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On the Interaction between Heterogeneity and Decay in Undirected Networks*

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Abstract

In this paper, we examine the role played by heterogeneity in the connection model. In sharp contrast to the homogeneous cases we show that under heterogeneity involving only two degrees of freedom, all networks can be supported as Nash or efficient. Moreover, we show that there does not always exist Nash networks. However, we show that on reducing heterogeneity, both the earlier "anything goes" result and the non existence problem disappear.

JEL Classification: C72, D85.

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1 Introduction

Galeotti, Goyal and Kamphorst (2006, [3]) consider the case of small levels of decay with heterogeneous players. Their insider-outsider model only has two groups of players, and thus two possible values of c. The notion of Nash networks was introduced by Bala and Goyal

(2000, [1]).[1] In their paper (2000, [1]), they study two frameworks. In the first one, links in the network never fail, and always transmit all information reliably. Given that link formation is costly, the authors find that Nash networks are always minimally connected. In the second one, Bala and Goyal introduce link imperfections in the form of information decay whereby direct links convey more information than indirect links.

In a recent paper in *Games and Economic Behavior*, Galeotti, Goyal and Kamphorst (2006, [3]) examine heterogeneity in Nash networks without taking any link imperfections into account. Their results are similar in spirit to those of Bala and Goyal (2000, [1]) in the sense that equilibrium networks now have components that are minimally connected. Heterogeneity in the presence of link imperfections has been analyzed by Haller and Sarangi (2005, [4]) who find that the homogeneity of the parameters plays a significant role in the two widely divergent results of Bala and Goyal (2000, [1], [2]). Haller and Sarangi (2005, [4]) allow different links to have different success probabilities and find that for any network $\boldsymbol{g}$, there exists a set of parameter values under which $\boldsymbol{g}$ is Nash – the model with heterogeneity can encompass the results of *both* Bala and Goyal papers.

In this paper we examine different possible heterogeneous Nash network formulations using the popular "connections model" introduced by Jackson and Wolinsky (1996, [5]) and studied extensively by Bala and Goyal (2000, [1], [2]). In the typical model players are endowed with some information which can be accessed by other players forming links with them. Link formation is costly and the cost of establishing a link is incurred by the initiating player. In these models heterogeneity manifests itself in the payoff function and can occur through three different variables: (i) the value of information held by players, (ii) the rate at which information decays or loses value as it traverses the network, and (iii) the cost of forming a link. Thus by introducing heterogeneity and decay we are able to generalize the results of Bala and Goyal (2000, [1]) where heterogeneity is not taken into consideration.

We focus on the two-way flow models introduced by Bala and Goyal (2000, [1]). The two-way flow model allows bi-directional flow of information through a link regardless of who

[1]In the following, we use Nash networks to refer to networks that satisfy Nash equilibrium as stability concept instead of Jackson and Wolinsky's (1996, [5]) notion of pairwise stability.

establishes it. Here we examine Nash networks, efficient networks and the existence of strict Nash networks under different possible heterogeneous frameworks arising from combinations of the above variables. Our main results can be summarized as follows: in models with decay and heterogeneity it is possible to support any network as a strict Nash or efficient network. Moreover, the existence of Nash networks can fail in situations where we have decay and heterogeneity together.

The paper is organized as follows. In Section 2, we present the model setup. Section 3 contains results about models that incorporate heterogeneity and decay. In Section 4 we discuss the relationship between the probabilistic models and the decay models.

2 Model Setup

In this section we define the formal elements of the strategic form network formation game. Let $N = \{1, \dots, n\}$, $n \geq 3$, denote the set of with generic elements i, j, k. For ordered pairs $(i, j) \in N \times N$, the shorthand notation $i\,j$ is used and for non-ordered pairs $\{i, j\} \subset N$ the shorthand $[i\,j]$ is used.

Strategies. For player i a pure strategy is a vector $\boldsymbol{g}_i = (g_{i,1}, \dots, g_{i,i-1}, g_{i,i+1}, \dots, g_{i,n}) \in \{0, 1\}^{n-1}$. Since our aim is to model network formation, $g_{i,j} = 1$ implies that there exists a direct link between i and j initiated by player i, whereas $g_{i,j} = 0$ means that i does not initiate this link. Regardless of what player i does, player j can always choose to initiate a link with i or set $g_{j,i} = 0$. Here we focus only on pure strategies. The set of all pure strategies of player i is denoted by $\mathcal{G}_i$ and consists of 2^{n-1} elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. Note that there is a one-to-one correspondence between $\mathcal{G}$ and the set of all directed graphs or networks with vertex set N. Namely, to a strategy profile $\boldsymbol{g} = (\boldsymbol{g}_1, \dots, \boldsymbol{g}_n) \in \mathcal{G}$ corresponds the graph $(N, E(\boldsymbol{g}))$ with edge set $E(\boldsymbol{g}) = \{(i, j) \in N \times N |\, i \neq j,\ g_{i,j} = 1 \text{ and } g_{i,i} = 0\}$. To simplify the notation, we denote by $i\,j$ the link formed by player i with j. In the sequel, we identify a joint strategy $\boldsymbol{g}$ by its corresponding graph and use the terminology directed graph or directed network $\boldsymbol{g}$ for it.

Payoffs. Payoffs of player i are given by the difference between benefits $B_i(\boldsymbol{g})$ and costs $c_i(\boldsymbol{g})$.

Hence the payoff of player i in network $\boldsymbol{g}$ is given by

$$u_i(\boldsymbol{g}) = B_i(\boldsymbol{g}) - c_i(\boldsymbol{g}). \tag{1}$$

Next we define various types of heterogeneity in networks by introducing different cost and benefit formulations.

(i) Link Costs. Players incur costs only for the direct links they establish. The cost of each link is assumed to be the same and the cost of forming links for player i is given by:

$$c_i(\boldsymbol{g}) = \sum_{j \neq i} g_{i,j} c \tag{2}$$

In this paper we focus only on homogeneous costs. Note that in our context heterogeneous costs would only increase the set of potential strict Nash networks and would weaken the possibility of existence of Nash networks.

(ii) Link Benefits. Decay models were introduced by Jackson and Wolinsky (1996, [5]) under the name of the "connections model". In such models links always transmit the information, but information acquired through indirect links is less valuable. Since information loses value as it travels along a sequence of links it captures the idea that "it is better to have the facts straight from the horse's mouth". In the Nash networks setting decay models were analyzed by Bala and Goyal (2000, [1]) who assumed that the value of information, the costs of link formation, and the decay parameter were identical across all players and links. In other words, they analyzed the case of *homogeneous decay.* We propose two different frameworks to study the interaction between heterogeneity and decay.

A link between players i and j allows for **two-way flow of information**. So the benefits from network $\boldsymbol{g}$ are derived from its closure $\overline{\boldsymbol{g}} \in \mathcal{G}$, defined by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for $i \neq j$. Moreover, since information is acquired through direct and indirect links we say information flows from player j to player i, when i and j are linked by means of a path in $\overline{\boldsymbol{g}}$. A **path** of length m in $\boldsymbol{g} \in \mathcal{G}$ from player i to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$, and $g_{i_k, i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let $\mathcal{C}_{i,j}(\boldsymbol{g})$ be the set of paths from j to i in the network $\boldsymbol{g}$, and let $C_{i,j}(\boldsymbol{g})$ be a typical element of $\mathcal{C}_{i,j}(\boldsymbol{g})$. We denote by $N_i(\boldsymbol{g}) = \{j \in N |\ j \neq i,$ there exists a path in $\overline{\boldsymbol{g}}$ between i and $j\}$, the set of other players whom i can access or "observe" in network $\boldsymbol{g}$. Information received from j is worth $V_{i,j}$ to player i. Therefore, player i's benefits from a network $\boldsymbol{g}$ is given by:

Note that $\overline{\boldsymbol{g}}$ belongs to the set $\mathcal{H} = \{\boldsymbol{h} \in \mathcal{G} | h_{i,j} = h_{j,i} \text{ for } i \neq j\}$. There is a one-to-one correspondence between the elements of $\mathcal{H}$ and the non-directed networks with node set N. Namely, for $\boldsymbol{h} \in \mathcal{H}$ and $i \neq j$, $[i\ j]$ is an edge of the corresponding non-directed network if and only if $h_{i,j} = h_{j,i} = 1$. In what follows, we identify $\boldsymbol{h}$ with the corresponding non-directed network. Hence, the notation $[i\ j] \in \boldsymbol{h}$ stands for "$[i\ j]$ is an edge of $\boldsymbol{h}$". Also, for $\boldsymbol{k} \in \mathcal{H}$, $\boldsymbol{k} \subset \boldsymbol{h}$ means that $\boldsymbol{k}$ is a subnetwork of $\boldsymbol{h}$.

Decay with Heterogeneous Players. Here we use the homogeneous decay assumption in conjunction with the heterogeneous players framework of Galeotti, Goyal and Kamphorst (2006, [3]), i.e., we assume that there exists $(i, j) \neq (k, \ell)$ such that $V_{i,j} \neq V_{k,\ell}$. Then the benefits function can be written as:

$$B_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \delta^{d_{i,j}(\boldsymbol{g})} V_{i,j} \tag{3}$$

where δ is the decay parameter and $d_{i,j}(\boldsymbol{g})$ is the distance in the shortest path between i and j in $\boldsymbol{g}$.[2] Let $V^m = \max_{(i,j) \in N \times N}\{V_{i,j}\}$ and $V^m = \min_{(i,j) \in N \times N}\{V_{i,j}\}$.

Decay with Heterogeneous Links. Here we assume that decay associated with the link $[i\ j]$ is not identical to decay associated with the link $[\ell\ k]$ for $[\ell\ k] \neq [i\ j]$. This assumption captures the fact that the quantity of information a link can convey is not the same across all links under decay. In other words, some channels of information or paths are "better" than others.

We measure decay associated with a link $[i\ j]$ by the parameter $\delta_{i,j} \in (0, 1)$. Given a network $\boldsymbol{g}$, it is assumed that if player i has a link with player j, then she receives information of value $\delta_{i,j}$ from j. For this model we retain the symmetry assumption, that is $\delta_{i,j} = \delta_{j,i}$. Without loss of generality we assume that the value of a link is $V = 1$. The benefits of player i in the network $\boldsymbol{g}$ is then given by:

$$B_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} \left(\prod_{[\ell\ k] \in C^*_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right), \tag{4}$$

[2]In Bala and Goyal (2000, [1]), it is assumed that players can always access their own information.

where $C^*_{i,j}(\boldsymbol{g}) = \arg\max_{C_{i,j}(\boldsymbol{g}) \in \mathcal{C}_{i,j}(\boldsymbol{g})} \left\{ \prod_{[\ell\ k] \in C_{i,j}(\boldsymbol{g})} \delta_{\ell,k} \right\}$.

Note that this expression fundamentally differs from the previous one because it does not use the geodesic distance between players to determine the value of information obtained. Let $\delta^M = \max_{(i,j) \in N \times N}\{\delta_{i,j}\}$ and $\delta^m = \min_{(i,j) \in N \times N}\{\delta_{i,j}\}$.

Network Definitions. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network that remains when all of player i's links have been removed. Clearly, $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$, where the symbol $\oplus$ indicates that $\boldsymbol{g}$ is composed of the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$ (similarly the symbol $\ominus$ is used to indicate removal of links). A strategy $\boldsymbol{g}_i$ is a *best response* of player i to $\boldsymbol{g}_{-i}$ if

$$u_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq u_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

Let $\mathcal{BR}_i(\boldsymbol{g}_{-i})$ denote the set of player i's best responses to $\boldsymbol{g}_{-i}$. A network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n)$ is said to be a *Nash network* if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$. A *strict Nash network* is a network where all players are playing a strict best response.

A network $\boldsymbol{g}$ is *efficient* if the total utility of players is maximum, that is $W(\boldsymbol{g}) = \sum_{i=1}^n u_i(\boldsymbol{g}) \geq \sum_{i=1}^n u_i(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{G}$.

Graph-theoretic Concepts. A network $\boldsymbol{g}$ is called a *star* if there is a vertex i_s, such that for all $j \neq i_s$, $\max\{g_{i_s,j}, g_{j,i_s}\} = 1$ and for all $k \notin \{i_s, j\}$, $\max\{g_{k,j}, g_{j,k}\} = 0$. Moreover a star, where $g_{i_s,j} = 1$ for all $j \neq i_s$ is a *center-sponsored star*, and a star, where $g_{i_s,j} = 0$ for all $j \neq i_s$, is a *periphery-sponsored star.* A network $\boldsymbol{g}$ is *connected* if there is a path in $\overline{\boldsymbol{g}}$ between all players $i, j \in N$. A network $\boldsymbol{g}$ is *minimally connected* if it is connected and for all $i\ j \in E(\boldsymbol{g})$, $\boldsymbol{g} \ominus i\ j$ is not connected. Finally, a network $\boldsymbol{g} \in \mathcal{G}$ is *essential* if $g_{i,j} = 1$ implies $g_{j,i} = 0$. Note that if $\boldsymbol{g} \in \mathcal{G}$ is a Nash network, then it must be essential. This follows from the fact that each link is costly, but allows for two-way flow of information regardless of who initiates (and pays) for the link.

3 Models with Decay

In this section we investigate Nash networks and examine efficient networks in models with decay. To begin with, we illustrate that parameter through which heterogeneity is introduced in the model of decay can have different implications. The next example shows that there are subtle differences between the model of heterogeneous players and heterogeneous links.

Example here

3.1 Decay with Heterogeneous Players

In this section we obtain two main results. First, we demonstrate that all networks can be supported as strict Nash and efficient. Next, we show that there exist parameter values for which there is no Nash network in pure strategies.

Theorem 1 *Let $\boldsymbol{g}$ be an essential network. If the benefits function satisfies equation (3), then there exist a link cost $c > 0$ and an array $\boldsymbol{V} = [V_{i,j}]$ of values such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game. Moreover this network is also strict Nash.*

Proof. We prove successively the two parts of the proposition.

1. Suppose $\boldsymbol{g}$ is an essential network. Let $V^1 = 1$, $c = (n - 3/2)/n^2$, $\delta = 1/n$, $V^0 = 1/(3n)$. We construct a symmetric $n \times n$-matrix $[V_{i,j}]$ of value as follows. If $i \neq j$ and i and j are linked, i.e. $g_{i,j} = 1$ or $g_{j,i} = 1$ set $V_{i,j} = V^1$. Otherwise set $V_{i,j} = V^0$. Now consider $i \neq j$. Let $g_{i,j} = 0$. Then, either $g_{j,i} = 1$ or $g_{j,i} = 0$. In the first case, agent i receives zero marginal benefits but incurs an additional positive cost when forming the link $i\,j$. It follows that $g_{i,j} = 0$ is the unique optimal choice for i given $\boldsymbol{g}_{-i}$. For $g_{j,i} = 0$, $V_{i,j} = V^0 = 1/(3n)$. If player i forms a link with j, then she obtains at most marginal benefits equal to $\delta V^0 + (n-2)\delta^2 V^1$, that is $1/(3n^2) + (n-2)/n^2$. We show that $\delta V^0 < c - (n-2)\delta^2 V^1$. We have $\delta V^0 = 1/(3n^2) < 1/(2n^2) = (n-3/2)/n^2 - (n-2)/n^2 = c - (n-2)\delta^2 V^1$. Therefore regardless of other links, not initiating the link $i\,j$ is optimal for agent i. Now let $g_{i,j} = 1$. Then by essentiality of $\boldsymbol{g}$, $g_{j,i} = 0$. Further, $V_{i,j} = V^1$. If player i removes the link $i\,j$, then she obtains at most a payoff equal to $\delta^2 V_{i,j}$ from player j. It follows that due to the link $i\,j$ player i obtains marginal benefits equal to at

least $V^1(\delta - \delta^2) = 1/n - 1/n^2 = (n-1)/n^2 > (n - (3/2))/n^2 = c$. Therefore regardless of other links, player i has no incentive to remove the link $i\,j$.

2. Suppose $\boldsymbol{g}$ is an essential network. Let $V^1 = 1$, $c = (n^4 + n^2 + 1)/(2n^8)$, $\delta = 1/n^4$, $V^0 = 1/n^4$. We construct a symmetric $n \times n$-matrix $[V_{i,j}]$ of value as follows. If $i \neq j$ and i and j are linked, i.e. $g_{i,j} = 1$ or $g_{j,i} = 1$ set $V_{i,j} = V^1$. Otherwise set $V_{i,j} = V^0$. Now consider $i \neq j$. Suppose $g_{i,j} = 0$ and $g_{j,i} = 0$ (if $g_{j,i} = 1$, then the proof is straightforward). If player i forms a link with j, then the players obtain total marginal benefits bounded by $2\delta(V^0) + \delta^2 n^2 V^1 = 2/(n^8) + 1/n^6 = (2 + n^2)/n^8 < (n^4 + n^2 + 1)/(2n^8) = c$, for all $n \geq 3$. Therefore regardless of other links, not initiating the link $i\,j$ is optimal. Further, $V_{i,j} = V^1$. If player i removes the link $i\,j$, then she obtains at most a payoff equal to $\delta^2 V_{i,j}$ from player j. Due to the link $i\,j$, player i obtains marginal benefits equal to at least $V^1(\delta - \delta^2) = 1/n^4 - 1/n^8 = (n^4 - 1)/n^8 > (n^4 + n^2 + 1)/(2n^8) = c$, for all $n \geq 3$. Since all values are positive, other players obtain a non negative payoff from this link. Therefore regardless of other links, maintaining the link $i\,j$ is optimal for agents.

□

With homogenous players, Nash networks are either empty or connected (see Bala and Goyal, 2000, [1]). With heterogenous values of players it is possible to obtain this result when the values of players are sufficiently close.

Proposition 1 *Suppose benefits function satisfies equation (3) and that* $V^M - V^m < \delta V^m/(1 + (n-3)\delta)$. *Then a strict Nash network is either empty or connected.*

Proof. Let $D^*(\boldsymbol{g}, i\,j)$ be the set of players $\ell \in N \setminus \{i, j\}$ such that the shortest chain between i and ℓ goes through the link $i\,j$ in $\boldsymbol{g}$. Consider a strict Nash network $\boldsymbol{g}$. Suppose $\boldsymbol{g}$ is neither empty nor connected. Then there exists three agents i, j and k such that i and j belong to one connected component C_1 and k belongs to a different component C_2 in $\boldsymbol{g}$. Moreover, wlog let $g_{i,j} = 1$. Then the incremental benefits to player i of having the direct link to j is given by:

$$\begin{aligned} \mathcal{A} &= \delta V_{i,j} + \textstyle\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \left(\delta^{d_{i,\ell}(\boldsymbol{g})} - \delta^{d_{i,\ell}(\boldsymbol{g} \ominus i\, j)}\right) V_{i,\ell} \\ &\leq \delta V_{i,j} + \textstyle\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V_{i,\ell} \\ &\leq \delta V^M + \textstyle\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V^M \end{aligned}$$

with the convention $\delta^{d_{i,\ell}(\boldsymbol{g} \ominus i\, j)} = 0$, if $\ell \notin N_i(\boldsymbol{g} \ominus i\, j)$. Clearly, we have $\mathcal{A} \geq c$.

If player k forms a link with player j, then the incremental benefits to player i of having the direct link to j is:

$$\begin{aligned} \mathcal{B} &= \delta V_{k,j} + \delta^2 V_{k,i} + \textstyle\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V_{k,\ell} \\ &\geq \delta V^m + \delta^2 V^m + \textstyle\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V^m. \end{aligned}$$

It is worth noting that:

$$\sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V^M - \sum_{\ell \in D^*(\boldsymbol{g}, i\, j)} \delta^{d_{i,\ell}(\boldsymbol{g})} V^m \leq (n-3)\delta^2 (V^M - V^m).$$

Since $V^M - V^m < \delta V^m / (1 + (n-3)\delta)$, we have $\mathcal{B} > \mathcal{A} \geq c$. It follows that player k has a incentive to form a link with j and $\boldsymbol{g}$ is not strict Nash. $\square$

From the above proposition it follows that in the homogeneous parameter model with decay not every network can be supported as a strict Nash network.

Polar cases. We now deal with some familiar architectures cases. More precisely, we give conditions which allow to obtain the complete network the empty network and the star networks as strict Nash networks.

Proposition 2 *Suppose benefits function satisfies equation (3).*

1. *If $\delta V^M < c$, then the empty network is strict Nash*
2. *If $(\delta - \delta^2) V^m > c$, then the complete network is strict Nash.*
3. *If $\delta V^m > c$ and $(\delta - \delta^2) V^M < c$, then any center-sponsored star is a strict Nash network. Moreover, if $(n-1)\delta V^m > c$ and $(\delta - \delta^2) V^M < c$, then any periphery-sponsored star is a strict Nash network.*

Proof. We prove successively the three parts of the proposition.

1. If $\delta V^M < c$, then it will not be worthwhile for any agent i to form a link. Hence the empty network is Nash.

2. If $(\delta - \delta^2)V^m > c$, then each player i has an incentive to form a link with player j if she does not obtain the resources of j thanks to a direct link. Hence the complete network is Nash.

3. If $\delta V^m > c$ and $(\delta - \delta^2)V^M < c$, then player i has no incentive to delete any link in the center sponsored star where she is the center. Moreover, no player $j \neq i$ has an incentive to form a link in the center sponsored star where i is the center since $(\delta - \delta^2)V^M < c$. the result follows. Finally, if $(n-1)\delta V^m > c$ and $(\delta - \delta^2)V^M < c$, then no player $j \neq i$ has an incentive to delete any link in the periphery sponsored star where player $i \in N$ is the center. Moreover, no player $j \neq i$ has an incentive to form a link with $j' \in N \setminus \{i, j\}$ in the periphery sponsored star where i is the center since $(\delta - \delta^2)V^M < c$. the result follows.

□

Existence of Nash networks. In this context we begin by showing that if heterogeneity is not "too high", more precisely if $V_{i,j} = V_i$ for all $i \in N$, then a Nash network always exists. It follows that there always exist a Nash network in the model of Bala and Goyal (2000, [1]).

Proposition 3 *If the benefits function satisfies equation (3) and, for all $i \in N$, $V_{i,j} = V_i$, for all $j \in N \setminus \{i\}$, then a Nash network always exists.*

Proof. Let $\mathcal{Z}_0 = \{j \in N \mid \delta V_j \geq c\}$ be the set of players who has an incentive to form a link with any player j with whom they are not (indirectely) linked; and let z be the maximal value player, that is the player such that $V_z \geq V_i$ for all $i \in N$. Moreover, let $\mathcal{Z}_1 = \{j \in N \mid (\delta - \delta^2)V_j \geq c\}$ be the set of players who has an incentive to form a link with any player j with whom they are not directely linked. Clearly if $\mathcal{Z}_1 \neq \emptyset$, then $z \in \mathcal{Z}_1$. Further if $\mathcal{Z}_1 = \emptyset$, then no player i will form a link with a player j in $\boldsymbol{g}$ whenever $d_{i,j}(\boldsymbol{g}) \leq 2$. If $\mathcal{Z}_0 = \emptyset$, then the empty network is a Nash network. If $\mathcal{Z}_0 \neq \emptyset$ and $\mathcal{Z}_1 = \emptyset$, then we let player z form links with all other players. We obtain a center-sponsored star which is a Nash network. Indeed, the distance between all players i and j is bounded by 2 and $\mathcal{Z}_1 = \emptyset$, no player has an incentive to form a link. If $\mathcal{Z}_0 \neq \emptyset$ and $\mathcal{Z}_1 \neq \emptyset$, then we create network $\boldsymbol{g}$ where player z forms links with all other players and where $g_{j,i} = 0$ implies $g_{i,j} = 1$ for all $i \in \mathcal{Z}_1$ and for all $j \in N \setminus \{z\}$. Clearly, $\boldsymbol{g}$ is Nash since player $z \in \mathcal{Z}_1$ has no incentive to delete one of her links, each player $i \in \mathcal{Z}_1$ has no incentive to delete any link by construction and no player $i' \notin \mathcal{Z}_1$ has any incentive to form an additional link (otherwise she would belong to $\mathcal{Z}_1$). □

Corollary 1 *If the benefits function satisfies equation (3) and, for all $i \in N$, $V_{i,j} = V$, for all $j \in N \setminus \{i\}$, then a Nash network always exists.*

The following example shows that non-existence can occur when we introduce higher player heterogeneity.

Example 1 *(Non-existence of Nash networks.)* Let $N = \{1, \ldots, 5\}$ be the set of players, and assume that:

1. $V_{1,2}\left(\delta - \delta^4\right) + V_{1,3}(\delta^2 - \delta^3) > c$, $\delta V_{1,3} < \delta V_{1,2} < c$, and for all $j \neq 2$, $\delta V_{1,j} + \delta^2 \sum_{k \neq j} V_{1,k} < c$.
2. $V_{2,3}\left(\delta - \delta^4\right) + V_{2,4}(\delta^2 - \delta^3) < c$, $\delta V_{2,3} + \delta^2 V_{2,4} + \delta^3 V_{2,5} + \delta^4 V_{2,1} > c$, and for all $j \neq 3$, $\delta V_{2,j} + \delta^2 \sum_{k \neq j} V_{2,k} < c$.
3. $\delta V_{3,4} > c$ and $\delta \sum_{k \neq 4} V_{3,k} + \delta^2 V_{3,4} < c$.
4. $\delta V_{4,5} > c$ and $\delta \sum_{k \neq 5} V_{4,k} + \delta^2 V_{4,5} < c$.
5. $\delta V_{5,1} > c$ and $\delta \sum_{k \neq 1} V_{5,k} + \delta^2 V_{5,1} < c$.

These five points provide a list of the players with whom the others have no incentives to form links, as well as those with whom they would like to form links. For example, item 1 implies that player 1 will never form a link with players 3, 4 and 5. Moreover, a Nash network must contain the links 3 4, 4 5, 5 1. From all of this, it follows that there is four possible Nash networks: $E(\boldsymbol{g}^1) = (3\,4, 4\,5, 5\,1, 1\,2, 2\,3)$, $E(\boldsymbol{g}^2) = (3\,4, 4\,5, 5\,1, 1\,2)$, $E(\boldsymbol{g}^3) = (3\,4, 4\,5, 5\,1)$, $E(\boldsymbol{g}^4) = (3\,4, 4\,5, 5\,1, 2\,3)$. We know from item 2 that player 2 prefers the network $\boldsymbol{g}^2$ to the network $\boldsymbol{g}^1$, so $\boldsymbol{g}^1$ is not Nash. Likewise, player 1 prefers the network $\boldsymbol{g}^3$ to the network $\boldsymbol{g}^2$ by point 1, so $\boldsymbol{g}^2$ is not Nash. Player 2 prefers the network $\boldsymbol{g}^4$ to the network $\boldsymbol{g}^3$ by point 2, so $\boldsymbol{g}^3$ is not Nash. Finally, by point 1, player 1 prefers the network $\boldsymbol{g}^1$ to the network $\boldsymbol{g}^4$. Hence $\boldsymbol{g}^4$ is not Nash.

3.2 Decay with Heterogeneous Links

In this section we consider situations where players have homogeneous values while the decay through each link is different. We obtain the following result.

Proposition 4 *Let $\boldsymbol{g}$ be an essential network. If the benefits function satisfies equation (4) and costs of forming links are homogeneous, then there exist $c > 0$ and an array $\boldsymbol{\delta} = [\delta_{i,j}]$ of decay such that:*

1. *$\boldsymbol{g}$ is a strict Nash network in the corresponding network formation game;*
2. *$\boldsymbol{g}$ is an efficient network in the corresponding network formation game. Moreover this network is also strict Nash.*

Proof. We prove succesively the two parts of the proof.

1. The proof of the first part of this proposition is an adaptation of the proof given by Haller and Sarangi (pg.186, 2005, [4]). Suppose $\boldsymbol{g}$ is an essential network. Let $\delta^1 = 1/(4n)$, $c = \delta^1/3$, $\delta^0 = c/n$. We construct a symmetric $n \times n$-matrix $[\delta_{i,j}]$ of decay as follows. If $i \neq j$ and i and j are linked, i.e. $g_{i,j} = 1$ or $g_{j,i} = 1$ set $\delta_{i,j} = \delta^1$. Otherwise set $\delta_{i,j} = \delta^0$. Now consider $i \neq j$. If $g_{i,j} = 0$, then either $g_{j,i} = 1$ or $g_{j,i} = 0$. In the first case, agent i would receive zero benefits but incurs a positive cost when forming the link $i\,j$. It follows that $g_{i,j} = 0$ is the unique optimal choice for i given $\boldsymbol{g}_{-i}$. In case $g_{j,i} = 0$, $\delta_{i,j} = \delta^0 = c/n$. It follows that $c > n\delta^0 > (n-1)\delta^0$, where $(n-1)\delta^0$ is the maximal i's benefits from the direct link $i\,j$ that player i can obtain. Therefore regardless of other links, not iniating the link $i\,j$ is optimal for agent i. If $g_{i,j} = 1$, then by essentiality of g, $g_{i,j} = 0$. Further, $\delta_{i,j} = \delta^1$. Without the link $i\,j$, the information flows between i and j via other links is at most $\left(\delta^1\right)^2 = \delta^1/(4n) < \delta^1/2$. Hence regardless of other links in $\boldsymbol{g}$, the benefits of player i from initiating the link with player j is at least $\delta^1 - \delta^1/2 = \delta^1/2$ which exceeds $c = \delta^1/3$. Therefore regardless of other links, iniating the link $i\,j$ is optimal for agent i.

2. Suppose $\boldsymbol{g}$ is an essential network. Let $\delta^1 = 1/(4n)$, $c = \delta^1/3$, $\delta^0 = c/(n+1)^2$. We construct a symmetric $n \times n$-matrix $[\delta_{i,j}]$ of decay as follows. If $i \neq j$ and i and j are linked, i.e. $g_{i,j} = 1$ or $g_{j,i} = 1$ set $\delta_{i,j} = \delta^1$. Otherwise set $\delta_{i,j} = \delta^0$. Now consider $i \neq j$. If $g_{i,j} = 0$, then either $g_{j,i} = 1$ or $g_{j,i} = 0$. In the first case, agents would receive zero benefits but agent i incurs a positive cost when forming the link $i\,j$. It follows that $g_{i,j} = 0$ is the unique optimal choice for i given $\boldsymbol{g}_{-i}$. In case $g_{j,i} = 0$, $\delta_{i,j} = \delta^0 = c/(n+1)^2$. It follows that $c > n^2\delta^0 > n(n-1)\delta^0$, where $n(n-1)\delta^0$ is an upper bound for the benefits from the direct link $i\,j$ that players can obtain. Therefore regardless of other links, not iniating the link $i\,j$ is optimal. If $g_{i,j} = 1$, then we know that agent i has no incentive to delete this link. Since the decay factor is positive, players $j \in N \setminus \{i\}$ obtain a non negative benefits from this link. It follows that this link must be preserved in an efficient network. The result follows.

□

Next, if we assume that decay begins only with indirect neighbors (instead of direct neighbors), then we can show that regardless of the value of the parameters, some essential networks are neither Nash nor efficient.

Example 2 Let $N = \{1, 2, 3\}$ be the set of players, let $\boldsymbol{g}$ be a network such that $E(\boldsymbol{g}) = \{1\,2\}$. Then, $\boldsymbol{g}$ is not a Nash network. Indeed, if player 1 has an incentive to form a link with player 2, then $V < c$. In that case, player 3 has an incentive to form a link with player 1. Likewise $\boldsymbol{g}$ is not an efficient network.

We are now interested in situations which allow to obtain the same kind of results than those established in the homogeneous decay framework. Hence, we give a threshold concerning the heterogeneity of decays which allows to obtain that Nash networks are either empty or connected.

Proposition 5 *Suppose the benefits function satisfies equation (4) and* $(\delta^M - \delta^m) < \delta^m/(1 + (n-3)\delta^M)$*. Then a Nash network is either empty or connected.*

Proof. Let $D^{**}(\boldsymbol{g}, i\,j)$ be the set of players $\ell \in N \setminus \{i, j\}$ such that $\prod_{[\ell'\;\ell''] \in C^*_{i,\ell}(\boldsymbol{g})} \delta_{\ell',\ell''} > \prod_{[\ell'\;\ell''] \in C^*_{i,\ell}(\boldsymbol{g} \ominus i\,j)} \delta_{\ell',\ell''}$. Consider a Nash network $\boldsymbol{g}$. Suppose $\boldsymbol{g}$ is neither empty nor connected.

Then there exist three agents i, j and k such that i and j belong to one connected component C_1 and k belongs to a different component C_2 in $\boldsymbol{g}$. Moreover, we suppose wlog that $g_{i,j} = 1$. Then the incremental benefits to player i of having the direct link to j is given by:

$$\begin{aligned}
\mathcal{A}' &= \delta_{i,j} + \sum_{\ell \in D^{**}(\boldsymbol{g}, i\ j)} \left(\delta_{i,j} \prod_{[\ell'\ \ell''] \in C^*_{j,\ell}(\boldsymbol{g})} \delta_{\ell'\ \ell''} - \prod_{[\ell'\ \ell''] \in C^*_{i,\ell}(\boldsymbol{g} \ominus i\ j)} \delta_{\ell'\ \ell''} \right) \\
&\leq \delta^M + \underbrace{\sum_{\ell \in D^{**}(\boldsymbol{g}, i\ j)} \delta^M \prod_{[\ell'\ \ell''] \in C^*_{j,\ell}(\boldsymbol{g}} \delta_{\ell', \ell''}}_{= \mathfrak{A}'},
\end{aligned}$$

with the convention $\prod_{[\ell'\ \ell''] \in C^*_{i,\ell}(\boldsymbol{g} \ominus i\ j)} \delta_{\ell'\ \ell''} = 0$, if $\ell \notin N_i(\boldsymbol{g} \ominus i\ j)$. Clearly, we have $\mathcal{A}' \geq c$.

If player k forms a link with player j, then the incremental benefits to player i of having the direct link to j is such that:

$$\begin{aligned}
\mathcal{B}' &\geq \sum_{\ell \in D^{**}(\boldsymbol{g}, i\ j)} \delta_{k,j} \prod_{[\ell'\ \ell''] \in C^*_{j,\ell}(\boldsymbol{g})} \delta_{\ell'\ \ell''} + \delta_{k,j} + \delta_{k,j}\delta_{i,j} \\
&\geq \underbrace{\sum_{\ell \in D^{**}(\boldsymbol{g}, i\ j)} \delta^m \prod_{[\ell'\ \ell''] \in C^*_{j,\ell}(\boldsymbol{g})} \delta_{\ell'\ \ell''}}_{= \mathfrak{B}'} + \delta^m + (\delta^m)^2
\end{aligned}$$

Clearly, we have:

$$\mathfrak{A}' - \mathfrak{B}' \leq (n-3)\delta^M(\delta^M - \delta^m).$$

Since $(\delta^M - \delta^m) < \delta^m/(1 + (n-3)\delta^M)$, we have $\mathcal{B}' > \mathcal{A}' \geq c$. It follows that player k has a incentive to form a link with j and $\boldsymbol{g}$ is not strict Nash. □

Polar cases. We now deal with some familiar architectures cases. More precisely, we give conditions which allow to obtain the complete network the empty network and the star networks as strict Nash networks.

Proposition 6 *Suppose benefits function satisfies equation (3).*

1. *If $\delta^M < c$, then the empty network is strict Nash*
2. *If $(\delta^m - (\delta^m)^2) > c$, then the complete network is strict Nash.*

3. *If $\delta^m < c$ and $(\delta^M - (\delta^M)^2) < c$, then any center sponsored star is a strict Nash network. Moreover, if $(n-1)\delta^m < c$ and $(\delta^M - (\delta^M)^2) < c$, then any periphery sponsored star is a strict Nash network.*

Proof. The two first parts of the proposition are straightforward. We now deal with the third part. Suppose $\delta^m < c$ for all $j \in N$ and $(\delta^M - (\delta^M)^2) < c$. Since $\delta^m < c$, then player i has an incentive to maintain all her links in the center sponsored star where she is the center, and since $(\delta^M - (\delta^M)^2) < c$, no player $j \neq i_0$ has an incentive to add a link in this network. Suppose $\delta^m < c$ and $(\delta^M - (\delta^M)^2) < c$. Since $\delta^m < c$, no player $j \neq i$ has an incentive to remove her link in a periphery sponsored star where i is the center and since $(\delta^M - (\delta^M)^2) < c$ no player has an incentive to add a link in such a network. □

Existence of Nash networks. As in the model with heterogeneous players, we begin by showing that if heterogeneity is not "too high", more precisely if $\delta_{i,j} = \delta_i$ for all $i \in N$, then a Nash network always exists.

Proposition 7 *Suppose the benefits function satisfies equation (4) and, for all $i \in N$, $\delta_{i,j} = \delta_i$, for all $j \in N \setminus \{i\}$, then a Nash network always exists.*

Proof. The proof is very similar as the proof of Proposition 3, hence we only give the sets which allow to construct the proof. Let $\mathcal{Z}'_0 = \{j \in N \mid \delta_j \geq c\}$ be the set of players who has an incentive to form a link with any player j with whom they are not (indirectely) linked; and let z' be the "minimal decay player", that is the player such that $\delta_{z'} \geq \delta_i$ for all $i \in N$. Moreover, let $\mathcal{Z}'_1 = \{j \in N \mid (\delta_j - \delta_j^2) \geq c\}$ be the set of players who has an incentive to form a link with any player j with whom they are not directely linked. □

The following example shows that non-existence can occur when we introduce decay heterogeneity.

Example 3 Let $N = 1, 2, 21, 3, 31, .., 36, 4, 41$ be the set of players. We assume that $c = 1.95$, $\delta_{1,2} = \delta_{2,1} = 0.6$, $\delta_{1,4} = \delta_{4,1} = 0.5$, $\delta_{4,3} = \delta_{3,4} = 0.28$, $\delta_{2,3} = \delta_{3,2} = 0.256$, $\delta_{2,21} = \delta_{21,2} = 1$, $\delta_{4,41} = \delta_{41,4} = 1$, $\delta_{3,k} = \delta_{k,3} = 1$, for all $k \in \{31, \ldots, 36\}$, and $\delta_{i,j} = 0$ for all remaining $i\,j$. Obviously, none of the links with $\delta_{i,j} = 0$ will be established. Clearly player 3 does not form the link 3 2. Moreover player 4 will never establish the link 4 1, and player 2 will never establish the link 2 1. Further, player 1 does not establish simultaneously the two links 1 4 and 1 4. It follows that the link [1 4] will always be formed. We obtain:

- if player 1 forms the link 1 4, then player 2 forms the link 2 3;
- if player 1 does not forms the link 1 4, then player 2 does not form the link 2 3;
- if player 2 forms the link 2 3, then player 1 forms the link 1 2 and does not form 1 4;
- if player 2 does not form the link 2 3, then player 1 forms the link 1 4 and does not form 1 2.

4 Discussion

It is noteworthy that we find in our paper some results which are qualitatively similar to the results find in the probabilistic models with heterogeneity. In particular Haller and Sarangi (2005, [4]) find that there exist parameters such that all networks are strict Nash and the possibility of non existence of Nash networks. It follows that it is useful to compare the probabilistic and the decay models. Specifically, we focus on two questions: Can strict Nash networks in one class of models tell us anything about strict Nash networks in the other class of models?

In the probabilistic model each link has a probability p to work. Hence, the probabilistic model uses all the paths between two players for computing payoffs while the decay model only uses the shortest path between two players to determine payoffs. At first glance this suggests that decay models might be a subset of the probabilistic models. Hence we ask if information about strict Nash networks in probabilistic models give some sense about strict Nash networks in decay models. To address this question, we compare marginal payoffs of links in both types of models.[3] In order to make the models comparable we assume that starting from the empty network, the marginal payoffs of a link is the same in both models, that is we set $\delta = p$.

[3]Note that in the probabilistic model players' marginal payoffs are expected marginal payoffs. However, for both types of models, we use the term marginal payoffs to make reading easier. Moreover, we assume that players in the probabilistic model are risk neutral.

We first show that when the initial network is minimal, the marginal payoff of a link is always at least as great in the probabilistic model as in the decay model.

Indeed, suppose that in a minimal network $\boldsymbol{g}^1$, one player, say player i, forms a link with say player j. Let the resulting network be denoted by $\boldsymbol{g}^2$. Either j is not observed by player i in $\boldsymbol{g}^1$ and it is obvious that the marginal payoff of the link $i\,j$ is the same in both models, or j is observed by i in $\boldsymbol{g}^1$. In the latter case, in the decay model, player i being at distance 1 from player j in $\boldsymbol{g}^2$, she obtains an amount p of resources of player j. In the probabilistic model, i accesses to the resources of j in $\boldsymbol{g}^2$ if the link $i\,j$ works, that occurs with a probability p. She also accesses the resources of j even when the link $i\,j$ does not work. It is enough that all the links which were contained in a path from j to i in $\boldsymbol{g}^1$ work. So, the amount of resources of player j obtained by player i in $\boldsymbol{g}^2$ is greater than p. With the same type of reasoning, we can show that the part of the resources of players $k \neq j$, obtained by i in $\boldsymbol{g}^2$, is at least as great in the probabilistic model as in the decay model. The result follows. From this result, it is straightforward that *a minimally connected Nash network in the probabilistic model is also a Nash network in the decay model.*

Next what happens if the initial network is not minimal? The example which follows shows that the above result does not hold anymore.

Example 4 Let $N = \{1, 2, 3, 4\}$ be the set of players and let $\boldsymbol{g}^1$ be a network such that $E\{\boldsymbol{g}^1\} = (1\,2, 2\,3, 3\,4, 4\,1)$.

Suppose that in $\boldsymbol{g}^1$ player 1 forms a link with player 3. We can check that for some p, for instance $p = 0.8$, the marginal payoff of this link is greater in the probabilistic model, whereas the converse is true for some other p, for instance $p = 0.9$.

Recall that if the initial network is minimal, the marginal payoff of a link is always as great in the probabilistic model as in the decay model. This difference in the result can be explained as follows.

Suppose that the initial network, denoted by $\boldsymbol{g}^1$, is not minimal. Then, there exist at least two players in $\boldsymbol{g}^1$, say i and j, such that there are at least two paths between these two players. Let player i form a link with player j in $\boldsymbol{g}^1$ and denote by $\boldsymbol{g}^2$ the resulting network. Although the total payoff of player i in $\boldsymbol{g}^2$ is greater in the probabilistic model than in the decay model, this does not imply that the marginal payoff of the link $i\,j$ is greater in the probabilistic model than in the decay model. Indeed, it is easy to check that, in $\boldsymbol{g}^1$, player i also gets a greater payoff in the probabilistic model than in the decay model.

When the initial network is not minimal, the difference in the marginal payoff of a link $i\,j$ depends on the architecture of the initial network (in particular the number of paths that exist between player i and the other players from whom i obtains resources) and on the probability that a link works. This makes it difficult to find a general rule which orders the marginal payoff of a link in both models. *Thus, when the number of players is greater than* 3 *and the initial network is not minimal, information about strict Nash networks in one type of models does not provide any indication about strict Nash networks in the other type of models.*

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