SOLVING DYNAMIC GENERAL EQUILIBRIUM MODELS
USING A SECOND-ORDER APPROXIMATION TO THE POLICY FUNCTION

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ABSTRACT

This paper derives a second-order approximation to the solution of a general class of discrete- time rational expectations models. The main theoretical contribution of the paper is to show that for any model belonging to the general class considered, the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules. In addition, the paper presents a set of MATLAB programs designed to compute the coefficients of the second-order approximation. The validity and applicability of the proposed method is illustrated by solving the dynamics of a number of model economies.

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1 Introduction

Since the seminal papers of Kydland and Prescott (1982) and King, Plosser, and Rebelo (1988), it has become commonplace in macroeconomics to approximate the solution to non-linear, dynamic, stochastic, general equilibrium models using linear methods. Linear approximation methods are useful to characterize certain aspects of the dynamic properties of complicated models. In particular, if the support of the shocks driving aggregate fluctuations is small and an interior stationary solution exists, first-order approximations provide adequate answers to questions such as local existence and determinacy of equilibrium and the size of the second moments of endogenous variables.

However, first-order approximation techniques are not well suited to handle questions such as welfare comparisons across alternative stochastic or policy environments. For example, Kim and Kim (forthcoming) show that in a simple two-agent economy, a welfare comparison based on an evaluation of the utility function using a linear approximation to the policy function may yield the spurious result that welfare is higher under autarky than under full risk sharing. The problem here is that some second- and higher-order terms of the equilibrium welfare function are omitted while others are included. Consequently, the resulting criterion is inaccurate to order two or higher. The same problem arises under the common practice in macroeconomics of evaluating a second-order approximation to the objective function using a first-order approximation to the decision rules. For in this case, too, some second-order terms of the equilibrium welfare function are ignored while others are not.[1] In general, a correct second-order approximation of the equilibrium welfare function requires a second-order approximation to the policy function.

In this paper, we derive a second-order approximation to the policy function of a general class of dynamic, discrete-time, rational expectations models. A strength of our approach is not to follow a value function formulation. This allows us to tackle easily a wide variety of

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[1]See Woodford (1999) for a discussion of conditions under which it is correct up to second order to approximate the level of welfare using first-order approximations to the policy function.

model economies that do not lend themselves naturally to the value function specification. To obtain an accurate second-order approximation, we use a perturbation method that incorporates a scale parameter for the variance of the exogenous shocks as an argument of the policy function. In approximating the policy function, we take a second-order Taylor expansion with respect to the state variables as well as this scale parameter. This technique was formally introduced by Fleming (1971) and has been applied extensively to economic models by Judd and Judd and co-authors (see Judd, 1998, and the references cited therein).

The main theoretical contributions of the paper are: First, it shows analytically that in general the first derivative of the policy function with respect to the parameter scaling the variance/covariance matrix of the shocks is zero at the steady state regardless of whether the model displays the certainty-equivalence property or not.[2] Second, it proves that in general the cross derivative of the policy function with respect to the state vector and with respect to the parameter scaling the variance/covariance matrix of the shocks evaluated at the steady state is zero. This result implies that for any model belonging to the general class considered in this paper, the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules.

The usefulness of our theoretical results can be illustrated by relating them to recent work on second-order approximation techniques by Collard and Juillard (2001a,b) and Sims (2000b). We follow Collard and Juillard closely in notation and methodology. However, an important difference separates our paper from their work. Namely, Collard and Juillard apply a fixed-point algorithm, which they call 'bias reduction procedure,' to capture the fact that the policy function depends on the variance of the underlying shocks. Their procedure

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[2]Judd (1998, p. 477-480) obtains this result in the context of a simple one-sector, stochastic, discrete-time growth model. Thus, our theoretical finding can be viewed as a generalization of Judd's result to a wide class of rational expectations models.

makes the coefficients of the approximated policy rule that are linear and quadratic in the state vector functions of the size of the volatility of the exogenous shocks. By the main theoretical result of this paper, those coefficients are, up to second order, independent of the variance of the shocks. It follows that the bias reduction procedure of Collard and Juillard delivers an inaccurate second-order approximation to the decision rules.

Sims (2000b) also derives a second-order approximation to the policy function for a wide class of discrete-time models. In his derivation, Sims (2000b) correctly assumes that the coefficients on the terms linear in the state vector do not depend on the volatility of the shock and obtains a second-order approximation to the policy function that is valid only under this assumption. However, he does not provide the proof that this must be the case. Our paper presents the proof of Sims's guess in a general setting.

At a practical level, our paper contributes to the existing literature by providing MAT-LAB code to compute second-order approximations for any rational expectations model whose equilibrium conditions can be written in the general form considered in this paper. We demonstrate the ability of this code to deliver accurate second-order approximations by applying it to a number of example economies. The first example considered is the standard, one-sector, stochastic growth model. Sims (2000b) computes a second-order approximation to this economy, which we are able to replicate.

The second example is motivated by the fact that, as pointed out by Sims, his Matlab code has known bugs that may show up only in multi-state models. The example applies our code to the two-country growth model with complete asset markets studied by Kim and Kim (forthcoming). This economy features multiple state variables. Kim and Kim have derived analytically the second-order approximation to the policy function of this economy. We use this example to verify that our code delivers correct answers in a multi-state environment.

Finally, we apply our code to the asset-pricing model of Burnside (1998). This example is also analyzed in Collard and Juillard (2001b). Burnside solves this model analytically. Thus, we can derive the exact analytical second-order approximation to the policy function.

This example serves two purposes. First, it gives support to the validity of our code. Second, it allows us to quantify the inaccuracy introduced by the bias reduction procedure of Collard and Juillard (2001a,b).

The remainder of the paper is organized as follows. In the next section we present the model. In section 3 we derive first- and second-order approximations to the policy function. In section 4 we describe the Matlab computer code designed to implement the second-order approximation to the policy rules. Section 5 closes the paper with applications of the algorithm developed in this paper to three example economies.

2 The Model

The set of equilibrium conditions of a wide variety of dynamic general equilibrium models in macroeconomics can be written as

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0, \tag{1}$$

where E_t denotes the mathematical expectations operator conditional on information available at time t. The state vector x_t is of size $n_x \times 1$ and the co-state vector y_t is of size $n_y \times 1$. We define $n = n_x + n_y$. The function f maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into R^n. The state vector x_t can be partitioned as $x_t = [x_t^1; \ x_t^2]'$. The vector x_t^1 consists of endogenous predetermined state variables and the vector x_t^2 of exogenous state variables. Specifically, we assume that x_t^2 follows the exogenous stochastic process given by

$$x_{t+1}^2 = \Lambda x_t^2 + \tilde{\eta} \sigma \epsilon_{t+1}; \quad \epsilon_t \sim NIID(\emptyset, I),$$

where both the vector x_t^2 and the innovation ϵ_t are of order $n_\epsilon \times 1$. The vector ϵ_t is independently, identically, and normally distributed with mean zero and variance/covariance matrix I. The scalar $\sigma \geq 0$ and the $n_\epsilon \times n_\epsilon$ matrix $\tilde{\eta}$ are known parameters. All eigenvalues of the

matrix Λ are assumed to have modulus less than one.[3]

To see how the equilibrium conditions of a familiar model can be expressed in the form given by equation (1), consider the simple neoclassical growth model. Its equilibrium conditions are given by

$$c_t^{-\gamma} = \beta E_t c_{t+1}^{-\gamma} [\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta]$$

$$c_t + k_{t+1} = A_t k_t^\alpha + (1-\delta)k_t$$

$$\ln A_{t+1} = \rho \ln A_t + \sigma \epsilon_{t+1}$$

for all $t \geq 0$, given k_0 and A_0. Let $y_t = c_t$ and $x_t = [k_t; \ \ln A_t]'$. Then

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = E_t \begin{bmatrix} y_{1t}^{-\gamma} - \beta y_{1t+1}^{-\gamma} [\alpha e^{x_{2t+1}} x_{1t+1}^{\alpha-1} + 1 - \delta] \\ y_{1t} + x_{1t+1} - e^{x_{2t}} x_{1t}^\alpha - (1-\delta)x_{1t} \\ x_{2t+1} - \rho x_{2t} \end{bmatrix},$$

where x_{it} and y_{it} denote, respectively, the i-th element of the vectors x_t and y_t.

We now return to the general case. The solution to the model given in equation (1) is of the form:

$$y_t = g(x_t, \sigma) \tag{2}$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \epsilon_{t+1}, \tag{3}$$

where g maps $R^{n_x} \times R^+$ into R^{n_y} and h maps $R^{n_x} \times R^+$ into R^{n_x}. The matrix η is of order $n_x \times n_\epsilon$ and is given by

$$\eta = \begin{bmatrix} \emptyset \\ \tilde{\eta} \end{bmatrix}.$$

We wish to find a second-order approximation of the functions g and h around the non-

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[3]Note that our formulation allows for any number of lags in endogenous and exogenous state variables. Also, it is straightforward to accommodate a more general law of motion for x_t^2 of the form $x_{t+1}^2 = \Gamma(x_t^2) + \sigma \tilde{\eta} \epsilon_{t+1}$, where Γ is a non-linear function satisfying the condition that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle. Further, the size of the innovation ϵ_t need not equal that of x_t^2.

stochastic steady state, $x_t = \bar{x}$ and $\sigma = 0$. We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ such that

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$

It is clear that $\bar{y} = g(\bar{x}, 0)$ and $\bar{x} = h(\bar{x}, 0)$. To see this, note that if $\sigma = 0$, then $E_t f = f$.

3   Approximating the Solution

Substituting the proposed solution given by equations (2) and (3) into equation (1), we can define

$$
\begin{aligned}
F(x, \sigma) &\equiv E_t f(g(h(x,\sigma) + \eta\sigma\epsilon', \sigma), g(x,\sigma), h(x,\sigma) + \eta\sigma\epsilon', x) \qquad (4) \\
&= 0.
\end{aligned}
$$

Here we are dropping time subscripts. We use a prime to indicate variables dated in period $t+1$.

Because $F(x, \sigma)$ must be equal to zero for any possible values of x and σ, it must be the case that the derivatives of any order of F must also be equal to zero. Formally,

$$F_{x^k \sigma^j}(x, \sigma) = 0 \quad \forall x, \sigma, j, k, \qquad (5)$$

where $F_{x^k \sigma^j}(x, \sigma)$ denotes the derivative of F with respect to x taken k times and with respect to σ taken j times.

3.1   First-order approximation

We are looking for approximations to g and h around the point $(x, \sigma) = (\bar{x}, 0)$ of the form

$$g(x, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma$$

$$h(x, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma$$

As explained earlier,

$$g(\bar{x}, 0) = \bar{y}$$

and

$$h(\bar{x}, 0) = \bar{x}.$$

The remaining unknown coefficients of the first-order approximation to g and h are identified by using the fact that, by equation (5), it must be the case that:

$$F_x(\bar{x}, 0) = 0$$

and

$$F_\sigma(\bar{x}, 0) = 0.$$

Thus, using the first of these two expressions, g_x and h_x can be found as the solution to the system

$$
\begin{aligned}
[F_x(\bar{x}, 0)]_j^i &= [f_{y'}]_\alpha^i [g_x]_\beta^\alpha [h_x]_j^\beta + [f_y]_\alpha^i [g_x]_j^\alpha + [f_{x'}]_\beta^i [h_x]_j^\beta + [f_x]_j^i \\
&= 0; \quad i = 1, \dots, n; \quad j, \beta = 1, \dots, n_x; \quad \alpha = 1, \dots, n_y
\end{aligned}
$$

Here we are using tensor notation. So, for example, $[f_{y'}]_\alpha^i$ is the (i, α) element of the derivative of f with respect to y'. The derivative of f with respect to y' is an $n \times n_y$ matrix. Therefore, $[f_{y'}]_\alpha^i$ is the element of this matrix located at the intersection of the i-th row and α-th column.[4] Note that the derivatives of f evaluated at $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ are known. The above expression represents a system of $n \times n_x$ quadratic equations in the $n \times n_x$ unknowns given by the elements of g_x and h_x.[5]

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[4]This is a variation of the tensor notation found in the mathematical literature. Our notation follows Collard and Juillard (2001a).

[5]A number of authors have developed algorithms for finding solutions to the above equation associated with non-explosive paths for the state and control variables (e.g., Blanchard and Kahn (1980), Sims (2000a),

Similarly, g_σ and h_σ are identified as the solution to the following n equations:

$$
\begin{aligned}
[F_\sigma(\bar{x},0)]^i &= E_t\{[f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[\eta]^\beta_\phi[\epsilon']^\phi + [f_{y'}]^i_\alpha[g_\sigma]^\alpha + [f_y]^i_\alpha[g_\sigma]^\alpha \\
&\quad + [f_{x'}]^i_\beta[h_\sigma]^\beta + [f_{x'}]^i_\beta[\eta]^\beta_\phi[\epsilon']^\phi\} \\
&= [f_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [f_{y'}]^i_\alpha[g_\sigma]^\alpha + [f_y]^i_\alpha[g_\sigma]^\alpha + [f_{x'}]^i_\beta[h_\sigma]^\beta \\
&= 0; \quad i = 1,\ldots,n; \quad \alpha = 1,\ldots,n_y; \quad \beta = 1,\ldots,n_x; \quad \phi = 1,\ldots,n_\epsilon. \quad (6)
\end{aligned}
$$

Note that this equation is linear and homogeneous in g_σ and h_σ. Thus, if a unique solution exists, we have that

$$h_\sigma = 0.$$

and

$$g_\sigma = 0.$$

These two expressions represent our first main theoretical result. They show that in general, up to first order, one need not correct the constant term of the approximation to the policy function for the size of the variance of the shocks. This result implies that in a first-order approximation the expected values of x_t and y_t are equal to their non-stochastic steady-state values $\bar{x}$ and $\bar{y}$.

3.2   Second-order approximation

The second-order approximations to g and h around the point $(x,\sigma) = (\bar{x},0)$ are of the form

$$
\begin{aligned}
[g(x,\sigma)]^i &= [g(\bar{x},0)]^i + [g_x(\bar{x},0)]^i_a[(x-\bar{x})]_a + [g_\sigma(\bar{x},0)]^i[\sigma] \\
&\quad + \frac{1}{2}[g_{xx}(\bar{x},0)]^i_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
&\quad + \frac{1}{2}[g_{x\sigma}(\bar{x},0)]^i_a[(x-\bar{x})]_a[\sigma]
\end{aligned}
$$

and Klein (2000)). In the numerical applications presented at the end of the paper we used Klein's package to solve for g_x and h_x.

$$+\frac{1}{2}[g_{\sigma x}(\bar{x},0)]_a^i[(x-\bar{x})]_a[\sigma]$$

$$+\frac{1}{2}[g_{\sigma\sigma}(\bar{x},0)]^i[\sigma][\sigma]$$

$$[h(x,\sigma)]^j = [h(\bar{x},0)]^j + [h_x(\bar{x},0)]_a^j[(x-\bar{x})]_a + [h_\sigma(\bar{x},0)]^j[\sigma]$$

$$+\frac{1}{2}[h_{xx}(\bar{x},0)]_{ab}^j[(x-\bar{x})]_a[(x-\bar{x})]_b$$

$$+\frac{1}{2}[h_{x\sigma}(\bar{x},0)]_a^j[(x-\bar{x})]_a[\sigma]$$

$$+\frac{1}{2}[h_{\sigma x}(\bar{x},0)]_a^j[(x-\bar{x})]_a[\sigma]$$

$$+\frac{1}{2}[h_{\sigma\sigma}(\bar{x},0)]^j[\sigma][\sigma],$$

where $i = 1,\ldots,n_y$, $a,b = 1,\ldots,n_x$, and $j = 1,\ldots,n_x$. The unknowns of this expansion are $[g_{xx}]_{ab}^i$, $[g_{x\sigma}]_a^i$, $[g_{\sigma x}]_a^i$, $[g_{\sigma\sigma}]^i$, $[h_{xx}]_{ab}^j$, $[h_{x\sigma}]_a^j$, $[h_{\sigma x}]_a^j$, $[h_{\sigma\sigma}]^j$, where we have omitted the argument $(\bar{x},0)$. These coefficients can be identified by taking the derivative of $F(x,\sigma)$ with respect to x and σ twice and evaluating them at $(x,\sigma) = (\bar{x},0)$. By the arguments provided earlier, these derivatives must be zero. Specifically, we use $F_{xx}(\bar{x},0)$ to identify $g_{xx}(\bar{x},0)$ and $h_{xx}(\bar{x},0)$. That is,[6]

$$
\begin{aligned}
[F_{xx}(\bar{x},0)]_{jk}^i &= \left([f_{y'y'}]_{\alpha\gamma}^i[g_x]_\delta^\gamma[h_x]_k^\delta + [f_{y'y}]_{\alpha\gamma}^i[g_x]_k^\gamma + [f_{y'x'}]_{\alpha\delta}^i[h_x]_k^\delta + [f_{y'x}]_{\alpha k}^i\right)[g_x]_\beta^\alpha[h_x]_j^\beta \\
&+[f_{y'}]_\alpha^i[g_{xx}]_{\beta\delta}^\alpha[h_x]_k^\delta[h_x]_j^\beta \\
&+[f_{y'}]_\alpha^i[g_x]_\beta^\alpha[h_{xx}]_{jk}^\beta \\
&+\left([f_{yy'}]_{\alpha\gamma}^i[g_x]_\delta^\gamma[h_x]_k^\delta + [f_{yy}]_{\alpha\gamma}^i[g_x]_k^\gamma + [f_{yx'}]_{\alpha\delta}^i[h_x]_k^\delta + [f_{yx}]_{\alpha k}^i\right)[g_x]_j^\alpha \\
&+[f_y]_\alpha^i[g_{xx}]_{jk}^\alpha \\
&+\left([f_{x'y'}]_{\beta\gamma}^i[g_x]_\delta^\gamma[h_x]_k^\delta + [f_{x'y}]_{\beta\gamma}^i[g_x]_k^\gamma + [f_{x'x'}]_{\beta\delta}^i[h_x]_k^\delta + [f_{x'x}]_{\beta k}^i\right)[h_x]_j^\beta \\
&+[f_{x'}]_\beta^i[h_{xx}]_{jk}^\beta \\
&+[f_{xy'}]_{j\gamma}^i[g_x]_\delta^\gamma[h_x]_k^\delta + [f_{xy}]_{j\gamma}^i[g_x]_k^\gamma + [f_{xx'}]_{j\delta}^i[h_x]_k^\delta + [f_{xx}]_{jk}^i
\end{aligned}
$$

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[6]At this point, an additional word about tensor notation is in order. Take for example the expression $[f_{y'y'}]_{\alpha\gamma}^i$. Note that $f_{y'y'}$ is a three dimensional array with n rows, n_y columns, and n_y pages. Then $[f_{y'y'}]_{\alpha\gamma}^i$ denotes the element of $f_{y'y'}$ located at the intersection of row i, column α and page γ.

$$= 0; \quad i = 1, \ldots n, \quad j, k, \beta, \delta = 1, \ldots n_x; \quad \alpha, \gamma = 1, \ldots n_y.$$

Since we know the derivatives of f as well as the first derivatives of g and h evaluated at $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$, it follows that the above expression represents a system of $n \times n_x \times n_x$ linear equations in the $n \times n_x \times n_x$ unknowns given by the elements of g_{xx} and h_{xx}.

Similarly, $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ can be obtained by solving the linear system $F_{\sigma\sigma}(\bar{x}, 0) = 0$. More explicitly,

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x}, 0)]^i &= [f_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma\sigma}]^\beta \\
&\quad + [f_{y'y'}]^i_{\alpha\gamma} [g_x]^\gamma_\delta [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
&\quad + [f_{y'x'}]^i_{\alpha\delta} [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
&\quad + [f_{y'}]^i_\alpha [g_{xx}]^\alpha_{\beta\delta} [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&\quad + [f_{y'}]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
&\quad + [f_y]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
&\quad + [f_{x'}]^i_\beta [h_{\sigma\sigma}]^\beta \\
&\quad + [f_{x'y'}]^i_{\beta\gamma} [g_x]^\gamma_\delta [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&\quad + [f_{x'x'}]^i_{\beta\delta} [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
&= 0; \quad i = 1, \ldots, n; \quad \alpha, \gamma = 1, \ldots, n_y; \quad \beta, \delta = 1, \ldots, n_x; \quad \phi, \xi = 1, \ldots, n_\epsilon.
\end{aligned}
\tag{7}
$$

This is a system of n linear equations in the n unknowns given by the elements of $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$.

Finally, we show that the cross derivatives $g_{x\sigma}$ and $h_{x\sigma}$ are equal to zero when evaluated at $(\bar{x}, 0)$. We write the system $F_{\sigma x}(\bar{x}, 0) = 0$ taking into account that all terms containing either g_σ or h_σ are zero at $(\bar{x}, 0)$. Then we have,

$$
\begin{aligned}
[F_{\sigma x}(\bar{x}, 0)]^i_j &= [f_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma x}]^\beta_j + [f_{y'}]^i_\alpha [g_{\sigma x}]^\alpha_\gamma [h_x]^\gamma_j + [f_y]^i_\alpha [g_{\sigma x}]^\alpha_j + [f_{x'}]^i_\beta [h_{\sigma x}]^\beta_j \\
&= 0; \quad i = 1, \ldots n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.
\end{aligned}
\tag{8}
$$

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This is a system of $n \times n_x$ equations in the $n \times n_x$ unknowns given by the elements of $g_{\sigma x}$ and $h_{\sigma x}$. But clearly, the system is homogeneous in the unknowns. Thus, if a unique solution exists, it is given by

$$g_{\sigma x} = 0$$

and

$$h_{\sigma x} = 0.$$

These equations represent our second main theoretical result. They show that in general, up to second-order, the coefficients of the policy function on the terms that are linear in the state vector do not depend on the size of the variance of the underlying shocks.[7] We summarize our two main theoretical results in the following theorem:

Theorem 1 *Consider the model given by equation (1) and its solution given by the policy functions (2) and (3). Then*

$$
\begin{aligned}
g_{\sigma}(\bar{x}, 0) &= 0, \\
h_{\sigma}(\bar{x}, 0) &= 0, \\
g_{x\sigma}(\bar{x}, 0) &= 0, \quad and \\
h_{x\sigma}(\bar{x}, 0) &= 0.
\end{aligned}
$$

Theorem 1 shows that the second-order approximation to the policy function of a stochastic model belonging to the general class given in equation (1) differs from that of its non-stochastic counterpart only in a constant term given by $\frac{1}{2}g_{\sigma\sigma}\sigma^2$ for the control vector y_t and by the first $n_x - n_\epsilon$ elements of $\frac{1}{2}h_{\sigma\sigma}\sigma^2$ for the endogenous state vector x_t^1. Therefore, any second-order expansion of the policy function of a stochastic problem whose linear and quadratic coefficients do not coincide with those of the non-stochastic version of the model is

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[7]Chen and Zadrozny (2001) obtain similar results in the context of a linear-quadratic exponential Gaussian optimal control problem. Also, upon receiving a draft of our paper, Ken Judd communicated to us that in work in progress he came across results similar to those we obtain in this section.

necessarily inaccurate. This is the case, for instance, with the approximation resulting from the bias reduction procedure of Collard and Juillard (2001a,b).[8] In section 5.2 below, we quantify the inaccuracy introduced by the bias reduction procedure for a particular model economy.

3.3 Higher-order approximations

It is straightforward to apply the method described thus far to finding higher-order approximations to the policy function. For example, given the first- and second-order terms of the Taylor expansion of h and g, the third-order terms can be identified by solving a linear system of equations. More generally, one can construct sequentially the nth-order approximation of the policy function by solving a linear system of equations whose (known) coefficients are the lower-order terms and the derivatives up to order n of f evaluated at $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ (see also Collard and Juillard, 2001a; and Judd, 1998).

4 Matlab Codes

We prepared a set of Matlab codes that implements the second-order approximation developed above. The programs are publicly available on the world wide web.[9] The program gx_hx.m computes the matrices g_x and h_x. The inputs to the program are the first derivatives of f evaluated at the steady state. That is, f_y, f_x, $f_{y'}$, and $f_{x'}$. This step amounts to obtaining a first-order approximation to the policy functions. A number of packages are available for this purpose. We use the one prepared by Paul Klein of the University of Western Ontario, which consists of the three programs solab.m, qzswitch.m, and reorder.m.[10]

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[8]Collard and Juillard (2001b) maintain that the presence of stochastic shocks will in general affect the coefficients on the terms that are linear or quadratic in x_t in a second-order approximation. Specifically, on p. 984 they state that "[i]t should be clear to the reader that, as higher-order moments will be taken into account, values for f_0 and f_1 [in our notation, f_1 corresponds to the coefficient on $x_t - \bar{x}$ in the second-order expansion] will be affected. More particularly, both of them will now depend on volatilities."

[9]The URL is http://www.econ.upenn.edu/~uribe/2nd_order.htm.

[10]For a description of the technique used in this package, see Klein (2000).

The program gxx_hxx.m computes the arrays g_{xx} and h_{xx}. The inputs to the program are the first and second derivatives of f evaluated at the steady state and the matrices g_x and h_x produced by gx_hx.m.

The program gss_hss.m computes the arrays $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$. The inputs to the program are the first and second derivatives of f evaluated at the steady state, the matrices g_x and h_x produced by the program gx_hx.m, the array g_{xx} produced by the program gxx_hxx.m, and the matrix η.

4.1 Computing the derivatives of f

Computing the derivatives of f, particularly the second derivatives, can be a daunting task if the model is large. We approach this problem as follows. The MATLAB Toolbox Symbolic Math can handle analytical derivatives. We wrote programs that compute the analytical derivatives of f and evaluate them at the steady state. The program anal_deriv.m computes the analytical derivatives of f and the program num_eval.m evaluates the analytical derivatives of f at the steady state.

4.2 Examples

To illustrate the use of the programs described thus far, we posted on the website given above the programs needed to obtain the second-order approximation to the decision rules of the three model economies studied in section 5 below. For example, to obtain the second-order approximation to the policy functions of the neoclassical growth model discussed in sections 2 above and 5 below, run the program neoclassical_model_run.m. The output of this program are the matrices g_x and h_x and the arrays g_{xx}, h_{xx}, $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$. This program calls the program neoclassical_model.m, which produces the first- and second derivatives of f. More generally, neoclassical_model.m illustrates how to write down analytically the equations of a model belonging to the class given in equation (1) using the MATLAB Toolbox Symbolic Math.

5 Applications

In this section, we apply the second-order approximation method developed above and the computer code that implements it to solve numerically for the equilibrium dynamics of a number of models. These models were chosen because they are particularly well suited for evaluating the ability of the proposed algorithm to arrive at the correct second-order approximation to the decision rule. We begin with the one-sector neoclassical growth model for which Sims (2000b) has computed a second-order approximation. This is an economy with one endogenous predetermined state and one control variable. We then consider a two-country growth model with complete asset markets. In this case there are two endogenous predetermined variables and one control. For this economy, Kim and Kim (forthcoming) derive analytically the second-order approximation to the policy function in the case that the underlying shocks are iid. We close the section with an examination of an asset pricing model that has an exact closed form solution due to Burnside (1998). Recently, Collard and Juillard (2001b) have used this model as a benchmark to evaluate the accuracy of their perturbation method, which incorporates an iterative procedure to capture the effects of the presence of uncertainty on the coefficients of the second-order expansion. As we show above, this procedure introduces an inaccuracy in the second-order approximation. We present parameterization under which these inaccuracies are quantitatively large.

5.1 Example 1: The Neoclassical Growth Model

Consider the simple neoclassical model, described in section 2. We calibrate the model by setting $\beta = 0.95$, $\delta = 1$, $\alpha = 0.3$, $\rho = 0$, and $\gamma = 2$. We choose these parameter values to facilitate comparison with the results obtained by applying Sims's (2000b) method.[11] Here we are interested in a quadratic approximation to the policy function around the natural

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[11]See the MATLAB script sessionEG.m in Sims's website (http://eco-072399b.princeton.edu/yftp/gensys2/GrowthEG)

logarithm of the steady state. Thus, unlike in section 2, we now define:

$$x_t = \begin{bmatrix} \ln k_t \\ \ln A_t \end{bmatrix}$$

and

$$y_t = \ln c_t.$$

Then the non-stochastic steady-state values of y_t and x_t are, respectively:

$$\bar{y} = -0.8734.$$

and

$$\bar{x} = \begin{bmatrix} -1.7932 \\ 0 \end{bmatrix}$$

The coefficients of the linear terms are:

$$g_x = \begin{bmatrix} 0.2525 & 0.8417 \end{bmatrix}$$

and

$$h_x = \begin{bmatrix} 0.4191 & 1.3970 \\ 0.0000 & 0.0000 \end{bmatrix}$$

The coefficients of the quadratic terms are given by:

$$g_{xx}(:,:,1) = \begin{bmatrix} -0.0051 & -0.0171 \end{bmatrix}$$

$$g_{xx}(:,:,2) = \begin{bmatrix} -0.0171 & -0.0569 \end{bmatrix}$$

and

$$h_{xx}(:,:,1) = \begin{bmatrix} -0.0070 & -0.0233 \\ 0 & 0 \end{bmatrix}$$

$$h_{xx}(:,:,2) = \begin{bmatrix} -0.0233 & -0.0778 \\ 0 & 0 \end{bmatrix}$$

Finally, the coefficients of the quadratic terms in σ are:

$$g_{\sigma\sigma} = -0.1921$$

and

$$h_{\sigma\sigma} = \begin{bmatrix} 0.4820 \\ 0 \end{bmatrix}.$$

A more familiar representation is given by the evolution of the original variables. Let

$$\hat{c}_t \equiv \ln(c_t/\bar{c})$$

and

$$\hat{k}_t \equiv \ln(k_t/\bar{k}).$$

Then, the laws of motion of these two variables are given by

$$\hat{c}_t = 0.2525\hat{k}_t + 0.8417\hat{A}_t + \frac{1}{2}\left[-0.0051\hat{k}_t^2 - 0.0341\hat{k}_t\hat{A}_t - 0.0569\hat{A}_t^2 - 0.1921\sigma^2\right]$$

and

$$\hat{k}_{t+1} = 0.4191\hat{k}_t + 1.3970\hat{A}_t + \frac{1}{2}\left[-0.0070\hat{k}_t^2 - 0.0467\hat{k}_t\hat{A}_t - 0.0778\hat{A}_t^2 + 0.4820\sigma^2\right].$$

It can be verified that these numbers coincide with those obtained by Sims (2000b).

5.2 Example 2: A Two-Country Neoclassical Model With Complete Asset Markets

The following 2-country international real business cycle model with complete asset markets is taken from Kim and Kim (forthcoming). The competitive equilibrium real allocations associated with this economy can be obtained by solving the first-best problem. The planner's objective function is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_{1t}^{1-\gamma} - 1}{1 - \gamma} + \frac{C_{2t}^{1-\gamma} - 1}{1 - \gamma} \right],$$

where C_{it}, $i = 1, 2$, denotes consumption of the representative household of country i in period t. The planner maximizes this utility function subject to the budget constraint

$$C_{1t} + C_{2t} + K_{1t+1} - (1 - \delta)K_{1t} + K_{2t+1} - (1 - \delta)K_{2t} = A_{1t}K_{1t}^{\alpha} + A_{2t}K_{2t}^{\alpha},$$

where K_{it} denotes the stock of physical capital in country i and A_{it} is an exogenous technology shock whose law of motion is given by

$$\ln A_{it} = \rho_i \ln A_{it-1} + \sigma \epsilon_{it}, \quad i = 1, 2,$$

where $\epsilon_{it} \sim NIID(0, 1)$ and $\rho_i \in (-1, 1)$. The optimality conditions associated with this problem are the above period-by-period budget constraint and

$$C_{1t} = C_{2t}$$

$$C_{1t}^{-\gamma} = \beta E_t C_{1t+1}^{-\gamma} [\alpha A_{1t+1} K_{1t+1}^{\alpha-1} + (1 - \delta)]$$

$$C_{1t}^{-\gamma} = \beta E_t C_{1t+1}^{-\gamma} [\alpha A_{2t+1} K_{2t+1}^{\alpha-1} + (1 - \delta)]$$

We use the following parameter values: $\gamma = 2$; $\delta = 0.1$; $\alpha = 0.3$; $\rho = 0$; and $\beta = 0.95$.

Given this parameterization, the second-order approximation to the policy function is given by:

$$
\hat{K}_{1t+1} = \begin{bmatrix} 0.4440 & 0.4440 & 0.2146 & 0.2146 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix}
$$

$$
+ \frac{1}{2} \begin{bmatrix} \hat{K}_{1t} & \hat{K}_{2t} & \hat{A}_{1t} & \hat{A}_{2t} \end{bmatrix} \begin{bmatrix} 0.22 & -0.18 & -0.023 & -0.088 \\ -0.18 & 0.22 & -0.088 & -0.023 \\ -0.023 & -0.088 & 0.17 & -0.042 \\ -0.088 & -0.023 & -0.042 & 0.17 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix}
$$

$$
- \frac{1}{2} 0.166\sigma^2,
$$

$$
\hat{K}_{2t+1} = \hat{K}_{1t+1},
$$

and

$$
\hat{C}_{1t} = \begin{bmatrix} 0.2 & 0.2 & 0.097 & 0.097 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix}
$$

$$
+ \frac{1}{2} \begin{bmatrix} \hat{K}_{1t} & \hat{K}_{2t} & \hat{A}_{1t} & \hat{A}_{2t} \end{bmatrix} \begin{bmatrix} 0.1 & -0.08 & -0.0093 & -0.038 \\ -0.08 & 0.1 & -0.038 & -0.0093 \\ -0.0093 & -0.038 & 0.079 & -0.019 \\ -0.038 & -0.0093 & -0.019 & 0.079 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix}
$$

$$
+ \frac{1}{2} 0.406\sigma^2
$$

$$
\hat{C}_{2t} = \hat{C}_{1t},
$$

where a hat over a variable denotes the log-deviation from its steady state. The non-stochastic steady state is given by $[K_i; A_i; C_i] = [2.6257; 1; 1.0733]$. Kim and Kim (forthcoming) derive analytically a second-order approximation to the decision rule of the model considered here. Our numerical results match those implied by Kim and Kim's analytical second-order approximation, as can be checked by running the program kim_run.m.

Example 3: An Asset Pricing Model

Consider the following endowment economy analyzed by Burnside (1998) and Collard and Juillard (2001b). The representative agent maximizes the lifetime utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^\theta}{\theta},$$

subject to

$$p_t e_{t+1} + C_t = p_t e_t + d_t e_t$$

and a borrowing limit that prevents agents from engaging in Ponzi games. In the above expressions, C_t denotes consumption, p_t the relative price of trees in terms of consumption goods, e_t the number of trees owned by the representative household at the beginning of period t, and d_t the dividends per tree in period t. Dividends are assumed to follow an exogenous stochastic process given by

$$d_{t+1} = e^{x_{t+1}} d_t,$$

where x_t denotes the gross growth rate of dividends and is assumed to follow an exogenous AR(1) process

$$x_{t+1} = (1 - \rho)\bar{x} + \rho x_t + \sigma \eta \epsilon_{t+1},$$

and $\epsilon_t \sim NIID(0, 1)$.[12]

---

[12]To make the notation compatible with our previous analysis we call $\sigma\eta$ what Burnside calls σ.

The optimality conditions associated with the household's problem are the above budget constraint, the borrowing limit, and

$$p_t C_t^{\theta-1} = \beta E_t C_{t+1}^{\theta-1}(p_{t+1} + d_{t+1}).$$

In equilibrium we have that $C_t = d_t$ and $e_t = 1$. Defining the price-dividend ratio as $y_t = p_t/d_t$ yields the equilibrium condition

$$y_t = \beta E_t \left\{ e^{\theta x_{t+1}}[1 + y_{t+1}] \right\}.$$

Burnside (1998) shows that the non-explosive solution to this equation is of the form

$$y_t \equiv g(x_t, \sigma) = \sum_{i=1}^{\infty} \beta^i e^{a_i + b_i(x_t - \bar{x})}, \tag{9}$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2 \eta^2}{2(1-\rho)^2} \left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2} \right]$$

and

$$b_i = \frac{\theta \rho(1-\rho^i)}{1-\rho}.$$

It is immediate to see that $g_\sigma(\bar{x}, 0) = g_{x\sigma}(\bar{x}, 0) = 0$, in line with Theorem 1. Collard and Juillard (2001b) present an algorithm to compute a second-order approximation to the above policy function. Their method appends to a deterministic perturbation method a fixed-point algorithm involving an iterative procedure. This procedure introduces a dependence of the coefficients of the linear and quadratic terms of the expansion of the policy function on the volatility of the underlying shocks. Theorem 1 shows that in a second-order expansion the coefficients on the terms linear and quadratic in the state are independent of the volatility of the exogenous shocks. It follows that the fixed-point algorithm proposed by Collard and Juillard (2001b) yields an inaccurate second-order approximation.

A second-order approximation of (9) around $x_t = \bar{x}$ and $\sigma = 0$ yields

$$y_t \approx g(\bar{x}, 0) + g_x(x_t - \bar{x}) + \frac{1}{2}g_{xx}(x_t - \bar{x})^2 + \frac{1}{2}g_{\sigma\sigma}\sigma^2, \tag{10}$$

where

$$g_x = \frac{\theta\rho\beta e^{\theta\bar{x}}}{(1 - \beta e^{\theta\bar{x}})(1 - \beta e^{\theta\bar{x}}\rho)},$$

$$g_{xx} = \left(\frac{\rho\theta}{1-\rho}\right)^2 \left[\frac{\beta e^{\theta\bar{x}}}{1 - \beta e^{\theta\bar{x}}} - \frac{2\beta e^{\theta\bar{x}}\rho}{1 - \beta e^{\theta\bar{x}}\rho} + \frac{\beta e^{\theta\bar{x}}\rho^2}{1 - \beta e^{\theta\bar{x}}\rho^2}\right],$$

and

$$g_{ss} = \left(\frac{\theta\eta}{1-\rho}\right)^2 \left[\frac{\beta e^{\theta\bar{x}}}{(1 - \beta e^{\theta\bar{x}})^2} + \left(\frac{\rho^2}{1-\rho^2} - \frac{2\rho}{1-\rho}\right)\frac{\beta e^{\theta\bar{x}}}{1 - \beta e^{\theta\bar{x}}} + \frac{2\rho^2}{1-\rho}\frac{\beta e^{\theta\bar{x}}}{1 - \beta e^{\theta\bar{x}}\rho} - \frac{\rho^4}{1-\rho^2}\frac{\beta e^{\theta\bar{x}}}{1 - \beta e^{\theta\bar{x}}\rho^2}\right].$$

We follow Burnside (1998) and Collard and Juillard (2001b) and use the calibration $\beta = 0.95$, $\theta = -1.5$, $\rho = -0.139$, $\bar{x} = 0.0179$, and $\eta = 0.0348$. Then, evaluating the above expressions we obtain

$$y_t \approx 12.30 + 2.27(x_t - \bar{x}) + \frac{1}{2}0.42(x_t - \bar{x})^2 + \frac{1}{2}0.35\sigma^2.$$

This is precisely the equation one obtains using the perturbation algorithm developed in this paper, as can be verified by running the program asset_run.m.

Collard and Juillard (2001b) express the second-order approximation to y_t as

$$y_t = f_0 + f_1(x_t - \bar{x}) + \frac{1}{2}f_2(x_t - \bar{x})^2.$$

Relating their notation to ours, we have $f_0 = g(\bar{x}, 0) + \frac{1}{2}g_{\sigma\sigma}\sigma^2$, $f_1 = g_x$, and $f_2 = g_{xx}$. In their table 2, Collard and Juillard (2001b) report the numerical values for f_i $(i = 0, 1, 2)$ for three different calibrations of the above asset-pricing model. To facilitate comparison, we reproduce in our table 1 their numbers in the rows labeled fixed-point algorithm. We also report the correct coefficients, which can be obtained either by evaluating equation (10) or

Table 1: Second-Order Approximation to the Policy Function of the Asset-Pricing Model

	f_0	f_1	f_2
Benchmark Calibration:			
Exact 2nd-Order Approx.	12.48	2.27	0.42
Fixed-Point Algorithm	12.48	2.30	0.43
High curvature, $\theta = -10$:			
Exact 2nd-Order Approx.	4.79	4.83	6.07
Fixed-Point Algorithm	5.00	5.97	7.50
High Persistence, $\rho = 0.9$:			
Exact 2nd-Order Approx.	22.02	-99.07	976.84
Fixed-Point Algorithm	14.50	-115.40	1137.81

Note. The benchmark calibration is $\beta = 0.95$, $\theta = -1.5$, $\rho = -0.139$, $\bar{x} = 0.0179$, $\sigma = 1$, and $\eta = 0.0348$. The rows labeled 'Exact 2nd-Order Approx.' are obtained either by evaluating equation (10) or by running the program asset_run.m. The rows labeled 'Fixed-Point Algorithm' are taken from Collard and Juillard (2001b, table 2).

by running the programs implementing our proposed algorithm (asset_run.m). The table shows that the inaccuracy introduced by the fixed-point algorithm of Collard and Juillard can be significant. For example, when the intertemporal elasticity of substitution is low ($\theta = -10$) the coefficients associated with the constant, linear, and quadratic terms are, respectively, 4, 24, and 24 percent larger than those of the exact second-order expansion. Similarly, when the underlying shock is assumed to be highly persistent ($\rho = 0.9$), then the difference between the coefficients on the constant term is 34 percent and on the linear and quadratic terms is 16 percent.

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