FORECASTING VOLATILITY IN THE PRESENCE OF LEVERAGE EFFECT

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ABSTRACT. We define a simple and tractable method for adding the Leverage effect in general volatility predictions. As an application, we compare volatility predictions with and without Leverage on the SP500 Index during the period 2002-2010.

1. INTRODUCTION

In this note, we adress the following problem: starting from a quite general symmetrical model of returns, how does one perturb it naturally in order to get a model with Leverage Effect? In formulas (2.1), (2.2) below, we propose a simple framework in this direction. The main motivation for this work is the problem of forecasting volatility , which has applications in many fields of finance: risk management, option pricing, etc... The most interesting feature of the construction is precisely that one can derive from the symmetrical model simple prediction formulas (cf. formula (5.1) below) for the perturbed model.

The paper is organized as follows: Section 2 defines the model in the discrete case and states the main properties of the model: correlation functions, existence of moments,etc... Section 3 defines the model in the continuous case. Section 4 is devoted to the simulation problem. Section 5 adresses the issue of forecasting volatility. Finally, we gather in the appendix the proofs of Section 2,3,4.

2. THE DISCRETE CASE

2.1. **The symmetrical model.** We consider the general stationary model for the (log) returns given by:

$$r_i = \sigma_i \epsilon_i,$$

where $\epsilon = (\epsilon_i)_{i \in \mathbb{Z}}$ is a sequence of centered i.i.d. variables such that $E[\epsilon_i^2] = 1$ and the volatility $(\sigma_i)_{i \in \mathbb{Z}}$ is a random process that we will write under the form:

$$\sigma_i = \sigma(\gamma + X_i),$$

where $X = (X_i)_{i \in \mathbb{Z}}$ is a centered stationary process independent of ϵ with $E[X_i^2] < 1$ and $\gamma^2 = 1 - E[X_i^2]$. With these conventions, we get that $E[r_i^2] = \sigma^2$.

In financial applications, X will have long range correlations:

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- Power law: $E[X_i X_{i+j}] \approx A/(j+1)^\mu$. Typically, we will take $\mu \in [0,1]$.
- Multifractal ($\mu \to 0$): $E[X_i X_{i+j}] \approx \lambda^2 \ln^+(T/(j+1))$. Typically, we will take $\lambda^2 \in [0.02, 0.05]$ and $T \approx 2000$.

2.2. **The model with leverage effect.** We consider the sequences ϵ, X of the previous section. We introduce two extra parameters β, α respectively standing for the magnitude of the leverage effect and for the inverse of the relaxation time of the leverage effect. We want to consider the following model:

$$r_i = \sigma_i \epsilon_i, \tag{2.1}$$

where the volatility $(\sigma_i)_{i \in \mathbb{Z}}$ is a random process that satisfies the following recursive equation:

$$\sigma_i = \sigma(\gamma + X_i - \beta \sum_{k=-\infty}^{i-1} e^{-\alpha(i-k)} r_k), \tag{2.2}$$

where $\gamma^2 = 1 - E[X_i^2] - \frac{\sigma^2 \beta^2}{e^{2\alpha}-1}$. We introduce the filtration:

$$\mathcal{F}_i = \sigma\{(X_j)_{j \leqslant i}, (\epsilon_j)_{j \leqslant i-1}\}$$

This leads to the following natural definition:

Definition 2.1. *We say that a sequence $(\sigma_i, r_i)_{i \in \mathbb{Z}}$ solution of (2.1), (2.2) is non anticipative if $(\sigma_i)_{i \in \mathbb{Z}}$ is $\mathcal{F}_i$-adapted.*

We can now state the following existence theorem:

Theorem 2.2. *There exists a unique stationary non anticipative square-integrable solution $(\sigma_i, r_i)_{i \in \mathbb{Z}}$ of (2.1) and (2.2) if and only if $\frac{\sigma^2 \beta^2}{e^{2\alpha}-1} < 1$.*

In the sequel of this section, we will therefore consider the unique stationary and non anticipative solution of (2.1), (2.2). It is straightforward to compute the following quantities (see appendix):

- Average vol and Variance:

$$E[\sigma_i] = \gamma\sigma \text{ and } E[r_i^2] = E[\sigma_i^2] = \sigma^2.$$

- Volatility fluctuations: $E[\sigma_i \sigma_{i+j}] - E[\sigma_i]^2 = \sigma^2(E[X_i X_{i+j}] + \frac{\beta^2}{e^{2\alpha}-1} e^{-\alpha j})$.

- Leverage correlations: $\frac{E[r_i \sigma_{i+j}]}{E[r_i^2] E[\sigma_{i+j}]} = -\frac{\beta}{\gamma} e^{-\alpha j}$

Remark: Assuming the renormalisation relation $\gamma^2 = 1 - E[X_i^2] - \frac{\sigma^2 \beta^2}{e^{2\alpha}-1}$ is not a restriction. Indeed, if we are given a set of coefficients $(\sigma, \gamma, \beta, \alpha)$ and a stationary process X that do not satisfy this relation, it is possible to renormalize the coefficients without changing the solution of (2.1) and (2.2). It suffices to define

$$c = \sqrt{\frac{\gamma^2 + E(X_i^2)}{1 - \frac{\beta^2 \sigma^2}{e^{2\alpha}-1}}} \quad \text{and} \quad \widetilde{\gamma} = \frac{\gamma}{c}, \ \widetilde{\sigma} = c\sigma, \ \widetilde{\beta} = \frac{\beta}{c}, \ \widetilde{X}_i = \frac{X_i}{c}.$$

Then the new set of coefficients $(\widetilde{\sigma}, \widetilde{\gamma}, \widetilde{\beta}, \alpha)$ and the stationary process $\widetilde{X}$ satisfy the renormalisation relation. Moreover, the solution of (2.1) and (2.2) is clearly the same for the datas $(\sigma, \gamma, \beta, \alpha, X)$ or $(\widetilde{\sigma}, \widetilde{\gamma}, \widetilde{\beta}, \alpha, \widetilde{X})$.

3. THE CONTINUOUS CASE

A natural extension of our discrete model to the continuous case is the following. Given a standard Brownian motion $B = (B_t; t \in \mathbb{R})$ and a centered square-integrable stationary process $X = (X_t; t \in \mathbb{R})$ independent from B, we are interested in finding a stationary solution $(\sigma_t)_{t\in\mathbb{R}}$ to the equations:

$$dR_t = \sigma_t \, dB_t, \tag{3.1}$$

$$\sigma_t = \sigma\Big(\gamma + X_t - \beta \int_{-\infty}^{t} e^{-\alpha(t-r)}\sigma_r \, dB_r\Big), \tag{3.2}$$

where $\alpha, \beta, \gamma, \sigma$ are positive parameters satisfying the relation $\gamma^2 = 1 - E[X_t^2] - \frac{\sigma^2\beta^2}{e^{2\alpha}-1}$. Once again, we introduce the filtration:

$$\mathcal{F}_t = \sigma\{(X_s)_{s \leqslant t}, (\epsilon_s)_{s \leqslant t}\},$$

leading to the following notion of non-anticipativity

Definition 3.1. *We say that a sequence $(\sigma_t, r_t)_{t\in\mathbb{R}}$ solution of (3.1), (3.2) is non anticipative if $(\sigma_r)_{t\in\mathbb{R}}$ is $\mathcal{F}_t$-progressively measurable.*

We can now state the following existence theorem:

Theorem 3.2. *There exists a unique stationary non anticipative square-integrable solution $(\sigma_i, r_i)_{i\in\mathbb{Z}}$ of (2.1) and (2.2) if and only if $\frac{\sigma^2\beta^2}{2\alpha} < 1$.*

The unique stationary and non anticipative solution of (3.1), (3.2) satisfies:

- Average vol and Variance:

$$E[\sigma_t] = \gamma\sigma, \quad E[(R_t - R_0)^2] = t\sigma^2, \ \text{ and } \ E[\sigma_t^2] = \sigma^2.$$

- Volatility fluctuations: for $s > 0$,

$$E[\sigma_t\sigma_{t+s}] - E[\sigma_t]^2 = \sigma^2(E[X_tX_{t+s}] + \frac{\beta^2}{2\alpha}e^{-\alpha s}).$$

- Leverage correlations: for $s > h > 0$,

$$\frac{E[(R_{t+h} - R_t)\sigma_{t+s}]}{E[(R_{t+h} - R_t)^2]E[\sigma_{t+s}]} = -\frac{\beta}{\gamma}e^{-\alpha s}\Big(\frac{e^{\alpha h} - 1}{h}\Big).$$

4. Simulation

Since the solution of (2.1) and (2.2) is obtained by the Picard fixed point theorem over a space of stationary processes and since the evolution at time i of σ depends on the whole trajectory of σ between $-\infty$ and $i-1$, the simulation of such a process is not straightforward. We suggest the following method based on finite dependence. Our purpose is to simulate the process $(\sigma_i, r_i)_{i \in \mathbb{Z}}$ between dates 0 and $p > 0$.

We choose $N > 1$ and define the mapping T^N which maps a sequence $u = (u_i)_{i \in \mathbb{Z}}$ to the sequence $u = ((T^N u)_i)_{i \in \mathbb{Z}}$ by:

$$(T^N u)_i = \sigma\Big(\gamma + X_i - \beta \sum_{k=i-N}^{i-1} e^{-\alpha(i-k)} u_i \epsilon_i\Big).$$

We define recursively the sequence $(\sigma^{N,n})_{n \in \mathbb{N}}$ of random sequences by $\sigma^{N,0} = (0)_{i \in \mathbb{Z}}$ and

$$\forall n \geqslant 1, \quad \sigma^{N,n} = T^N(\sigma^{N,n-1}).$$

We get the following approximation result:

Theorem 4.1. *Given $N \geqslant 1$, the approximation error between $(\sigma_i^{N,n})_{i \in \mathbb{Z}}$ and the solution $(\sigma_i)_{i \in \mathbb{Z}}$ of equations (2.1) and (2.2) can be estimated by*

$$\sup_{i \in \mathbb{Z}} E[|\sigma_i - \sigma_i^{N,n}|^2] \leqslant \sigma^2 \Big(\frac{e^{-\alpha N}}{1 - C^{1/2}} + C^{n/2}\Big)^2, \tag{4.1}$$

where we have set $C = \frac{\sigma^2 \beta^2}{e^{2\alpha} - 1} < 1$.

We deduce a simple algorithm to simulate the process σ between 0 and p. We fix the parameters N and n to get the desired approximation error and we compute the sequence $(\sigma^{N,k})_{0 \leqslant k \leqslant n}$ by applying iteratively the mapping T^N. Note that it is sufficient to compute $(\sigma_i^{N,n})_{i \in \mathbb{Z}}$ over the only dates $i = -(n-k)N, \dots, p$.

In Figure. 1, we simulate both processes $(\sigma_i)_{i \in \mathbb{Z}}$ and $(r_i)_{i \in \mathbb{Z}}$ between the dates 0 and $p = 300$. $(X_n)_n$ is a stationary centered Gaussian process with covariation $E[X_i X_{i+j}] = \lambda^2 \ln^+(T/(j+1))$ with $\lambda^2 = 0.016$ and $T = 2000$. The other parameters are chosen equal to $\alpha = 0.1$, $\beta = \gamma = 0.89$, $\sigma^2 = 0.025$. We choose $N = 300$ and $n = 10$ in such a way that $\sup_{k \in \mathbb{Z}} E[|\sigma_k - \sigma_k^{N,n}|^2] \approx 10^{-12}$.

5. Forecasting volatility

5.1. **The general forecasting formula.** We suppose that we observe $(\sigma_k, r_k)_{i \leqslant 0}$ and we want to forecast σ_i for $i \geqslant 1$. For all $i \geqslant 1$, we want to compute the best least squares linear predictor of σ_i, i.e. the minimiser of:

$$\inf_{\alpha_i(j), \beta_i(k)} E[(\sigma_i - E[\sigma_j] - \sum_{j=-\infty}^{0} \alpha_i(j)(\sigma_j - E[\sigma_j]) - \sum_{k=-\infty}^{0} \beta_i(k) r_k)^2]$$
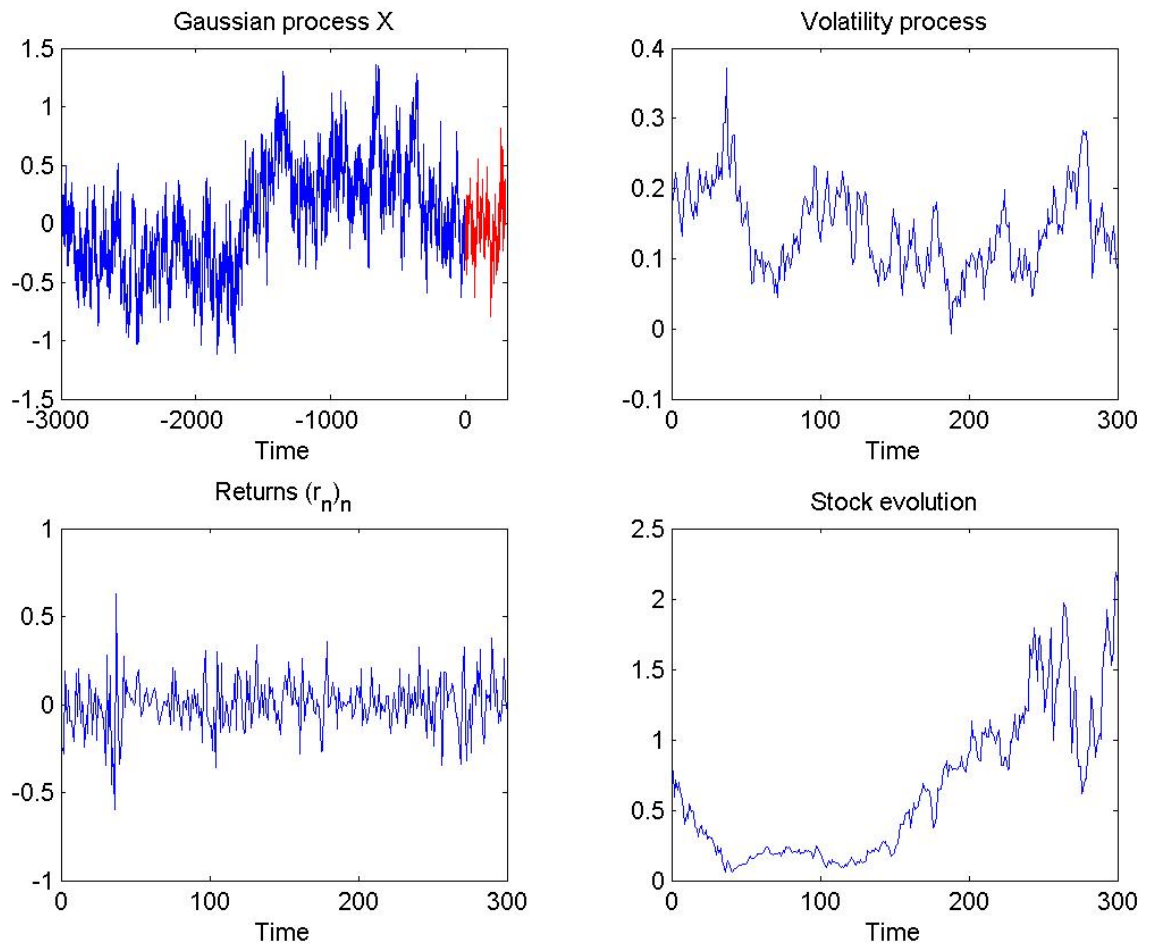
FIGURE 1. Example of simulation

Thus, we introduce for all i, the coefficients $(\alpha_i(j))_{j \leqslant 0}$ of the best least squares linear predictor of X_i, namely the ones that minimise:

$$\inf_{\alpha_i(j), \beta_i(k)} E[(X_i - \sum_{j=-\infty}^{0} \alpha_i(j) X_j].$$

We remind that one can find the $(\alpha_i(j))_{j \leqslant 0}$ by solving the system:

$$C(|i - k|) = \sum_{j=-\infty}^{0} \alpha_i(j) C(|j - k|), \quad k \leqslant 0$$

where $C(j) = E[X_i X_{i+j}]$ is the covariance function.

We can now state the main theorem of this section:

Theorem 5.1. *Let us denote by $\overline{\sigma}_i$ the best linear predictor of σ_i. Then we have:*

$$\overline{\sigma}_i = E[\sigma_i] + \sum_{j=-\infty}^{0} \alpha_i(j)(\sigma_j - E[\sigma_j]) + \beta\sigma \sum_{k=-\infty}^{0} \widetilde{\alpha}_i(k)r_k \qquad (5.1)$$

where the $\widetilde{\alpha}_i(k)$ (independent of β) are given by $\widetilde{\alpha}_i(0) = -e^{-\alpha i}$ and for $k \leqslant -1$:

$$\widetilde{\alpha}_i(k) = (\sum_{j=k+1}^{0} \alpha_i(j)e^{-\alpha(j-k)}) - e^{-\alpha(i-k)}. \qquad (5.2)$$

In conclusion, we get prediction formulas that depend on C, α, β (a function and 2 parameters which have a clear signification).

5.2. **An application: the long range correlation case.** As an application of formula (5.1), we will compare the quality of the forecasts with and without Leverage for the multifractal model, i.e. $E[X_i X_{i+j}] = \lambda^2 \ln^+(T/(j+1))$. We study the SP500 index on the period 2002-2010. Our data set will consist of the daily returns r_i and the following volatility proxy σ_i^{HL} on day i:

$$\sigma_i^{HL} = \ln(H_i/L_i),$$

where H_i and L_i are respectively the highest and the lowest value of the index on day i. In this context, we will use a filtering window of size $L = 1000$ (4 years) and use approximate formulas for the coefficients $(\alpha_i(j))_{-L+1 \leqslant j \leqslant 0}$ associated to the multifractal choice. More specifically, we choose the $(\alpha_i(j))_{-L+1 \leqslant j \leqslant 0}$ by discretization of continuous formulas first derived in [9] $(i \geqslant 1)$:

$$\alpha_i(j) = \frac{1}{\pi}\int_{j-1}^{j} \frac{\sqrt{i}\sqrt{(L+i)}}{\sqrt{-s}\sqrt{(L+s)}(t-s)}ds, \qquad (5.3)$$

where $L = 1000$. To keep the paper self-contained, we give in the appendix a new derivation of formula (5.3). We will compare two set of values for α and β:

(1) The values in the absence of Leverage: $\beta = 0$.
(2) Fixing $\beta = 5$ (a typical empirical value: cf. [6]), we try different values of the Leverage correlation length: setting $\alpha = \frac{1}{\text{Relaxtime}}$, we choose Relaxtime = $10, 30, 50, 200$.

With these two set of values, we will therefore compare the forecasts $\overline{\sigma_i^{HL}}$ derived from (5.1):

$$\overline{\sigma_i^{HL}} = <\sigma^{HL}> + \sum_{j=-999}^{0} \alpha_i(j)(\sigma_j^{HL} - <\sigma^{HL}>) + \beta\sqrt{<(\sigma^{HL})^2>}\sum_{k=-999}^{0}\widetilde{\alpha}_i(k)r_k,$$

where the $\alpha_i(j)$ are given by formula (5.3), the $\widetilde{\alpha}_i(j)$ by formula (5.2) and the $<\sigma^{HL}>$ and $<(\sigma^{HL})^2>$ denote the empirical means:

$$<\sigma^{HL}> = \frac{1}{1000}\sum_{j=-999}^{0}\sigma_j^{HL}, \quad <(\sigma^{HL})^2> = \frac{1}{1000}\sum_{j=-999}^{0}(\sigma_j^{HL})^2.$$

For the aforementionned set of values, we have plotted the renormalized empirical mean square error (MSE) as a function of the horizon (see figure 2 below where we set $\alpha = \frac{1}{\text{Relaxtime}}$):

$$i \to \frac{\sqrt{E[(\sigma_i^{HL} - \overline{\sigma_i^{HL}})^2]}}{E[\sigma_i^{HL}]}.$$

We see that, for short horizons (depending on α), adding the Leverage effect improves the renormalized (MSE) whereas only certain values of α (around 0.01) improve the renormalized (MSE) for all horizons. This can be troublesome since the precise value of α that one would derive by fitting the Leverage correlation function seems to depend on the period (not to mention that the Leverage effect is very noisy and therefore the error bars on the estimation of α are huge!).
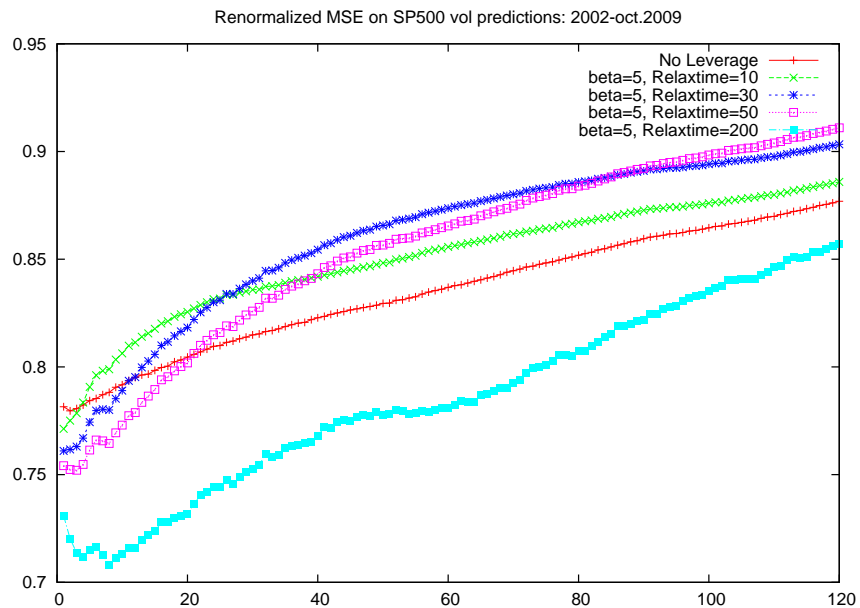

Figure 2. **Plot**: empirical MSE for the SP500 index on the period 2002-2010.

6. APPENDIX

6.1. **Proof of theorem 2.2.** Let σ, γ, β be positive parameters and $(X_i)_{i \in \mathbb{Z}}$ a square integrable stationary sequence. For $1 \leqslant p < +\infty$, we introduce the metric space (E_p, d_p) where:

$$E_p = \{(\sigma_i)_{i \in \mathbb{Z}};\ d_p(0, (\sigma_i)) < +\infty,\ (\sigma_i)\ \text{is}\ \mathcal{F}_i-\text{adapted and}\ (\sigma_i, X_i, \epsilon_i)_{i \in \mathbb{Z}}\ \text{is stationary}\}$$

and the distance function is defined by:

$$d_p((\sigma_i), (\sigma'_i)) = \sup_{i \in \mathbb{Z}} E[(\sigma_i - \sigma'_i)^p]^{1/p}$$

It is straightforward to check that (E_p, d_p) is a complete metric space. We then introduce the mapping T defined on E_p by:

$$(T\sigma)_i = \sigma(\gamma + X_i - \beta \sum_{k=-\infty}^{i-1} e^{-\alpha(i-k)} \sigma_k \epsilon_k). \tag{6.1}$$

For each element $(\sigma_i)_{i \in \mathbb{Z}}$, the sequence $(\sigma_i, X_i, \epsilon_i)_{i \in \mathbb{Z}}$ is stationary. It is plain to deduce that $((T\sigma)_i, X_i, \epsilon_i)_{i \in \mathbb{Z}}$ is stationary. Provided that $E(|X_i|^p) + E(|\epsilon_i|^p) < +\infty$, we deduce that T maps E_p into E_p.

In the case $p = 2$, the mapping T is strictly contractive if and only if $\frac{\sigma^2 \beta^2}{e^{2\alpha}-1} < 1$. This can be seen by expanding:

$$E\big[((T\sigma)_i - (T\sigma')_i)^2\big] = E\Big[\Big(\sigma\beta \sum_{k=-\infty}^{i-1} e^{-\alpha(i-k)}(\sigma_k - \sigma'_k)\epsilon_k\Big)^2\Big]$$

$$= \sigma^2 \beta^2 \sum_{k=-\infty}^{i-1} \sum_{q=-\infty}^{i-1} e^{-\alpha(i-k)} e^{-\alpha(i-q)} E\Big[(\sigma_k - \sigma'_k)(\sigma_q - \sigma'_q)\epsilon_k \epsilon_q\Big]$$

The last double sum reduces to

$$\sigma^2 \beta^2 \sum_{k=-\infty}^{i-1} e^{-2\alpha(i-k)} E\Big[(\sigma_k - \sigma'_k)^2\Big] E[\epsilon_k^2]$$

because of the independence of $(\epsilon_i)_{i \geqslant n}$ from $\mathcal{F}_n$. It is then plain to deduce that

$$E\big[((T\sigma)_i - (T\sigma')_i)^2\big] \leqslant \frac{\sigma^2 \beta^2}{e^{2\alpha}-1} d_2(\sigma, \sigma')^2.$$

Therefore T is a contractive map if $\frac{\sigma^2 \beta^2}{e^{2\alpha}-1} < 1$. By the fixed point theorem, there is therefore a unique solution to the equation $T(\sigma) = \sigma$. Conversely, if there exists a square integrable non anticipative solution (σ, r) to (2.1) and (2.2), we can compute $E[\sigma_i^2]$. By expanding the square as above, we obtain

$$E[\sigma_i^2] = \sigma^2(\gamma^2 + E[X_i^2] + \beta^2 \sum_{k=-\infty}^{i-1} e^{-2\alpha(i-k)} E[\sigma_k^2])$$

$$= \sigma^2(\gamma^2 + E[X_i^2] + \frac{\beta^2}{e^{2\alpha}-1} E[\sigma_i^2]).$$

Therefore we get

$$E[\sigma_i^2]\Big(1 - \frac{\sigma^2 \beta^2}{e^{2\alpha}-1}\Big) = \sigma^2(\gamma^2 + E[X_i^2]) \tag{6.2}$$

and we deduce $\frac{\sigma^2 \beta^2}{e^{2\alpha}-1} < 1$ (the process $(X_i)_i$ is assumed to be non trivial). Note that we have also proved that such a solution satisfies $E[\sigma_i^2] = \sigma^2$.

Remark: In the case when the processes $(\epsilon_i)_{i \in \mathbb{Z}}$ and $(X_i)_{i \in \mathbb{Z}}$ are of p-th power integrable for $p \in [1, +\infty[$, by using the Hölder inequality instead of expanding the square, we can show that for $p \in [1, +\infty[$ the mapping T is strictly contractive for the metric d_p if

$$\sigma^p \beta^p E[|\epsilon_0|^p] \left(\frac{1}{e^\alpha - 1} \right)^p < 1.$$

6.2. **Proof of theorem 3.2.** The proof is a simple adaptation of the proof in the discrete setting. For $1 \leqslant p < +\infty$, we introduce the metric space (E, d_p) where E is the set of all random progressively measurable processes $(\sigma_t)_{t \in \mathbb{R}}$ such that $d_p(0, (\sigma_t)) < +\infty$ and $(\sigma_t, X_t, (B_{t+h} - B_t))_{t \in \mathbb{R}}$ is stationary for each $h > 0$ and where d_p is the distance

$$d_p((\sigma_t), (\sigma'_t)) = \sup_{t \in \mathbb{R}} E[(\sigma_t - \sigma'_t)^p]^{1/p}$$

(E, d_p) is a complete metric space. We define the mapping $T_c : E_p \to E_p$ by

$$(T_c \sigma)_t = \sigma \left(\gamma + X_t - \beta \int_{-\infty}^t e^{-\alpha(t-r)} \sigma_r \, dB_r \right). \tag{6.3}$$

In the case $p = 2$, it is straightforward to check that the mapping T_c is strictly contractive if and only if $\frac{\sigma^2 \beta^2}{2\alpha} < 1$. In that case, existence and uniqueness of a solution to (3.1) and (3.2) results from the Picard fixed point theorem.

6.3. **Proof of theorem 4.1.** We define the sequence $(\sigma^n)_{n \geqslant 0}$ of elements of E obtained by iterating the mapping T from 0, that is $\sigma^0 = (0)_{i \in \mathbb{Z}}$ and

$$\forall n \geqslant 1, \quad \sigma^n = T(\sigma^{n-1})$$

and we set $C = \frac{\sigma^2 \beta^2}{e^{2\alpha} - 1}$. From classical arguments of fixed point theorem, we have the estimates

$$d_2(\sigma, \sigma^n)^2 \leqslant C^n d_2(\sigma, 0)^2. \tag{6.4}$$

Now we consider the mapping $T^N : (E, d_2) \to (E, d_2)$ defined in section 4. Given a sequence $u \in E$, we can estimate the quantity $d_2(Tu, T^N u)$ as in the proof of Theorem 2.2 to obtain

$$d_2(Tu, T^N u)^2 \leqslant C e^{-2\alpha N} d_2(u, 0)^2, \tag{6.5}$$

$$d_2(T^N u, T^N u')^2 \leqslant C d_2(u, u')^2. \tag{6.6}$$

Furthermore, by using (6.5) and (6.6), we have

$$\begin{aligned} d_2(\sigma^n, \sigma^{N,n}) &= d_2(T\sigma^{n-1}, T^N \sigma^{N,n-1}) \\ &\leqslant d_2(T\sigma^{n-1}, T^N \sigma^{n-1}) + d_2(T^N \sigma^{n-1}, T^N \sigma^{N,n-1}) \\ &\leqslant C^{1/2} e^{-\alpha N} d_2(\sigma^{n-1}, 0) + C^{1/2} d_2(\sigma^{n-1}, \sigma^{N,n-1}), \end{aligned}$$

so that we get

$$d_2(\sigma^n, \sigma^{N,n}) \leqslant \sum_{k=0}^{n-1} C^{k/2} e^{-\alpha N} d_2(\sigma^{n-1-k}, 0). \tag{6.7}$$

It remains to estimate $d_2(\sigma^k, 0)$. We use the recursive relation

$$d_2(\sigma^n, 0)^2 = \sigma^2(\gamma^2 + E[X_0^2]) + Cd_2(\sigma^{n-1}, 0)^2$$
$$= \sigma^2(1 - C) + Cd_2(\sigma^{n-1}, 0)^2,$$

which is easily derived from the definition of T. We deduce

$$d_2(\sigma^k, 0)^2 \leqslant \sigma^2. \tag{6.8}$$

By gathering (6.4), (6.7) and (6.8), we complete the proof.   □

6.4. **Proof of theorem 5.1.** Since for all i, j $E[X_i r_j] = 0$, we have:

$$\overline{\sigma}_i = E[\sigma_i] + \sigma \sum_{j=-\infty}^{0} \alpha_i(j)X_j - \beta \sum_{k=-\infty}^{0} e^{-\alpha(i-k)}r_k.$$

Since we have $X_j = \frac{\sigma_j}{\sigma} - \gamma + \beta \sum_{k=-\infty}^{j-1} e^{-\alpha(j-k)}r_k$, we get:

$$\sigma \sum_{j=-\infty}^{0} \alpha_i(j)X_j$$

$$= \sum_{j=-\infty}^{0} \alpha_i(j)(\sigma_j - E[\sigma_j]) + \beta\sigma \sum_{j=-\infty}^{0} \alpha_i(j)(\sum_{k=-\infty}^{j-1} e^{-\alpha(j-k)}r_k)$$

$$= \sum_{j=-\infty}^{0} \alpha_i(j)(\sigma_j - E[\sigma_j]) + \beta\sigma \sum_{k=-\infty}^{-1} (\sum_{j=k+1}^{0} \alpha_i(j)e^{-\alpha(j-k)})r_k$$

Finally, we get the prediction formula:

$$\overline{\sigma}_i = E[\sigma_i] + \sum_{j=-\infty}^{0} \alpha_i(j)(\sigma_j - E[\sigma_j]) + \beta\sigma \sum_{k=-\infty}^{0} \widetilde{\alpha}_i(k)r_k$$

where $\widetilde{\alpha}_i(0) = -e^{-\alpha i}$ and for $k \leqslant -1$, $\widetilde{\alpha}_i(k) = (\sum_{j=k+1}^{0} \alpha_i(j)e^{-\alpha(j-k)}) - e^{-\alpha(i-k)}$.

6.5. **The prediction formulas for fractional brownian motion and for the 1/f noise.** We remind in this subsection the prediction formula for fractional gaussian noise dB^H with Hurst index $H \in]0.5, 1[$. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of C^∞ rapidly decreasing functions. We remind that the fractional gaussian noise is a centered Gaussian mesure in $\mathcal{S}'(\mathbb{R})$ (the space of tempered distributions) with covariance formally given by:

$$E[dB^H(s)dB^H(t)] = H(2H - 1)\frac{dsdt}{|t - s|^{2(1-H)}}.$$

We restate the main theorem of [11] (theorem 3.1):

Theorem 6.1. *We get the following prediction formulas ($L \dot{\iota} 0$):*

$$E[dB^H(t)|(dB^H(s))_{-L<s<0}] = \int_{-L}^{0} g_{H,L}(s,t)dB^H(s)$$

where the kernel $g_{H,L}(s,t)$ is given by:

$$g_{H,L}(s,t) = \frac{\sin(\pi(H-\frac{1}{2}))}{\pi} \frac{t^{H-\frac{1}{2}}(L+t)^{H-\frac{1}{2}}}{(-s)^{H-\frac{1}{2}}(L+s)^{H-\frac{1}{2}}(t-s)}.$$

To define the $1/f$ noise, one must introduce the space $\mathcal{S}_0(\mathbb{R})$ of functions φ in $\mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi = 0$. The $1/f$-noise X is then the centered Gaussian measure in the quotient space $\mathcal{S}'(\mathbb{R})/\mathbb{R}$ defined by $(\varphi, \psi \in \mathcal{S}_0(\mathbb{R}))$:

$$E[\int_{\mathbb{R}} \varphi(s)X_s ds \int_{\mathbb{R}} \psi(t)X_t dt] = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s)\psi(t) \ln \frac{1}{|t-s|} ds dt.$$

Since for all $\varphi, \psi \in \mathcal{S}_0(\mathbb{R})$:

$$\frac{E[\int_{\mathbb{R}} \varphi(s)dB^H(s) \int_{\mathbb{R}} \psi(t)dB^H(t)]}{2(1-H)} \rightarrow E[\int_{\mathbb{R}} \varphi(s)X_s ds \int_{\mathbb{R}} \psi(t)X_t dt]$$

as $H \rightarrow 1$, we can therefore recover the following prediction formula of [9] by letting $H \rightarrow 1$:

Theorem 6.2. *We get the following prediction formula for the $1/f$-noise X:*

$$E[X_t|(X_s)_{-L<s<0}] = \int_{-\infty}^{0} g_L(s,t)X_s ds \qquad (6.9)$$

where the kernel $g(s,t)$ is given by:

$$g_L(s,t) = \frac{1}{\pi} \frac{\sqrt{t}\sqrt{(L+t)}}{\sqrt{-s}\sqrt{(L+s)}(t-s)}.$$

We can finally recover formula (5.3) by discretization of formula (6.9).

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