A Method for Implementing Counterfactual Experiments in Models with Multiple Equilibria

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Abstract

This paper proposes a method for implementing counterfactual experiments in estimated models that have multiple equilibria. The method assumes that the researcher does not know the equilibrium selection mechanism and wants to impose minimum restrictions on it. Our key assumption is that the equilibrium selection function does not jump discontinuously between equilibria as we change marginally the structural parameters of the model. Under this assumption, we show that, although the equilibrium selection function is unknown, the researcher can obtain an approximation of this function in a neighborhood of the estimated values of the structural parameters. Under the additional assumption that the counterfactual equilibrium is stable, this approximation can be combined with iterations in the equilibrium mapping to obtain the exact counterfactual equilibrium. We illustrate the differences between our approach and other methods, such as the selection of a counterfactual equilibrium that is closer to the equilibrium in the data, and equilibrium mapping iterations using the equilibrium in the data as the initial value. We show that, in general, these alternative methods are not consistent with the assumption that the equilibrium selection mechanism is continuous with respect to the structural parameters.

Keywords: Structural models with multiple equilibria; Counterfactual experiments; Equilibrium selection.

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1 Introduction

Multiplicity of equilibria is a prevalent feature in static and dynamic games and in general equilibrium models. Models with multiple equilibria do not have a unique reduced form, and this indeterminacy poses practical estimation problems. In the context of discrete games with incomplete information, recent papers have proposed two-step and sequential estimators that significantly simplify the estimation of these models (Aguirregabiria and Mira, 2007; Bajari, Benkard and Levin, 2007; and Pesendorfer and Schmidt-Dengler, 2008). Nevertheless, the indeterminacy problem associated with multiple equilibria still remains an issue when the researcher wants to use the estimated model to predict the effects of counterfactual changes in the structural parameters. Although we can use the data to identify which of the multiple equilibria is the one observed in the data, we do not know which equilibrium will be selected in a counterfactual scenario. In some contexts, a possible approach for dealing with this issue is to calculate all of the equilibria in the counterfactual scenario and then draw conclusions that are robust to whatever equilibrium is selected. However, this approach is of very limited applicability because the different equilibria typically provide ambiguous or even contradictory predictions for the effects we want to measure. Given that one of the most attractive features of structural models is the possibility of implementing counterfactual experiments, this is a very important issue in structural econometrics. This issue is clearly illustrated in the recent literature on empirical dynamic oligopoly games. Most applications in this area either do not present counterfactual experiments (Collard-Wexler, 2006, and Sweeting, 2007), or ignore the issue of multiple equilibria in the counterfactual model (Ryan, 2009, see footnote 32 in page 49; and Dunne et al, 2009, see pages 33-34).

This paper proposes a simple approach for dealing with multiple equilibria when undertaking counterfactual experiments with an estimated model. Under the assumption that the equilibrium selection mechanism, which is unknown to the researcher, is a smooth function of the structural parameters, we show how to obtain a Taylor approximation of the counterfactual equilibrium. More specifically, we show that, although the equilibrium selection function is unknown, the Jacobian matrix of that function, evaluated at the estimated equilibrium, depends on objects that the researcher knows.

As in any Taylor approximation, the approximation error has the same order of magnitude as the distance between the factual and the counterfactual parameters. Therefore, this approach can be inaccurate when the counterfactual experiment does not imply marginal changes in the parameters. For these cases, we propose to combine the Taylor approximation with iterations in the equilibrium mapping. The idea is that the Taylor approximation can be far away from the counterfactual equilibrium but close enough to lie within the dominion of attraction of that equilibrium.

We illustrate the differences between our approach and other methods, such as the selection of a counterfactual equilibrium that is closer to the equilibrium in the data, and equilibrium mapping iterations using the equilibrium in the data as the initial value. In general, these alternative methods are not consistent with the assumption that the equilibrium selection mechanism is continuous with respect to the structural parameters.

2 Model

Let $\mathbf{y} \in Y$ and $\mathbf{x} \in X$ be two vectors of random variables with discrete and finite support.[1] Let $\mathbf{P}_0 \equiv \{\Pr(\mathbf{y}|\mathbf{x}) : (\mathbf{y},\mathbf{x}) \in Y \times X\}$ be a vector with the probability distribution of $\mathbf{y}$ conditional to $\mathbf{x}$ in the population under study. The structural model is a parametric family of probability distributions $\pi(\mathbf{y}|\mathbf{x},\boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Theta$ is a vector of K parameters, and $\Theta \subset \mathbb{R}^K$ is a compact set. Let $\Pi(\boldsymbol{\theta})$ be the vector with the probability distribution of $\mathbf{y}$ conditional to $\mathbf{x}$ in the model for a value $\boldsymbol{\theta}$ of the structural parameters: i.e., $\Pi(\boldsymbol{\theta}) \equiv \{\pi(\mathbf{y}|\mathbf{x},\boldsymbol{\theta}) : (\mathbf{y},\mathbf{x}) \in Y \times X\}$. The probability distribution $\Pi(\boldsymbol{\theta})$ is implicitly defined as the solution of a fixed point problem. Let $\Psi(\boldsymbol{\theta},\mathbf{P})$ be a fixed-point or equilibrium mapping from $\Theta \times [0,1]^{|X||Y|}$ into $[0,1]^{|X||Y|}$ such that $\Psi(\boldsymbol{\theta},\mathbf{P}) \equiv \{\psi(\mathbf{y}|\mathbf{x},\boldsymbol{\theta},\mathbf{P}) : (\mathbf{y},\mathbf{x}) \in Y \times X\}$. The vector $\Pi(\boldsymbol{\theta})$ is a fixed point of $\Psi(\boldsymbol{\theta},.)$, i.e., $\Pi(\boldsymbol{\theta}) = \Psi(\boldsymbol{\theta},\Pi(\boldsymbol{\theta}))$. However, for some values of $\boldsymbol{\theta}$, the mapping $\Psi(\boldsymbol{\theta},.)$ can have more than one fixed point. That is, the model can have multiple equilibria for some values of the structural parameters. We use $\Gamma(\boldsymbol{\theta})$ to denote the set of equilibria associated with $\boldsymbol{\theta}$. We know that $\Pi(\boldsymbol{\theta})$ belongs to $\Gamma(\boldsymbol{\theta})$ but we do not make any additional assumption

[1] We describe our approach in the context of a class of models in which all the variables have a discrete and finite support. This is convenient because we can use standard derivatives to construct Taylor approximations. However, it is possible to extend this approach to models where variables have continuous support by using Banach spaces and Fréchet derivatives.

on how $\Pi(\boldsymbol{\theta})$ is selected within the set $\Gamma(\boldsymbol{\theta})$. This class of econometric models includes as particular cases discrete models with social interactions (Brock and Durlauf, 2001), quantal response games (McKelvey and Palfrey, 1995), and static and dynamic games of incomplete information (Bajari et al., 2009, and Doraszelski and Satterthwaite, 2009), among others.

Let $\boldsymbol{\theta}_0$ be the true value of $\boldsymbol{\theta}$ in the population under study. The model establishes that $\mathbf{P}_0 = \Pi(\boldsymbol{\theta}_0)$. Suppose that $\mathbf{P}_0$ and $\boldsymbol{\theta}_0$ are point-identified given a random sample on $\{\mathbf{y},\mathbf{x}\}$. Let $\hat{\boldsymbol{\theta}}_0$ and $\hat{\mathbf{P}}_0$ be our consistent estimates of $\boldsymbol{\theta}_0$ and $\mathbf{P}_0$, respectively. Let $\boldsymbol{\theta}_*$ be a value of the vector of structural parameters that is different to $\hat{\boldsymbol{\theta}}_0$. We denote $\boldsymbol{\theta}_*$ as the vector of counterfactual values of the structural parameters. The researcher wants to obtain the counterfactual equilibrium $\mathbf{P}_*$ associated with $\boldsymbol{\theta}_*$, i.e., $\mathbf{P}_* = \Pi(\boldsymbol{\theta}_*)$, and compare this equilibrium with the one estimated from the data, $\hat{\mathbf{P}}_0$. However, although the researcher can calculate the set of equilibria $\Gamma(\boldsymbol{\theta}_*)$ (i.e., the set of fixed points of the mapping $\Psi(\boldsymbol{\theta}_*, .)$), he does not know which of these equilibria is $\mathbf{P}_*$.

It is clear that we need additional information/structure to select $\mathbf{P}_*$ from among the set of equilibria $\Gamma(\boldsymbol{\theta}_*)$. A possible approach might be to impose restrictions on the characteristics of the equilibrium $\mathbf{P}_*$ (e.g., stability, symmetry, Pareto optimality, maximum payoffs for a certain player) that define a subset of $\Gamma(\boldsymbol{\theta}_*)$ such that all the equilibria in that subset provide similar predictions. That is, we may specify an *equilibrium selection mechanism* that selects a single equilibrium, or a very reduced set of equilibria, in $\Gamma(\boldsymbol{\theta}_*)$. However, in most of the applications, these restrictions may be difficult to justify. The researcher would want to have a method for implementing counterfactual experiments that does not require these additional assumptions.[2] In this paper, we consider that the researcher is not willing to impose these restrictions. We propose an approach that imposes minimum conditions on the characteristics of the equilibrium $\mathbf{P}_*$. Assumptions 1 and 2 specify our restrictions on the model.

ASSUMPTION 1: Ψ *is twice continuously differentiable in* $\boldsymbol{\theta}$ *and* $\mathbf{P}$.

ASSUMPTION 2: The equilibrium selection mechanism is such that $\Pi(\boldsymbol{\theta})$ *is a continuous differentiable function within a convex subset of* Θ *that includes* $\hat{\boldsymbol{\theta}}_0$ *and* $\boldsymbol{\theta}_*$.

[2] In fact, if these restrictions were plausible, the researcher would incorporate them into his model to obtain more precise estimates of the structural parameters.

Our approach is agnostic with respect to the equilibrium selection mechanism. We assume that there is such a mechanism, that it is a function, and that it does not "jump" between the possible equilibria when we move over the parameter space. However, we do not specify any particular form for the equilibrium selection mechanism $\Pi(.)$.

Figure 1 illustrates Assumption 2 for a simple model where $\mathbf{P}$ is a scalar. The three curves represent the equilibrium mapping for three different values of the vector of structural parameters, say $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\boldsymbol{\theta}_3$, such that $||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||$ and $||\boldsymbol{\theta}_3 - \boldsymbol{\theta}_1||$ are small, i.e., marginal changes in the parameters. In that figure, an equilibrium is a value of $\mathbf{P}$ for which the curve meets with the 45-degree line, i.e., $\mathbf{P} = \Psi(\boldsymbol{\theta}, \mathbf{P})$. The set of equilibria associated with $\boldsymbol{\theta}_1$ is $\Gamma(\boldsymbol{\theta}_1) = \{\mathbf{P}_{A1}, \mathbf{P}_{B1}, \mathbf{P}_{C1}\}$. Suppose that $\Pi(\boldsymbol{\theta}_1) = \mathbf{P}_{C1}$, i.e., when the vector of structural parameters is $\boldsymbol{\theta}_1$, the selected equilibrium is the one with the highest value of $\mathbf{P}$. Assumption 2 implies that $\Pi(\boldsymbol{\theta}_2) = \mathbf{P}_{C2}$ and $\Pi(\boldsymbol{\theta}_3) = \mathbf{P}_{C3}$. That is, the equilibrium selection mechanism does not jump discontinuously from the 'high-type' equilibrium to the 'low-type' (i.e., P_A) or to the 'middle-type' (i.e., P_B).[3]

Assumption 2 seems a reasonable condition when the researcher is interested in evaluating the effects of a change in the structural parameters but keeping in the counterfactual the same equilibrium type as the one that generates the data.

3 Counterfactual experiments

We want to obtain the counterfactual equilibrium associated with $\boldsymbol{\theta}_*$, that we denote $\mathbf{P}_*$. Under Assumption 2, we know that $\hat{\mathbf{P}}_0 = \Pi(\hat{\boldsymbol{\theta}}_0)$ and $\mathbf{P}_* = \Pi(\boldsymbol{\theta}_*)$. Although $\hat{\mathbf{P}}_0$, $\hat{\boldsymbol{\theta}}_0$, and $\boldsymbol{\theta}_*$ are known to the researcher, $\mathbf{P}_*$ and the function $\Pi(.)$ are unknown. Under Assumptions 1-2, we can use a first order Taylor expansion to obtain an approximation to the counterfactual equilibrium $\Pi(\boldsymbol{\theta}_*)$ around the estimated vector $\hat{\boldsymbol{\theta}}_0$. We do not know the function Π. However, it is possible to use the equilibrium condition to obtain the Jacobian matrix $\partial\Pi(\hat{\boldsymbol{\theta}}_0)/\partial\boldsymbol{\theta}'$ in terms of derivatives of the equilibrium mapping evaluated at $(\hat{\mathbf{P}}_0, \hat{\boldsymbol{\theta}}_0)$. A Taylor expansion

[3]Some equilibrium 'types' may disappear when we move along the parameter space. Therefore, Assumption 2 establishes that the type of equilibrium $\Pi(\hat{\boldsymbol{\theta}}_0)$ does not disappear when we move from $\hat{\boldsymbol{\theta}}_0$ to $\boldsymbol{\theta}_*$.

of $\Pi(\boldsymbol{\theta}_*)$ around $\hat{\boldsymbol{\theta}}_0$ implies that:

$$\Pi(\boldsymbol{\theta}_*) = \Pi(\hat{\boldsymbol{\theta}}_0) + \frac{\partial \Pi(\hat{\boldsymbol{\theta}}_0)}{\partial \boldsymbol{\theta}'}(\boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0) + O(\| \boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0 \|^2) \tag{1}$$

Note that $\Pi(\hat{\boldsymbol{\theta}}_0) = \hat{\mathbf{P}}_0$ that is known. Taking into account that $\Pi(\hat{\boldsymbol{\theta}}_0) = \Psi(\hat{\boldsymbol{\theta}}_0, \Pi(\hat{\boldsymbol{\theta}}_0))$, differentiating this expression with respect to $\boldsymbol{\theta}$, and solving for $\partial \Pi(\hat{\boldsymbol{\theta}}_0)/\partial \boldsymbol{\theta}'$, we can represent this Jacobian matrix in terms of Jacobians of $\Psi(\boldsymbol{\theta}, \mathbf{P})$ evaluated at the estimated values $(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)$. That is,

$$\frac{\partial \Pi(\hat{\boldsymbol{\theta}}_0)}{\partial \boldsymbol{\theta}'} = \left(\mathbf{I} - \frac{\partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)}{\partial \mathbf{P}'} \right)^{-1} \frac{\partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)}{\partial \boldsymbol{\theta}'} \tag{2}$$

where $\mathbf{I}$ is the identity matrix. Solving expression (2) into (1), we have that:

$$\Pi(\boldsymbol{\theta}_*) = \hat{\mathbf{P}}_0 + \left(\mathbf{I} - \frac{\partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)}{\partial \mathbf{P}'} \right)^{-1} \frac{\partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)}{\partial \boldsymbol{\theta}'}(\boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0) + O(\| \boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0 \|^2) \tag{3}$$

Therefore, when $\| \boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0 \|^2$ is small, the vector $\tilde{\mathbf{P}}_* \equiv \hat{\mathbf{P}}_0 + (\mathbf{I} - \partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)/\partial \mathbf{P}')^{-1}$ $\partial \Psi(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)/\partial \boldsymbol{\theta}'$ $(\boldsymbol{\theta}_* - \hat{\boldsymbol{\theta}}_0)$ provides a good approximation to the true counterfactual equilibrium $\mathbf{P}_*$. Note that all the elements in the expression that describes $\tilde{\mathbf{P}}_*$ are known to the researcher.

In some applications, the counterfactual experiments of interest are far from being marginal changes in the parameters. In such a situation, a first order Taylor approximation could be inaccurate. Higher-order approximations to $\Pi(\boldsymbol{\theta}_*)$ can be used. It is possible to show that higher-order derivatives of $\Pi(.)$ at $\hat{\boldsymbol{\theta}}_0$ depend only on derivatives of Ψ at $(\hat{\boldsymbol{\theta}}_0, \hat{\mathbf{P}}_0)$, which are known to the researcher. However, in applications where the dimension of the vector $\mathbf{P}$ is large (e.g., dynamic games with heterogeneous players), the numerical computation of high-order derivatives of Ψ with respect to $\mathbf{P}$ can be computationally very demanding. An alternative approach for improving the accuracy of the Taylor approximation is to combine it with iterations in the equilibrium mapping. Suppose that $\mathbf{P}_*$ is an stable equilibrium. This implies that there is a neighborhood of $\mathbf{P}_*$, say $\mathcal{N}$, such that if we iterate in the equilibrium mapping $\Psi(\boldsymbol{\theta}_*, .)$ starting with a $\mathbf{P} \in \mathcal{N}$, then we converge to $\mathbf{P}_*$, i.e., if $\mathbf{P}_1 \in \mathcal{N}$ and $\mathbf{P}_{k+1} = \Psi(\boldsymbol{\theta}_*, \mathbf{P}_k)$ for $k \geq 1$, then $\lim_{k \to \infty} \mathbf{P}_k = \mathbf{P}_*$ (Judd, 1998, Theorem 5.4.2). The neighborhood $\mathcal{N}$ is called the *dominion of attraction* of the stable equilibrium $\mathbf{P}_*$. Suppose that the Taylor approximation is precise enough such that $\tilde{\mathbf{P}}_*$ belongs to the dominion of

attraction of $\mathbf{P}_*$. Then, by iterating in the equilibrium mapping $\Psi(\boldsymbol{\theta}_*, .)$ starting at $\tilde{\mathbf{P}}_*$ we will obtain the counterfactual equilibrium $\mathbf{P}_*$.

It is important to explain the differences between the method that we propose here and two alternative methods for calculating a counterfactual equilibrium. The first alternative method consists of iteration of the equilibrium mapping $\Psi(\boldsymbol{\theta}_*, .)$ starting with the equilibrium in the data. This approach will return the counterfactual equilibrium $\mathbf{P}_*$ *if only if* $\hat{\mathbf{P}}_0$ belongs to the dominion of attraction of $\mathbf{P}_*$. It should be clear that this condition is stronger than the one establishing that the Taylor approximation $\tilde{\mathbf{P}}_*$ belongs to the domination of attraction of $\mathbf{P}_*$.

A second alternative approach consists of calculating all the equilibria of the mapping $\Psi(\boldsymbol{\theta}_*, .)$ and then selecting as the counterfactual the equilibrium with the smallest Euclidean distance to $\hat{\mathbf{P}}_0$. In general, this approach is not consistent with Assumption 2 which establishes that the equilibrium selection mechanism does not jump. We illustrate this issue in Figures 2 and 3. By Assumption 2, the counterfactual equilibrium $\mathbf{P}_*$ has the same type as $\hat{\mathbf{P}}_0$. In Figures 2 and 3, $\mathbf{P}_*$ and $\hat{\mathbf{P}}_0$ are 'high-type' equilibria. In the example presented in Figure 2, $\mathbf{P}_*$ is also the equilibrium in $\Gamma(\boldsymbol{\theta}_*)$ that is closest (in Euclidean distance) to $\hat{\mathbf{P}}_0$. Therefore, our method and the "closest-equilibrium" method coincide in that case. However, in Figure 3, the closest equilibrium to $\hat{\mathbf{P}}_0$ is not $\mathbf{P}_*$ but rather the 'middle-type' equilibrium. In this example, the "closest-equilibrium" method implies an equilibrium selection function that is not continuous over $\langle \hat{\boldsymbol{\theta}}_0, \boldsymbol{\theta}_* \rangle$.

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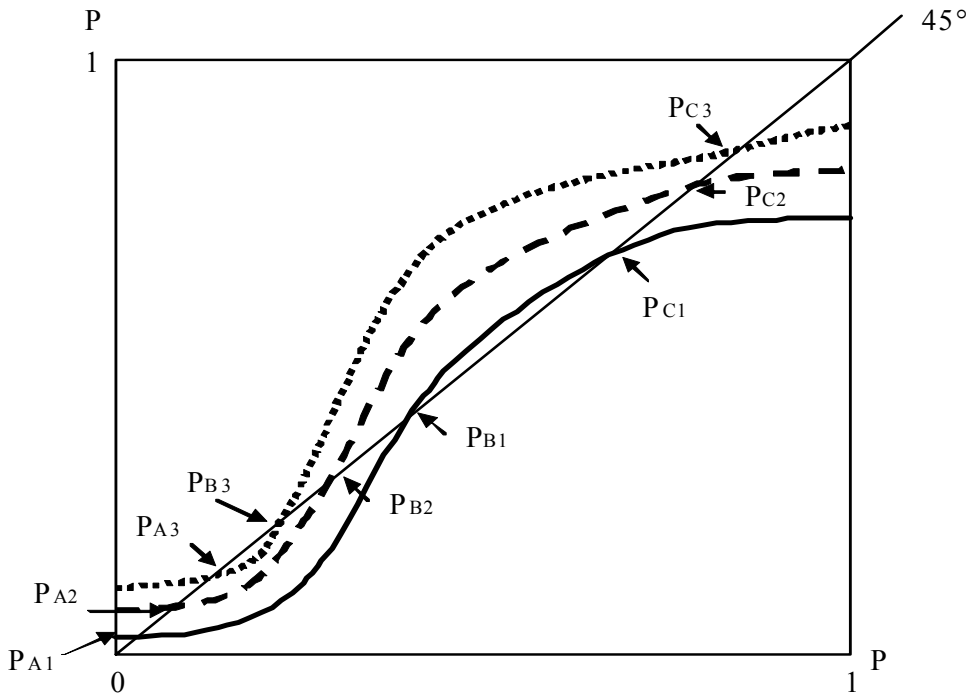
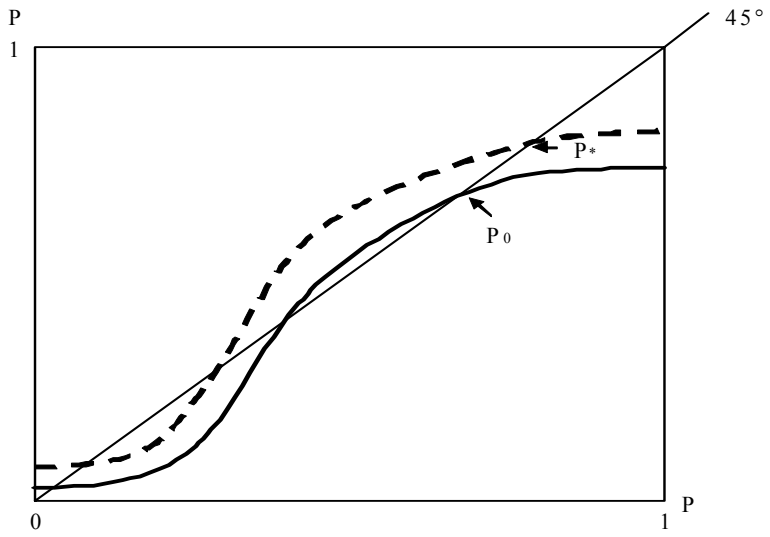
Figure 1



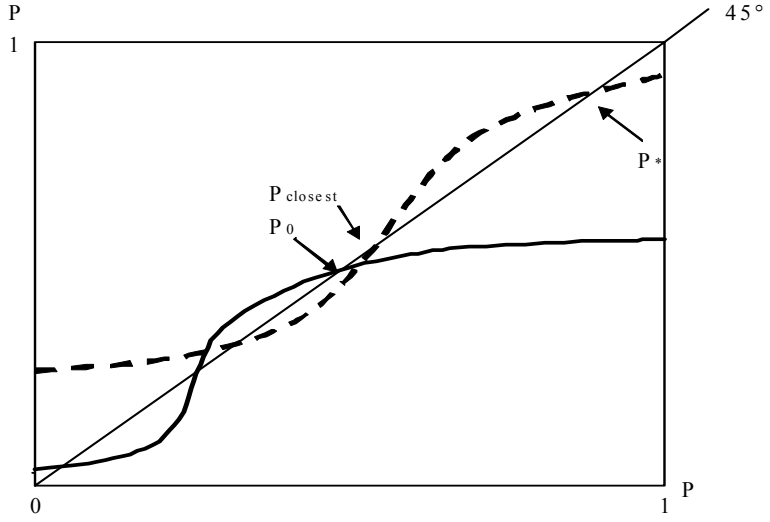
_____ Equilibrium mapping 1 (Equilibria: P_{A1}, P_{B1}, and P_{C1})
_ _ _ Equilibrium mapping 2 (Equilibria: P_{A2}, P_{B2}, and P_{C2})
…… Equilibrium mapping 3 (Equilibria: P_{A3}, P_{B3}, and P_{C3})

Figure 2



_____ Factual equilibrium mapping
_ _ _ Counterfactual equilibrium mapping
P_0 is the factual equilibrium
P_* is the counterfactual equilibrium

Figure 3



_____ Factual equilibrium mapping
_ _ _ Counterfactual equilibrium mapping
P_0 is the factual equilibrium
P_* is the counterfactual equilibrium
$P_{closest}$ is the equilibrium of the counterfactual mapping that is closest to P_0