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MONETARY POLICY AS A SOURCE OF UNCERTAINTY

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Abstract

This paper proposes a model in which control variations induce an increase in the uncertainty of the system. The aim of our paper is to provide a stochastic theoretical model that can be used to explain under which uncertainty conditions monetary policy rules should be less or more aggressive, or, simply, applied or not.

Keywords: Uncertainty, Monetary Policy.
JEL: C61, E42, E52, E58

1 Introduction

This paper examines a model of systems for which the control action has an ensuing impact on the uncertainty that pervades the system. For systems that have origin in physical, chemical, biological or engineering phenomena, the inherent system uncertainty is not dependent upon the controller action, whereas in some macroeconomic systems considerable changes in the policy can create an impact on the confidence of the various economic agents and aggravate the overall system uncertainty. If the action is interpreted as a change of course, for example a new insight from the central bank into the health of the economy, the economic agents may have uncertain reactions to

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the new view. Thus, any change of course induced by a change in a policy is bound to create further uncertainty, and policy-makers should act with extra caution. There is no parallel to this reinforcing effect in the study of physical phenomena, and to our knowledge, this behavior is not dealt with in the field of control theory either.

Recently there has been a great deal of research on monetary policy making under uncertainty. This research can be explained by the need to reach beyond the classical linear-quadratic framework to model accurately the policy-maker's problem. During the sixties and the seventies, with the development of modern optimal control theory, monetary models were constructed under the assumption that the parameters of the model would remain fixed over time, i.e., they are not affected by political decisions. Hence, the main idea was that it is possible to deal with uncertainty using, for example, the Bellman's certainty equivalence. In a classic paper, Brainard (1967) already criticized this point of view. He argued that the certainty equivalence can be employed only in the case of additive uncertainty (e.g. shock uncertainty). Brainard highlighted the fact that, in the case of multiplicative uncertainty (e.g. model uncertainty), uncertainty about the parameters of a model would lead to less aggressive policies than those used when uncertainty is ignored. This result is known as the "Brainard conservatism principle". Another conclusion from this work is that we need robustness. This conclusion is not a new one. One can find a similar conclusion in Rosenbrock and McMorran (1971). For Rosenbrock and McMorran a distinction exists between *Good, Bad* and *Optimal* decision rules. It may happen that optimal rules are not *Good* rules since the optimality characteristic comes from an exact specification of the optimization problem (including the constraints). Hence, any deviations of the constraints, for example, may lead to poor performances of the ex-ante optimal rules. As a consequence, and as argued by Onatski and Williams (2003), model uncertainties has to be explicitly described.

Based on the use of Bayesian decision theory and robust optimal control theory, there have recently been many research papers challenging the need for robustness (see for example Edge, Laubach, and Williams (2007) or Onatski and Williams (2003)). Among these approaches, Markov switching models have also provided an interesting tool. The use of Markov switching models in the optimal monetary policy literature is explained by their ability to model both exogenous and endogenous regime changes. Blake and Zampolli (2006), for example, provide two algorithms to compute the solution of a model with regime shifts. They show that these methods can be applied to decision-making processes in order to incorporate the case of different beliefs between the policy-maker and the private sector. In a recent paper, Svensson and Williams (2007) combine Bayesian learning theory and Markov jump-linear-quadratic (MJLQ) systems to handle the problem of a policy-maker who does not know the structure of the economy.

While interesting, these papers are far from having reached any consensus on the theoretical aggressiveness of the monetary policy rules. As noticed by Issing (2002), uncertainty about the persistence of the inflation process can lead policy-makers to adjust interest rates more vigorously, since this can reduce uncertainty about the future development of inflation. Similarly, when such uncertainty arises from imperfect credibility, policy-makers may be well-advised to act more decisively. Furthermore, empirical monetary policies, such as official interest rate changes, tend to be adjusted relatively infrequently and in small steps. Usually, theoretical policies do not exhibit such a smooth behavior.

As noticed by Cagliarini and Heath (2000), interest rates tend to move in a sequence of steps in a given direction, or remain constant for some time, rather than experiencing the frequent reversals that commonly arise from optimal policy simulations. In other words, models of optimal monetary policy behaviour tend to generate much more volatile paths of interest rates, for which policy reversals are frequent. Cagliarini and Heath (2000) develop a model based on Bewley preferences with Knightian uncertainty which has the potential to explain the inertia in the level of the interest rate observed in actual interest rate paths. This Knightian uncertainty is directly related to the fact that policy-makers may be uncertain not only about the parameters, but also about the general specification of the model being used. For Cagliarini and Heath (2000), such a Knightan/risk approach is preferable to some robust approaches since monetary policy-makers usually communicate in terms of the balance of risks rather than in terms of avoiding worst-case scenarios.

Our paper aims to take in account the step behaviour of the monetary policy-makers through a different approach. We assume that the policy-makers will choose an interest rate that decreases the amount of uncertainty they are likely to face in the subsequent period. This can be used to explain the collective decision of the FED, ECB and BOE to decrease all the federal funds rates in the subprime crisis. Rather than restoring trust in the financial markets, this decision was followed by a worsening of this situation.

The effect described in the previous paragraph is well known. When dealing with monetary policy rules, or more generally with economic systems, the actions of the policy-maker are observed by the distinct economic agents, who react according to their interests. As a consequence, a change of course represented by a change in a policy is bound to create further uncertainty, and policy-makers should act with extra caution. This has been noted in a seminal paper by Lucas (1976) for whom political decisions have an impact on the economy and induce changes in the value of parameters. Within this context, the aim of our paper is to provide a stochastic theoretical model that can be used to explain under which uncertainty conditions a policy rule should be less or more aggressive, or, simply applied or not. The paper is structured as follows. Section 2 first surveys the literature on monetary policy and uncertainty. Then we introduce a model that endogenizes uncertainty. Section 3 applies this model to a general macroeconomic monetary policy model. We show that, over 1964-2008, the FED may have been more aggressive than predicted in our model. Then we conclude.

2 Monetary Policy and Uncertainty

Since the seminal work of Brainard (1967) the link between uncertainty and aggressiveness of monetary policy has been considerably studied. We briefly highlight some of the main results in the following section. We then present the CVIU approach developed by Calmon, Vallée, and do Val (2009) that endogenizes uncertainty.

2.1 Dynamic Uncertainty

2.1.1 The System

Consider a discrete-time system described by the state equation:

$$x_{k+1} = A_k x_k + b_k u_k + \omega_k, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$ and $\omega_k \in \mathbb{R}^n$ are respectively, the state, the input and a the noise (a stochastic process).

Suppose that in a given time window $0 \leq k \leq N$, the system performance is evaluated by means of the cost functional,

$$J(x_0, \mu) = E\Big[\sum_{k=0}^{N-1} C_k(x_k, u_k) + C_N(x_N)\Big], \tag{2}$$

where $E[\cdot]$ stands for the expected value of the corresponding random variable, C_k is non-negative for each $0 \leq k < N-1$ and convex in both arguments. Also, the terminal cost C_N is non-negative and convex. The following quadratic cost function is a possible candidate

$$J(x_0, \mu) = E\left\{\sum_{k=0}^{N-1} (x_k^T Q x_k + u_k P^T x + u_k^T R u_k) + x_N^T F x_N\right\}$$

We assume that at each time instant k the policy maker can determine the input control, u_k, having perfect state information, and $\mu = \{u_0, u_1, \ldots, u_{N-1}\}$ stands for an admissible policy.

2.1.2 Results on Uncertainty

Following Onatski and Stock (2000), different sources of uncertainty exist that affect the formulation of monetary policy :

1. Shock uncertainty: Future events, shocks and disturbances.
2. Parameter/Model uncertainty :
   - The actual workings of the economy;
   - Market reactions to central bank policy;
   - Market expectations of central bank policy;
3. Econometric estimation error: Limitations of the date.

If we refer to our dynamic discrete system, these sources become

- Additive uncertainty → Shock: uncertainty on ω_k;
- Parameter uncertainty
  - → Multiple uncertainty (e.g. policy effectiveness): uncertainty about the vector b,
  - → Persistence uncertainty: uncertainty about the matrix A.
  - → Objective uncertainty: uncertainty about the weights of the loss function, Q,R and P.
- Econometric uncertainty: for example uncertainty about the final date N.

Among some of the results (see Sahuc and Bihan (2002) for a survey), we know that additive uncertainty does not impact the monetary policy because of the certainty equivalence principle (Brainard (1967)). The same author showed that multiplicative uncertainty leads to caution unless a negative correlation between additive and multiple uncertainties exists. But, if persistence uncertainty is more important than the one about policy effectiveness (multiplicative uncertainty), then optimal policy may be aggressive (see Craine (1979) or Mercado and Kendrick (2000,2006)). As Söderström (2000) says when there is uncertainty about the persistence of inflation, it is optimal for the central bank to respond more aggressively to shocks than under certainty equivalence, since, this way, the central bank reduces uncertainty about future changes in inflation. Finally, if the cost of target variability (e.g. matrix Q of the loss function) is directly related to certain structural parameters of the model (as B), the classic (Brainard) attenuation of monetary policy can be overturned (see, for example, Levin and Williams (2003)).

One of the main results is that there is no consensus about the link between uncertainty and the optimal degree of aggressiveness of monetary policy. Furthermore, these results also depend on the "optimization strategy" adopted.[1].

[1]It is well known now that using an optimal robust control approach will generally increase the aggressiveness.

2.2 Historical Federal Funds and Endogenous Uncertainty

As previously discussed, the sources of uncertainty are many and their consequences on the degree of aggressiveness of monetary policy are varied. When looking at empirical data, (see figure 1), one can easily conclude that the ECB is less aggressive in its conduct of monetary policy than the FED.

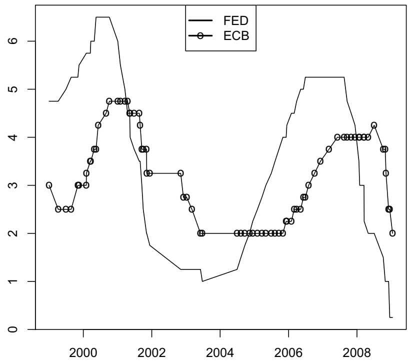


Figure 1: Evolution of the FED and ECB funds rate

Based on different assumptions concerning uncertainty, two different explanations of this figure are possible:

- The ECB is facing more uncertainty than the FED, and, their conservatism is compatible with a negative correlation between uncertainty and the aggressiveness of their policy.
- The uncertainty was higher in USA, and this uncertainty is positively correlated with the aggressiveness of their policy.

If we go back to a constant parameter case of our general state equation (1):

$$x_{k+1} = Ax_k + bu_k + \omega_k, \tag{3}$$

The existence of uncertainty about some parameters of A and B will lead to

$$x_{k+1} = (A + \tilde{A})x_k + (b + \tilde{b})u_k + \omega_k, \tag{4}$$

which can be reformulated as

$$x_{k+1} = Ax_k + bu_k + \omega_k + \tilde{A}x_k + \tilde{b}u_k, \tag{5}$$

If x_k is a known vector at time k, $\tilde{A}x_k$ is a shift in the initial additive uncertainty ω_k, while $\tilde{b}u_k$ becomes an *endogeneous* uncertainty due to the unknown exact impact of u_k on the state. So it is likely that any change in monetary policy can impact the overall level of uncertainty.

One framework used to analyze such an interdependence is the one of signalling game (see for example Canzoneri (1985) and Walsh (1999)) where any change in monetary policy will be interpreted by private agents as a positive or a negative signal. In the case of a loss of credibility (bad signal), the monetary policy may increase GDP and inflation variability, and thus increase the level of uncertainty. Although interesting, this approach generally assumes that the central bank knows the way its signal will impact private agents' expectations. Let assume, that the monetary authority knows only that any particular change of its monetary policy, such as an increase of the federal funds rate, may increase the level of uncertainty on the system and, as a consequence, the future effectiveness of its policy. Following Brainard's conservatism principle, a risk neutral policy-maker may found optimal to keep unchanged the federal funds rate.

We know that the ECB, like the FED, defines its optimal policy by looking at both the inflation and the output gap dynamics. As a consequence, many monetary policy models, such as the well known Rudebush-Svensson model (Rudebusch and Svensson (1999)), integrate these two dynamics. So, two sources of uncertainty are taken into account. Uncertainty concerning future inflation dynamics, and uncertainty on other economic dynamics like the output gap changes. Of course, uncertainty faced by the policy-makers can influence these both dynamics with the same strength or may impact them differently. In a similar way, the policy used can feedback on both uncertainties or only one.

In the next section, we will introduce the principle of a system for which *control variations increase state uncertainty* (CVIU). With such a model, one can endogenize the level of uncertainty of the different dynamics of the system.

2.3 The CVIU Approach

Traditionally, in stochastic control problems, the imbedded system uncertainty is modeled by means of additive or multiplicative disturbances (Bertsekas (2005)) and of parameter uncertainty, as in the Markov Jump Linear Systems (Costa, Fragoso, and Marques (2004)). However, these models may not suffice to describe uncertainty in many situations. Consider, again, the problem faced by a National Central Bank (NCB) when defining monetary policy (Onatski and Stock (2002), Svensson and Williams (2005)). As it increases or decreases the interest rate, the NCB has an uncertainty about the expectations of the economic agents. Significant course change in the monetary policy may induce unexpected and undesired consequences such as an increase in inflation or a reduction in GDP. On the other hand, if the variation of the interest rate is too small, the NCB objectives may not be accomplished. This is an example of a system for which any change of policy leads to an increase in system uncertainty.

The monetary policy problem is an example of a *system where control variations increase state uncertainty (CVIU)*. From a control theory perspective, CVIU systems can be used to control systems with complex, underdetermined dynamics, for which the behavior near a given point and for a given control policy can be fairly well described by a linear model. However, large variations of the control action can drift the system to regions where the linear approximation error is too large. In this case, the approximation error corresponds to the uncertainty generated by policy variations.

We consider that the magnitude of the control action acts as a source of system uncertainty in such a way that the noise sequence ω_k is modulated by the *absolute value of the control* $|u_k|$, as follows:

$$\omega_k = (\overline{\sigma}_k + \sigma_k |u_k|)\, \varepsilon_k,$$

where $\overline{\sigma}_k > 0, \sigma_k > 0$, and ε_k is an i.i.d. random vector with a normalized covariance matrix $\mathrm{cov}(\tilde{\varepsilon}_k) = I_{n\times n}$.

We aim at the dynamic programming method, and in a preliminary step we are interested in characterizing the function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $V^* : \mathbb{R}^n \to \mathbb{R}$, defined as

$$V(x, u) = C(x, u) + E\left[F(x_1)\right], \tag{6}$$

and

$$V^*(x) = \inf_{u\in\mathbb{R}^m} V(x, u), \tag{7}$$

where $C : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$ are both convex, non-negative, Lipschitz functions. The random vector x_1 is determined by (1) with $x_0 = x$ and $u_0 = u$. Note that the system is time homogeneous in the sense that if $x_k = x$ and $u_k = u$, one evaluates equivalently the expected value in (6) of x_{k+1}.

The following lemma is important for the characterization of V^*.

Lemma 1. *The functions $V(x, u)$ and $V^*(x)$ given by (6) and (7) , respectively, are convex.*

Proof. See Calmon, Vallée, and do Val (2009). □

Even though V^* is defined as a piecewise function, it will be differentiable if V is strictly convex and differentiable. Moreover, based on V, we can characterize the generalized gradient of V^*. These facts are stated in the following lemma.

Lemma 2. *With V as defined as in (6) and V^* as in (7), and $u^*(x)$ defined as*

$$u^*(x) = \arg\min_{u\in\mathbb{R}} V(x, u). \tag{8}$$

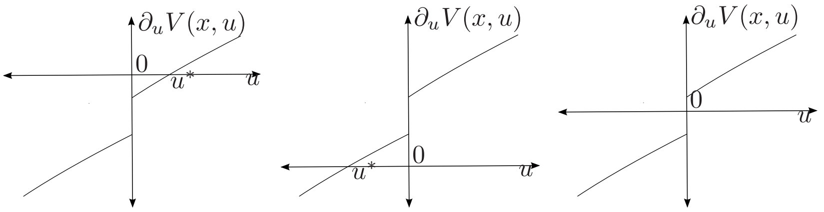
then

1. $\partial V^*(x) = \text{co}\{\partial_x V(x,u) : u \in u^*(x)\}$;

2. $V^*(x)$ *will be differentiable if* V *is a strictly convex differentiable function.*

Proof. See Calmon, Vallée, and do Val (2009). □

With the fact that the value function is convex, we can determine the sign of u^* based solely on the value of the state x (see figure 2). Assume that a function $(x,u) \to f(x,v)$ is differentiable. Provided that f is also convex in u, one can obtain the sign of the minimum in u by analyzing $\nabla_u f|_{u=0}$ for each x. If $\nabla_u f|_{u=0} > 0$ (< 0), then the function is increasing (decreasing) at the origin and, consequently, the minimum will be in the negative (positive) half-plane. Of course, if $\nabla_u f|_{u=0} = 0$ the optimal solution is $u^* = 0$. Note that this analysis can not be applied to V in (6) since, even though V is convex, it will not necessarily be differentiable at $u = 0$ (in fact, this will never be the case). The following Lemma presents a result concerned with this issue and shows that there will be a region in the state space where $u^* = 0$ if the cost function is Lipschitz. Furthermore, for the case where C and F are differentiable, we will show that a region where $u^* = 0$ will always exist.



(a) $x \in \mathcal{R}_1$ (b) $x \in \mathcal{R}_2$ (c) $x \in \mathcal{R}_3$

Figure 2: Behaviour of $\partial_u V(x,u)$ for different x. In (c), $u^* = 0$.

Lemma 3. *For the function* V *described in* (6) *and* u^* *given by* (8) *we have*

$$\begin{cases} u^*(x) > 0, & \text{if } x \in \mathcal{R}_1(V), \\ u^*(x) < 0, & \text{if } x \in \mathcal{R}_2(V), \\ u^*(x) = 0, & \text{if } x \in \mathcal{R}_3(V). \end{cases} \tag{9}$$

where

$$\mathcal{R}_1(V) = \{x : x \in \mathbb{R}^n, \lim_{u_i \downarrow 0} \nabla V(x,u_i) < 0\}, \tag{10}$$

$$\mathcal{R}_2(V) = \{x : x \in \mathbb{R}^n, \lim_{u_i \uparrow 0} \nabla V(x,u_i) > 0\}, \tag{11}$$

$$\mathcal{R}_3(V) = \overline{\mathcal{R}_1 \cup \mathcal{R}_2}. \tag{12}$$

Proof. See Calmon, Vallée, and do Val (2009). □

We can now provide the following theorem.

Theorem 1. *Suppose that for each $0 \leq k < N$ we have that $x \to C_k(x,u), u \to C_k(x,u)$ and $x \to C_N(x)$ are convex functions and that (2) is limited for all $x_0 \in \mathbb{R}^n$ and μ. For the system in (1) and evaluated by means of the cost function (2), the optimal policy can be obtained recursively as follows.*

1. *Define $J_N^*(x) = C_N(x),\ x \in \mathbb{R}^n$ and set $k = N-1$;*

2. *Define $J_k(x,u)$ for each $x \in \mathbb{R}^n$, as*

$$J_k(x,u) = C_k(x,u) + E[J_{k+1}^*(A_k x + b_k u + (\overline{\sigma}_k + \sigma_k|u|)\varepsilon_k)],$$

3. *For each $x \in \mathbb{R}^n$, determine if the optimal action u_k^* will be positive, negative or zero with*

$$\begin{cases} u_k^*(x) > 0, & \text{if } x \in \mathcal{R}_1(J_k), \\ u_k^*(x) < 0, & \text{if } x \in \mathcal{R}_2(J_k), \\ u_k^*(x) = 0, & \text{if } x \in \mathcal{R}_3(J_k). \end{cases} \tag{13}$$

   If $u_k^* \in \mathcal{R}_1(J_k)$ or $u_k^* \in \mathcal{R}_2(J_k)$, determine u_k^* such that $J_k(x,u_k^*) \leq J_k(x,u)$ for each $u \in \mathbb{R}$. This is equivalent to requiring that

$$0 \in \partial_u C_k(x,u_k^*) + \partial_u E[J_{k+1}^*(A_k x + b_k u_k^* + (\overline{\sigma}_k + \sigma_k|u_k^*|)\varepsilon_k)]$$

4. *Define the function $J_k^*(x)$ by*

$$J_k^*(x) = J_k(x,u_k^*(x)) = C(x,u_k^*(x)) + E[J_{k+1}^*(x_{k+1})]$$

   with $x_{k+1} = A_k x + b_k u_k^*(x) + (\overline{\sigma}_k + \sigma_k|u_k^*(x)|)\varepsilon_k$. If $k = 0$, stop. If else, return to step 2.

The optimal policy u_k for each $0 \leq k < N$ is thus obtained, in such a way that

$$J_k^*(x) = J_k(x,u_k^*(x)) \leq J_k(x,u),$$

holds for each (x,u) and, in particular, $J^*(x) = J_0(x,u_0^*(x)) \leq J_0(x,u),\ \forall (x,u)$.

Proof. See Calmon, Vallée, and do Val (2009) □

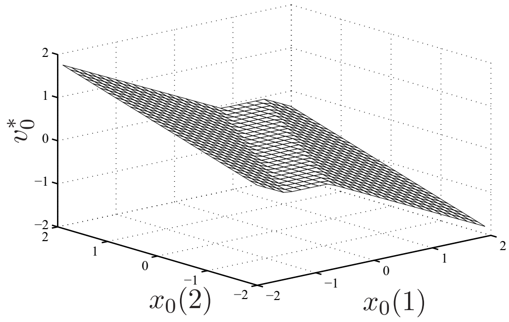


Figure 3: Optimal input u^* as a function of the state value for the system described in the example.

2.4 A two Dimensional Example

In order to represent graphically the region in the state-space where the optimal policy is to maintain the same control as before, i.e., $v^* = 0$, consider a two dimensional system described by

$$x_{k+1} = Ax_k + Bv_k + (1 + |v_k|)\varepsilon_k, \quad \varepsilon_k \sim N(0, I_{n\times n})$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Suppose that the system is evaluated through the quadratic cost function:

$$J(x_0, \pi) = \sum_{k=0}^{4} (x_k^T x_k + 2v_k P^T x + 0.5v_k^2) + x_5^T x_5$$

with $P = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Figure 3 shows v_0^* as a function of the initial state x_0. The region where $v_0^* = 0$ can be clearly seen in the center of the graph. In the other regions, the optimal input will be a linear function of the initial state.

3 The CVIU Approach in a Monetary Model

3.1 The Model

Let consider a backward-looking model of the US economy given by:

$$\pi_{k+1} = \sum_{j=0}^{n_1} \alpha_j \pi_{k-j} + \delta y_k + \epsilon_{\pi,k}, \tag{14}$$

$$y_{k+1} = \sum_{j=0}^{n_2} \beta_j \pi_{k-j} + \sum_{j=0}^{n_3} \lambda_j y_{k-j} + \sum_{j=0}^{n_4} \gamma_j i_{k-j} + \epsilon_{y,k}, \tag{15}$$

where π_k is the quarterly annualized inflation, y_k is the output gap and i_k is the quarterly average federal funds rate in percentage points given at an annual rate.

The two shocks, $\epsilon_{\pi,k}$ and $\epsilon_{y,k}$, are responsible for state uncertainty. This model is derived from the model proposed by Rudebusch and Svensson (1999) (see also Svensson and Williams (2005)). The first equation is a version of the Phillips curve, which relates inflation to a lagged output gap term and to three lags of inflation. The second equation represents an aggregate-demand relation that relates the output gap to its own lag, to inflation over the previous three quarters and to two lags in the interest rate.

It is interesting to observe that large variations in the interest rate lead to a high state uncertainty. Consequently, this system fits into our CVIU approach with $\epsilon_{y,k}$ defined as a function of the interest rate variations. Denoting the variations in the interest rate from instant k to instant $k+1$ as v_k, we have:

$$i_k = i_{k-1} + v_k$$

Using a model similar to the one presented in Section 2.3, we obtain:

$$\epsilon_{k,\pi} = (\overline{\sigma}_\pi + \sigma_\pi |v_k|)\, \xi_{k,\pi} \tag{16}$$

$$\epsilon_{k,y} = (\overline{\sigma}_y + \sigma_y |v_k|)\, \xi_{k,y} \tag{17}$$

where $\xi_{k,\pi} \sim N(0,1)$ and $\xi_{k,y} \sim N(0,1)$.

To determine the optimal decision policy, we must first describe the system in a suitable form. First, by rewriting (15), we obtain the new system:

$$\pi_{k+1} = \sum_{j=0}^{n_1} \alpha_j \pi_{k-j} + \delta y_k + \epsilon_{\pi,k} \tag{18}$$

$$y_{k+1} = \sum_{j=0}^{n_2} \beta_j \pi_{k-j} + \sum_{j=0}^{n_3} \lambda_j y_{k-j} + \gamma_0 (v_k + i_{k-1}) + \sum_{j=1}^{n_4} \gamma_j i_{k-j} + \epsilon_{y,k} \tag{19}$$

This system can be put in a state-space form:

$$x_{k+1} = Ax_k + Bv_k + \mathcal{E}_k(v_k), \tag{20}$$

where

$$n_{max} = \max\{n_1, n_2\}$$

$$x_k \equiv [\pi_k \;\; \ldots \;\; \pi_{k-n_{max}} \; y_k \;\; \ldots \;\; y_{k-n_3} \; i_{k-1} \;\; \ldots \;\; i_{k-n_4}]^T,$$

$$\mathcal{E}_k(v_k) \equiv \left[\epsilon_{\pi,k+1} \quad \mathbf{0}_{1\times n_{max}} \quad \epsilon_{y,k+1} \quad \mathbf{0}_{1\times(n_3+n_4)}\right]^T.$$

Defining:

$$a := \begin{bmatrix}\alpha_0 & \dots & \alpha_{n_1}\end{bmatrix}^T, \quad b := \begin{bmatrix}\beta_0 & \dots & \beta_{n_2}\end{bmatrix}^T,$$
$$c := \begin{bmatrix}\lambda_0 & \dots & \lambda_{n_3}\end{bmatrix}^T, \quad d := \begin{bmatrix}\gamma_0 + \gamma_1 & \gamma_2 & \dots & \gamma_{n_4}\end{bmatrix}^T,$$

the transition matrix A will be given by:

$$A \equiv \begin{bmatrix} A_{11} & A_{12} & \mathbf{0}_{n_{max}+1 \times n_4} \\ A_{21} & A_{22} & A_{23} \\ \mathbf{0}_{n_4 \times n_{max}+1} & \mathbf{0}_{(n_4)\times(n_3+1)} & A_{33} \end{bmatrix}$$

where

$$A_{11} = \begin{cases} \begin{bmatrix} a^T \\ [\mathbf{I}_{n_1 \times n_1} \quad \mathbf{0}_{n_1 \times 1}] \end{bmatrix}, & \text{if } n_{max} = n_1 \\ \\ \begin{bmatrix} a^T & \mathbf{0}_{1 \times n_2 - n_1} \\ [\mathbf{I}_{n_2 \times n_2} & \mathbf{0}_{n_2 \times 1}] \end{bmatrix}, & \text{if } n_{max} = n_2 \end{cases}$$

and

$$A_{21} = \begin{cases} \begin{bmatrix} \begin{bmatrix} b^T & \mathbf{0}_{1 \times n_1 - n_2} \end{bmatrix} \\ \mathbf{0}_{n_3 \times n_1 + 1} \end{bmatrix}, & \text{if } n_{max} = n_1 \\ \\ \begin{bmatrix} b^T \\ \mathbf{0}_{n_3 \times n_2 + 1} \end{bmatrix}, & \text{if } n_{max} = n_2 \end{cases}$$

The other terms of A are:

$$A_{12} = \begin{bmatrix} \begin{bmatrix} \delta & \mathbf{0}_{1 \times n_3} \end{bmatrix} \\ \mathbf{0}_{n_1 \times n_3 + 1} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} c^T \\ \begin{bmatrix} \mathbf{I}_{n_3 \times n_3} & \mathbf{0}_{n_3 \times 1} \end{bmatrix} \end{bmatrix},$$
$$A_{23} = \begin{bmatrix} d^T \\ \mathbf{0}_{n_3 \times n_4} \end{bmatrix}, \quad A_{33} = \begin{bmatrix} \begin{bmatrix} 1 & \mathbf{0}_{1 \times n_4 - 1} \end{bmatrix} \\ \begin{bmatrix} \mathbf{I}_{n_4 - 1 \times n_4 - 1} & \mathbf{0}_{n_4 - 1 \times 1} \end{bmatrix} \end{bmatrix}$$

The matrix B is given by

$$B \equiv \begin{bmatrix} \mathbf{0}_{1 \times n_{max}+1} & \gamma_0 & \mathbf{0}_{1 \times n_3} & 1 & \mathbf{0}_{1 \times (n_4 - 1)} \end{bmatrix}^T$$

In this example, the policy-maker's goals are to reduce inflation and output gap analyzing a period of N quarters. In the monetary literature (see Dennis and Söderström (2006) for example), two different loss functions are used. The first one does not include the interest rate smoothing, focusing mainly on the two policy objectives of the Federal Reserve, while the second one takes the rate smoothing parameter into account. Since it is widely believed that central banks smooth interest rates, that is, they aim at limiting interest rate volatility, following Rudebusch and Svensson (1999) and Svensson and Williams (2005), we consider the following criterion[2] to minimize at each current period k:

$$L = \pi_k^2 + y_k^2 + \frac{1}{2}(i_k - i_{k-1})^2 \tag{21}$$

The cost function can be rewritten as a function of v, leading to:

$$J(x) = E\Big[x'_N Q x_N + \sum_{k=0}^{N-1} x'_k Q x_k + 2P x_k v_k + r v_k^2\Big] \tag{22}$$

Using (20) and (22) we can apply Theorem 1 directly to obtain the optimal input policy.

3.2 Empirical State Space Form

In order to simulate the model, we suppose that the model's parameters are as shown in Table 1, with $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ and $n_4 = 1$. This estimation is taken from Pardo, Rautureau, and Vallée (2009). They used quarterly data for the US economy, from the first quarter of 1960 to the fourth quarter of 2008. The interest rate (i_k), is four-quarter average federal funds rate from the Board of Governors. Inflation (π_k) is the GDP chain-type price index in percent at an annual rate, i.e. $400(\ln p_k - \ln p_{k-1})$. The output gap (y_k) is built as $100(q_k - q_k^*)/q_k^*$, where q_k is the actual real GDP and q_k^* is the potential GDP. The data used are available from BEA and CBO. All the variables were de-meaned prior to estimation.

Parameters	Estimation	Parameters	Estimation
α_0	0.554	β_0	0.015
α_1	0.145	β_1	-0.033
α_2	0.256	β_2	-0.02
δ	0.115	γ_0	0.133
λ	0.902	γ_1	-0.197

Table 1: Estimates of the US quarterly model

Using the parameters of Table 1, the system we are considering is:

$$x_{k+1} = Ax_k + Bv_k + (\overline{\sigma} + \sigma|v_k|)\xi_k, \quad \xi_k \sim N(0, \Sigma) \tag{23}$$

[2] We will show in section 3.5 that relaxing the interest rate smoothing goal by setting $r = 0$ will not change our general results.

$$x_k = \begin{bmatrix}\pi_k & \pi_{k-1} & \pi_{k-2} & y_k & i_{k-1}\end{bmatrix}^T, \qquad v_k = \begin{bmatrix}i_k - i_{k-1}\end{bmatrix},$$

$$A = \begin{bmatrix} 0.554 & 0.145 & 0.256 & 0.115 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.015 & -0.033 & -0.020 & 0.902 & (0.133 - 0.197) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix}0 & 0 & 0 & 0.133 & 1\end{bmatrix}^T,$$

$$\Sigma = \begin{bmatrix} I_{4\times4} & 0 \\ 0 & 0 \end{bmatrix}$$

and with $\overline{\sigma}$ and σ being diagonal matrices where:

$$\overline{\sigma}_k = \begin{bmatrix}\overline{\sigma}_\pi & 0 & 0 & \overline{\sigma}_y & 0\end{bmatrix}^T,$$

$$\sigma_k = \begin{bmatrix}\sigma_\pi & 0 & 0 & \sigma_y & 0\end{bmatrix}^T$$

Finally, the system is evaluated through the cost function (22) with

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$P = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T;$$

$$r = 0.5.$$

We will consider two numerical approaches. In the first one, we will look at the impact of σ starting from a given state value x_0. We will compare the results with standard Linear Quadratic Regulator (LQR) simulations. Then, in a second approach, we will compare our numerical solutions with some real data.

Remark: for all the simulations we add a constraint on the non negative value of $i(k)$, $\forall k$. That is, if at some time k the measure of change $v(k)$ is such that $i(k) = v(k) + i(k-1) < 0$ then we set $v(k) = -i(k-1)$. That is, the maximum decrease of the interest rate is fixed to the step between the previous positive interest rate and 0.

3.3 LQR versus CVIU

We start the simulation with the historical values for the year 2008 (last quarter), that is $x_0 = \begin{bmatrix}0.4869 & 3.8615 & 1.1101 & -4.280 & 1.94\end{bmatrix}^T$. We will look at the impact of adding uncertainty, that is additive and multiplicative shocks. As stated earlier, when the uncertainty generated by input variation (represented by σ) rises, the region in the state space where $v_k^* = 0$ will become larger. Conversely, if $\sigma = 0$ the problem becomes the traditional LQR problem and the optimal input variation will be a linear function of the state. In order to concentrate on the impact of rising uncertainty, we assume that the shocks impact identically both economic equations (inflationary and GDP), in that

$$\overline{\sigma}_k = \begin{bmatrix} \overline{\sigma} & 0 & 0 & \overline{\sigma} & 0 \end{bmatrix}^T,$$
$$\sigma_k = \begin{bmatrix} \sigma & 0 & 0 & \sigma & 0 \end{bmatrix}^T$$

where σ and $\overline{\sigma}$ are some constant scalar.

We simulate realizations of the system for $\sigma = 0.3$ and $\sigma = 0$, with $\overline{\sigma} = 0.8$. The results are shown in Figure 4. In the case where interest rate variations increase system uncertainty ($\sigma = 0.3$) there exists many time instants where $v_k^* = 0$, leading to a smoother behaviour of the interest rate (Figures 4a-4b). At the opposite, when $\sigma = 0$ the interest rate varies more frequently. If we set $\sigma = 0.1$, as shown in Figures 4c-4d, then reaction is more frequent, and the descent towards 0 is faster. Although the CVIU monetary policy involves a **small steps** behaviour, this policy can be stronger than the LQR one, as shown by Figure 4c, if we are far away from $\mathcal{R}_3(J_k)$.

3.4 Comparison with Real Data

In order to study the impact of changing the historical starting date, Figure 5 shows the change in the interest rate based on one specific run (5a) and based on an average evolution on 200 runs (5b). We compare these changes with historical data when the simulations start at two different initial dates: $t_0 \in \{2006.1, 2008.4\}$[3]. As one can check, the descent in the last quarter of 2008 is faster than the one in the first quarter of 2006, which is consistent with the historical data. The difference with the historical rate of change can be explained either by a different level of uncertainty than the one used in the simulations (*ceteris paribus* an increase of σ will generate conservatism), or by a different gap on objective and current state value (*ceteris paribus* an increase of this gap will increase the aggressiveness of the monetary policy).

Moreover, in a time horizon from 1962 to 2008, we compare at each period of time k what would have been the next period CVIU monetary policy and the LQR one. We compare these values with the historical ones. We made two CVIU's run defined by $\overline{\sigma} = 0.8$ and $\sigma \in \{0.3, 0.8\}$. The same vector of gaussian noises ϵ were used for each simulation. The results are shown in Figure 6.

[3]We used an empirical state space estimated up to the first quarter of 2006 for the run started at 2006.1: $\alpha_0 = 0.608$, $\alpha_1 = 0.108$, $\alpha_2 = 0.236$, $\beta_0 = 0.038$, $\beta_1 = -0.075$, $\beta_2 = 0.000$, $\delta = 0.106$, $\gamma_0 = 0.121$,$\gamma_1 = -0.195$ and $\lambda = 0.896$.

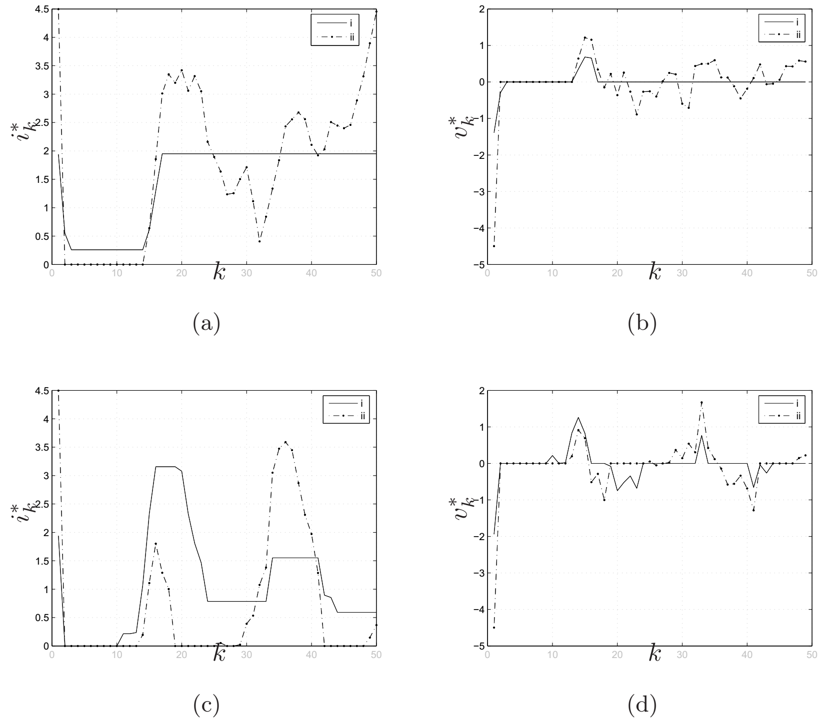
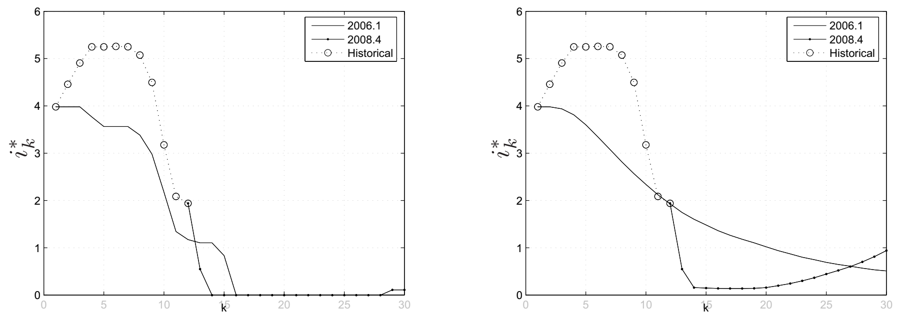


Figure 4: Comparison between realizations of the system considering the following scenarios: (i) $\overline{\sigma} = 0.8$ and $\sigma = 0.3$ (a,b) or $\sigma = 0.1$ (c,d); (ii) $\overline{\sigma} = 0.8$ and $\sigma = 0$ (a,b,c,d), which is the standard LQR case.

Observe first that, as expected, the LQR policy does worst than the CVIU one for all σ. Second, the CVIU interest rates are closer to the real ones when uncertainty is high. This result means that the Federal Reserve Bank monetary policy was most likely conducted in an highly uncertain world, mainly in the 70's, up to the middle of the 80's, where the inflation and interest rates were high and increasing. Third, although the CVIU policy is close to the real ones in the low inflation regime (mainly from the middle of the 80's to the 00's), this is not true in the 70's. In order to improve these results one must add non constant and asymmetric standard deviations σ, and/or modify the cost weights (Q).

3.5 A Sensitivity Analysis

From our previous results, one could ask whether or not the FED was running its monetary policy under high or low level of uncertainty. Although our theoretical model is not directly formulated as an econometric model, we try to answer this question based on the following steps. First, we ran



(a) (b)

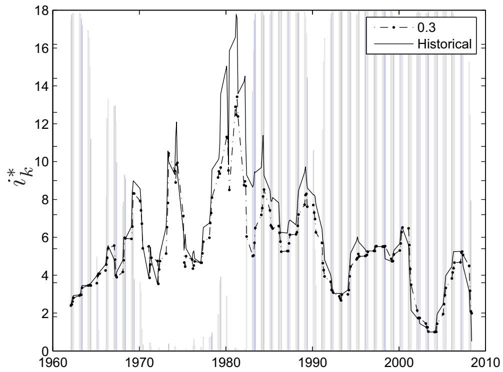
Figure 5: Comparison between realizations of the system for $n = 50$ considering $t_0 \in \{2006.1, 2008.4\}$: one isolated run (left), average run (right).

a Monte Carlo analysis (10000 runs), where at each run we randomly draw $\sigma_\pi, \sigma_y, \overline{\sigma}_\pi, \overline{\sigma}_y \sim N[0, 1]$. Then we compare[4] the results of our theoretical model with real data taken from 1962 (first quarter) to 2008 (fourth quarter). That is, we test the values of the uncertainty parameters, $\overline{\sigma}_i$ and σ_i, that allow the CVIU solution to be the closest of the real ones. We add the following constraint to our previous simulations: $\epsilon_k = \epsilon$. We assume that only one similar shock exists that affects both equations in a similar way. This last assumption does not affect the general results.

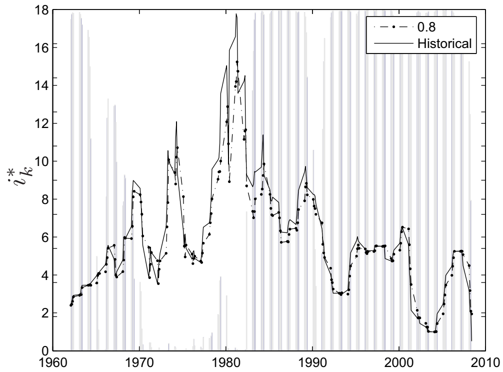
The results of the Monte Carlo simulations are plotted using boxplots (see Figures on Appendix A) were we underline the mean trend over different quintile of the uncertainty parameters. Furthermore, we carried out some regressions in order to analyze more precisely the impacts of the different sources of uncertainty ($\overline{\sigma}_i$, σ_i) on the discrepancy between real data and our theoretical estimations (see Appendix B). From Tables 2-4 and Figures 8-10 the following remarks can be made.

- Additive uncertainty appears to be less important than multiplicative uncertainty.
- We observe that smaller values in the interest rate errors are linked to highest values of both additive and multiplicative uncertainty parameters. This confirms the result of Figure 6b. This may primarily be interpreted as the fact that the interest rates policy is conducted in a world of uncertainty.
- One should also notice that the reduction in the variance of the errors when the value of the multiplicative uncertainty parameter increases

[4]The comparison is made based on the measure of the total square of error of the one-period ahead forecast: $\sum_{t=1}^{N}(\xi_t)^2$, with $\xi_t = x_{t|t-1} - x_t$



(a)



(b)

Figure 6: Comparison between theoretical interest rate values and historical ones considering the following scenarios: $\overline{\sigma} = 0.8$ and (a) $\sigma = 0.3$; (b) $\sigma = 0.8$ and, (c) $\sigma = 0$. Grey bars highlight lower historical inflation regime.

is mainly due to the property of our theoretical model that says to do less (and sometimes nothing).

- The inflation rate has a better fit when uncertainty is high on GDP rather on inflation. Increasing the multiplicative inflation uncertainty increases the errors. A possible interpretation is that the activism of the FED was stronger than it should be according to the CVIU solutions. This is a possible interpretation we have already noted in Section 2.2.

- Similarly, the errors on GDP gap are smaller when uncertainties lay on inflation rather on GDP.

As shown by Figures 11-14 in Appendix C, these results are not modified by a change in the loss function parameters Q and r.

3.6 Sensitivity Analysis à la Diebold-Mariano

Our previous results are based on mean square errors minimization. We analyze how these results are modified when using a more sophisticated method. That is, we rank in a vector the different simulations using an ordinal procedure. Then, we analyze a possible correlation between the value of a given parameter and the order of the simulation in this rank vector. We use the test of predictive accuracy proposed by Diebold and Mariano (1995) in order to construct the rank vector.

Assume that two different values for σ's vector involves two different models. The interest rate forecast errors from the two models are

$$\xi_{1,t+1|t} = i_{t+1} - i_{1,t+1|t} \tag{24}$$

$$\xi_{2,t+1|t} = i_{t+1} - i_{2,t+1|t} \tag{25}$$

These one-step forecast are computed for $t = t_0, .., T$, where $t_0 = 1961.1$ and $T = 2008.4$, for a total of $T_0 = 188$ forecasts. The accuracy of each forecast is measured by a particular loss function. We restrict ourself to the absolute error loss:

$$L(\xi_{i,t+h|t}) = |\xi_{i,t+h|t}| \tag{26}$$

As in Diebold and Mariano (1995), one may test the null hypothesis of equality performance

$$H_0 : E[L(\xi_{1,t+h|t})] = E[L(\xi_{2,t+h|t})] \tag{27}$$

against the alternative

$$H_0 : E[L(\xi_{1,t+h|t})] \neq E[L(\xi_{2,t+h|t})] \tag{28}$$

Using

$$d_t = L(\xi_{1,t+h|t}) - L(\xi_{2,t+h|t}) \tag{29}$$

one can rewrite the null hypothesis as

$$H_0 : E[d(t)] = 0 \tag{30}$$

The Diebold-Mariano test statistic is

$$S = \frac{\bar{d}}{\sqrt{\frac{\hat{V}(\bar{d})}{T}}} \tag{31}$$

where

$$\overline{d} = \frac{1}{T}\sum_{t=1}^{T} d_t \quad (32)$$

$$\hat{V}(\overline{d}) = \gamma_0 + 2\sum_{j=1}^{T} \gamma_j \quad (33)$$

with $\gamma_j = cov(d_t, d_{t-j})$.

Rather than testing the null hypothesis using the result of S values in a normal law, we compare the S values between two models. When a model fits better than another one, one should rank this model in a better place. So, we construct a rank vector based on the results of the Diebold-Mariano test. The Pseudo-Code of the algorithm we used is as follows, where a total number of runs N was done:

- BEGIN
- Initializations:
  - 1) Creation of a rank vector $(0_{1\times N})$.
  - 2) The loop counter is set to $k = 1$.
  - 3) A vector of run's number is created $VecNumb = [1, 2, .., N]$.
- Loop:
  - 1) Initialization: the best run is set to $i = 1$.
  - 2) While $i \leq N - k$ do
    - a) Test the run $VecNumb(i)$ against other runs $VecNumb(j)$, with $j > i$ and $j \leq N - k + 1$.
    - b) If $Test(VecNumb(i)) < Test(VecNumb(j))$, the best run is set to j.
    - c) Go back to a) with $i \leftarrow j$.
  - 3) Set the loop counter to $k \leftarrow k + 1$.
  - 4) Set the score $N - k + 1$ at the position $VecNumb(i)$ in the rank vector.
  - 5) Reduce the order of $VecNumb$: $VecNumb \leftarrow VecNumb - Vecnumb(i)$.
  - 6) Go Back to 1).
- END

First notice that the higher the number in the rank vector, the higher the accuracy of the model. Figure 7 shows the link between the values in the rank vector and the different values of σ's vector. This underlines the accuracy of the model with a high level of uncertainty since highest uncertainties involve highest rank order. The main positive impact of increasing uncertainty is due to multiple uncertainty on GDP. That is, the FED policy is compatible with our CVIU approach if, and only if, the uncertainty facing the FED is higher for GDP than for inflation. A natural counterintuitive corollary of this result is that the FED thinks it is more able to manage inflation than GDP.

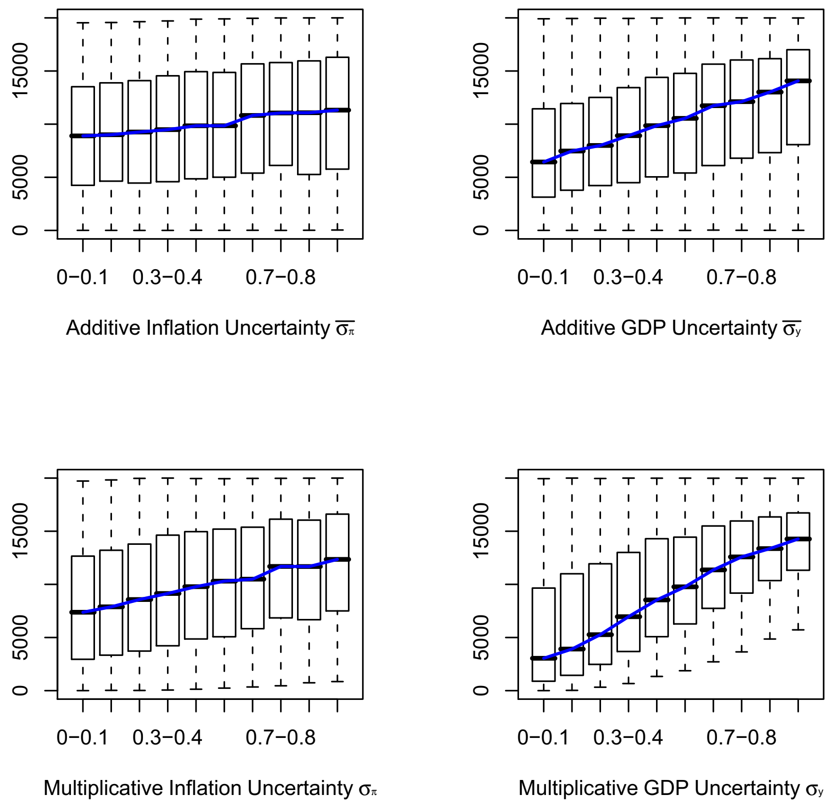


Figure 7: Relationship between Diebold-Mariano test value and the σ's vector values.

4 Conclusion

In this paper, we developed a theoretical framework and the optimal control strategy for CVIU systems associated with a convex cost functional. The convexity of the cost-to-go functions was asserted, making it simpler to obtain the optimal policy using dynamic programming. Since the state equation is not differentiable, an algorithm for determining the optimal policy was described using generalized gradients. The optimal strategy yields that the state-space will be divided into three disjoint regions, representing the regions where the optimal control policy is to increase, decrease, or maintain the previous input. Furthermore, for the case where the Cost Functional is differentiable, it was asserted that a region where no variation is optimal will always exist. This characteristic of the optimal policy is intuitively sound, since in many real-world problems, in face of the uncertainty generated by changing the control policy, the best strategy is to maintain the same policy as before. The presented CVIU model and analysis can be directly applied to numerous practical scenarios, ranging from monetary policy problems to medicine and biology and, in general, to problems for which a complete dynamic model is too complex to be feasible.

From our numerical analysis, we obtain two main results. First, and on the last two decades, monetary policy was mainly conducted on the basis of an uncertain (multiple) world. Second, the activism of the FED was stronger that it should be following our CVIU approach unless we admit that uncertainty mainly occurs in GDP dynamics rather than in inflation one. Our result is similar to the one of Tillmann (2008) who shows that the optimal monetary policy under parameter uncertainty can motivate a non-linear interest rate rule which involves that the policy response to inflation becomes stronger, the higher the inflation rate and the larger the output gap.

Finally, a comparison between our theoretical CVIU approach and the real data lead to two general remarks. First, we known from regime switching modelization (see Pardo, Rautureau, and Vallée (2009)) that there have been at least two historical regimes of inflation: a high level one (mainly in the 70's) and a low one (mainly the 90's and 00's). One should question how our results are modified if evaluated in such a switching regime framework. Second, and obviously, a link exists between our CVIU approach and signalling monetary game theory (Canzoneri (1985), Walsh (1999)). A characteristic of these "political business cycle models" is that the central bank knows the signaling effects of its monetary policy on private agents' expectations. By losing credibility, monetary policy can increase GDP and inflation variability, and thus increase the level of uncertainty. Proposing a link between regime switching modelization (e.g. jump Markov model) and monetary signalling game is the focus of our current research.

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A Figures of the Sensitivity Analysis

A.1 Inflation Rate

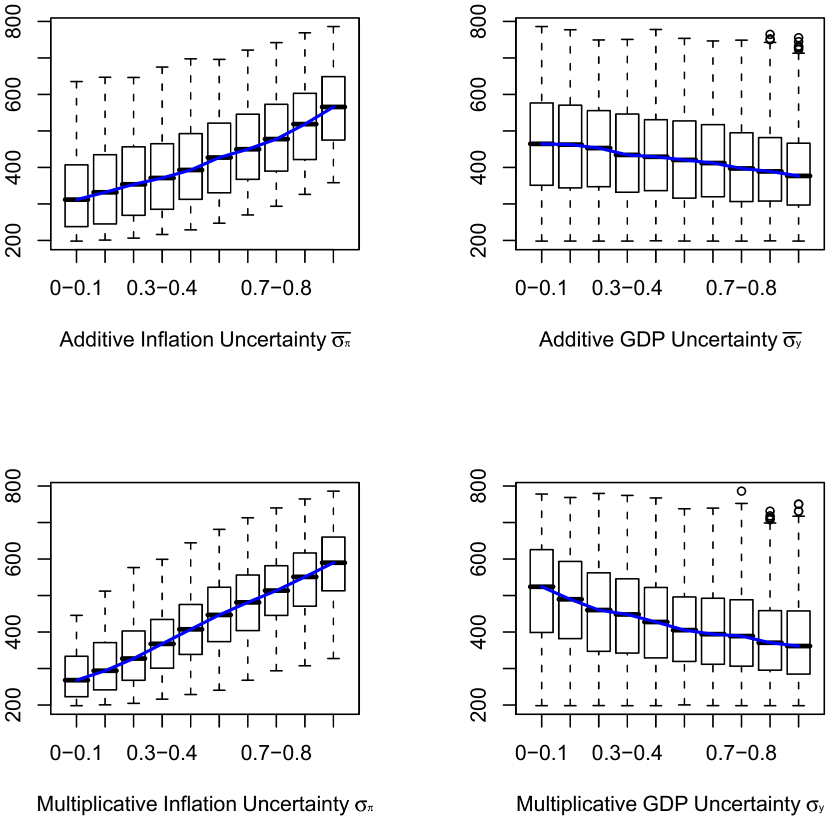

Figure 8: Impact of uncertainty on the sum of the total square difference to the real inflation rate.

A.2 Interest Rate

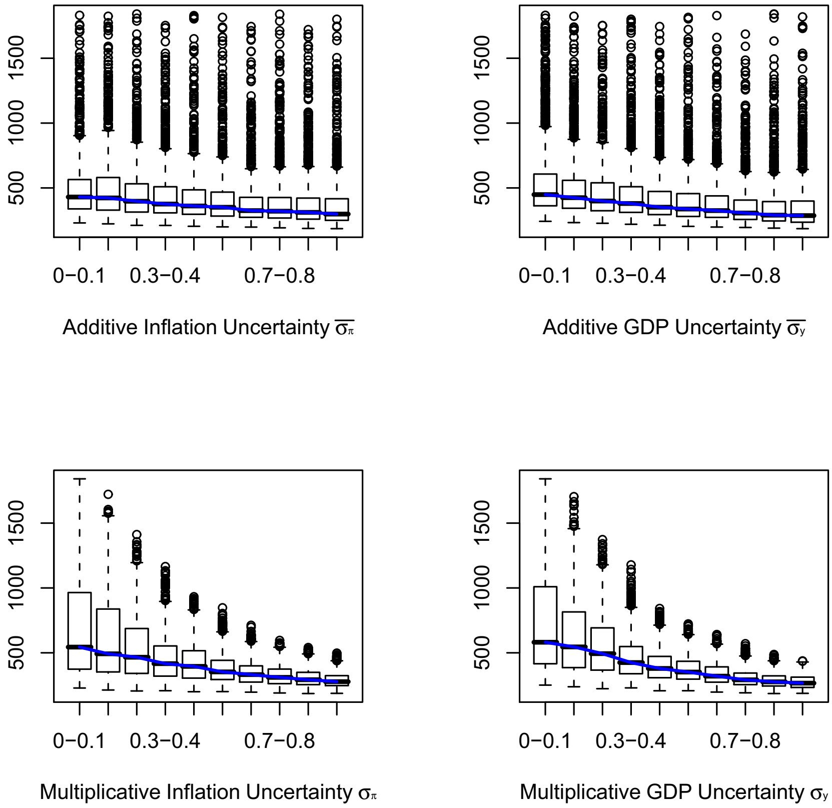

Figure 9: Impact of uncertainty on the sum of the total square difference to the real interest rate.

A.3 Output Gap

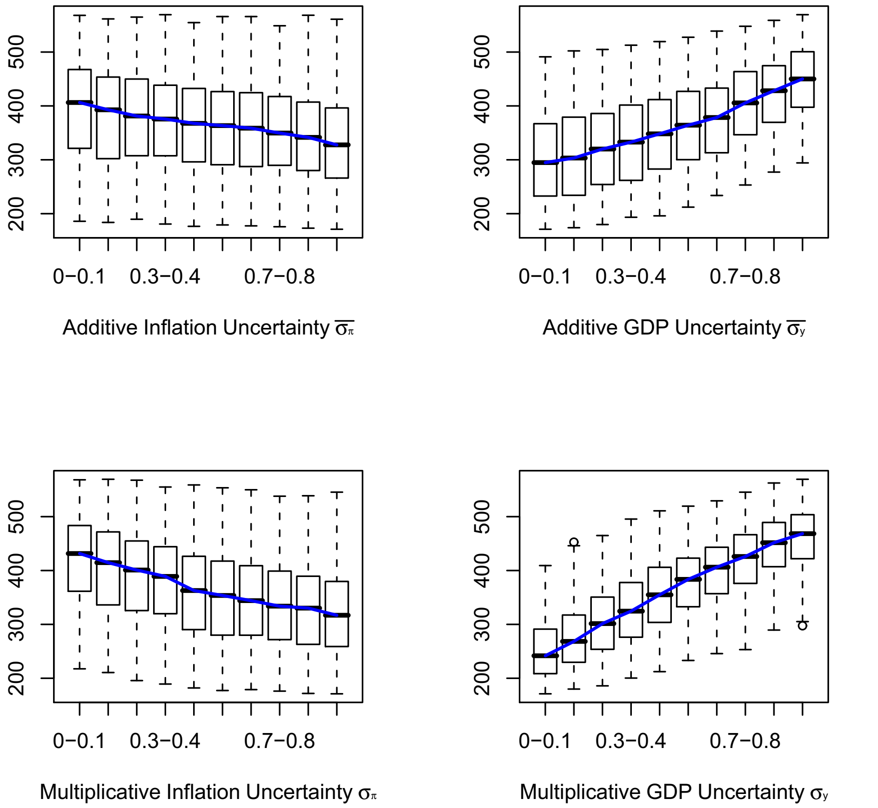

Figure 10: Impact of uncertainty on the sum of the total square difference to the real output gap.

B Some Regression Statistics

All the regressions have been carried out using the "lm" procedure of the R software.

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Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
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B.1 Inflation Rate Errors: $\Sigma_\pi = \sum_{t=1}^{N}(\pi_{t|t-1} - \pi_t)^2$.

Σ_π	Estimate	Std. Error	t-value	$Pr(>\|t\|)$	Adjusted R-squared
Constant case					0.955
(Intercept)	240.303	1.012	237.5	$< 2e-16***$	
$\overline{\sigma}_\pi$	253.120	0.977	259.1	$< 2e-16***$	
$\overline{\sigma}_y$	-82.055	0.968	-84.8	$< 2e-16***$	
σ_π	344.315	0.981	350.9	$< 2e-16***$	
σ_y	-140.103	0.982	-142.6	$< 2e-16***$	

Table 2: Impact of the values of the uncertainty parameters on Σ_π

B.2 Interest Rate Errors: $\Sigma_i = \sum_{t=1}^{N}(i_{t|t-1} - i_t)^2$.

Σ_i	Estimate	Std. Error	t-value	$Pr(>\|t\|)$	Adjusted R-squared
Constant case					0.676
(Intercept)	1083.82	5.11	212.2	$< 2e-16***$	
$\overline{\sigma}_\pi$	-160.63	4.93	-32.6	$< 2e-16***$	
$\overline{\sigma}_y$	-192.79	4.89	-39.5	$< 2e-16***$	
σ_π	-444.89	4.95	-89.8	$< 2e-16***$	
σ_y	-492.80	4.96	-99.4	$< 2e-16***$	

Table 3: Impact of the values of the uncertainty parameters on Σ_i

B.3 GDP Gap Errors: $\Sigma_y = \sum_{t=1}^{N} (y_{t|t-1} - y_t)^2$.

Σ_y	Estimate	Std. Error	t-value	$Pr(>\|t\|)$	Adjusted R-squared
Constant case					0.967
(Intercept)	256.174	0.588	436	$< 2e-16***$	
$\overline{\sigma}_\pi$	-61.808	0.567	-109	$< 2e-16***$	
$\overline{\sigma}_y$	154.532	0.562	275	$< 2e-16***$	
σ_π	-111.043	0.570	-195	$< 2e-16***$	
σ_y	233.825	0.571	410	$< 2e-16***$	

Table 4: Impact of the values of the uncertainty parameters on Σ_y

C Impact of Q and r

We ran 20000 simulations with $Q_{\pi,\pi} = 1$. At each run we randomly drew $\sigma_\pi, \sigma_y, \overline{\sigma}_\pi, \overline{\sigma}_y \sim [0, 1]$, $Q_{\pi,\pi} = 1$, $Q_{y,y} \in [0.1, 10]$, and $r \in [0, 1.5]$.

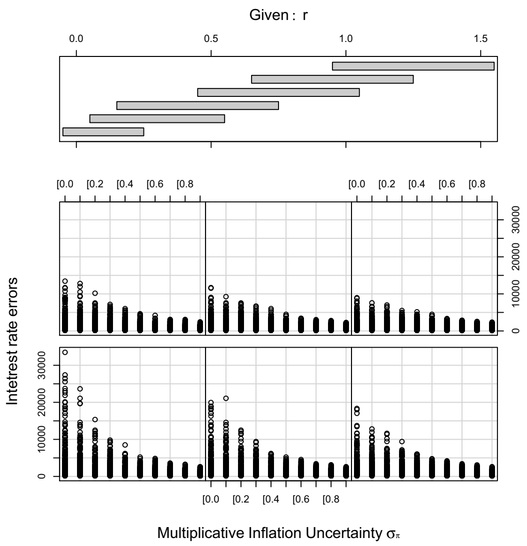

Figure 11: Sensitivity Analysis: Impact of r and σ_π on Σ_i.

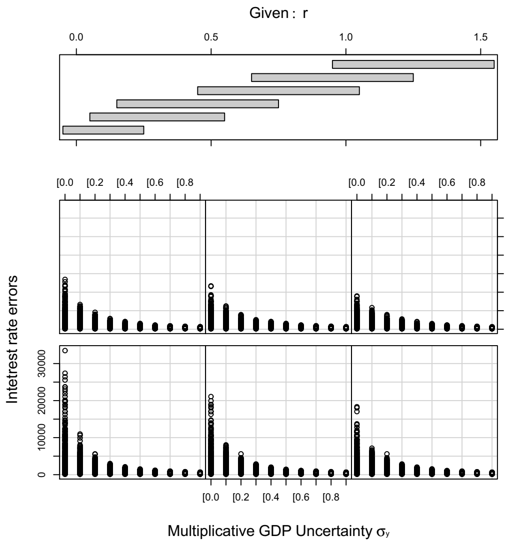

Figure 12: Sensitivity Analysis: Impact of r and σ_y on Σ_i.

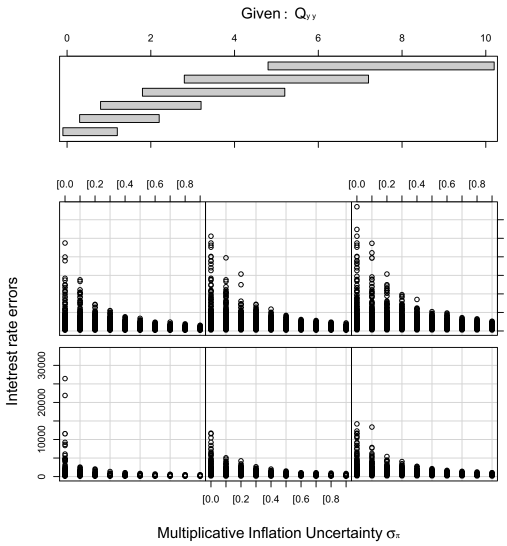

Figure 13: Sensitivity Analysis: Impact of $Q_{y,y}$ and σ_π on Σ_i.

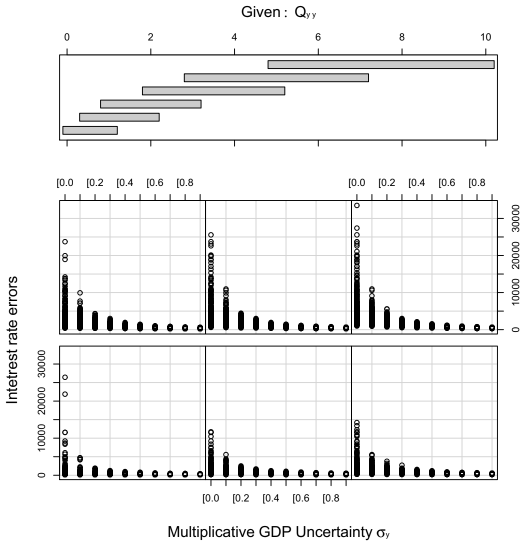

Figure 14: Sensitivity Analysis: Impact of $Q_{y,y}$ and σ_y on Σ_i.