

# A Mean-Variance Frontier in Discrete and Continuous Time

Paul A. Bekker

Department of Economics

University of Groningen

2004

Department of Economics, University of Groningen, P.O.Box 800, 9700 AV Groningen, The Netherlands.

Email: [P.A.Bekker@eco.rug.nl](mailto:P.A.Bekker@eco.rug.nl)

## Abstract

The paper presents a mean-variance frontier based on dynamic frictionless investment strategies in continuous time. The result applies to a finite number of risky assets whose price process is given by multivariate geometric Brownian motion with deterministically varying coefficients. The derivation is based on the solution for the frontier in discrete time. Using the same multiperiod framework as Li and Ng (2000), I provide an alternative derivation and an alternative formulation of the solution. It allows for a nice asymptotic formulation of the efficient hyperbola and its underlying efficient processes that applies in continuous time.

**Keywords:** multiperiod mean-variance frontier, discrete time, continuous time

# 1 Introduction

Dynamic mean-variance optimal solutions have been used, mainly in a univariate context, to study asset allocation and derivative pricing by, among many others, Richardson (1989), Duffie and Richardson (1991), Schäl (1994), Schweizer (1995), Bajoux-Besnainou and Portait (2002). In particular Li and Ng (2000) generalized Markowitz (1952, 1959). They provided an analytical expression for the mean-variance frontier along with optimal portfolio policy for a multivariate multiperiod portfolio selection problem where the returns of risky assets are assumed to be independent over time. The solution was hard to find due to problems of nonseparability in the sense of dynamic programming. Their solution scheme was to embed the original problem into a tractable auxiliary separable problem and to investigate the relationship between the solution sets. Leippold et al. (2004) use this approach to cover portfolios consisting of both assets and liabilities. They emphasize the use of orthogonal projections and provide a financial interpretation of the desired optimal policies.

Here I start by considering the same problem and derive the solution in a more direct way. Similar to the standard single-period problem the mean-variance frontier is fully described by three of the first two moments of two efficient strategies. As the moments of the returns of the risky assets are not path dependent, the strategy that minimizes the expected squared return is easily found. Its solution does not depend on the number of periods. So the problem reduces to finding a single moment of another efficient strategy. If there is no riskless asset present this problem is nontrivial. Still, relatively simple steps lead to its solution. The overall solution describes the three moments in terms of the mean-variance parameters that apply in the separate periods.

In a next step the limits of the relevant expressions are found as the number of periods increases, while the variances and expectations in the separate periods are shrunk at the same time. The underlying price process converges to geometric Brownian motion, with deterministically varying coefficients, and the mean-variance solution converges to the solution in continuous time. It generalizes the continuous-time efficient frontier of Zhou and Li (2000) to the case where a riskless asset need not be present. In a separate paper the solution is used to model the current

yield curve<sup>1</sup>. The implied mean-variance parameters, which are found by fitting yield curves, are shown to be closely related to the business cycle and to the equity risk premium.

## 2 The model and the solution in discrete time

Consider a finite number of assets whose price vector at time  $t_i$  is given by  $S_{t_i}$ ,  $i = 0, 1, \dots, n$ . Let the vector of gross returns over the separate periods be given by  $\theta_{t_i} = \text{Diag}(S_{t_{i-1}})^{-1} S_{t_i}$ ,  $i = 1, \dots, n$ . The returns are assumed to be independent over time with finite first and second moments. Consider frictionless trading and self-financing strategies. That is, if equity positions at time  $t_i$  are given in the vector  $\phi_{t_i}$ , and the portfolio value is given by  $V_{t_i} = \iota' \phi_{t_i}$ , where  $\iota$  is a vector of ones, then  $V_{t_i} = \iota' \phi_{t_i} = \theta'_{t_i} \phi_{t_{i-1}}$ ,  $i = 1, \dots, n$ . The returns of the strategies are given by  $R_{t_n} = V_{t_n} / V_{t_0}$ . Consider strategies with finite first and second moments. They form a linear space. Consequently, standard one-period mean-variance analysis and its results (cf. Cochrane 2001) apply to the dynamic framework as well.

### 2.1 The frontier

Let the first two moments be denoted by  $m = E(R_{t_n})$ ,  $s^2 = E(R_{t_n}^2)$  and let the variance be given by  $\sigma^2 = \text{Var}(R_{t_n})$ . The efficient frontier is usually formulated in terms of the Global Minimum Variance (GMV) portfolio. Let the strategy that minimizes  $\sigma^2$  have moments  $m^{\text{GMV}}$  and  $\sigma^{\text{GMV}2}$  and let  $\Gamma$  be a third parameter, then the frontier for efficient strategies, where  $\sigma^2$  is minimal conditional on  $m$ , can be formulated as

$$\sigma^2 = \sigma^{\text{GMV}2} + \left( \frac{m - m^{\text{GMV}}}{\Gamma} \right)^2. \quad (1)$$

Another formulation of the frontier is in terms of the Least Squared Return (LSR) strategy with moments  $m^{\text{LSR}}$  and  $s^{\text{LSR}2}$ , that minimizes  $s^2$ . It is given by

$$s^2 = s^{\text{LSR}2} + \left( \frac{m - m^{\text{LSR}}}{F} \right)^2. \quad (2)$$

---

<sup>1</sup>Based on approximate bond-replicating strategies that hedge against constant claims at varying maturities and by using appropriate risk premia for the residual risk, I arrive at a model for the current yield curve without making assumptions about the dynamic evolution of yield curves.

The relations between the two sets of parameters are easily found by using the optimality of GMV and LSR in (2) and (1), respectively:

$$\begin{aligned} m^{\text{GMV}} &= \frac{m^{\text{LSR}}}{1 - F^2}, \\ \sigma^{\text{GMV}2} &= s^{\text{LSR}2} - \frac{m^{\text{LSR}2}}{1 - F^2}, \\ \Gamma^2 &= \frac{F^2}{1 - F^2}. \end{aligned} \tag{3}$$

Another useful efficient strategy is found by minimizing  $\sigma^2/m^2$  and will be referred to as the Minimum Risk (MR) strategy. Its expected return is given by

$$m^{\text{MR}} = m^{\text{LSR}} + \frac{s^{\text{LSR}2}}{m^{\text{LSR}}} F^2. \tag{4}$$

## 2.2 The solution

Define the first two moments of the returns  $m_{t_i} \equiv E(\theta_{t_{i+1}})$  and  $\Omega_{t_i} \equiv E(\theta_{t_{i+1}}\theta'_{t_{i+1}})$ , which is assumed to be nonsingular, and let single period mean-variance parameters, for  $i = 0, \dots, n-1$ , be given by

$$s_{t_i}^{\text{lsr}2} = \frac{1}{l'\Omega_{t_i}^{-1}l}, \quad m_{t_i}^{\text{lsr}} = \frac{m'_{t_i}\Omega_{t_i}^{-1}l}{l'\Omega_{t_i}^{-1}l}, \quad f_{t_i}^2 = m'_{t_i} \left( \Omega_{t_i}^{-1} - \frac{\Omega_{t_i}^{-1}ll'\Omega_{t_i}^{-1}}{l'\Omega_{t_i}^{-1}l} \right) m_{t_i}. \tag{5}$$

The solution for the multi-period mean-variance frontier parameters of frontier (2), which will be derived in the next subsection, is given by

$$m^{\text{LSR}} = \prod_{i=0}^{n-1} m_{t_i}^{\text{lsr}}, \tag{6}$$

$$s^{\text{LSR}2} = \prod_{i=0}^{n-1} s_{t_i}^{\text{lsr}2}, \tag{7}$$

$$F^2 = \sum_{j=0}^{n-1} \left( f_{t_j}^2 \prod_{i=j+1}^{n-1} \frac{m_{t_i}^{\text{lsr}2}}{s_{t_i}^{\text{lsr}2}} \right), \tag{8}$$

where  $\prod_{i=0}^{n-1} m_{t_i}^{\text{lsr}2}/s_{t_i}^{\text{lsr}2} = 1$ . For  $n = 1$  the multi-period solution coincides with the one-period solution.

## 2.3 The derivation

Describing the LSR strategy and its moments is easy. As the returns are independent the minimization of  $s^2$  amounts to minimization in each period separately, i.e.

$$E(R_{t_n}^2) = \prod_{i=1}^n E \left\{ \left( \frac{\theta'_{t_i} \phi_{t_{i-1}}}{l' \phi_{t_{i-1}}} \right)^2 \right\}$$

is minimized by

$$\phi_{t_i}^{\text{LSR}} = \frac{\Omega_{t_i}^{-1} l}{l' \Omega_{t_i}^{-1} l} V_{t_i}^{\text{LSR}}, \quad (9)$$

$i = 0, \dots, n-1$ . Consequently,

$$\begin{aligned} m^{\text{LSR}} &= \prod_{i=1}^n E \left( \frac{\theta'_{t_i} \Omega_{t_{i-1}}^{-1} l}{l' \Omega_{t_{i-1}}^{-1} l} \right) = \prod_{i=1}^n \frac{m'_{t_{i-1}} \Omega_{t_{i-1}}^{-1} l}{l' \Omega_{t_{i-1}}^{-1} l} = \prod_{i=0}^{n-1} m_{t_i}^{\text{LSR}}, \\ s^{\text{LSR}2} &= \prod_{i=1}^n E \left\{ \left( \frac{\theta'_{t_i} \Omega_{t_{i-1}}^{-1} l}{l' \Omega_{t_{i-1}}^{-1} l} \right)^2 \right\} = \prod_{i=1}^n \frac{1}{l' \Omega_{t_{i-1}}^{-1} l} = \prod_{i=0}^{n-1} s_{t_i}^{\text{LSR}2}, \end{aligned}$$

which amounts to (6) and (7).

To derive (8) consider the strategy that minimizes  $E\{(V_{t_n} - 1)^2\}$ . As  $E V_{t_n} = V_{t_o} m$  and  $\text{Var } V_{t_n} = V_{t_o}^2 \sigma^2$ , it follows that  $E\{(V_{t_n} - 1)^2\} = (V_{t_o} m - 1)^2 + V_{t_o}^2 \sigma^2$ , which is minimized for fixed  $m$  and  $\sigma^2$  by  $V_{t_o} = m/s^2$ . For this optimal value of  $V_{t_o}$  the minimum of  $E\{(V_{t_n} - 1)^2\} = \sigma^2/s^2$  is found by investing  $V_{t_o}$  in the MR strategy that minimizes both  $\sigma^2/m^2$  and  $\sigma^2/s^2$ . Therefore, the portfolio value at time  $t_i$  will be indicated as  $V_{t_i}^{\text{MR}}$ . Cochrane (2001) refers to its return as the constant-mimicking return.

Alternatively, the initial value  $V_{t_o}^{\text{MR}} = m^{\text{MR}}/s^{\text{MR}2}$  can be found as follows. Due to the optimality of the portfolio,  $V_{t_n}^{\text{MR}} - 1$  is orthogonal to any other portfolio, in particular to  $R_{t_n}^{\text{LSR}}$ , i.e.

$$E \{ R_{t_n}^{\text{LSR}} (V_{t_n}^{\text{MR}} - 1) \} = 0. \quad (10)$$

Similarly, due to its optimality,  $R_{t_n}^{\text{LSR}}$  is orthogonal to any portfolio with initial value equal to 0, in particular to  $V_{t_n}^{\text{MR}}/V_{t_o}^{\text{MR}} - R_{t_n}^{\text{LSR}}$ ,

$$E \left\{ R_{t_n}^{\text{LSR}} \left( \frac{V_{t_n}^{\text{MR}}}{V_{t_o}^{\text{MR}}} - R_{t_n}^{\text{LSR}} \right) \right\} = 0. \quad (11)$$

Combining (10) and (11) the initial value is found as

$$V_{t_o}^{\text{MR}} = \frac{m^{\text{LSR}}}{s^{\text{LSR}2}}. \quad (12)$$

Consequently, the expected portfolio value at maturation is given by

$$\text{E}(V_{t_n}^{\text{MR}}) = \frac{m^{\text{MR}} m^{\text{LSR}}}{s^{\text{LSR}2}}. \quad (13)$$

Due to (4) the problem is solved once this expectation is expressed in the mean-variance parameters of the separate periods.

To achieve this goal the equity positions  $\phi_{t_i}^{\text{MR}}$  of the portfolio have to be found. To start with  $t_o$ ,  $\phi_{t_o}^{\text{MR}}$  satisfies  $\iota' \phi_{t_o}^{\text{MR}} = V_{t_o}^{\text{MR}}$  as given by (12). Now, the steps that led to this result can be repeated with  $\iota$  replaced by another, arbitrary vector  $\rho$ , say. That is, consider the LSR- $\rho$  portfolio that minimizes  $\text{E}\{(V_{t_n}^{\text{LSR}-\rho})^2\}$  under the restriction  $\rho' \phi_{t_o}^{\text{LSR}-\rho} = 1$ . Similar to (6) and (7) the first two moments of this portfolio are found as

$$\text{E}(V_{t_n}^{\text{LSR}-\rho}) = \frac{m_{t_o}' \Omega_{t_o}^{-1} \rho}{\rho' \Omega_{t_o}^{-1} \rho} \prod_{i=1}^{n-1} m_{t_i}^{\text{LSR}}, \quad (14)$$

$$\text{E}\{(V_{t_n}^{\text{LSR}-\rho})^2\} = \frac{1}{\rho' \Omega_{t_o}^{-1} \rho} \prod_{i=1}^{n-1} s_{t_i}^{\text{LSR}2}. \quad (15)$$

Furthermore, (10) holds if  $R_{t_n}^{\text{LSR}}$  is replaced by  $R_{t_n}^{\text{LSR}-\rho}$  and similar to (11),

$$\text{E}\left\{R_{t_n}^{\text{LSR}-\rho} \left(\frac{V_{t_n}^{\text{MR}}}{\rho' \phi_{t_o}^{\text{MR}}} - R_{t_n}^{\text{LSR}-\rho}\right)\right\} = 0 \quad (16)$$

must hold, and so, similar to (12),

$$\rho' \phi_{t_o}^{\text{MR}} = \rho' \Omega_{t_o}^{-1} m_{t_o} \prod_{i=1}^{n-1} \frac{m_{t_i}^{\text{LSR}}}{s_{t_i}^{\text{LSR}2}}.$$

As  $\rho$  is arbitrary, the initial equity positions of the MR portfolio are given by

$$\phi_{t_o}^{\text{MR}} = \Omega_{t_o}^{-1} m_{t_o} \prod_{i=1}^{n-1} \frac{m_{t_i}^{\text{LSR}}}{s_{t_i}^{\text{LSR}2}}. \quad (17)$$

To derive the equity positions for  $t_i$ ,  $i = 1, \dots, n-1$ , first consider the LSR- $i$  and MR- $i$  portfolios that are defined similar to the LSR and MR portfolios with the difference that the former portfolios start at  $t_i$  instead of  $t_o$ . Similar to (9) and (17) the initial equity positions are given by

$$\phi_{t_i}^{\text{LSR}-i} = \frac{\Omega_{t_i}^{-1} \iota}{\iota' \Omega_{t_i}^{-1} \iota}, \quad \phi_{t_i}^{\text{MR}-i} = \Omega_{t_i}^{-1} m_{t_i} \prod_{j=i+1}^{n-1} \frac{m_{t_j}^{\text{LSR}}}{s_{t_j}^{\text{LSR}^2}}. \quad (18)$$

The positions of the MR portfolio at time  $t_i$  will be different in general from  $\phi_{t_i}^{\text{MR}-i}$  due to the fact that the initial value of the MR- $i$  portfolio is not restricted, whereas  $\phi_{t_i}^{\text{MR}}$  should satisfy  $\iota' \phi_{t_i}^{\text{MR}} = V_{t_i}^{\text{MR}}$ .

So, the MR portfolio for  $t_j$ ,  $j = i, \dots, n$ , can be defined as the solution to the problem of minimizing  $E(V_{t_n} - 1)^2$  conditional on  $V_{t_i} = V_{t_i}^{\text{MR}}$ . As  $V_{t_n} = V_{t_n}^{\text{MR}-i} + (V_{t_n} - V_{t_n}^{\text{MR}-i})$  and due to the orthogonality of  $V_{t_n}^{\text{MR}-i} - 1$  to any other portfolio, in particular to  $V_{t_n} - V_{t_n}^{\text{MR}-i}$ , the definition of the MR portfolio for  $t_j$ ,  $j = i, \dots, n$  amounts to minimizing

$$E(V_{t_n} - 1)^2 = E(V_{t_n}^{\text{MR}-i} - 1)^2 + E(V_{t_n} - V_{t_n}^{\text{MR}-i})^2,$$

conditional on  $V_{t_i} = V_{t_i}^{\text{MR}}$ . Then, clearly,  $V_{t_i} - V_{t_i}^{\text{MR}-i}$  should be invested in the LSR- $i$  strategy, i.e.  $V_{t_j}^{\text{MR}} - V_{t_j}^{\text{MR}-i} = V_{t_j}^{\text{LSR}-i} (V_{t_i}^{\text{MR}} - V_{t_i}^{\text{MR}-i})$  for  $j = i, \dots, n$ . As a result we find the equity positions of the MR portfolio for all  $t_i$  equal to

$$\begin{aligned} \phi_{t_i}^{\text{MR}} &= \phi_{t_i}^{\text{MR}-i} + \phi_{t_i}^{\text{LSR}-i} (V_{t_i}^{\text{MR}} - V_{t_i}^{\text{MR}-i}) \\ &= \Omega_{t_i}^{-1} m_{t_i} \prod_{j=i+1}^{n-1} \frac{m_{t_j}^{\text{LSR}}}{s_{t_j}^{\text{LSR}^2}} + \frac{\Omega_{t_i}^{-1} \iota}{\iota' \Omega_{t_i}^{-1} \iota} \left( V_{t_i}^{\text{MR}} - \prod_{j=i}^{n-1} \frac{m_{t_j}^{\text{LSR}}}{s_{t_j}^{\text{LSR}^2}} \right), \\ &= v_{t_i} \prod_{j=i+1}^{n-1} \frac{m_{t_j}^{\text{LSR}}}{s_{t_j}^{\text{LSR}^2}} + w_{t_i} V_{t_i}^{\text{MR}}, \end{aligned} \quad (19)$$

where (18) has been used for the second equality and

$$v_{t_i} \equiv \left( \Omega_{t_i}^{-1} - \frac{\Omega_{t_i}^{-1} \iota \iota' \Omega_{t_i}^{-1}}{\iota' \Omega_{t_i}^{-1} \iota} \right) m_{t_i}, \quad (20)$$

$$w_{t_i} \equiv \frac{\Omega_{t_i}^{-1} \iota}{\iota' \Omega_{t_i}^{-1} \iota}. \quad (21)$$



Finally, (19) can be used to describe the MR portfolio values recursively:

$$V_{t_{i+1}}^{\text{MR}} = \theta'_{t_{i+1}} v_{t_i} \prod_{j=i+1}^{n-1} \frac{m_{t_j}^{\text{lsr}}}{s_{t_j}^{\text{lsr}^2}} + \theta'_{t_{i+1}} w_{t_i} V_{t_i}^{\text{MR}}.$$

Consequently, the final portfolio value is given by

$$V_{t_n}^{\text{MR}} = \prod_{i=0}^{n-1} \left( \theta'_{t_{i+1}} w_{t_i} \frac{m_{t_i}^{\text{lsr}}}{s_{t_i}^{\text{lsr}^2}} \right) + \sum_{j=1}^n \left( \theta'_{t_i} v_{t_{j-1}} \prod_{i=j}^{n-1} \theta'_{t_{i+1}} w_{t_i} \frac{m_{t_i}^{\text{lsr}}}{s_{t_i}^{\text{lsr}^2}} \right).$$

As the returns are independent and  $E(\theta'_{t_{i+1}} v_{t_i}) = f_{t_i}^2$  and  $E(\theta'_{t_{i+1}} w_{t_i}) = m_{t_i}^{\text{lsr}}$ , as in (5), the expectation of the final portfolio value is given by

$$E(V_{t_n}^{\text{MR}}) = \prod_{i=0}^{n-1} \frac{m_{t_i}^{\text{lsr}^2}}{s_{t_i}^{\text{lsr}^2}} + \sum_{j=1}^n \left( f_{t_{j-1}}^2 \prod_{i=j}^{n-1} \frac{m_{t_i}^{\text{lsr}^2}}{s_{t_i}^{\text{lsr}^2}} \right).$$

Using (4) and (13) the result (8) follows.

### 3 Mean-variance efficiency in continuous time

In order to formulate the efficient mean-variance frontier in continuous time, along with two efficient strategies, consider the returns  $\theta_{t+\Delta t}$  over a period  $\Delta t$ , where  $\Delta t = (T - t_0)/n$  and  $T = t_n$  is fixed as  $\Delta t \rightarrow 0$ . Let the first two moments of the returns be given by

$$\Omega_t = m_t m_t' + \Sigma_t \Delta t \quad \text{and} \quad m_t = \iota + \mu_t \Delta t,$$

and the limit process by

$$dS_t = \text{Diag}(S_t)(\mu_t dt + \Sigma_t^{1/2} dW_t),$$

where  $W_t$  is multivariate Brownian motion and  $\mu_t$  and  $\Sigma_t$  vary deterministically over time and  $\Sigma_t$  is assumed to be nonsingular for all  $t \in [t_0, T]$ . Finally, define instantaneous mean-variance parameters

$$\alpha_t^2 = \frac{1}{\iota' \Sigma_t^{-1} \iota}, \quad \beta_t = \frac{\mu_t' \Sigma_t^{-1} \iota}{\iota' \Sigma_t^{-1} \iota}, \quad \gamma_t^2 = \mu_t' \left( \Sigma_t^{-1} - \frac{\Sigma_t^{-1} \iota \iota' \Sigma_t^{-1}}{\iota' \Sigma_t^{-1} \iota} \right) \mu_t, \quad (22)$$

which are assumed to be bounded and integrable over  $[t_0, T]$ .

After some algebraic manipulations I find

$$\begin{aligned} m'_t \Omega_t^{-1} \iota &= \frac{1 + \beta_t \Delta t}{\alpha_t^2 \Delta t (1 + \gamma_t^2 \Delta t) + (1 + \beta_t \Delta t)^2}, \\ \iota' \Omega_t^{-1} \iota &= \frac{1 + \gamma_t^2 \Delta t}{\alpha_t^2 \Delta t (1 + \gamma_t^2 \Delta t) + (1 + \beta_t \Delta t)^2}, \\ m'_t \Omega_t^{-1} m_t &= \frac{\alpha_t^2 \gamma_t^2 (\Delta t)^2 + (1 + \beta_t \Delta t)^2}{\alpha_t^2 \Delta t (1 + \gamma_t^2 \Delta t) + (1 + \beta_t \Delta t)^2}. \end{aligned}$$

Thus, the parameters (5) can be expressed as

$$\begin{aligned} s_t^{\text{LSR}^2} &= 1 + (2\beta_t + \alpha_t^2 - \gamma_t^2) \Delta t + o(\Delta t), \\ m_t^{\text{LSR}} &= 1 + (\beta_t - \gamma_t^2) \Delta t + o(\Delta t), \\ f_t^2 &= \gamma_t^2 \Delta t + o(\Delta t), \end{aligned}$$

and the the frontier parameters (6), (7) and (8) are given in continuous time by

$$s^{\text{LSR}^2} = \exp \left\{ \int_{t_0}^T (2\beta_t + \alpha_t^2 - \gamma_t^2) dt \right\}, \quad (23)$$

$$m^{\text{LSR}} = \exp \left\{ \int_{t_0}^T (\beta_t - \gamma_t^2) dt \right\}, \quad (24)$$

$$F^2 = \int_{t_0}^T \gamma_t^2 \exp \left\{ - \int_t^T (\alpha_u^2 + \gamma_u^2) du \right\} dt. \quad (25)$$

Notice that, although  $\Sigma_t$  was assumed to be nonsingular in the derivation, the solution allows  $\alpha_t$  to be arbitrary close to 0. Due to continuity the expressions where  $\alpha_t$  is replaced by 0 represent the parameters that describe the frontier in the presence of a riskless asset.

If  $\alpha_t$ ,  $\beta_t$  and  $\gamma_t$  are constant, the parameters (23), (24) and (25) equal

$$\begin{aligned} s^{\text{LSR}^2} &= \exp \left\{ (2\beta + \alpha^2 - \gamma^2)(T - t_0) \right\}, \\ m^{\text{LSR}} &= \exp \left\{ (\beta - \gamma^2)(T - t_0) \right\}, \\ F^2 &= \frac{\gamma^2}{\alpha^2 + \gamma^2} \left[ 1 - \exp \left\{ -(\alpha^2 + \gamma^2)(T - t_0) \right\} \right], \end{aligned}$$

respectively. For example, the problem of finding the minimum variance for a portfolio value

satisfying  $E(V_T) = 1$ , which is given by

$$\frac{\sigma^{\text{MR}2}}{m^{\text{MR}2}} = \frac{s^{\text{LSR}2}}{m^{\text{MR}}m^{\text{LSR}}} - 1,$$

converges to the finite value  $\alpha^2/\gamma^2$  as  $T \rightarrow \infty$ .

To describe the LSR and MR processes, for which the discrete versions have been given in (9) and (19), respectively, consider the limits of (20) and (21):

$$\begin{aligned} v_t &\rightarrow \left( \Sigma_t^{-1} - \frac{\Sigma_t^{-1} \iota \iota' \Sigma_t^{-1}}{\iota' \Sigma_t^{-1} \iota} \right) \mu_t, \\ w_t &\rightarrow \frac{\Sigma_t^{-1} \iota}{\iota' \Sigma_t^{-1} \iota} - \left( \Sigma_t^{-1} - \frac{\Sigma_t^{-1} \iota \iota' \Sigma_t^{-1}}{\iota' \Sigma_t^{-1} \iota} \right) \mu_t. \end{aligned}$$

Consequently, the following limits are found:

$$\begin{aligned} \mu_t' v_t &\rightarrow \gamma_t^2, \\ \mu_t' w_t &\rightarrow \beta_t - \gamma_t^2, \end{aligned}$$

and

$$\begin{aligned} v_t' \Sigma_t^{-1} v_t &\rightarrow \gamma_t^2, \\ v_t' \Sigma_t^{-1} w_t &\rightarrow -\gamma_t^2, \\ w_t' \Sigma_t^{-1} w_t &\rightarrow \alpha_t^2 + \gamma_t^2. \end{aligned}$$

Based on these moments the processes are given by

$$\begin{aligned} dV_t^{\text{LSR}} &= V_t^{\text{LSR}} \left\{ (\beta_t - \gamma_t^2) dt + \sqrt{\alpha_t^2 + \gamma_t^2} dW_t^{(1)} \right\}, \\ dV_t^{\text{MR}} &= V_t^{\text{MR}} \left\{ (\beta_t - \gamma_t^2) dt + \sqrt{\alpha_t^2 + \gamma_t^2} dW_t^{(1)} \right\} \\ &\quad + (\gamma_t^2 dt + \gamma_t dW_t^{(2)}) \exp \left\{ - \int_t^T (\beta_t + \alpha_t^2) dt \right\}, \end{aligned}$$

where  $W_t^{(1)}$  and  $W_t^{(2)}$  are Brownian motions with deterministically varying correlation equal to  $-\sqrt{\frac{\gamma_t^2}{\alpha_t^2 + \gamma_t^2}}$ . Notice the correlation equals  $-1$  if  $\alpha_t = 0$ ,  $t \in [t_o, T]$ . In that case the MR strategy reduces to investment in a riskless asset.

## References

- Bajoux-Besnainou, I. and Portait, R. (2002), “Dynamic, deterministic and static optimal portfolio strategies in a mean-variance framework under stochastic interest rates,” in *New Directions in Mathematical Finance*, eds. Willmott, P. and Rasmussen, H., New York: Wiley.
- Cochrane, J. (2001), *Asset pricing*, Princeton: Princeton University Press.
- Duffie, D. and Richardson, H. (1991), “Mean-variance hedging in continuous time,” *Annals of Applied Probability*, 1, 1–15.
- Leippold, M., Trojani, F., and Vanini, P. (2004), “A geometric approach to multiperiod mean variance optimization of assets and liabilities,” *Journal of Economic Dynamics & Control*, 28, 1079–1113.
- Li, D. and Ng, W.-L. (2000), “Optimal dynamic portfolio selection: multiperiod mean-variance formulation,” *Mathematical Finance*, 10, 387–406.
- Markowitz, H. (1952), “Portfolio Selection,” *Journal of Finance*, 7, 77–91.
- (1959), *Portfolio selection: Efficient diversification of investments*, New York: Wiley.
- Richardson, H. (1989), “A minimum variance result in continuous trading portfolio optimization,” *Management Science*, 35, 1045–1055.
- Schäl, M. (1994), “On quadratic cost criteria for option hedging,” *Mathematics of Operations Research*, 19, 121–131.
- Schweizer, M. (1995), “Variance-optimal hedging in discrete time,” *Mathematics of Operations Research*, 20, 1–32.
- Zhou, X. and Li, D. (2000), “Continuous-time mean-variance portfolio section: a stochastic LQ framework,” *Applied Mathematics & Optimization*, 42, 19–33.