϕ which maps the nodes of a game tree onto the nodes of another game tree. It is not difficult to imagine what it means that ϕ ***respects*** the structures of the trees, i.e., the successor relation and the partitions - or else consult [6]. We then quote

Definition 1.7. *A game tree* $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ *is* ***isomorphic*** *to a game tree* $(E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p')$ *if there is a bijective mapping* $\phi : E \longrightarrow E'$ *(an* ***isomorphism between*** *the game trees) which satisfies the following properties.*

(1) The mapping ϕ *respects* $(\prec, \prec')$, $(\mathbf{P}, \mathbf{P}')$, *and* $(\mathbf{Q}, \mathbf{Q}')$.

(2) $\phi(P_0) = P'_0$ *and* $p'^{\phi(\xi)}(\phi(\xi')) = p^{\xi}(\xi')$ *holds true for* $\xi \in P_0$ *and* $\xi' \in C(\xi)$.

(3) For $Q \in \mathbf{Q}$ *the mapping* ϕ *respects* $(\mathbf{C}(Q), \mathbf{C}'(Q'))$, *where* $Q' \in \mathbf{Q}'$ *is the unique information set which satisfies* $\phi(Q) = Q'$.

Note that *(2)* makes sense in view of *(1)*, the bijectivity of ϕ, and the underlying tree structure. Now turn to extensive preforms. The underlying game trees being isomorphically mapped into each other, we also want to adapt the assignment of partitions to players *consistently.*

Definition 1.8. *Let* $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota)$ *and* $\epsilon' = (N, E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p'; \iota')$ *be extensive preforms. A* ***isomorphism*** *between* ϵ *and* ϵ' *is a pair of mappings* (π, ϕ) *such that* π *is a permutation of* $N, \phi : E \rightarrow E'$ *with the following properties.*

(1) ϕ *is an isomorphism between the underlying game trees (cf. Definition 1.7).*

(2) $\pi(\iota(P)) = \iota'(\phi(P))$ $(P \in \mathbf{P})$

One can imagine that a pair of bijective mappings (an isomorphism) (π, ϕ) acts separately on all the objects constituting an extensive preform. The exact definitions are more or less canonical and will not be explicitely mentioned. To represent it in a closed form, the action (π, ϕ) on ϵ is described as follows:

$$\begin{aligned}(\pi, \phi)\epsilon &= (\pi, \phi)(N, E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p; \iota) \\ &:= (\pi(N), \phi E, \prec^{\phi}, \phi\mathbf{P}, \phi\mathbf{Q}, \phi\mathbf{C}, p_{\phi^{-1}}; \pi \circ \iota \circ \phi^{-1}) \\ &= (N, E', \prec', \mathbf{P}', \mathbf{Q}', \mathbf{C}', p'; \iota')\end{aligned} \tag{1.12}$$

Again let us emphasize that isomorphisms respect the ordering of the nodes. This reflects the notion of 'focality' as discussed in the introductory section.

As to isomorphisms of game forms, our next object of interest is the alphabet of outcomes. Again we first demonstrate our goal in the range of strategic territory. Consider the following two matrices, each of them representing a strategic game form.

$$F = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \qquad G = \begin{pmatrix} c & a \\ b & c \end{pmatrix}$$

The alphabet or outcome set being the same in both cases, we feel that F and G are structurally equal, i.e., *isomorphic* - nevertheless this cannot be obtained by reshuffling the strategies and/or the players. In addition to the familiar operations we should admit a bijection of the alphabet; in the above case this would be expressed by a bijection $\rho : A \to A,\ a \to c,\ b \to a,\ c \to b$. This consideration motivates the following definition.

Definition 1.9. *Let $g = (e; A, h)$ and $g' = (e'; A', h')$ be strategic game forms. An* ***isomorphism*** *of g and g' is a triple $(\pi, \varphi; \rho)$ such that (π, φ) is an isomorphism between e and e' while $\rho : A \to A'$ is a bijection satisfying*

$$h' = \rho \circ h \circ (\varphi^\pi)^{-1}.$$

Here, φ^π is the mapping defined in (1.10). That is, we have

$$(\pi, \varphi; \rho)(e; A, h) \ := \ ((\pi, \varphi)e; \rho \circ h \circ (\varphi^\pi)^{-1}) \ = \ (e'; A', h')$$

An isomorphism is ***outcome preserving*** *if $A = A'$ and ρ is the identity.*

Inevitably we have to discuss isomorphisms for the third group of objects under consideration, i.e., for games. The difference with respect to game forms is constituted by the presence of utility functions instead of the outcome function. Therefore we have to explain the kind of action a permutation (renaming) of the players induces on n-tuples of utility functions, i.e. on u or υ respectively.

Clearly, if (π, φ) is an isomorphism between the strategic preforms e and e' and u and u' are tuples of utilities defined on S and S' respectively, then the utility of i's image $\pi(i) \in N$ should be given by

$$u'_{\pi(i)}(\varphi^\pi(s)) = u_i(s) \quad (i \in N), \tag{1.13}$$

thus indicating that we rename players and strategies simultaneously.

This defines the action of the pair (π, φ) on tuples of utility functions via

$$((\pi, \varphi)u)_{\pi(i)}(\varphi^\pi(s)) \ := \ u_i(s). \tag{1.14}$$

We have thus explained

$$(\pi, \varphi)u : S' \to I\!R^N. \tag{1.15}$$

Analogously, within the extensive set-up, if we have an isomorphism (π, ϕ) of preforms ϵ and ϵ', cf. Definition 1.8, and if $\upsilon : \partial E \to I\!R^N$ is a utility N-tuple defined on the endpoints of $(E, \prec)$, the action of (π, ϕ), i.e.

$$(\pi, \phi)\upsilon \ : \ \partial E' \ \to I\!R^N \tag{1.16}$$

is given by

$$((\pi, \phi)\upsilon)_{\pi(i)}(\phi(\xi)) := \upsilon_i(\xi) \ (\xi \in \partial E,\ i \in N). \tag{1.17}$$

Thus we have

Definition 1.10. *(1) Let $G = (e; u)$ and $G' = (e'; u')$ be strategic games. An **isomorphism** between G and G' is a pair (π, φ) such that (π, φ) is an isomorphism between e and e' (see Definition 1.6 and (1.11)) and $u'_{\pi(i)}(\varphi^{\pi}(s)) = u_i(s)$ $(i \in N, s \in S)$. That is, we have*

$$(\pi, \varphi)G = (\pi, \varphi)(e; u) = ((\pi, \varphi)e; (\pi, \varphi)u) = (e', u') \tag{1.18}$$

(see (1.14) and (1.15)).

*(2) Let $\Gamma = (\epsilon, \upsilon)$ and $\Gamma' = (\epsilon', \upsilon')$ be extensive games. An **isomorphism** between Γ and Γ' is a pair (π, ϕ) such that (π, ϕ) is an isomorphism between ϵ and ϵ' (see Definition 1.8) and $\upsilon'_{\pi(i)}(\phi(\xi)) = \upsilon_i(\xi)$ $(\xi \in \partial E, i \in N)$. That is, we write*

$$(\pi, \phi)\Gamma = (\pi, \phi)(\epsilon; \upsilon) = ((\pi, \phi)\epsilon, (\pi, \phi)\upsilon) = (\epsilon', \upsilon') \tag{1.19}$$

(see (1.16) and (1.17)).

We have now finished the discussion of the various concepts of isomorphisms as appropriate for preforms, game forms, and games - in each family with consideration of the strategic as well as the extensive version. The next and basic concept is the one of symmetry. This concept applies to games only. As we have stressed in Peleg, Rosenmüller, and Sudhölter ([6]), it does not work to consider just *automorphisms* of a game G, say in strategic form, i.e., isomorphisms (π, φ), $\pi: N \to N$, $\varphi_i : S_i \to S_{\pi(i)}$ $(i \in N)$, such that the game is preserved, i.e.,

$$(\pi, \varphi)G = G$$

holds true. The reasons were explained in [6] , we refer the reader in particular to Example 3.6 of this paper. Essentially there are two objections to taking the group of automorphisms as to be the group of symmetries: the first is that the only game that is preserved under the full group of automorphisms is the constant game whereas we definitely feel that e.g. the 'Battle of Sexes', though not constant, is symmetric. The second stems from the opposite observation: A player whose payoff does not depend on his actions may reshuffle his strategies arbitrarily, thus a rather substantial group of automorphisms is generated – nevertheless there might be no obvious symmetry in view of another player's situation.

These considerations lead to the idea of 'disregarding' the 'impersonal' automorphisms which mathematically amounts to factorizing them out or forming the quotient group. More precisely we quote the relevant definitions from [6] as follows.

Definition 1.11. *A **motion** of a strategic game G is an automorphism (π, φ) of G. A motion (π, φ) is **impersonal** if π is the identity (and $\varphi_i : S_i \to S_i (i \in N)$).*

Remark 1.12. *Motions form a **group**. To see this, define the* **unit motion** *by*

$$(id, id) = (id_N, (id_{S_i})_{i\in N})$$

and the product of motions (π, φ) *and* (σ, ψ) *by*

$$(\sigma, \psi)(\pi, \varphi) = (\sigma\pi, \psi \otimes \varphi) \tag{1.20}$$

where $\otimes$ *is given in a natural manner via*

$$(\psi \otimes \varphi)_i := \psi_{\pi(i)} \circ \varphi_i. \tag{1.21}$$

Clearly, the impersonal motions constitute a ***subgroup.*** *This subgroup is normal and therefore 'disregarding' or 'factorizing out' the impersonal part formally amounts to constructing the quotient group as to constitute the group of symmetries of the game.*

The precise version is given as follows:

Definition 1.13. *Let* G *be a strategic game.* $\mathcal{M} = \mathcal{M}(G)$ *denotes the group of* ***motions****.* $\mathcal{I} = \mathcal{I}(G) \subseteq \mathcal{M}$ *denotes the subgroup of* G *constituted by the* ***impersonal*** *motions. The group of* ***symmetries*** *of* G *is the quotient group*

$$\mathcal{S} = \mathcal{S}(G) = \mathcal{M}/\mathcal{I} \ = \ \frac{\mathcal{M}(G)}{\mathcal{I}(G)} \tag{1.22}$$

A game G *is* ***symmetric*** *if its symmetry group is isomorphic to the full group of permutations of* N *called* $\Sigma(N)$*, i.e., if*

$$\mathcal{S}(G) \cong \Sigma(N) \tag{1.23}$$

holds true.

The reader is referred to Example 3.15 of [6] in order to sharpen his intuition regarding symmetries and to appreciate or debate our notion of symmetry. We finish this section by adding the completely analogous definitions for extensive games. Clearly, the group of automorphisms of an extensive game is a much more involved and less easier to view object compared to its strategic counterpart. Nevertheless, the formal definitions are based on the same type of arguments.

Definition 1.14. *Let* $\Gamma = (\epsilon, \upsilon)$ *be an extensive game. A* ***motion*** (π, ϕ) *of* Γ *is an automorphism of* Γ *and* $\mathcal{M} = \mathcal{M}(\Gamma)$ *denotes the group of motions.* $\mathcal{I} = \mathcal{I}(\Gamma)$ *is the normal subgroup of* ***impersonal motions****, i.e., automorphisms of shape* (id, ϕ)*. The* ***group of symmetries*** *is the quotient group*

$$\mathcal{S} = \mathcal{S}(\Gamma) = \mathcal{M}/\mathcal{I} = \frac{\mathcal{M}(\Gamma)}{\mathcal{I}(\Gamma)} \tag{1.24}$$

and Γ *is symmetric if* $\mathcal{S}(\Gamma) \cong \Sigma(N)$ *holds true.*

Thus far we have rehearsed the contents of the first part of our venture, i.e., [6]. We now turn to the more involved task of lying the foundations of representation theory. Eventually we shall construct to every strategic game a uniquely defined extensive game 'representing' it and admitting for exactly the same group of symmetries. This is the task described in the following sections.

2 Faithful Representations

Roughly speaking, a representation of a strategic object (a preform, a game form, or a game) is an extensive object which yields the same von Neumann-Morgenstern strategic version as the one we startet out with. However, 'the same' is not well defined, as there is no canonical ordering of the strategies when one constructs the von Neumann-Morgenstern strategic version. This notion has to be made precise. We then continue to introduce faithful representations, that is, representations that preserve the group of symmetries.

Let ϵ be an extensive preform. A ***pure strategy*** of player i is a mapping that selects a choice at each information set of player i. If we define for $i \in N$

$$\tilde{S}_i := \{\sigma_i \mid \sigma_i \text{ is a pure strategy of player } i\} \tag{2.1}$$

then we obtain a strategic preform (the von Neumann-Morgenstern strategic preform)

$$\mathtt{N}(\epsilon) = (N, \tilde{S}) = (N, (\tilde{S}_i)_{i \in N}) \tag{2.2}$$

Next, if $\upsilon : \partial E \to I\!R^N$ is a vector of utility functions, then to any $\sigma \in \tilde{S}$ there is a corresponding random variable X^σ choosing plays in accordance with the distribution induced by σ and p. We may define the payoff at each realization of X^σ to be the one at the endpoint, thus the expectation

$$\tilde{u}_i(\sigma) = E\upsilon_i(X^\sigma) \tag{2.3}$$

is well defined. This generates a strategic game

$$\mathcal{N}(\epsilon; \upsilon) = \mathcal{N}(\Gamma) := (N, \tilde{S}; \tilde{u}) = (\mathtt{N}(\epsilon); \tilde{u}(\epsilon, \upsilon)) \tag{2.4}$$

which is the 'von Neumann-Morgenstern' ('vNM') strategic game of Γ.

Definition 2.1. $\mathtt{N}(\epsilon)$ *as defined by 2.1 and 2.2 is called the* ***vNM*** *strategic preform of* ϵ*;* $\mathcal{N}(\Gamma)$ *as defined by 2.2 and 2.4 is the* ***vNM*** *strategic game of* Γ*. (The term normal form of an extensive game is common in the literature. This we shall not employ in our present context, because the term 'form' in our context is used for a particular type of structure (strategic and extensive game* ***forms*** *and pre****forms****)).*

Our next purpose is to define the relation between isomorphisms of extensive forms and strategies. To this end, the reader should review Definition 1.7. In this context, when we are given the partitions $\mathbf{P}, \mathbf{Q}, \mathbf{C}$ of a game tree (a game form, a game), we would like to refer to the player dependent elements only. Thus we introduce the following notations. We write

$$\mathbf{P}_{-0} := \mathbf{P} - \{P_0\}, \mathbf{Q}_{-0} := \mathbf{Q} - \mathbf{Q}_0 \text{ where } \mathbf{Q}_0 := \{Q \in \mathbf{Q} | Q \subseteq P_0\}. \tag{2.5}$$

Now, if (π, ϕ) is an isomorphism between our game tree and some other game tree with partitions $\mathbf{P}', \mathbf{Q}', \mathbf{C}'$, then we want to consider the ***induced mappings*** given by

$$\begin{gathered}\phi^{\mathbf{P}} : \mathbf{P}_{-0} \longrightarrow \mathbf{P}'_{-0}, \quad \phi^{\mathbf{Q}} : \mathbf{Q}_{-0} \longrightarrow \mathbf{Q}'_{-0}, \\ \text{where } \phi^{\mathbf{P}}(P) = P', \text{ whenever } P = \phi^{-1}(P'), \\ \text{and } \phi^{\mathbf{Q}}(Q) = Q', \text{ whenever } Q = \phi^{-1}(Q'). \end{gathered} \tag{2.6}$$

Note that both mappings are well defined and bijective, because ϕ is bijective and respects the partitions. Also note that

$$\phi^{\mathbf{C}(Q)} : \mathbf{C}(Q) \longrightarrow \mathbf{C}'(Q') \tag{2.7}$$

can be defined analogously to $\phi^{\mathbf{P}}$.

The influence of isomorphism on strategies is then explained as follows.

Remark 2.2. *Note that an isomorphism (π, ϕ) between extensive preforms ϵ and ϵ' induces an isomorphism between the corresponding vNM preforms. Intuitively, a strategy of a player may be thought of as a set of 'arrows' indicating actions of this player at his information sets. Clearly, reshuffling the nodes and the players in the extensive preform also rearranges the arrows. More precisely, there exists a mapping Ξ which to any (π, ϕ) assigns a (π, φ), where φ is defined by*

$$(\varphi_i(\sigma))(\phi^{\mathbf{Q}}(Q)) = \phi^{\mathbf{C}(Q)}(\sigma(Q)) \quad (Q \in \mathbf{Q}, Q \subseteq P_i, i \in N, \sigma \in \tilde{S}_i). \tag{2.8}$$

For the particular case that ϵ equals ϵ', the mapping Ξ can be seen to respect compositions of isomorphisms, i.e.,

$$\Xi(\sigma \circ \pi, \chi \circ \phi) = (\sigma \circ \pi, \psi \otimes \varphi) = \Xi(\sigma, \chi)\Xi(\pi, \phi)$$

for automorphisms $(\sigma, \chi), (\pi, \phi)$ of ϵ, where $\Xi(\sigma, \chi) = (\sigma, \psi)$ and $\Xi(\pi, \phi) = (\pi, \varphi)$.

Let Γ and Γ' be games and consider an isomorphism (π, ϕ) between them. Then, of course, (π, ϕ) is an isomorphism between the underlying extensive preforms, hence $\Xi(\pi, \phi)$ is an isomorphism between the resulting vNM strategic preforms. Now we have

Lemma 2.3. *Let (π, ϕ) be an isomorphism between the extensive games $\Gamma = (\epsilon; \upsilon)$ and $\Gamma' = (\epsilon'; \upsilon')$. Then $\Xi(\pi, \phi)$ is an isomorphism between $\mathcal{N}(\Gamma)$ and $\mathcal{N}(\Gamma')$.*

Proof: The proof is rather straightforward as all mappings involved are bijective. In view of this fact we restrict ourselves to a mere sketch. The probability at each chance move of ϵ is fully transported to the corresponding probability at the image node in ϵ', which is also a chance move. Therefore the expectations of payoffs are preserved. **q.e.d.**

With game forms the situation is more involved. On the other hand here is the clue to the decisive role game forms play in our treatment of symmetries. Therefore we have to start with the following definition which emphasizes the importance of the game form with respect to all games it may induce. To this purpose, we introduce a set of outcomes A and utilities $U : A \to I\!R^N$. Recall Remark 1.5 concerning composition of a game form g or γ and U to obtain a game $U * g$ or $U * \gamma$.

Definition 2.4.

*(1) Let $g = (e; A, h)$ and $\gamma = (\epsilon; A, \eta)$ be game forms (with identical N and A). Then γ is called a **representation** of g if there is a family of bijections $\psi = (\psi_i)_{i \in N}$, $\psi_i : S_i \to \tilde{S}_i$ such that for every $U : A \to I\!R^N$ (id, ψ) is an (impersonal) isomorphism between $U * g$ and $\mathcal{N}(U * \gamma)$.*

*(2) A representation γ of g is said to be **faithful** if for every $U : A \to I\!R^N$ the symmetry groups coincide, i.e., if*

$$\mathcal{S}(U * g) \cong \mathcal{S}(U * \gamma) \quad (U \ : \ A \to I\!R^N). \tag{2.9}$$

Our first observation is that representations can only occur if the extensive game form is of a nature which avoids the introduction of 'lotteries' for the computation of outcomes resulting from strategies. To this end, for any pure strategy σ_0 of chance and any strategy profile (n-tuple) σ of the players the resulting play is denoted by $X^{\sigma_0,\sigma} = (X_0^{\sigma_0,\sigma}, \ldots, X_T^{\sigma_0,\sigma})$. The outcome induced is $\eta(X_T^{\sigma_0,\sigma})$. However, it turns out that, given a representation, the outcome does not depend on σ_0. More precisely, we have

Theorem 2.5. *Let γ be a representation of g. Then for all $\sigma \in \tilde{S}$ the outcome $\eta(X_T^{\sigma_0,\sigma})$ does not depend on σ_0.*

Proof: Let (id, ψ) be the isomorphism mentioned in Definition 2.4. Let $a \in A$. Let $s \in S$ be such that the outcome (in g) is a and let σ be the image under ψ^{id} of s, i.e.

$$\sigma \in \psi^{id}(h^{-1}(\{a\})).$$

We want to show that $\eta(X_T^{\sigma_0,\sigma}) = a$ for all pure strategies σ_0 of chance. To this purpose define $U : A \to I\!R^N$ by $U_i(a) = 1 \ \ (i \in N)$ and $U_i(b) = 0 \ \ (i \in N, b \in A, b \neq a)$. Consider the games $\Gamma = U * \gamma$ and $G = U * g$ which is impersonally isomorphic to $\mathcal{N}(U * \gamma)$. The first game possesses only payoffs 0 and 1 and so does the latter one. Hence it follows that $E(U_i \circ \eta)(X^{\sigma}) = 1 \ \ (i \in N)$ (cf. 2.3). But this necessarily implies that all plays $X^{\sigma_0,\sigma}$ yield a payoff 1. **q.e.d.**

Corollary 2.6. *Let γ be a representation of g. If g is general, then the impersonal isomorphism (id, ψ) given by Definition 2.4 is uniquely defined.*

Proof: To see this observe that 'mixing' (taking expectations) can be avoided in computing the outcomes resulting from strategy profiles in the framework of γ. Hence an outcome can be associated to any entry of g. As g is general this association defines a unique mapping. **q.e.d.**

The same consideration motivates the introduction of a *vNM strategic game form of* an extensive game form g, even if g does not happen to be general. For, the above mentioned association can be performed in any case, hence the following definition is noncontradictory.

Definition 2.7. *Let $\gamma = (\epsilon; A, \eta)$ be an extensive game form.*

(1) If, for every $\sigma \in \tilde{S}$, the expression

$$\eta(X_T^{\sigma_0,\sigma}) =: h^\eta(\sigma) \tag{2.10}$$

is a constant independently of the pure strategy σ_0 of chance, then γ is ***nonmixing****.*

(2) If γ is nonmixing, then

$$\mathfrak{N}(\gamma) := (\mathtt{N}(\epsilon); A, h^\eta)$$

is the ***vNM strategic game form*** *of γ.*

Intuitively within the framework of a nonmixing extensive game form taking lotteries (or expectations for that matter) is avoided. Every play which chance can generate yields the outcome determined by the strategy profile of the players.

Remark 2.8. *Let γ be an extensive game form. Then the following are equivalent.*

(1) γ is nonmixing.

(2) γ is a representation of some strategic game form g.

Every representation of a strategic game form is by Theorem 2.5 nonmixing. Conversely, if γ is nonmixing, then it is straightforward to verify that it represents $\mathfrak{N}(\gamma)$.

Corollary 2.9. *Let γ be a nonmixing extensive game form and g be a strategic game form.*

(1) γ represents g, if and only if there is an ***i****mpersonal* ***o****utcome* ***p****reserving* ***(IOP)*** *isomorphism (id, ψ, id) between g and $\mathfrak{N}(\gamma)$.*

(2) If g is general and γ represents g, then the IOP isomorphism mentioned above is uniquely determined.

3 Square General Game Forms

Our next task is to exhibit faithful representations. This will be done in the context of game forms that, in principle, allow for symmetries, i.e., game forms with an equal number of strategies for each player. Since for two persons the strategic versions of such game forms resemble square matrices, we call such versions square as well. The following definition provides a formal approach.

Definition 3.1. *A strategic preform e, game form $(e; A, h)$, or game $(e; u)$ respectively is called **square**, if we have $r(e) = r = |S_i|$ $(i \in N)$.*

Lemma 3.2. *Let $\gamma = (\epsilon; A, \eta)$ be a nonmixing extensive game form and let $U : A \to I\!R^N$ be a utility profile. Also, let (π, ϕ) be an automorphism of ϵ. Then $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ if and only if $\Xi(\pi, \phi) \in \mathcal{M}(U * \mathfrak{N}(\gamma))$.*

The proof is easy and shall be omitted.

Remark 3.3. *The situation is essentially the same if the vNM game form $\mathfrak{N}(\gamma)$ is replaced by a strategic game form g of which γ is a representation; however we have to observe the IOP isomorphism (id, ψ, id) (cf. Corollary 2.9). Indeed, for fixed γ and g, Ξ induces a mapping $\Theta = \Theta^{\epsilon,e}$ which carries automorphisms of ϵ into automorphisms of e via*

$$\Theta(\pi, \phi) = (\pi, \psi^{-1} \otimes \varphi \otimes \psi), \text{ where } \Xi(\pi, \phi) = (\pi, \varphi). \tag{3.11}$$

Theorem 3.4. *Let γ be a faithful representation of the square general strategic game form g and let $U : A \to I\!R^N$ be a utility profile. Then*

$$\Theta : \mathcal{M}(U * \gamma) \longrightarrow \mathcal{M}(U * g) \tag{3.12}$$

is surjective.

Proof: For $r = r(e) = 1$ the assertion is obvious, thus we assume $r \geq 2$.

1st Step: Let $\gamma = (\epsilon; A, \eta)$ and $g = (N, S; A, h)$ be game forms with the desired properties. We can assume without loss of generality that $S_i = S_j = \{1, \ldots, r\}$ $(i, j \in N = \{1, \ldots, n\})$ holds true. Indeed, we are going to show that

$$\Theta : Aut(\epsilon) \longrightarrow Aut(e) \tag{3.13}$$

(here Aut denotes the group of automorphisms) is surjective. On first sight this might seem to be a more comprehensive statement, however in view of our subsequent proof it will become clear that every automorphism can occur as a motion of a suitable game; hence both claims are in fact equivalent.

2nd Step: First of all consider a utility profile $U : A \to I\!R^n$ specified as follows. We take

$$U_i(s) = i \cdot r + s_i \quad (i \geq 3, s \in S)$$

in order to avoid any symmetries between players $i, j \geq 3$. Furthermore put

$$U_1(s) = U_1(s_1, s_2) \text{ and } U_2(s) = U_2(s_1, s_2)$$

(meaning that U_1, U_2 depend on the first two coordinates only). In addition U_1, U_2 is specified by

$$U_1(s_1, s_2) = U_2(s_2, s_1) = \begin{cases} 0, & \text{if } s_1 \in \{1, 2\} \text{ and } s_2 \geq 3 \\ s_1, & \text{if } s_1 \geq 3 \end{cases}$$

and

$$U_1(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{array} \qquad U_2(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \tag{3.14}$$

for $(\cdot,\cdot) \in \{1,2\} \times \{1,2\}$.

Now we are going to discuss the group of motions corresponding to $U * g$. To this end, let $\pi = (1,2)$ be the transposition of the first two players. Also let $\tau^{ij} : S_i \to S_j$ be defined by $\tau^{ij} : 1 \mapsto 2 \mapsto 1$ and let $id^{ij} : S_i \to S_j$ be the identity mapping for $i, j \in N$. Then we have

$$\mathcal{M}(U*g) = \left\{ \begin{array}{cc} (id, (id^{11}, id^{22}, id^{(n-2)})), & (\pi, (\tau^{12}, id^{21}, id^{(n-2)})), \\ (\pi, (id^{12}, \tau^{21}, id^{(n-2)})), & (id, (\tau^{11}, \tau^{22}, id^{(n-2)})) \end{array} \right\} = \left\{ \begin{array}{cc} c^0, & c, \\ c^3, & c^2 \end{array} \right\} \tag{3.15}$$

where id^{n-2} is self-explaining.

As γ is faithful, there exists ϕ such that $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ and Ξ throws (π, ϕ) on either c or c^3. As the group is cyclic, the powers of (π, ϕ) are thrown onto all of $\mathcal{M}(U * g)$.

3rd Step: The next utility profile we have to consider is indicated by

$$U_1(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \end{array} \qquad U_2(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \end{array} \tag{3.16}$$

(using the convention established in the 2nd Step). Here the group of motions can easily be computed as

$$\mathcal{M}(U * g) = \{c^0, (\pi, (id^{12}, id^{21}, id^{(n-2)}))\} = \{d^0, d\}. \tag{3.17}$$

Again using faithfulness it is at once established that d necessarily has to be the image of some $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ under Θ.

4th Step: Now c^0, c, c^2, c^3, and d occured as motions in a suitable context but, of course, they are automorphisms of e as well. We may generate similar automorphisms as images under Θ by exchanging any two strategies of players 1 and 2 or, for that matter, of any two players. The reader has now to convince himself that the family of automorphisms created this way generates the full group of automorphisms of e. **q.e.d.**

The following example shows that Theorem 3.4 is false if the square game form is not **general**. Moreover, it turns out in the next section that 'square' cannot be dropped as a condition.

Example 3.5. *(1) Let $g = (e; A, h)$ be the square* **nongeneral** *strategic 3-person game form given by $S_1 = \{u,\ d\}$, $S_2 = \{t,\ b\}$, $S_3 = \{l,\ r\}$, and $A = \{a\}$. Then g can be*

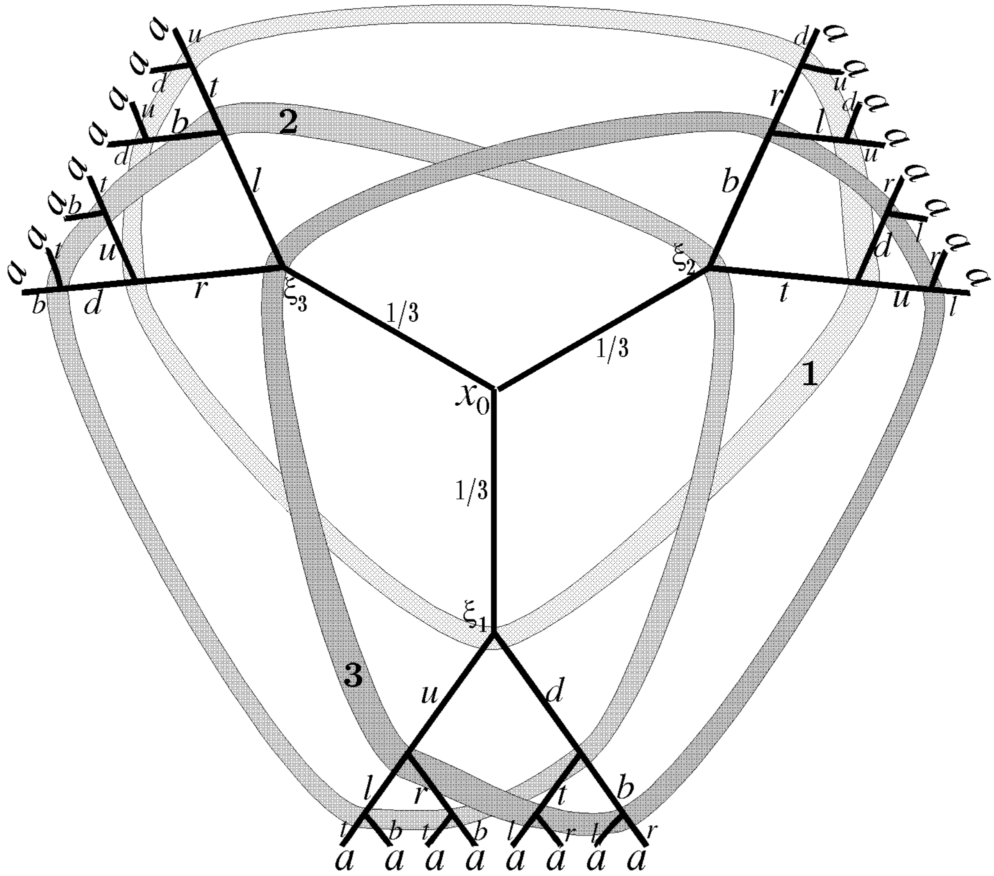

Figure 3.1: A faithful representation of a nongeneral form

represented by the extensive game form $\gamma = (\epsilon; A, \eta)$ *as indicated in Fig. 3.1. Note that for every* $\pi \in \Sigma(N)$ *there is a unique* ϕ *such that* $(\pi, \phi) \in Aut(\epsilon)$*. Indeed,* π *uniquely determines* $\phi(\xi_i)$ $(i \in N)$ *and thus* ϕ*. Hence* $|Aut(\epsilon)| = 6$*. Let* $U : A \to I\!R$ *be a utility profile. In order to show that* γ *is faithful it is sufficient to distinguish 3 cases.*

(a) If $U_i(a)$ $(i \in N)$ *are pairwise distinct, then the symmetry groups of* $U * \gamma$ *and* $U * g$ *consist of the identity permutation only. However,* $U * g$ *possesses 8 motions, whereas the identity mapping is the unique motion of* $U * \gamma$*.*

(b) If $U_i(a) = U_j(a) \neq U_k(a)$*, where* $\{i, j, k\} = N$*, let us say* $i = 1,\ j = 2,$ *and* $k = 3$*, then*

$$\mathcal{S}(U * \gamma) \cong \mathcal{S}(U * \gamma) \cong \{\pi, id\},$$

(Here π *denotes the transposition of players* 1 *and* 2.) *whereas* $|\mathcal{M}(U * g)| = 16$ *and* $|\mathcal{M}(U * \gamma)| = 2$.

(c) If $U_1(a) = U_2(a) = U_3(a)$*, then*

$$\mathcal{S}(U * g) \cong \Sigma(N) \cong \mathcal{S}(U * \gamma),$$

whereas $|\mathcal{M}(U * g)| = 48$ *and* $|\mathcal{M}(U * \gamma)| = 6$.

This example shows that the mapping $\Theta : \mathcal{M}(U * \gamma) \to \mathcal{M}(U * g)$ *is not necessarily surjective, if* g *is not general. In the present case* Θ *is injective.*

(2) *The situation described in the preceding example cannot occur, if the strategic game is general. Indeed, if the strategic game form* g *is 'the' corresponding general game form represented by the extensive game form* γ *sketched in Fig. 3.2, then a utility*

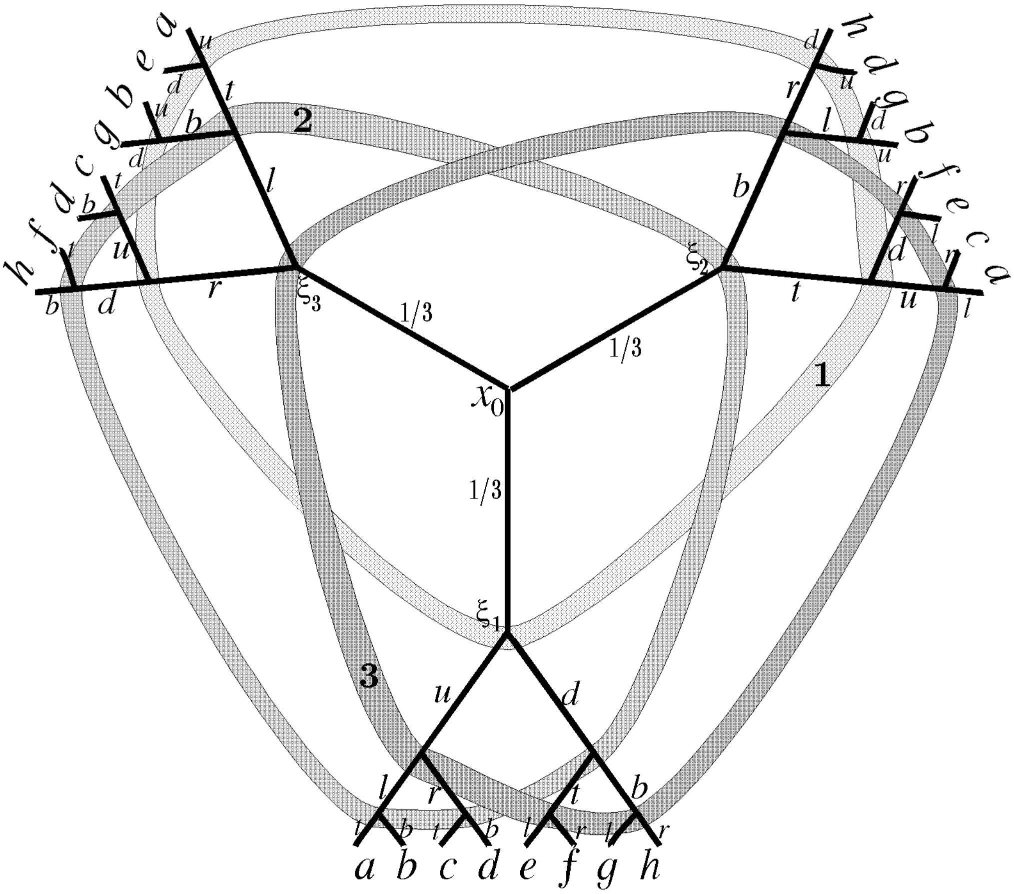

Figure 3.2: A nonfaithful representation of a general form

profile used in the 3rd step of the preceding proof shows that γ *is not faithful. (Note that the utility profiles defined in the 2nd step do not yield a contradiction.)*

For a strategic general game form $g = (e; A, h)$ every automorphism (π, φ) of e induces an automorphism $(\pi, \varphi, h \circ \varphi^{\pi} \circ h^{-1})$ of g. Analogously, for any nonmixing extensive game form $\gamma = (\epsilon; A, \eta)$ every automorphism (π, ϕ) of ϵ induces an automorphism (π, ϕ, ρ) of γ, where ρ is essentially given by $\eta \circ \phi \circ \eta^{-1}$; meaning that $\rho(a) = \eta(\phi(\xi))$ for all $\xi \in \eta^{-1}(a)$ is welldefined independently of ξ $(a \in A)$. This fact and the last proof enables us to reformulate Theorem 3.4.

Corollary 3.6. *Let* $\gamma = (\epsilon; A, \eta)$ *be a representation of the general square strategic game form* $g = (e; A, h)$ *and let*

$$\tilde{\Theta} : Aut(\gamma) \to Aut(g)$$

be defined by

$$\tilde{\Theta}(\pi,\phi,\rho) = (\Theta^{\epsilon,e}(\pi,\phi),\rho) \quad ((\pi,\phi,\rho) \in Aut(\gamma)).$$

Then γ is a faithful representation of g, if and only if $\tilde{\Theta}$ is surjective.

Indeed, note that $\tilde{\Theta}(\pi,\phi,\rho)$ is an automorphism, because $\rho = h \circ \varphi^{\pi} \circ h^{-1}$, where $\Theta^{\epsilon,e}(\pi,\phi) = (\pi,\varphi)$, is satisfied.

The above development suggests to briefly consider automorphisms of extensive preforms that leave the corresponding vNM preforms untouched. This kind of automorphisms is described by the following definition.

Definition 3.7. *An automorphism (π,ϕ_0) of an extensive preform ϵ is said to be* ***chance related*** *if the following holds true:*

(1) $\pi = id$

(2) $\phi_0^{\mathbf{Q}}(Q) = Q \quad (Q \in \mathbf{Q_{-0}})$ *(cf. Definition 1.7)*

(3) $\phi_0^{\mathbf{C}(Q)}(C) = C \quad (C \in \mathbf{C}(Q), Q \in \mathbf{Q_{-0}})$

A motion (π,ϕ_0) of a game Γ is ***chance related*** *if conditions (1), (2), and (3) are satisfied. $\mathcal{C}(\Gamma)$ denotes the subgroup of chance related motions of $\mathcal{M}(\Gamma)$. Note that formula 2.8 of Remark 2.2 implies that $\Xi(\pi,\phi_0)$ is the identity, i.e., the strategies of the corresponding vNM preform are not disturbed.*

Theorem 3.8. *The chance related automorphisms of an extensive preform constitute a normal subgroup and, for every extensive game Γ, the subgroup $\mathcal{C}(\Gamma) \subseteq \mathcal{I}(\Gamma)$ is normal.*

Proof: We have to show that for any automorphism (π,ϕ) and any chance related automorphism (id,ϕ_0) we can find a chance related automorphism (id,ϕ_0') such that

$$(\pi,\phi)(id,\phi_0) = (id,\phi_0')(\pi,\phi)$$

holds true. To this end it suffices to show that

$$\phi_0' = \phi\ \phi_0\ \phi^{-1}$$

is chance related. Indeed, we have for $i \in N$ and $Q \subseteq P_{\pi(i)}$

$$\phi^{-1}(Q) \subseteq P_i$$

that is

$$\phi_0(\phi^{-1}(Q)) = \phi^{-1}(Q)$$

and hence

$$\phi(\phi_0(\phi^{-1}(Q)) = Q,$$

and analogously for (3) of Definition 3.7 **q.e.d.**

Theorem 3.9. *Let γ be a faithful representation of the square general strategic game form g. Then, for any utility profile $U : A \to \mathbb{R}^N$ it follows that*

$$\mathcal{M}(U \star \gamma)/\mathcal{C}(U \star \gamma) \cong \mathcal{M}(U \star g) \tag{3.18}$$

holds true.

Proof: Theorem 3.4 implies that Θ is a surjective mapping which respects composition (Remark 2.2). It suffices to show that $\mathcal{C}(U \star \gamma)$ is the kernel of this mapping.

Clearly, if $(\pi, \phi) \in \mathcal{C}$, then $\Theta(\pi, \phi) = (id, id)$. On the other hand if $\Theta(\pi, \phi) = (id, id)$, then ϕ has to satisfy conditions (2) and (3) of Definition 3.7, for otherwise we can construct a strategy $\sigma_i \in \tilde{S}_i$ that suffers under the influence of ϕ as defined in (4.5). **q.e.d**

4 Atoms

Our previous results provide us with a possibility to discuss 'square games' as a first attempt to introduce the symmetric canonical extensive version. In order to approach this program we shall first of all discuss the simplest (and 'nonsymmetric') version of a representation: an atom.

Definition 4.1.

*(1) A game tree is **atomic**, if there are no chance moves and the player partition coincides with the information partition. In this case we write $(E, \prec, \mathbf{P}, \mathbf{C})$. In particular an atomic game tree is said to be **square** if $|C(\xi)| = |C(\xi')| = r$ for $\xi, \xi' \in E - \partial E$ holds true.*

*(2) A preform $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota)$ is **atomic**, if $(E, \prec, \mathbf{P}, \mathbf{C})$ is an atomic game tree.*

*(3) An extensive game form $\alpha = (\epsilon; A, \eta)$ is an **atom**, if ϵ is an atomic preform and $\eta : \partial E \to A$ is bijective.*

*(4) An atom α and its preform and game tree is **time structured** if every nonvoid level $\mathcal{L}(E, \prec, t)$ coincides with one player set $P \in \mathbf{P}$. In this case α is called **T-atom**.*

Remark 4.2. *Let $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota)$ be a preform of an atom and let $e = (N, S)$ be a strategic preform of a square game form such that $|S_i| = |\tilde{S}_i|$ $(i \in N)$ is satisfied, where $\mathtt{N}(\epsilon) = (N, \tilde{S})$. Moreover, let (id, ψ) be an isomorphism between e and $\mathtt{N}(\epsilon)$ (which exists because the corresponding strategy sets have coinciding sizes).*

*(1) If $h : S \to A$ is a bijection, then there exists a unique (bijective) mapping $\eta : \partial E \to A$ such that (id, ψ, id) is an IOP isomorphism between $g = (e; A, h)$ (a general strategic game form) and $\mathfrak{N}(\alpha)$ (where $\alpha = (\epsilon; A, \eta)$), i.e. α is a representation of g. An atom which represents g is said to be an **atom of** g.*

(2) If $\eta : \partial E \to A$ is a bijection, then there exists a unique (bijective) mapping $h : S \to A$ such that (id, ψ, id) is an isomorphism between $g = (e; A, h)$ and $\mathfrak{N}(\alpha)$ (where $\alpha = (\epsilon; A, \eta)$), i.e. the atom α is a representation of the general game form g.

Example 4.3.

(1) For two persons consider g as indicated by

$$\begin{array}{cc} & \begin{array}{cc} l & r \end{array} \\ \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \tag{4.19}$$

There are two atoms as indicated by Figure 4.3.

Figure 4.3: Atoms for a $(2; 2 \times 2)$ game

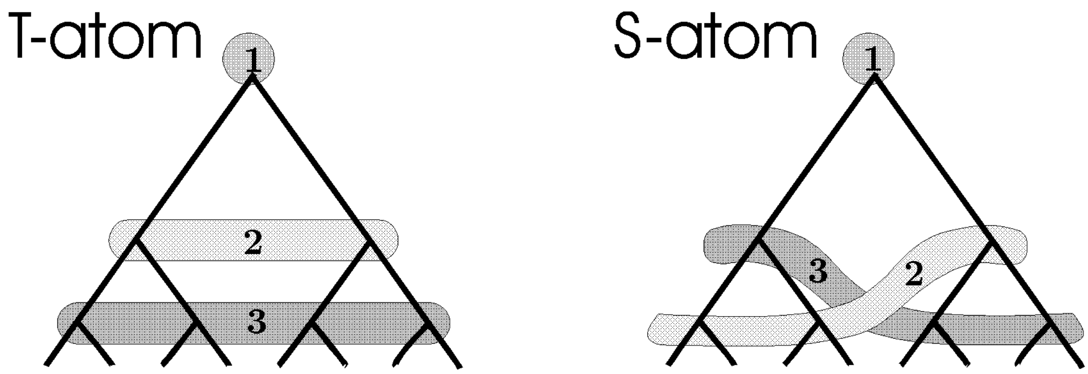

Figure 4.4: Atomic preforms for a $(3; 2 \times 2 \times 2)$ game

(2) For 3 persons and $r = |S_i| = 2$ for all i, consider the preforms in Figure 4.4 which may be augmented to game forms representing appropriate strategic forms.

The T-atom is 'time structured'. Assuming that the game is 'Common Knowledge', player i is aware that he moves 'at instant i'. The $S-$atom seems to exhibit some symmetry between players 2 and 3.

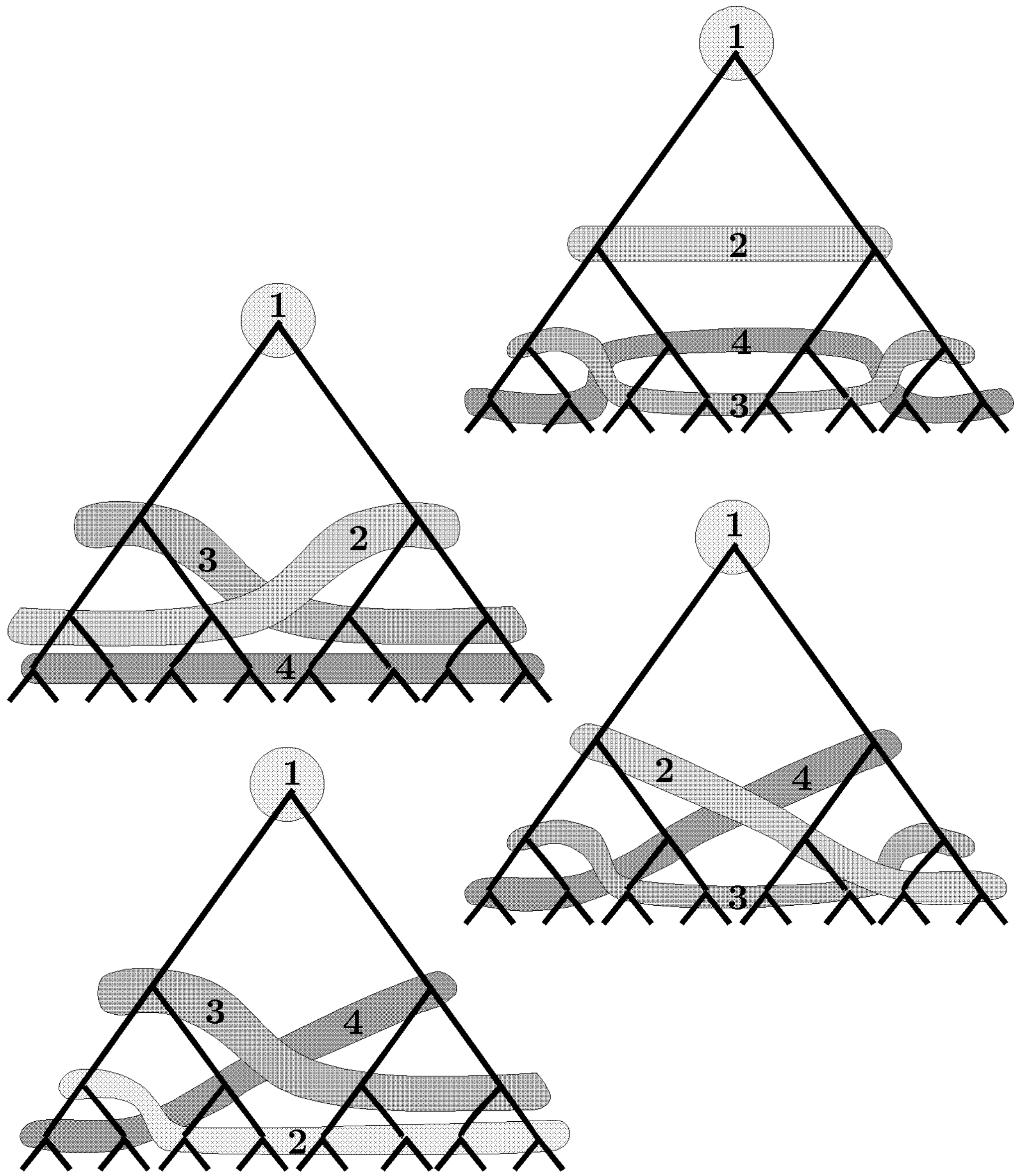

Figure 4.5: Atomic preforms for a $(4; 2 \times 2 \times 2 \times 2)$ game

(3) For 4 persons and $r = 2$, examples of atoms can be seen in Figure 4.5.

Remark 4.4.

(1) *Figure 4.5 suggests that the underlying* **tree** *$(E, \prec)$ of an atom α of a square general game form $g = (N, S; A, h)$ is essentially (i.e. up to order respecting bijective mappings, cf. Section 1) uniquely determined. Clearly, the pair $(E, \prec)$ is a tree of a square atom $\alpha = (E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$ of $\mathfrak{N}(\alpha)$ which is general (in the sense of Definition 1.3) iff the maximal rank coincides with the number $|N|$ of players and at every node $\xi \in E - \partial E$ there are exactly $r = |S_i|$ alternatives.*

(2) *Also, α is an atom of g if and only if it represents g and its total rank is minimal.*

(3) *In addition note that to any square atom α we can at once construct isomorphic ones by permuting the players and renaming the outcomes accordingly. E.g., there are at once 4! different but isomorphic atoms corresponding to each one suggested by Figure 4.5; all of them being obtained by permuting the players arbitrarily and renaming the outcomes accordingly.*

(4) *An atom of a general square game form g cannot be a faithful representation of g, because there is no automorphism of the preform which replaces the 'owner of the root' by any other player.*

5 Symmetrizations

In order to obtain faithful representations of square general game forms, we will now construct 'symmetrizations' of square atoms. To this end we shall shortly describe a further operation acting on game trees called restriction.

A game tree $(E^*, \prec^*, \mathbf{P}^*, \mathbf{Q}^*, \mathbf{C}^*, p^*)$ is a ***restriction*** of the game tree $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ (**the** restriction **to** $(E^*, \prec^*)$), if

(1) $(E^*, \prec^*)$ is a subtree of $(E, \prec)$ (i.e., $E^* \subseteq E$, $\prec^* := \prec|_{E^*} := \prec \cap (E^* \times E^*)$, $\partial E^* \subseteq \partial E$),

(2) $\mathbf{P}^* := \{P \cap E^* \mid P \in \mathbf{P}\}$, $\mathbf{Q}^* := \{Q \cap E^* \mid Q \in \mathbf{Q}\}$,

(3) $\mathbf{C}^*(Q \cap E^*) := \{S \cap E^* \mid S \in \mathbf{C}(Q)\}$ $(Q \in \mathbf{Q})$,

(4) $C(\xi) \subseteq E^*$ and $p^{*\xi} = p^\xi$ $(\xi \in P_0^*)$ (i.e., a chance move together with its choices and probability distribution is either completely preserved or disappears). Note that a possible generalization of this notion (the probabilities p^* should be the conditional probabilities given E^*) is not needed in our present approach.

Restrictions of extensive game forms and games are defined in an obvious way. Also, for the strategic versions the definition of a restriction is the straightforward one, e.g. a strategic preform (N, S^*) is a restriction of (N, S), if $S^* \subseteq S$ holds true.

Definition 5.1. *Let α be a square atom. A game form $\gamma = (N, E, \prec, \mathbf{P}, \mathbf{C}, p; \iota; A, \eta)$ (i.e. $\mathbf{P} = \mathbf{Q}$) is a **symmetrization** of α, if the following conditions are satisfied.*

(1) *γ is nonmixing.*

(2) *The root x_0 of γ is the only chance move and p^{x_0} is uniform distribution, i.e., every edge at x_0 has the same probability.*

(3) *For every $\xi \in C(x_0)$ the restricted game form $\gamma^\xi := (N, E^\xi, \prec^\xi, \mathbf{P}^\xi, \mathbf{C}^\xi; \iota^\xi; A, \eta^\xi)$ of γ obtained by restricting γ to the subtree with root ξ generated by the edge (x_0, ξ) is isomorphic to α.*

(4) For every atom β which represents $\mathfrak{N}(\alpha)$ and is isomorphic to α there exists a unique $\xi \in C(x_0)$ such that β is IOP isomorphic to γ^ξ.

Example 5.2. *Consider the case of two persons each of them having two strategies. Two atoms of the general game form, i.e., of g represented by*

$$\begin{array}{cc} & \begin{array}{cc} l & r \end{array} \\ \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \tag{5.20}$$

have been indicated in Figure 4.3. Clearly, they are isomorphic. A symmetrization is indicated as follows.

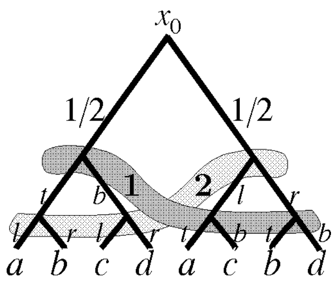

Figure 5.6: The symmetrization

Thus for the simple $(2; 2 \times 2)$-case, the symmetrization described in Figure 5.6 suggests the structure of the 'canonical' representation we have in mind.

Already for 3 persons, this is not so obvious. As Figure 4.4 suggests, there are essentially 2 nonisomorphic atoms: the 'time structured' or 'T-atom' and the 'S-atom' which seems to exhibit more symmetry with respect to the players not called upon in the first move, i.e., players 2 and 3 in 4.4.

Both allow for symmetrizations and at this stage it is not clear which of them will be a candidate for the canonical version.

Example 5.3. *Figure 5.7 shows a symmetrization of the T-atom in Figure 4.4, that could be called $TSYM^3_{2\times2\times2}$. Figure 5.8 is the analogous version with respect to the S-atom of Figure 4.4. At this state of affairs it may become conceivable that there is a problem arising from the question as to which version of an extensive game represents the $(3; 2 \times 2 \times 2)$ case 'appropriately' in view of symmetry considerations.*

The next result shows that symmetrizations exist and are faithful.

Theorem 5.4. *Let α be a square atom. Then*

(1) α possesses a symmetrization,

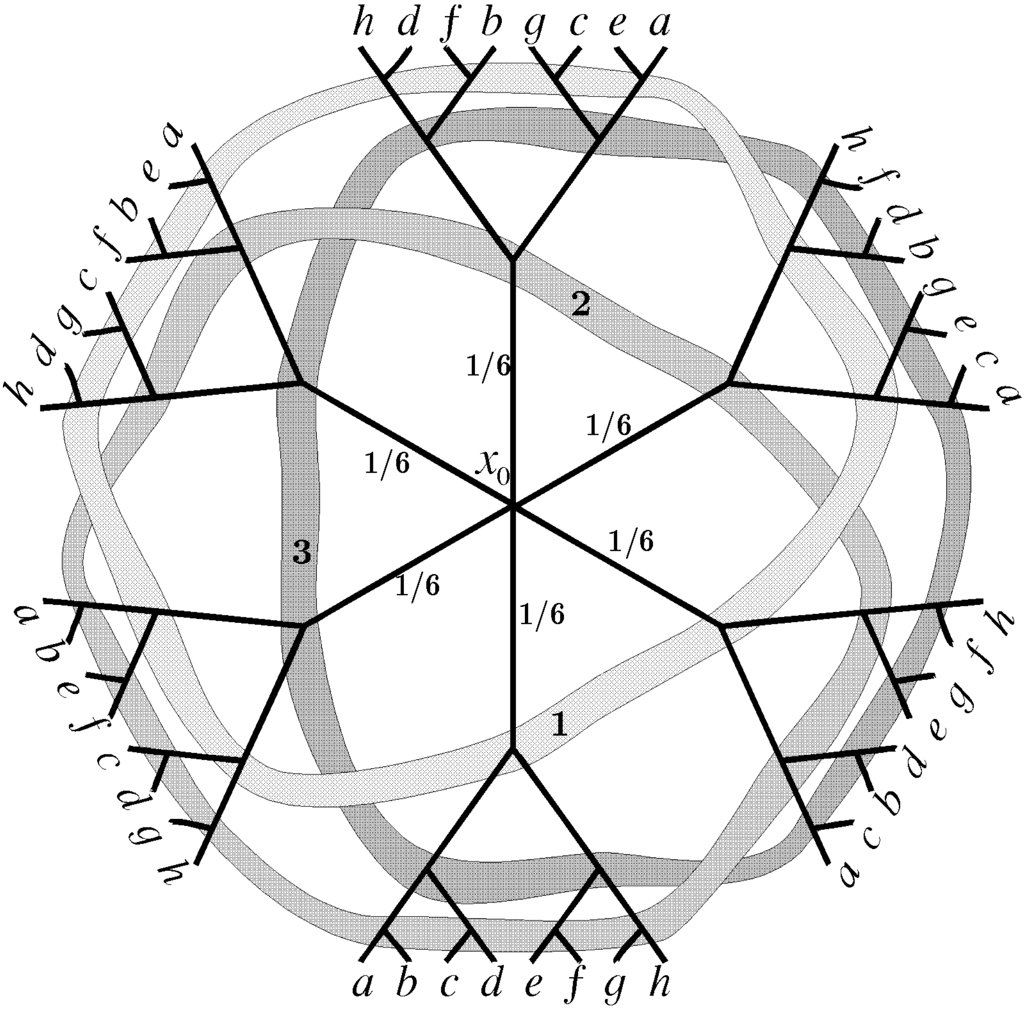

Figure 5.7: The symmetrization of a T-atom ($\mathrm{TSYM}^3_{2\times 2\times 2}$)

(2) every two symmetrizations of α are IOP isomorphic,

(3) a symmetrization of α is a faithful representation of every strategic game form g represented by α.

The proof is decomposed according to the three items claimed.

Proof of item (1): Let $\alpha = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$ and $\mathfrak{N}(\alpha) = g$. Furthermore, define

$$\mathcal{B} = \{\beta \mid \beta \text{ is an atom of } g \text{ isomorphic to } \alpha \text{ with tree } (E, \prec)\}.$$

Every atom of g which is isomorphic to α is isomorphic to some atom of the finite set $\mathcal{B}$. Choose a maximal subset $\mathcal{A} \subseteq \mathcal{B}$ of atoms which are not IOP isomorphic. Indeed, $\mathcal{B}$ can be partitioned into the equivalence classes of IOP isomorphic atoms. The set $\mathcal{A}$ contains precisely one representative of each equivalence class. Moreover, for every $\beta = (N, E, \prec, \mathbf{P}^\beta, \mathbf{C}^\beta; \iota^\beta; A, \eta^\beta) \in \mathcal{A}$ take an IOP isomorphism (id, ψ_β, id) between g and

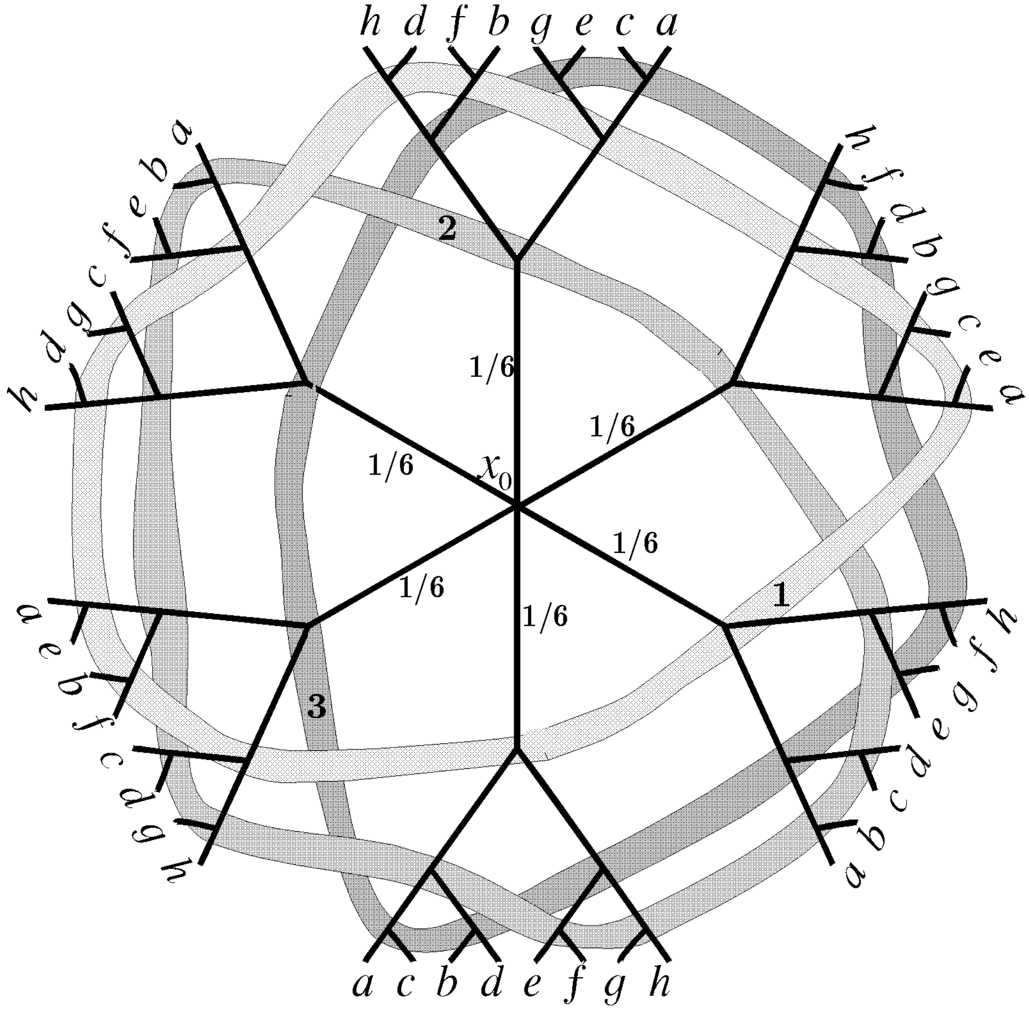

Figure 5.8: The symmetrization of an 'S-atom'

$\mathfrak{N}(\beta)$. In view of Corollary 2.9 ψ_β exists. The extensive game form

$$\gamma = (N, \breve{E}, \breve{\prec}, \breve{\mathbf{P}}, \breve{\mathbf{C}}, p; \breve{\iota}; A, \breve{\eta})$$

is defined as follows.

(1) $\breve{E} = \{0\} \cup (E \times \mathcal{A})$,

(2) $0 \breve{\prec} (x_0, \beta), p^0(x_0, \beta) = |\mathcal{A}|^{-1} \quad (\beta \in \mathcal{A})$, where x_0 is the root of $(E, \prec)$,

(3) $(\xi, \beta) \breve{\prec} (\xi', \beta')$, iff $\beta = \beta'$ and $\xi \prec \xi' \quad (\beta, \beta' \in \mathcal{A}, \xi, \xi' \in E)$.

(4) $\breve{P}_i = \bigcup_{\beta \in \mathcal{A}} P_i^\beta \times \beta, \breve{P}_0 = \{0\} \quad (i \in N)$,

(5) $\breve{C}(\breve{P}_i) = \{\bigcup_{\beta \in \mathcal{A}} (\psi_{\beta,i}(s_i), \beta) | s_i \in S_i\} \quad (i \in N)$,

(6) $\breve{\eta}(\xi, \beta) = \eta^\beta(\xi) \quad (\xi \in \partial E, \beta \in \mathcal{A})$.

By construction γ is a symmetrization of α.

Proof of item (2): Let γ and δ be two symmetrizations of α. By Definition 5.1 (3) there is a bijection between the atoms in γ and δ which maps every atom in γ to an IOP isomorphic atom in δ. These IOP isomorphisms together induce an IOP isomorphism between γ and δ in a straightforward manner.

Proof of item (3): Let γ be a symmetrization of α which represents g. The extensive game form γ represents g, because γ is nonmixing and every atom in γ represents g. Let $\beta = (\epsilon; A, \eta)$, where $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}, \iota)$, be an atom of $g = (N, S; A, h) = (e; A, h)$ such that β is isomorphic to α. Assume without loss of generality that $S_i = \{1, \ldots, r\}$ $(i \in N)$ holds true. Moreover, let (π, φ) be an automorphism of e and (id, ψ, id) the IOP isomorphism between g and $\mathfrak{N}(\beta)$. Then ρ, defined by $\rho = h \circ \varphi \circ h^{-1}$, generates an automorphism (π, φ, ρ) of g. Define $\beta' = (N, E, \prec, \mathbf{P}', \mathbf{C}', \iota'; A, \eta')$ and $\phi : E \to E$ as follows.

(1) Define $\phi(x_0) = x_0$ and assume that $\phi(\xi) \in \mathcal{L}(E, \prec, t)$ $(\xi \in \mathcal{L}(E, \prec, t))$ is already defined for some $0 < T \le n = |N|$ and $0 \le t < T$. If $\xi \in \mathcal{L}(E, \prec, T)$, let us say $\xi \in C(\xi')$ and $\xi' \in P_i$ for some $i \in N$, then take the unique strategy $s_i \in S_i$ such that $\xi' \in \psi_i(s_i)$ and determine $\zeta \in C(\phi(\xi'))$ which satisfies $\zeta \in \psi_k(\varphi(s_i))$, where $\phi(\xi') \in P_k$. Define $\phi(\xi) = \zeta$ and observe that ϕ is bijective and respects $(\prec, \prec)$.

(2) Put $P'_{\pi(i)} = \phi(P_i)$ $(i \in N)$.

(3) Put $\mathbf{C}'(P'_{\pi(i)}) = \phi(\mathbf{C}(P_i))$ $(i \in N)$.

(4) Put $\eta'(\xi) = (\rho \circ \eta \circ \phi^{-1})(\xi)$ $(\xi \in \partial E)$, observe that (π, ϕ, ρ) is an isomorphism between β and β', and that β' represents g. Indeed, with $\psi'_i(s_i) = \{\psi_j(s_i) | P_j \cap P'_i \neq \emptyset\} \cap C(P'_i)$ $(i \in N, s_i \in S_i)$ the triple (id, ψ', id) is an IOP isomorphism between g and $\mathfrak{N}(\beta')$.

This procedure applied to every restricted game γ^ξ (where ξ is a successor of the root of γ) yields an automorphism $(\pi, \tilde{\phi}, \rho)$ of γ (note that β' is, up to an IOP isomorphism, a restricted game of γ). Clearly $\Theta(\pi, \tilde{\phi}) = (\pi, \varphi)$ (cf. Remark 3.3 for the definition of Θ), thus the proof is finished. **q.e.d.**

Remark 5.5. *It should be noted that the cardinality of the set $\mathcal{A}$ as defined in the proof of item (1) varies according to the shape of the atom involved. Most conspicuously the T-atom requires fewer isomorphic copies than a more complex version as explained by the following sketches (and proved later on).*

Figure 5.9 shows 3 copies of non-IOP isomorphic atoms, each of them from a different equivalence class of $\mathcal{B}$ as mentioned in item (1). Each of these atoms admits of $3! = 6$ *further isomorphic atoms which are obtained by permuting the players. Of course all of these again stem from different equivalence classes, because permuting players forbids IOP isomorphism. Hence there are alltogether 18 non-IOP isomorphic atoms which, similarly as indicated in Figure 5.7, are then glued together in order to generate the symmetrization.*

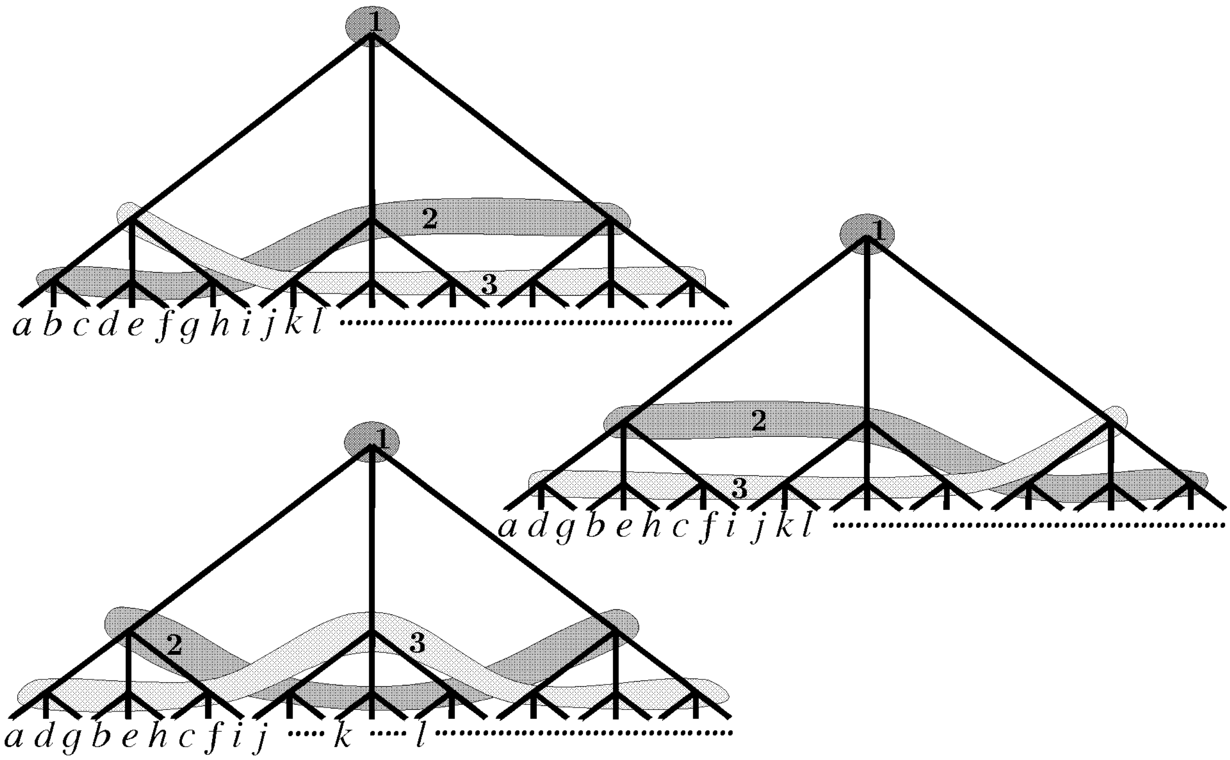

Figure 5.9: Non-IOP isomorphic atoms

By contrast, the T-atom for the same general $3 \times 3 \times 3$ *strategic game form, as represented by Figure 5.10, generates a symmetrization which is obtained by glueing together only the* 6 *non-IOP isomorphic copies obtained by permuting the players.*

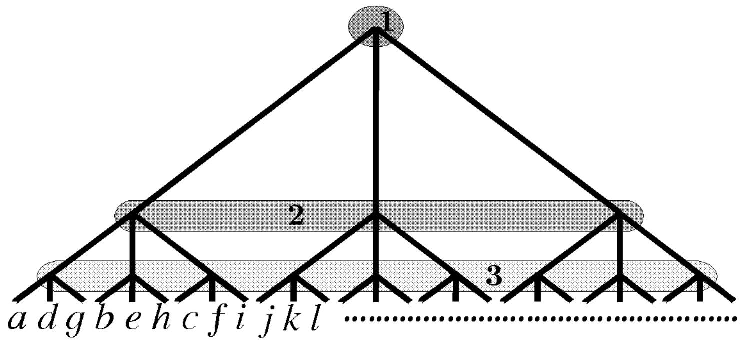

Figure 5.10: The $3 \times 3 \times 3$ T-atom

Corollary 5.6. *Let* γ *be a symmetrization of some square atom which represents* g*. Then every restriction of* γ *which faithfully represents* g *coincides with* γ*.*

Proof: To verify this assertion a part of the proof of item (3) has to be repeated. Clearly at least one of the atoms in γ, say β, has to occur in the faithful restriction (otherwise the restriction is not a representation of g). Moreover, for another atom β' which occurs in

γ there is an isomorphism (π, ϕ) between both atoms. Applying Ξ and the IOP isomorphisms between the normalizations of the atoms and g yields an automorphism (π, φ, ρ) of g. In view of the proof of Theorem 3.4 there must be an automorphism of γ which is mapped to this automorphism of g. Definition 5.1 (3) completes the proof. **q.e.d.**

Different atoms of a square general game form possess isomorphic trees (e.g., the number of nodes coincides). This is no longer true for 'the' symmetrizations as shown in Remark 5.5, i.e. symmetrizations of different atoms of a square general game form may have different numbers of plays and, thus, endpoints. Hence the total ranks may differ. Nevertheless the following result holds true.

Theorem 5.7. *Let $g = (e; A, h)$ be a general square strategic game form and let γ be a faithful representation of g with minimal total rank. Then γ is the symmetrization of an atom of g. Moreover, the symmetrization of a T-atom of g possesses minimal total rank.*

Proof:
1st Step: Let $r := |S_i|$ $(i \in N)$. First of all consider the case that γ is the symmetrization of a T-atom of g. Clearly γ possesses exactly $n!$ (where $n = |N|$) atoms. Now all plays have the same length (i.e. rank of the endpoint) which is $n+1$. In each of the $n!$ atoms there are r^n such paths. Hence the total rank of any of these atoms is $r^n n$. With respect to γ, the corresponding rank originating from each atom is $r^n(n+1)$, because there is an additional edge joining the atom to the root of γ. There are $n!$ atoms, hence the total rank of the graph $(E, \prec)$ of γ is $r^n(n+1)n! = r^n(n+1)!$.
2nd Step: Next, we are going to show that a representation which is faithful has total rank which is at least $r^n(n+1)!$. To this end let γ be a faithful representation of g which has minimal total rank. Without loss of generality it can be assumed that $\mathfrak{N}(\gamma) = g$ hols true. For every $s \in S$ and every $\pi \in \Sigma(N)$ choose $\varphi(\pi, s) = \varphi$ and $\rho(\pi, s) = \rho$ such that (π, φ, ρ) is an automorphism of g and $\varphi^\pi(s) = s$ is satisfied. Let $\tilde{\Theta}$ be defined as in Corollary 3.6. Let $\phi = \phi(\pi, s)$ be an automorphism of the game tree of γ such that (π, ϕ, ρ) is in the inverse image of (π, φ, ρ), i.e., $\tilde{\Theta}(\pi, \phi, \rho) = (\pi, \varphi, \rho)$. The existence of ϕ is guaranteed by faithfulness. Fix a pure strategy σ_0 of chance and let $X^{\sigma_0, s} = (x_0, x_1^s, \ldots, x_{T(s)}^s)$ be the play generated by (σ_0, s). For every permutation π the outcome $\eta(\phi(\pi, s)(x_{T(s)}^s))$ coincides with $h(s)$, because $\varphi(\pi, s)^\pi$ keeps the strategy profile s. Different strategy profiles lead to different outcomes, because g is assumed to be general. Counting the number of strategy profiles and the number of permutations yields $r^n n!$ different plays with endpoints $\phi(\pi, s)(x_{T(s)}^s)$.
The length $T(s)$ of every play is at least $n+1$, because every play intersects an information set of every player and of chance. Indeed, if a player is not involved, then a 'row' of g does not depend on the player's strategy (which is impossible, because g is general). Moreover, the root of γ cannot belong to the information set of some player.
3rd Step: The total rank of $(E, \prec)$ is therefore at least $n! r^n (n+1)$ (recall that the length of each play is at least $n+1$ due to the 2nd step). By minimality of the total rank and the 1st step it follows that the total rank is exactly equal to this number and the root is the only chance move. Hence the restriction of γ to every subtree generated by a successor of the root is an atom and Theorem 3.4 completes the proof. **q.e.d.**

The 'minimal total rank property' will be crucial in the next section. Of course there are many faithful representations of a general square game form; one is presented in the following example.

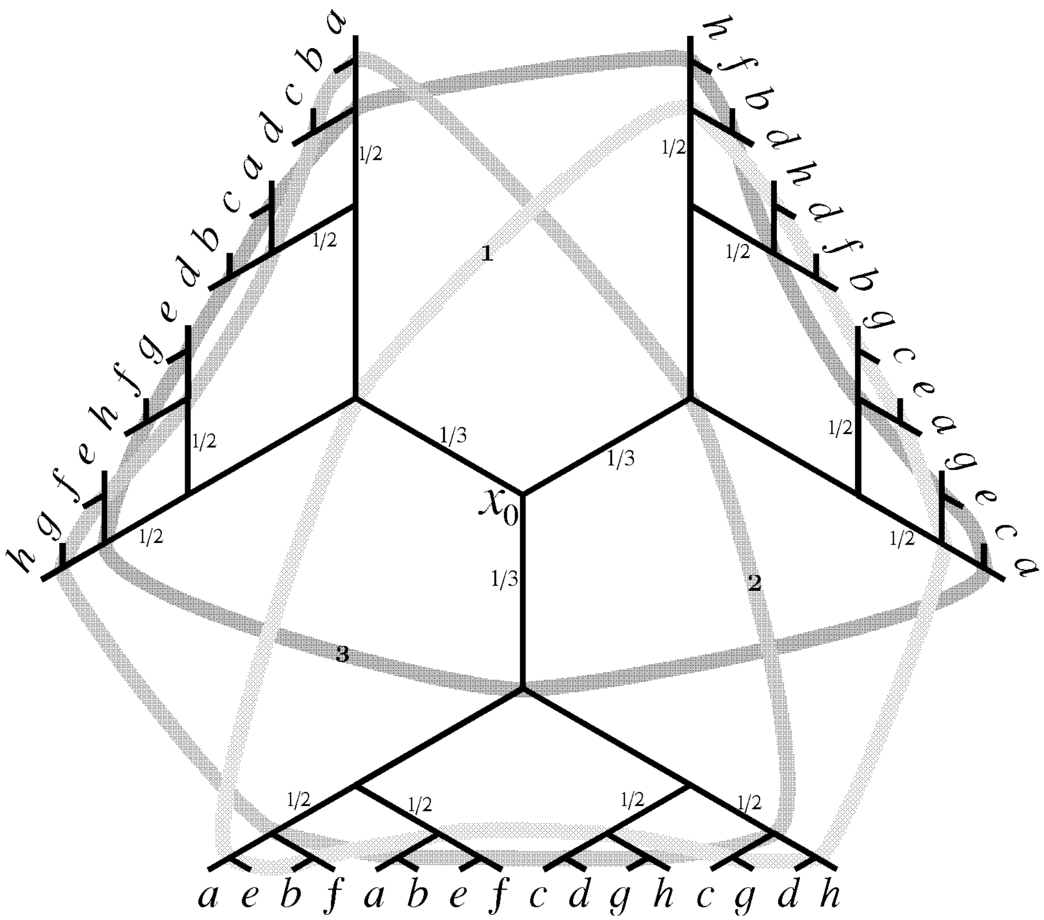

Figure 5.11: A faithful representation with multiple chance moves

Example 5.8. *A faithful representation γ^g of a general square game form $g = (N, S; A, h)$ can be constructed recursively on the number n of players. For $n = 2$ 'the' symmetrization of its atom (see Figure 5.6) is taken. For $n \geq 3$ the faithful representation is constructed as follows. The root x_0 is constructed to be a chance move. It possesses n successors ξ_i belonging to the different player sets and p^{x_0} is uniform distribution. There is a bijection from the set of successors κ_i of ξ_i to player i's strategy set S_i (let us say $\kappa_i \mapsto s_i$). The restriction to the subtree generated by κ_i is the faithful representation $\gamma^{g^{s_i}}$ of the $n-1$ person* ***reduced*** *strategic game form $g^{s_i} = (N \setminus \{i\}, \Pi_{j \neq i} S_j; A, h^{s_i})$, defined by*

$$h^{s_i}((s_j)_{j \neq i}) = h(s_i, (s_j)_{j \neq i}).$$

Of course every player has only one information set. The straightforward proof of faithfulness is left to the reader. Figure 5.11, which sketches the $2 \times 2 \times 2$ case, should be compared with Figures 5.7 and 5.8.

6 The Canonical Representation

Within the previous section we have characterized the symmetrizations of square atoms as the only representations of square strategic games that respect the symmetries and satisfy a minimality condition. Apart from the fact that the result holds true only in the case that all players have the same number of strategies, the assignment of an extensive game form to a given strategic game form is not unique. For the class of square atoms (and their symmetrizations) is still remarkably large: compare e.g. Figure 4.5; here we see various nonisomorphic atoms that are capable of representing a $2 \times 2 \times 2 \times 2$-game.

The final task is, therefore, to introduce symmetrizations of time structured atoms that yield representations in the case of game forms that are not necessarily square. In addition we show that this construction admits of an axiomatically defined unique mapping, the 'canonical' representation.

Definition 6.1. *Let α be a T-atom. An extensive game form $\gamma = (N, E, \prec, \mathbf{P}, \mathbf{C}, p; \iota; A, \eta)$ is a **symmetrization** of α, if the following conditions are satisfied.*

(1) γ is nonmixing.

(2) The root x_0 of γ is the only chance move and p^{x_0} is uniform distribution, i.e., every edge at x_0 has the same probability.

(3) For every $\xi \in C(x_0)$ the restricted game form γ^{ξ} is a T-atom.

(4) For every T-atom β which represents $\mathfrak{N}(\alpha)$ there is a unique $\xi \in C(x_0)$ such that β is IOP isomorphic to γ^{ξ}.

Note that the (unique up to IOP isomorphisms) symmetrization of a T-atom α has $n!$ branches at the root corresponding to as many T-atoms, which all belong to the vNM strategic game form $\mathfrak{N}(\alpha)$.

Generally we will have to accept that a representation can only be defined up to outcome preserving impersonal isomorphisms. On the other hand the variety offered by all atoms is too large. Moreover, we should additionally have faithful representations of nonsquare game forms.

Clearly the preservation of symmetries as formulated so far cannot help in a general nonsquare game, for even in the case of two players there are no symmetries of a general game at all since there are no bijective mappings of the strategy sets. However, as our discussion in Section 0 shows, there are symmetries of restricted versions which should be preserved. Verbally, if two strategies/actions of a player result in the same payoff no matter what his opponents choose to do, then this game is in a well defined sense reducible and the restricted version may well have symmetries the preservation of which should be satisfied by a 'canonical' representation. And if we construct nongeneral game forms with the above property, then the symmetries obtained this way may indeed be used to further reduce the family of representations and hence result in a canonical representation.

Thus, it will be the interplay of restriction and symmetries that characterizes the canonical representation of a strategic game form (minimality assumed). Therefore we shall add the notion of 'consistency' (with respect to restriction) to our requirements concerning representation.

Arbitrary restrictions however, as defined at the beginning of Section 5 cannot be admitted. We shall call a restriction $(E^*, \prec^*, \mathbf{P}^*, \mathbf{Q}^*, \mathbf{C}^*, p^*)$ of a game tree $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ ***proper*** if the root is preserved. Recall that all chance moves together with their choices (and the probabilities) are either fully preserved or completely disappear. The notion is at once extended to game forms and games.

There is a further, more formal obstacle to be tackled before we can reach a rigorous formulation of the 'canonical' representation. This is presented by the aim to precisely define a mapping which represents the choice of a canonical representation. Mappings should be defined on a nice domain - of game forms in our present context. However, if we speak about the 'set of all game forms' we might encounter unpleasant surprises common in elementary set theory, for game forms so far are defined with arbitrary (finite) outcome sets.

More than that, if we look closer, we made no restrictions on the underlying sets of strategies (in a strategic form) neither concerning the elements of the underlying graph (in an extensive form). Thus, when speaking about the set of e.g. strategic games, at the present state of affairs, we will be forced to speak about the set of all finite sets several times.

In order to avoid such footangels we should restrict ourselves to a fixed infinite set $\boldsymbol{U}$ (the ***alphabet*** or ***universe*** of *letters* or *outcomes*) which intuitively first of all is a list of all possible outcomes admitted for game forms (strategic and extensive). I.e., we shall always tacitly assume that for any game form mentioned, the outcome set satisfies $A \subseteq \boldsymbol{U}$; thus the admissible outcome sets are subsets of $\boldsymbol{U}$.

It is no loss of generality to assume in addition that any strategy set S_i mentioned as well as the set of nodes E of a graph involved in our consideration is also a subset of $\boldsymbol{U}$. For the present section we set out under this additional hypothesis.

We feel that this kind of intricacies should be mentioned but not overstressed. Thus, we fix the *set of general strategic game forms* $\boldsymbol{G}$ and *the set of extensive game forms* $\boldsymbol{\Gamma}$ and define a mapping $\mathcal{F} : \boldsymbol{G} \to \boldsymbol{\Gamma}$ always assuming that the outcomes, nodes, strategies ... involved are given by subsets of $\boldsymbol{U}$.

Definition 6.2. *Let* $\mathcal{F} : \boldsymbol{G} \to \boldsymbol{\Gamma}$ *be a mapping.*

(1) $\mathcal{F}$ *is called a* ***representation*** *(of strategic game forms) if, for any* $g \in \boldsymbol{G}$ *it follows that* $\mathcal{F}(g)$ *is a representation of* g *(cf. Definition 2.4 (1)).*

(2) A representation $\mathcal{F}$ *is said to be* ***faithful*** *if it preserves symmetries and respects isomorphisms. More precisely, for any* $g \in \boldsymbol{G}$*, it should follow that* $\mathcal{F}(g)$ *is a faithful*

representation (cf. Definition 2.4 (2)) and whenever g and g' are isomorphic, then so are $\mathcal{F}(g)$ and $\mathcal{F}(g')$.

(3) A faithful representation $\mathcal{F}$ is said to be ***consistent*** *if it respects proper restriction up to impersonal isomorphisms. More precisely, for any $g \in \boldsymbol{G}$ and any restriction $\tilde{g}$ of g it is true that $\mathcal{F}(\tilde{g})$ is IOP isomorphic to a proper restriction of $\mathcal{F}(g)$. That is, '$\mathcal{F}$ and the restriction operation commute'.*

(4) A faithful representation $\mathcal{F}$ is said to be ***minimal****, if for every square $g \in \boldsymbol{G}$, the total rank of $\mathcal{F}(g)$ is minimal.*

(5) A faithful, consistent, and minimal representation is said to be ***canonical****.*

Remark 6.3. *Given our present state of development, we are in the position to construct a canonical representation. To this end, assign to every $g \in \boldsymbol{G}$ the symmetrization of a time structured atom (cf. Definition 6.1). This mapping is not uniquely defined; given $g \in \boldsymbol{G}$, we may apply an impersonal and outcome preserving isomorphism to $\mathcal{F}(g)$ without 'essentially' changing the nature of the mapping thus defined. In this sense a 'time structured' representation is defined uniquely 'up to impersonal outcome preserving isomorphisms'. It is canonical, because a symmetrization of a T-atom of a restricted game form of a strategical game form g is a proper restriction of a symmetrization of a T-atom of g.*

Definition 6.4. *The* ***time structured*** *canonical representation as described by Remark 6.3 is denoted by $\mathcal{T}$.*

Clearly our next aim is to show that $\mathcal{T}$ is 'the' only canonical representation. As it stands now the development in Section 5 and in particular Theorem 5.7 point to symmetrizations of atoms but not necessarily to the time structured version. As a first result we shall now prove that the time structure appears necessarily for general square game forms with at least three strategies for each player.

Theorem 6.5. *Let $g = (N, S; A, h)$ be a square general game form such that $|S_i| = r \geq 3$ $(i \in N)$. Let α be an atom of g. Then the symmetrization of α is a totally rank minimal faithful representation of g, if and only if α is a T-atom.*

Proof: Without loss of generality we assume that $S_i = \{1, \ldots, r\}$ $(i \in N)$ holds true. The atom is denoted by $\alpha = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$, the symmetrization is γ; we may assume without loss of generality that α occurs as some γ^{ξ} in the sense of Definition 5.1 (3).

1st Step: Now we attach labels according to strategies at all nodes of γ except the root and its successors. To this end observe first that ψ_i identifies elements of S_i and of $\tilde{S}_i$ as explained in Definition 2.4. Therefore, if player i is in command at node ξ and chooses $s_i \in S_i$, this leads to a well specified successor ζ of ξ which now carries the label s_i. The process of labeling constitutes a mapping from S_i into the successors of nodes at which player i is in command. This reflection of a strategy by labels is commom in Game Theory and can be viewed in Figure 3.1, where the labels u, d, t, b, l, r are drawn at the corresponding edges for more clarity. We will not formally define the mapping but frequently refer to it.

2nd Step: The labeling induces an identification of plays in γ as well as of all the atoms in γ as follows. First of all any $s \in S$ corresponds to a unique play in α (just follow the labels). Next consider the automorphism (π, id^*) of the preform of g. Here id^* is the natural family of 'identities' $id_i^* : S_i \to S_{\pi(i)}$. To this automorphism there corresponds a unique automorphism (π, ϕ) of the preform of γ (cf. Corollary 3.6). ϕ transforms the play in α labeled by s into some other play carrying the same label. In particular consider $\pi \neq id$ and $s = (1, \ldots, 1)$ or $s = (2, \ldots, 2)$ or etc. Then the second play cannot run through α, because it leads to the same outcome as the first one - but there is exactly one play carrying an outcome in each atom. From this we see immediately that ϕ carries α bijectively to some other atom in γ, say α^π, and that, indeed, $n!$ atoms can be identified by the permutations (id corresponding to α). As the faithful representation of g by γ is totally rank minimal, we conclude in view of Theorem 5.7 that γ has exactly the $n!$ atoms $\alpha^\pi \quad (\pi \in \Sigma(N))$.

3rd Step: We focus the attention on α and recall the definition of $\mathbf{r}$ (cf. Section 1). For $\zeta \in E$ and $\mathbf{r}(\zeta) < t$ let $\mathcal{L}^\zeta(t)$ denote the set of nodes on level t that have ζ as a common ancestor with respect to the completion of $\prec$. Call a level $\mathcal{L}(E, \prec, t)$ ***intact***, if it is contained in some P_i (hence equals P_i) and ***broken*** otherwise. (The $n-$th level is broken!) We introduce

$$\bar{t} = \min\{t \mid \mathcal{L}(E, \prec, t) \text{ is broken}\}.$$

It is our final aim to show that $\bar{t}$ equals n. Assume, on the contrary, that $\bar{t} < n$ holds true. Let ζ be such that

(1) $\mathcal{L}^\zeta(\bar{t}) \not\subseteq P_i \ (i \in N)$ and $\mathbf{r}(\zeta) < \bar{t}$

(2) the rank $\mathbf{r}(\zeta)$ is maximal with respect to (1).

Then every successor of ζ is the common ancestor of all nodes of $\mathcal{L}^\xi(\bar{t})$. By maximality of $\mathbf{r}(\zeta)$ the set $\mathcal{L}^\xi(\bar{t})$ belongs to one player set. We now claim that for different $\xi, \xi' \in C(\zeta)$ the corresponding $\mathcal{L}^\xi(\bar{t})$ and $\mathcal{L}^{\xi'}(\bar{t})$ belong to different player sets, hence at least r players are in command on level $\bar{t}$. This claim will be confirmed in the 4th Step. Figure 6.1 indicates the procedure to be followed during the remaining steps of the proof.

4th Step: To this end let $\xi_1, \xi_2, \xi_3 \in C(\zeta)$ be different successors of ζ (recall that $r = |S_i| \geq 3$). Assume without loss of generality that $\mathcal{L}^{\xi_1}(\bar{t}) \subseteq P_1$ and $\mathcal{L}^{\xi_2}(\bar{t}) \subseteq P_2$. It suffices to show that $\mathcal{L}^{\xi_3}(\bar{t})$ is not contained in P_1. Let $\mathcal{L}^{\xi_3}(\bar{t})$ be contained in P_i. The player who is in command at node ζ assigns two labels to ξ_2 and ξ_3 as described in the first step; let these labels be s^2 and s^3 respectively. There is an impersonal automorphism of the preform of g which just transposes s^2 and s^3. To this automorphism there exists the corresponding automorphism of the preform of γ given by Corollary 3.6; we call it

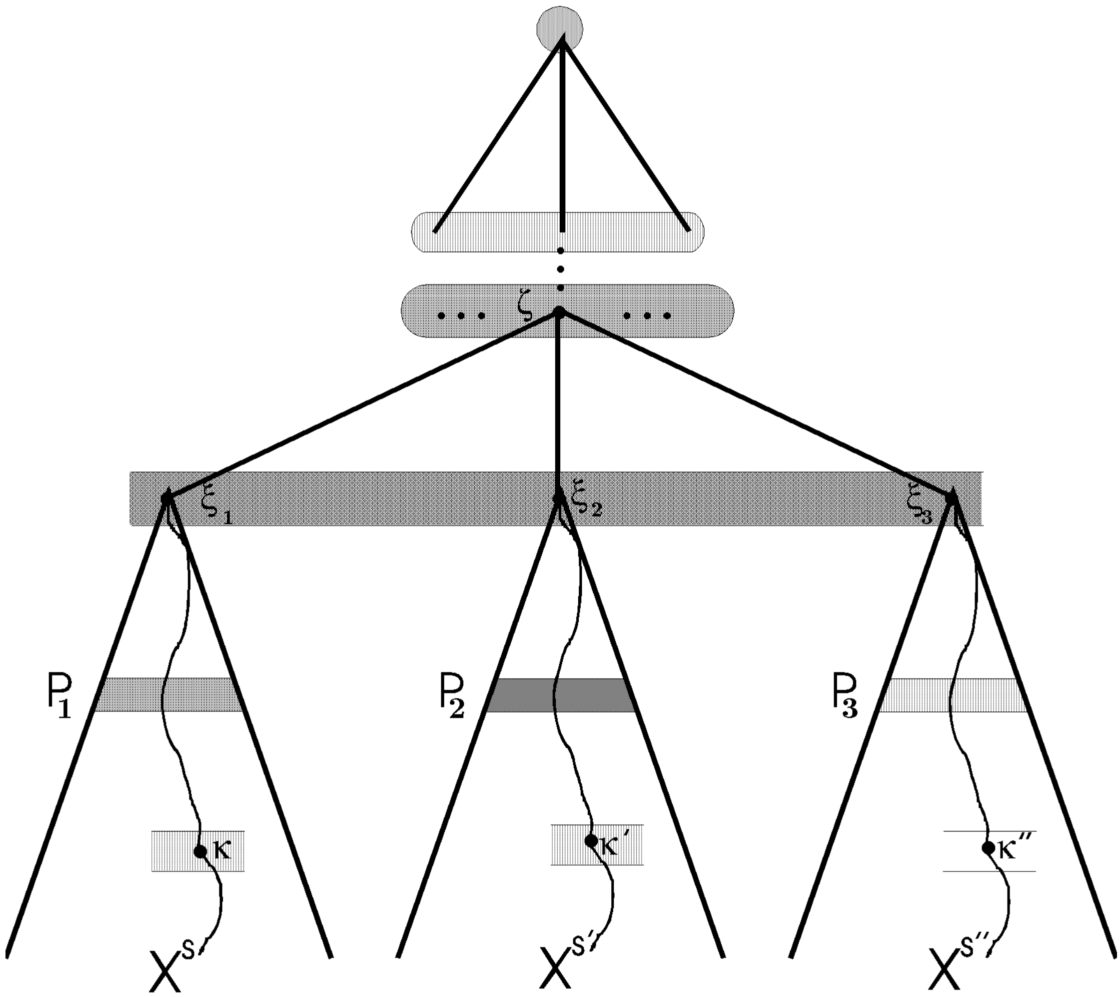

Figure 6.1: The preform of α

(id, ϕ^{23}). In view of the fact that automorphisms transfer atoms to atoms, we can single out an atom which is the image of α under ϕ^{23}. By the 2nd Step this atom is of the form α^{π} for some $\pi \in \Sigma(N)$. We refer to the levels within α^{π} by subscript $_{\pi}$.

Again consider the successor ξ_3 of ζ. We have

$$\mathcal{L}_{\pi}^{\phi^{23}(\xi_3)}(\bar{t}) = \phi^{23}\mathcal{L}^{\xi_3}(\bar{t}) \subseteq \phi^{23}P_i = P_i, \tag{6.1}$$

because ϕ^{23} respects the ancestor relation and (id, ϕ^{23}) is impersonal. On the other hand, $\phi^{23}(\xi_3)$ carries the label s^2, because ξ_3 carries the label s^3 and ϕ^{23} has been constructed according to the transposition of s^2 and s^3. Hence $\mathcal{L}_{\pi}^{\phi^{23}(\xi_3)}(\bar{t})$ has to be a subset of $P_{\pi(2)}$. From this and from formula (6.1) we conclude that $\pi(2) = i$.

Next perform the same operation for the successor ξ_2 of ζ in order to show that $\pi(i) = 2$ holds true as well. And, if ξ_1 is considered, it follows that $\pi(1) = 1$. Clearly this shows that $i \neq 1$ is true. Hence we have proved the claim raised at the end of the 3rd Step.

5th Step: Without loss of generality we assume $i = 3$. Player 3 appears the first time on level $\bar{t}$ (in α), as all previous levels are intact. Therefore ξ_1 is the ancestor of some $\kappa \in P_3$ with $\mathbf{r}(\kappa) > \bar{t}$. Let ϕ^{12} be generated by the exchange of ξ_1 and ξ_2 analogously to the construction of ϕ^{23} by the exchange of ξ_2 and ξ_3 in the 4th Step. The corresponding atom is $\alpha^{\pi^{12}}$ (as $\alpha^{\pi} = \alpha^{\pi^{23}}$ was specified above). Analogously to the 4th Step we conclude that $\pi^{12}(1) = 2, \pi^{12}(2) = 1$, and $\pi^{12}(3) = 3$. Let $s \in S$ be a strategy profile which generates a play X^s in α passing through ξ_1 and κ (use the labeling of the 2nd Step). The automorphism (id, φ^{12}) that induces (id, ϕ^{12}) via Θ throws s into some s' and, hence, specifies a play $X^{s'}$ in α. On level $\mathbf{r}(\kappa)$ we find exactly one node κ' on $X^{s'}$. The labeling s corresponds to $\phi(X^{s'})$ in $\alpha^{\pi^{12}}$. As $\pi^{12}(3) = 3$, the play corresponding to label s in $\alpha^{\pi^{12}}$ will pass through P_3 on level $\mathbf{r}(\kappa)$, i.e. $\phi(\kappa') \in P_3$. Therefore $\kappa' \in P_3$ holds as well. Note that $\kappa' \in \mathcal{L}^{\xi_2}(r(\kappa))$ due to the construction of s'.

6th Step: The same procedure argued with the automorphism ϕ^{13} (constructed analogously again) is now applied to the play X^s in α. The play X^s passes through P_1 first and reaches P_3 at κ, Because κ is a member of X^s and ξ_1 is an ancestor of $\kappa \in P_3$. We conclude that the play $X^{s''}$ (obtained by using φ^{13}) passes the level $\mathbf{r}(\kappa)$ at some node κ'' which is an element of P_1. By interchanging the rôles of player 1 and 2 (i.e., application of ϕ^{23} to κ' or $X^{s'}$ respectively) we have to conclude that $\kappa'' \in P_2$. This is a contradiction which shows that the assumption $\bar{t} < n$ raised in the 3rd Step cannot be true. Hence α is time structured. **q.e.d.**

The main theorem can now be stated as follows.

Theorem 6.6. *There is a unique (i.e. up to impersonal outcome preserving isomorphisms) canonical representation of strategic games (over a given universal alphabet) and this is the time structured mapping $\mathcal{T}$.*

Proof:

As we have seen in Remark 6.3 the mapping $\mathcal{T}$ has the desired properties (of course $\mathcal{T}$ again is only defined up to impersonal outcome preserving isomorphisms). Thus it remains to show uniquenes.

To this end, let $\mathcal{F}$ be a representation enjoying the desired properties. Fix a strategic game $g = (N, S; A, h) \in \boldsymbol{G}$. Let $S^* = \prod_{i \in N} S_i^*$ be such that $S_i^* \supseteq S_i$ yields $|S_i^*| = r \geq \max_{j \in N} |S_j|$ $(i \in N)$ for some $r \geq 3$ and let $g^* = (N, S^*; A^*, h^*) \in \boldsymbol{G}$ be such that g is a restriction of g^*. The existence of $g^* \in \boldsymbol{G}$ is ensured by the choice of $\boldsymbol{U}$ which renders $\boldsymbol{G}$ to be sufficiently large. By Theorem 6.5 it follows that $\mathcal{F}(g^*) =: \gamma^* = \mathcal{T}(g^*)$ holds true (up to an IOP isomorphism). Let (id, ψ^*, id) be the IOP isomorphism between g^* and $\mathfrak{N}(\mathcal{T}(g^*))$. This automorphism in particular carries the subset S of strategies available in g into the strategies available for $\mathfrak{N}(\mathcal{T}(g^*))$, called $\tilde{S}^*$. We now define an extensive game form γ with the aid of γ^* and ψ^* : All we have to do is to take all plays of γ^* that are images of strategies $s \in S$ under ψ^*, i.e. all plays generated by $\psi^{id}(S) \subseteq \tilde{S}^*$. (This amounts to taking all plays X^s in all atoms α^{π} of γ^* as discussed in the proof of Theorem 6.5.)

The nodes of γ^* obtained by persuing all these plays together with the obvious binary relation constitute a tree to which all further data of γ^* may be restricted in the obvious way. The time structured nature of a symmetrization which is characteristically for γ^* allows for an easy verification of the fact that the restriction is, indeed, proper. Call the resulting extensive game form γ. As $\mathcal{T}$ respects proper restriction it follows that γ is a faithful representation of g. However, consistency applies as well for $\mathcal{F}$, hence $\mathcal{F}(g)$ is IOP isomorphic to $\mathcal{T}(g)$. **q.e.d.**

Remark 6.7. *Definition 6.2 can be generalized to mappings* $\mathcal{F} : \boldsymbol{H} \rightarrow \boldsymbol{\Gamma}$ *for any subset* $\boldsymbol{H} \subseteq \boldsymbol{G}$ *of general strategic game forms without changes; however* $\boldsymbol{H}$ *has to comply with a few additional requirements. This is so because a set* $\boldsymbol{H}$ *which is too small may not allow for sufficiently many games, thus the existence requirement of Definition 6.2 (3) could be damaged. To avoid this possibility, call* $\boldsymbol{H}$ ***hereditary****, if every restriction of a game form of* $\boldsymbol{H}$ *belongs to* $\boldsymbol{H}$*. For a hereditary* $\boldsymbol{H}$ *the time structured representation* $\mathcal{T}$ *restricted to* $\boldsymbol{H}$ *is clearly canonical. Uniqueness can be guaranteed (repeat the proofs of Theorems 6.5 and 6.6) provided the following condition is satisfied.*

$$\begin{aligned}&\textit{For any } g = (N, S; A, h) \in \boldsymbol{H} \textit{ there is a square strategic game}\\&\textit{form } g^* = (N, S^*; A^*, h^*) \in \boldsymbol{H} \textit{ such that } g \textit{ is a restriction of } g^*.\end{aligned} \tag{6.2}$$

The remainder of this section is devoted to an example which shows that (6.2) cannot be dropped as a prerequisite of uniqueness.

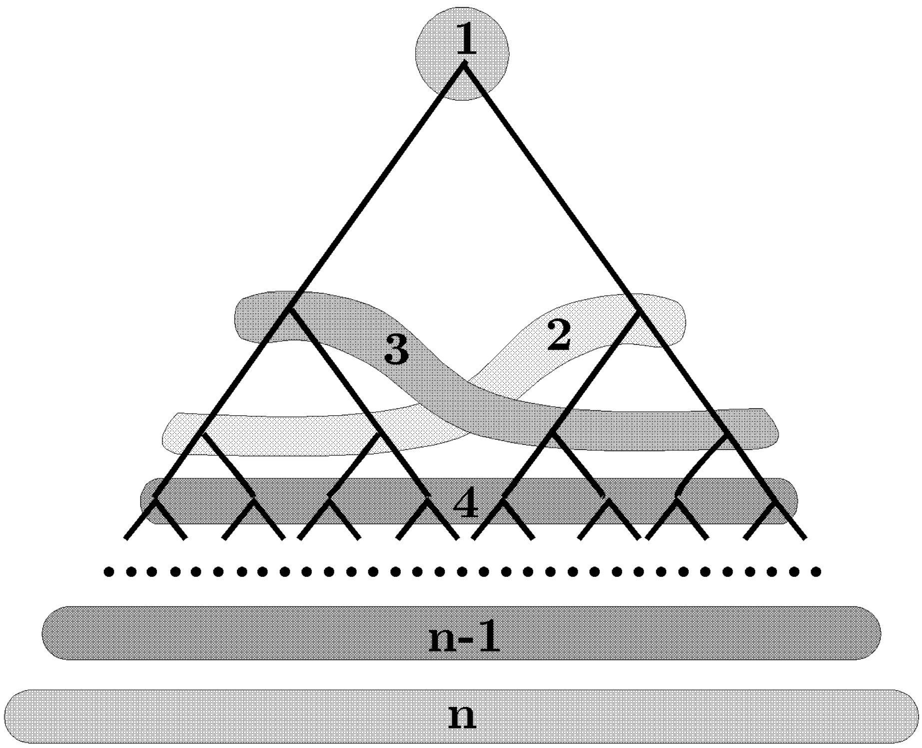

Figure 6.2: The cross over example

Example 6.8. *Let* $n \geq 3$ *and* $\boldsymbol{H}$ *be a hereditary subset of* $\boldsymbol{G}$ *consisting of game forms with strategy sets of cardinality* 1 *or* 2. *For* $N = \{1, \ldots, n\}$ *and* $(N, S; A, h) \in \boldsymbol{H}$ *satisfying* $|S_i| = 2 \quad (i \in N)$, *define the atom* α *as follows. The nodes of* E *are given by* $\{(t,l) | 0 \leq t \leq n, 1 \leq l \leq 2^t\}$. *The player sets are specified via* $(t,l) \in P_{t+1} \quad (t = 0$ *or* $3 \leq t < n); (1,l) \in P_{l+1}; (2,l) \in P_3 \quad (l \leq 2)$ *and* $(2,l) \in P_2 \quad (l \leq 3)$. *The choices are indicated in Figure 6.2.*

Let $\mathcal{F} : \boldsymbol{H} \to \boldsymbol{\Gamma}$ *be the mapping that assigns the symmetrization of* α *to* g *and is arranged consistently otherwise. This mapping is canonical. The clue is found by an inspection of Figure 6.2 and of the proof of Theorem 6.5. As the 3rd strategy is missing, the overcrossing of player sets* P_2 *and* P_3 *cannot be avoided by the construction supplied in the 5th Step.*

References

[1] van Damme, E. (1987). *Stability and Perfection of Nash Equilibria.* Springer-Verlag, Berlin.

[2] Harsanyi, J.C. and Selten, R. (1988). *A General Theory of Equilibrium Selection in Games.* MIT Press, Cambridge, Mass.

[3] Kohlberg, E. and Mertens, J.-F. (1986). 'On the Strategic Stability of Equilibria'. *Econometrica* **54**, 1003-1037.

[4] Myerson, R.B. (1991). *Game Theory.* Harvard University Press, Cambridge, Mass.

[5] von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior.* Princeton University Press, Princeton.

[6] Peleg, B., Rosenmüller, J., and Sudhölter, P. (1996). *The Canonical Extensive Form of a Game Form: Part I - Symmetries.* WP 253, Institute of Mathematical Economics, University of Bielefeld, Germany

[7] Rapoport, A. (1994). 'Order of Play in Strategically Equivalent Games in Extensive Form'. University of Arizona, Tucson, AZ 85721.