Equilibrium Properties of Finite Binary Choice Games

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Abstract

In this paper I derive a complete characterization for the equilibria that may arise in a binary choice interaction model with a finite number of interacting agents. In particular, the correspondence between the interaction strength, the number of agents and the set of equilibria is derived.

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1 Introduction

Over the last decade, the study of the role of social interactions in economic behavior has become an important area of research. One reason for this is the power of these models to explain large variations in aggregate behavior over time and space. Differences in crime rates over states (Glaeser, Sacerdote and Scheinkman, 1996), large variation in educational attainment across school classes (Hoxby, 2000), fashion cycles (Pesendorfer, 1995), fluctuations in stock prices (Bikhchandani, Hirshleifer and Welch, 1998) and herd behavior (Banerjee, 1992) are all examples of phenomena that cannot be fully accounted for by economic or cultural differences, but for which interactions-based models offer an explanation. Almost all of these models use the notion of *strategic complementarity*, implying that an increase in the action of all agents except agent i increases the marginal return to agent i's action. A main reason for the popularity of these models is that they are capable of generating the social multiplier or multiple equilibria that creates large variations in aggregate behavior.

In models with *strategic substitutes*, an increase in the action of all agents except agent i *decreases* the marginal return to agent i's action. In contrast to models with complements, models with strategic substitutes do not generate multiplier effects or multiple equilibria *at the aggregate level*.[1] One can think of many instances in which people derive utility from being different than others. To give an example, most school classes inhabit some pupils that purposely try to distinguish themselves from the others. Depending on context, games with best-response functions that are decreasing in the ac-

[1] As far as I am aware, Glaeser and Scheinkman (2003) present the only example in which strategic substitutes create multiple equilibria. However, instead of showing how multiple equilibria can be generated within one group, they introduce a second group which negatively interacts with the first group such that in equilibrium, aggregate behavior may differ between the two groups. That is, agents in different groups can make different choices in equilibrium, but the choices of agents in the same group have to be the same. Aggregated at the society-level however, all equilibria exhibit the same behavior.

tions of other players are also refered to as games with negative externalities or games with negative interactions.

In a general analysis of games with strategic complementarities, Cooper and John (1988) prove that in games with homogenous agents, strategic complementarity is a necessary condition for the existence of multiple (symmetric) Nash equilibria at the aggregate level. Brock and Durlauf (2001a, 2001b, 2003) derive conditions under which strategic complementarity between the choices of individual agents leads to multiple equilibria at the aggregate level in the class of discrete choice interactions models. The literature on the effects of strategic complements further includes studies on conformity (Bernheim, 1994), peer effects (Kremer and Levy, 2001) or the upholding of social norms (Becker and Murphy, 2001).

In this study, instead of considering the consequences of social interactions for aggregate behavior, I will focus on the consequences of social interactions at the level of the individual. The main contribution of the paper is a number of propositions on equilibrium behavior in a binary simultaneous discrete response model with a limited number of agents. In this model, an individual agent's behavior is dependent on the observed behavior of other agents. This model is closely related to the model estimated in Kooreman and Soetevent (2002). In that paper we use the model to empirically analyze interaction effects among teenagers in school classes. Questions that are answered in the present paper concern equilibrium existence and multiplicity of equilibria. Not only do I look at the case with strategic complements, but I purposely extend the discussion to encompass also the case with strategic substitutes. It turns out that the latter case has some remarkable implications.

For both cases, equilibrium existence is proved. In empirical work on binary choice interaction models, it is particularly important to account for the possibility of multiple equilibria. Assumptions concerning equilibrium selection are more pivotal when multiple equilibria occur more frequently. For this reason, the derivation of tight upper bounds on the number of equilibria given a certain level of complementarity (substitutability) and a number of agents N is useful. I find that the upper bound for the number of equilibria grows linearly in N for the case with strategic complements, but exponentially for the case with strategic substitutes. Besides the formulation of upper bounds, analytical expressions on the *expected* number of equilibria are obtained. Surprisingly, for the model with strategic substitutes (complements), the expected number of equilibria is non-increasing (non-decreasing) when $N \leq 3$, but much more whimsical when $N > 3$. Another result is that in the case of strategic substitutes, two outcomes can only belong both to the set of equlbria if in both outcomes the aggregate number of agents choosing a particular action is the same. Finally, I derive some results concerning the differences in model behavior for an even and an odd number of agents. For an even number of agents, the upper bound on the number of equilibria is always reached for a high enough level of substitutability; for an odd number, this is crucially dependent on the distribution of private utilities of the agents.

In the analysis, I will assume that the number of agents that comprises the reference group is limited. In its focus on small groups, the paper is complementary to much of the existing literature. As Moffitt (2001) notes: "The crude proxies for neighborhood effects that are used in the empirical literature, which are solely the result of data limitations, should not lead to a conclusion that no social interactions are present in smaller geographic areas. More generally, the theory is consistent with a small intervention affecting only a small number of individuals." Other studies that pay attention to small groups are Krauth (2001), who adapts the Brock-Durlauf model to a small group environment and Ioannides (2003) who derives a number of general results for bounded social structures using graph theory.

In a related branch of literature, Tamer (2003) does explicitly allow for negative externalities in a 2×2 binary choice game. His objective is not to derive equilibrium properties as well as to find an unbiased estimation procedure in the presence of multiple equilibria. In another paper, Tamer (2002) extends this procedure to situations with more than two agents. In an empirical application he estimates interactions in the decisions of airline companies whether or not to enter a given market. A noteworthy difference with Tamer's approach and earlier work on simultaneous discrete choice models is that I categorize binary choices as $\{-1, 1\}$, instead of $\{0, 1\}$. Whereas in standard binary choice models the categorization is immaterial, it is not in the present context. I will argue that a $\{0, 1\}$-support is a natural choice in simultaneous discrete response models that study firm's entry decisions to a certain market – the focus of Tamer's application – but that the $\{-1, 1\}$-support may be more appropriate when one studies social interactions in small groups.

Understanding the effects of negative externalities on equilibrium behavior is a prerequisite for the incorporation of more general interaction patterns into economic models. Until now, interactions-based models have focused on either positive or negative interactions within a social group. In reality however, it is likely that within a group, some agents are affected by conformity and others by the desire to distinguish themselves. This leads to a process in which agent A wants to resemble person B, but person B wants to behave different than person A. In this case, there does not exist a Nash equilibrium in pure strategies. Analysis of these kind of models is left for future research.

The paper proceeds as follows. The model is presented in the next section. Differences in behavior when using choices with support $\{0, 1\}$ instead of $\{-1, 1\}$ are indicated. Results on the equilibrium properties of the model are derived in section 3. Section 4 briefly discusses a slightly more general model with gender-based interaction terms. Section 5 concludes.

2 The model

Consider a population of N individuals indexed by i, $i = 1, 2, \ldots, N$. Each player i faces a binary choice and these choices are denoted by an indicator variable y_i which has support $Y_i = \{-1, 1\}$. Y_i is the strategy set of player i and $Y = \times_{i=1}^{N} Y_i$. Elements of Y are called strategy profiles. A full strategy profile is denoted by $\mathbf{y} = (y_i, \mathbf{y_{-i}})$, where $\mathbf{y_{-i}} = (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)'$. Note that the number of elements in Y is 2^N. Each individual makes a choice in order to maximize a payoff function $V : Y \to \mathbb{R} \cup \{-\infty\}$.

In the standard economic approach, the payoff function is dependent on individual characteristics. Following the notation in Brock and Durlauf (2001b), I assume that these characteristics can be divided into an observable vector $\mathbf{x_i}$ and a random shock $\epsilon_i(y_i)$ that is unobservable to the modeller but observable to agent i. Moreover, in interactions-based models explicit attention is given to the influence of the behavior of others on each individual's choice. Each choice is then described as

$$\max_{y_i \in Y_i} V(y_i, \mathbf{x_i}, \mathbf{y_{-i}}, \epsilon_i(y_i)). \tag{1}$$

Similar to Brock and Durlauf (2001b), I assume that the payoff function V can be additively decomposed into three terms:

$$V(y_i, \mathbf{x_i}, \mathbf{y_{-i}}, \epsilon_i(y_i)) = u(y_i,\, \mathbf{x_i}) + S(y_i, \mathbf{x_i}, \mathbf{y_{-i}}) + \epsilon_i(y_i), \tag{2}$$

where the first term $u(y_i,\, \mathbf{x_i})$ denotes deterministic private utility, $S(y_i, \mathbf{x_i}, \mathbf{y_{-i}})$ denotes deterministic social utility and $\epsilon_i(y_i)$ denotes random private utility. The social utility term is assumed to have the following form

$$S(y_i, \mathbf{x_i}, \mathbf{y_{-i}}) = \frac{\gamma}{2(N-1)} y_i \sum_{j \neq i} y_j.$$

Define $\mathbf{y_{-ij}} = \mathbf{y}\backslash\{y_i, y_j\}$ so that $(y_i, \mathbf{x_i}, y_j, \mathbf{y_{-ij}})$. Note that

$$\{V(1, \mathbf{x_i}, 1, \mathbf{y_{-ij}}, \epsilon_i(y_i)) - V(-1, \mathbf{x_i}, 1, \mathbf{y_{-ij}}, \epsilon_i(y_i))\} -$$
$$\{V(1, \mathbf{x_i}, -1, \mathbf{y_{-ij}}, \epsilon_i(y_i)) - V(-1, \mathbf{x_i}, -1, \mathbf{y_{-ij}}, \epsilon_i(y_i))\}$$
$$= \{S(1, \mathbf{x_i}, 1, \mathbf{y_{-ij}}) - S(-1, \mathbf{x_i}, 1, \mathbf{y_{-ij}})\} -$$
$$\{S(1, \mathbf{x_i}, -1, \mathbf{y_{-ij}}) - S(-1, \mathbf{x_i}, -1, \mathbf{y_{-ij}})\}$$
$$= \frac{2\gamma}{(N-1)}$$

Thus, for $\gamma > 0$ the utility of choosing $y_i = 1$ (versus $y_i = -1$) when another individual j chooses $y_j = 1$ as well is larger than the utility of choosing $y_i = 1$ (versus $y_i = -1$) when another individual chooses $y_j = -1$. In this case the parameter γ measures the strategic complementarity when $\gamma > 0$ between the choice of any pair of individuals, or the extent to which the choices are strategic substitutes when $\gamma < 0$.[2] In fact, for $\gamma > 0$ ($\gamma < 0$), the model falls into the class of supermodular (submodular) games. Supermodular (submodular) games are games in which each player's strategy set is partially ordered and the marginal returns to increasing one's strategy rise (decrease) with increases in the competitors' strategies.[3]

Conditional on the choice by individual i, deterministic private utility is assumed to be a linear function of exogenous characteristics $\mathbf{x_i}$, such that:

$$u(1,\, \mathbf{x_i}) = \beta_1'\mathbf{x_i}; \qquad u(-1,\, \mathbf{x_i}) = \beta_{-1}'\mathbf{x_i}.$$

The best response function of individual i given the choices of the other individuals can now be represented as

$$\begin{cases} y_i^* = \beta'\mathbf{x_i} + \frac{\gamma}{N-1}\sum_{j\neq i} y_j + \epsilon_i & \\ y_i = 1 & \text{if } y_i^* > 0 \\ y_i = -1 & \text{if } y_i^* \leq 0 \end{cases} \tag{3}$$

[2] When $\gamma = 0$, the model reduces to the standard binary choice formulation without externalities.

[3] Milgrom and Roberts (1990, p. 1255). See also Vives (1990) and the recent textbook treatments of Topkis (1998) and Vives (1999). Topkis (p. 11) defines a partially ordered set as a set X on which there is a binary relation $\succeq$ that is reflexive, antisymmetric, and transitive.

where y_i^* denotes the difference between the utility individual i derives from choosing $y_i = 1$ and the utility he derives from choosing $y_i = -1$, conditional on $\mathbf{y_{-i}}$, that is,

$$y_i^* = V(1, \mathbf{x_i}, \mathbf{y_{-i}}, \epsilon_i(1)) - V(-1, \mathbf{x_i}, \mathbf{y_{-i}}, \epsilon_i(-1)),$$

with $\beta \equiv \beta_1 - \beta_{-1}; \epsilon_i \equiv \epsilon_i(1) - \epsilon_i(-1)$.

Define $\mathbf{x} \equiv (\mathbf{x_1'}, \mathbf{x_2'}, \ldots, \mathbf{x_N'})'$ and $\epsilon \equiv (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)'$. A strategy profile $\mathbf{y}$ is a pure Nash equilibrium profile if and only if it is consistent with (3) for all i, i.e. if after substitution of these values of y_i in S_i we have $y_i^* > 0$ for all i with $y_i = 1$, and $y_i^* \leq 0$ for all i with $y_i = -1$. Let $Q(\beta, \gamma, \mathbf{x}, \epsilon, N)$ denote the number of pure Nash equilibria given $\{\beta, \gamma, \mathbf{x}, \epsilon\}$ and the population size N. That is, for $N \geq 2$,

$$Q(\beta, \gamma, \mathbf{x}, \epsilon, N) = \tag{4}$$
$$\sum_{t=1}^{2^N} \left[\prod_{i=1}^{N} I\left(-\epsilon_i < \beta'\mathbf{x_i} + \frac{\gamma}{N-1} \sum_{j \neq i} y_{jt} \right)^{\frac{1+y_{it}}{2}} I\left(\epsilon_i \leq -\beta'\mathbf{x_i} - \frac{\gamma}{N-1} \sum_{j \neq i} y_{jt} \right)^{\frac{1-y_{it}}{2}} \right]$$

with $I(\cdot)$ an indicator function.[4] For each element y_{it} of a strategy profile, the indicator functions evaluate the relevant condition on ϵ_i: $-\epsilon_i < \beta'\mathbf{x_i} + (\gamma \sum_{j \neq i} y_{jt})/(N-1)$ if $y_{it} = 1$, and $\epsilon_i \leq -\beta'\mathbf{x_i} - (\gamma \sum_{j \neq i} y_{jt})/(N-1)$ if $y_{it} = -1$. When the relevant condition is satisfied for all elements of a certain strategy profile $\mathbf{y_t}$, the product in (4) is one. In all other cases, the product is zero. Finally summing over all 2^N strategy profiles gives the number of pure Nash equilibria. In the model without social interactions (i.e. $\gamma = 0$) each combination of $\{\beta, \gamma = 0, \mathbf{x}, \epsilon\}$ obviously defines a unique equilibrium, and thus $Q(\beta, 0, \mathbf{x}, \epsilon, N) = 1$.

An important feature of the model with social interactions is that, for a given combination of $\{\beta, \gamma \neq 0, \mathbf{x}, \epsilon\}$, several strategy profiles may be consistent with (3). If $\gamma = 1$ and $\beta'\mathbf{x_1} + \epsilon_1 = \beta'\mathbf{x_2} + \epsilon_2 = -\frac{1}{2}$, for example,

[4] I follow the convention $0^0 \equiv 1$.

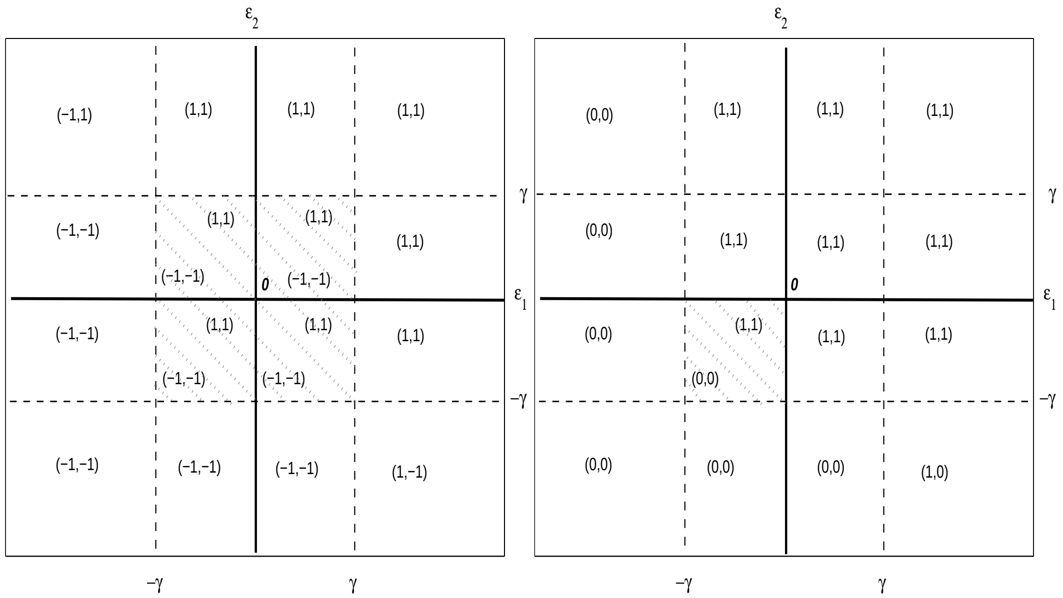

Figure 1: Multiple equilibria in ϵ-space ($N = 2, \gamma > 0, \beta'\mathbf{x_1} = \beta'\mathbf{x_2} = 0$) for support Y_i (left panel) and support $\tilde{Y}_i$ (right panel).

profiles $\mathbf{y} = (1,1)'$ and $\mathbf{y} = (-1,-1)'$ are both consistent with (3). In the left panel of figure 1, equilibrium profiles for the two-person game are drawn in ϵ-space. The shaded area is the area with multiple equilibria.

The main objective of this paper is to analyze the relationship between different values of γ and N on the number and nature of equilibria. I will derive a number of results regarding the existence and maximum number of equilibria that can occur, for both the case with a positive as with a negative value of γ.

2.1 Choice of support $\{-1, 1\}$ versus $\{0, 1\}$

It is of some importance to discuss the choice of support $Y_i = \{-1, 1\}$ instead of the alternative $\tilde{Y}_i = \{0, 1\}$. The latter is the common choice in standard binary choice models where the difference is just a matter of scaling and therefore immaterial. In this section, I will show that the specific choice of support *does* affect the equilibrium properties of binary choice interaction models. This fact has hitherto not been explicitly recognized in the literature. Krauth (2001) for example, taciturnly switches to $\tilde{Y}_i$ as support in his development of the small sample analog of the Brock-Durlauf model, whereas these authors themselves employ Y_i. Key idea is that in using support Y_i, the model is symmetric and therefore invariant with respect to interchanging the two choices. This is not the case with $\tilde{Y}_i$.

This difference between the two models becomes clear when one compares the equilibria for the two-person game in ϵ-space under the assumption that exogenous variables are irrelevant ($\beta'\mathbf{x_1} = \beta'\mathbf{x_2} = 0$). The left panel of figure 1 uses support Y_i and is symmetric with respect to the line $\epsilon_1 + \epsilon_2 = 0$. The right panel, which uses support $\tilde{Y}_i$, is not.

Compared to the left panel of figure 1, one observes that in the right panel the shaded area with multiple equilibria is reduced and restricted to the points where the private utility difference of smoking for both players is negative ($\beta'\mathbf{x_i} + \epsilon_i = \epsilon_i < 0, i = 1, 2$). When using $\tilde{Y}_i$, one implicitly assumes that only positive choices have a social effect. One justification for this is from an evolutionary point of view, for example by arguing that everybody starts as a non-smoker, such that only the teenagers that start smoking give a signal and the number of non-smokers is irrelevant. Note, however, that the decision not to smoke can convey just as strong a signal to others, especially in environments with many smokers.[5]

In other contexts however, $\tilde{Y}_i$ may be the preferred support. Consider for example the context in which firms have to make a decision to enter a certain market (Tamer, 2002). It is plausible that this decision is only

[5]To give an example, suppose that in a class with 9 teenagers, 3 of them would smoke were social interactions absent ($\gamma = 0$), that is, $y_i^* = \beta'\mathbf{x_i} + \epsilon_i > 0$ for three of them and $y_i^* \leq 0$ for the others. How would one interpret in this instance the observation of zero smokers in this class? A natural explanation is that due to a social effect, the six non-smokers keep the potential smokers from smoking. Support Y_i allows for this explanation, since the difference in social utility of smoking when nobody else smokes equals $\frac{\gamma}{N-1}\sum_{j\neq i} y_j = \gamma\frac{-8}{8} < 0$ for $\gamma > 0$. On the contrary, with $\tilde{Y}_i$ as underlying support, $\frac{\gamma}{N-1}\sum_{j\neq i} y_j = 0$ irrespective of γ, such that social interactions cannot offer an explanation.

dependent on how many other firms decide to enter the market and that the number of firms that decide not to enter is irrelevant. All results in the sequel are derived while working with support Y_i.

3 Equilibrium properties

3.1 Existence

Define $z_i \equiv \beta'\mathbf{x_i} + \epsilon_i$ and $k \equiv \sum_{i=1}^{N} y_i$, that is, k is the net number of agents choosing $y = 1$.[6] Rank observations on basis of the values of z_i. Denote the ordered values as $z_{[1]} \geq z_{[2]} \geq \ldots \geq z_{[N]}$. Denote the corresponding values of y for the agent with $z_{[j]}$ as $y_{[j]}$. Note that the latter are not ordered, such that it is not precluded that e.g. $y_{[j]} < y_{[j+1]}$. The following proposition guarantees equilibrium existence for model (3) in case of strategic complements as well as strategic substitutes.

Proposition 1 Equilibrium existence
For every combination $\{\beta, \gamma, \mathbf{x}, \epsilon\}$ there exists at least one vector $\mathbf{y} \equiv (y_1, y_2, \ldots, y_N)'$ for which (3) holds.

Proof: See the Appendix.

In the proof, explicit equilibria are derived for every combination $\{\beta, \gamma, \mathbf{x}, \epsilon\}$, although for the case with strategic complements, an alternative proof can be given, based on the supermodular character of the game. It is worth mentioning that for the equilibria given in the proof, $|k| = |\sum_{i=1}^{N} y_i|$ decreases monotonically to 0 (1) as $\gamma \to -\infty$ for N even (N odd). In the next subsection, it will become clear that this result holds more generally: in equilibrium, the difference between the number of agents choosing $y = 1$ and

[6] Note that given N, only those values of k for which $N + k$ is an even number are possible. This follows from the observation that $k = a \cdot 1 - (N - a), a \in \{0, 1, \ldots, N\}$ can be rewritten as $N + k = 2a$.

the number of agents choosing $y = -1$, is smaller when γ is more negative, other things equal.

3.2 Multiple equilibria

In section 2, we observed that multiple equilibria may occur for certain combinations of variables and parameter values. In this section, I will derive strict upper bounds for $Q(\beta, \gamma, \mathbf{x}, \epsilon, N)$. It turns out that the situation with strategic substitutes ($\gamma < 0$) is characterized by fundamentally different equilibrium behavior than the one with strategic complements ($\gamma > 0$). Moreover, it makes a difference whether the population has an even or an odd number of members. In section 3.3, two examples are provided which illustrate the results for the case with strategic substitutes.

The proof of proposition 2 for strategic complements uses the following lemma that was first formulated in Kooreman and Soetevent (2003).

Lemma 1 *Let $\gamma > 0$. Suppose model (3) has an equilibrium $\mathbf{y}$. Then*

$$\min_{\{i|y_i=1\}} z_i - \max_{\{i|y_i=-1\}} z_i > \frac{2\gamma}{N-1},$$

where $z_i \equiv \beta' \mathbf{x_i} + \epsilon_i$.

The lemma's effect is that it restricts the maximum number of potential equilibria to $N + 1$. The following observation is an immediate consequence of lemma 1:

1 *In any equilibrium the agents with $y_i = 1$ are those with the largest values for z_i.*

Now consider two vectors $\mathbf{y}$ and $\tilde{\mathbf{y}}$ that differ in one element only. Without loss of generality, assume that $y_i = 1$ and $\tilde{y}_i = -1$ for some i. Define $\mathbf{y_{-i}} \equiv (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)'$ and $\tilde{\mathbf{y}}_{\mathbf{-i}} \equiv (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{i-1}, \tilde{y}_{i+1}, \ldots, \tilde{y}_N)'$. Since $\mathbf{y_{-i}} = \tilde{\mathbf{y}}_{\mathbf{-i}}$, it follows that $y_i^* = z_i + \frac{\gamma}{N-1} \sum_{j \neq i} y_j = z_i + \frac{\gamma}{N-1} \sum_{j \neq i} \tilde{y}_j =$

$\tilde{y}_i^*$ given a combination of $\{\beta, \gamma, \mathbf{x}, \epsilon\}$. This implies that $y_i = \tilde{y}_i$ and we arrive at a contradiction. Note that this result holds irrespective of γ being positive or negative. The following observation is thus obtained:

2 *Two vectors* $\mathbf{y}$ *and* $\tilde{\mathbf{y}}$ *that differ in only one element cannot belong both to the set of equilibria.*

The following proposition states that for a situation with strategic complements, the maximal number of equilibria grows linearly in N. The second part ensures that the upper bound on the number of equilibria is strict.

Proposition 2 Maximum number of equilibria (strategic complements)
For every combination $\{\beta, \gamma > 0, \mathbf{x}, \epsilon\}$*, the discrete interaction model (3) with* N *agents can have at most* $d(N)$ *distinct equilibria, with*

$$d(N) = \lfloor \frac{N}{2} + 1 \rfloor. \tag{5}$$

Moreover, for every number N*, there exists a combination of* $\{\beta, \gamma > 0, \mathbf{x}, \epsilon\}$ *for which* $Q(\beta, \gamma, \mathbf{x}, \epsilon, N) = d(N)$.

Proof Proposition 2: From observations 1 and 2 it directly follows that the number of equilibria for a given combination of $\{\beta, \gamma > 0, \mathbf{x}, \epsilon\}$ can be at most $d = \lfloor \frac{N}{2} + 1 \rfloor$, where $\lfloor w \rfloor$ denotes the largest integer not larger than w. To give an example, assume that the number of agents $N = 8$. Due to observations 1 and 2, the strategy profiles of the different equilibria must be strictly ordered and differ in at least two elements. This leaves the following five strategy profiles as the only candidates:

$$(1,1,1,1,1,1,1,1)';\ (1,1,1,1,1,1,-1,-1)';\ (1,1,1,1,-1,-1,-1,-1)';$$
$$(1,1,-1,-1,-1,-1,-1,-1)';\ (-1,-1,-1,-1,-1,-1,-1,-1)'.$$

According to the proposition, the maximum number of equilibria can indeed be at most 5, since $\lfloor \frac{N}{2} + 1 \rfloor = 5$ when $N = 8$.

The proof of the second part – the upper bound on the number of equilibria is strict – runs as follows. Denote the d equilibria that are to be sustained as[7]

$$
\begin{aligned}
\mathbf{y^1} &= (1, 1, \ldots, 1)' \\
\mathbf{y^2} &= \begin{cases} (1, \ldots, 1, -1, -1)' & \text{if } N \text{ is even,} \\ (1, \ldots, 1, -1, -1, -1)' & \text{if } N \text{ is odd,} \end{cases} \\
&\quad \vdots \\
\mathbf{y^j} &= \begin{cases} (1, \ldots, 1_{N-2(j-1)}, -1_{N-2(j-1)+1}, \ldots, -1)' & \text{if } N \text{ is even,} \\ (1, \ldots, 1_{N-2(j-1)-1}, -1_{N-2(j-1)}, \ldots, -1)' & \text{if } N \text{ is odd,} \end{cases} \\
&\quad \text{with } j = 3, \ldots, d-1. \\
\mathbf{y^d} &= (-1, -1, \ldots, -1)'.
\end{aligned}
$$

First note that $\mathbf{y^1}$ can be sustained as an equilibrium outcome if and only if $z_{[N]} > -\gamma$ and that $\mathbf{y^d}$ can be sustained as an equilibrium outcome if and only if $z_{[1]} \leq \gamma$. Further note that $\mathbf{y^{d-i}}$, $i = 1, \ldots, d-2$ can be sustained as equilibria if and only if $z_{[2i]} > \gamma \frac{N-4i+1}{N-1}$ and $z_{[2i+1]} \leq \gamma \frac{N-4i-1}{N-1}$. The fact that these necessary and sufficient conditions on the values of z can be satisfied simultaneously completes the proof. □

In the proof of the corresponding proposition for the case with strategic substitutes, I will make use of the following two lemma's:

Lemma 2 *If for a given combination $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$, $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are both equilibria of (3), then $\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} \tilde{y}_i$.*

The proof of lemma 2 uses the following result:

Lemma 3 *If for a given combination $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$ there exists an equilibrium $\mathbf{y}$ with $y_{[j]} = -1$ and $y_{[j+1]} = 1$, then there also exists an equilibrium $\tilde{\mathbf{y}}$ with $\tilde{y}_{[j]} = 1$ and $\tilde{y}_{[j+1]} = -1$ and $\tilde{y}_{[i]} = y_{[i]}$ for $i \neq j, j+1$.*

[7] When N is odd, there has to be one equilibrium that differs in at least three elements when compared to any of the other equilibria. Without loss of generality I assume the last three elements of $\mathbf{y}$ to be the three elements that move together.

Proofs Lemma 2 and 3: See the Appendix.

The message of lemma 2 is that for a given value of $\gamma < 0$, two different equilibria $\mathbf{y}$ and $\tilde{\mathbf{y}}$ can co-exist only if $\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} \tilde{y}_i$. That is, both equilibria must have the same number of subjects with outcome $+1$ and with outcome -1.

Repeated application of lemma 3 shows that a strategy profile $\mathbf{y}$ with $\sum_{i=1}^{N} y_i = k$ can only be an equilibrium if the ordered (with respect to the z_i's) strategy profile $\mathbf{y} = (1_1, 1_2, \ldots, 1_k, -1_{k+1}, \ldots, -1_N)'$ is an equilibrium. This result will prove to be useful later on in deriving upper bounds for the number of equilibria that may be sustained for a given value of γ.

Proposition 3 Maximum number of equilibria (strategic substitutes)

For every combination $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$, the discrete interaction model (3) with N agents can have at most $d(N)$ distinct equilibria, with

$$\begin{aligned} d(N) = d^e(N) &= \frac{N!}{(N/2)!(N/2)!} && \text{if } N \text{ is even, and} \\ d(N) = d^o(N) &= \frac{N!}{\{(N+1)/2\}!\{(N-1)/2\}!} && \text{if } N \text{ is odd.} \end{aligned}$$

Moreover, for every even (odd) number N, there exists a combination of $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$ for which $Q(\beta, \gamma, \mathbf{x}, \epsilon, N) = d^e(N)$ ($Q(\beta, \gamma, \mathbf{x}, \epsilon, N) = d^o(N)$).

Proof Proposition 3: The first part follows from lemma 2 by noting that the maximum number of possible equilibria subject to the condition $\sum_{i=1}^{N} y_i = k$ is obtained when k is chosen to equal 0 ($+1$ or -1) when N is even (odd). In that case, there are $N/2$ ($N/2+1$ or $N/2-1$) agents choosing $+1$ and the others choosing -1, giving the upper bounds on the number of possible equilibria as given by $d(N)$ in proposition 3.

What is left to show is that there exists a combination of $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$ for which the maximum number of equilibria is obtained. From lemma 2 we know that, given a combination of $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$, every element in the equilibrium set must have the same number of agents choosing $y = 1$. For N is even, the set can thus only have $d^e(N)$ elements when the set contains *all* strategy profiles for which the number of agents choosing $y = 1$ equals the number of agents choosing $y = -1$. For each of these profiles to be an equilibrium, it must be optimal for *each* agent i to choose $y_i = 1$ given that $\sum_{j \neq i} y_j = -1$ and to choose $y_i = -1$ given that $\sum_{j \neq i} y_j = 1$. In particular, it must hold that

$$\begin{aligned} z_{[1]} + \gamma \frac{1}{N-1} &\leq 0 \text{ and;} \\ z_{[N]} + \gamma \frac{-1}{N-1} &> 0. \end{aligned}$$

For γ negative enough, this condition is satisfied irrespective of the values of $z_{[1]}, \dots z_{[N]}$.

For N is odd, the equilibrium set can only contain $d^o(N)$ elements when the set contains all strategy profiles for which $\sum_{i=1}^{N} y_i = 1$ or all strategy profiles for which $\sum_{i=1}^{N} y_i = -1$. The necessary and sufficient conditions for each of the profiles with $\sum_{i=1}^{N} y_i = 1$ to be an equilibrium, are

$$z_{[1]} + \gamma \frac{2}{N-1} \leq 0 \qquad \text{and} \qquad z_{[N]} > 0, \tag{6}$$

and the corresponding conditions for the strategy profiles with $\sum_{i=1}^{N} y_i = -1$ are

$$z_{[1]} \leq 0 \qquad \text{and} \qquad z_{[N]} + \gamma \frac{-2}{N-1} > 0. \tag{7}$$

From these conditions it follows that the equilibrium set with $d^o(N)$ elements for which $\sum_{i=1}^{N} y_i = 1$ ($\sum_{i=1}^{N} y_i = -1$) is only obtainable when all z values are positive (non-positive). Together this proves proposition 3. □

Proposition 3 states that for the case with strategic substitutes — as in the case with strategic complements — the upper bound on the number of equilibria is strict. However, a notable difference with proposition 2 is that the maximal number of equilibria does not grow linearly but exponentially in N. Note that $\frac{d^e(N)}{d^o(N-1)} = 2$ for all even N and $\lim_{N\to\infty} \frac{d^o(N)}{d^e(N-1)} \uparrow 2$ for N odd. That is, (in the limit) adding one agent to the population doubles the upper bound on the number of equilibria.

Lemma 2 and the observation that for the equilibria in the proof of proposition 1, $|k| = |\sum_{i=1}^N y_i|$ monotonically decreases as $\gamma \to -\infty$, together lead to the corollary[8] that for all equilibria, $|k|$ decreases monotonically to 0 (1) as $\gamma \to -\infty$, given N even (odd). This result is consonant with intuition: variation in behavior increases when the utility derived from being different increases.

Corollary 1 *For the equilibria* $\mathbf{y}$ *of the discrete choice interaction model given by (3),*

$$|\sum_{i=1}^{N} y_i| \searrow 0 \quad \textit{as } \gamma \to -\infty \textit{ and } N \textit{ is even,}$$
$$|\sum_{i=1}^{N} y_i| \searrow 1 \quad \textit{as } \gamma \to -\infty \textit{ and } N \textit{ is odd.}$$

3.3 Examples

In order to provide some further intuition for the results derived for strategic substitutes, I will present two examples in this section.

Example 1 Consider four agents with private utilities $z_{[1]} = 4, z_{[2]} = 2.5, z_{[3]} = 2$ and $z_{[4]} = -3$. Thus, without interactions ($\gamma = 0$), the unique equilibrium is $\mathbf{y} = (1, 1, 1, -1)$ and $k = \sum_{i=1}^4 y_i = 2$. How does the set of equilibria change when γ decreases? From corollary 1 we know that $|k|$ decreases monotonically to zero (since N is even) as $\gamma \to -\infty$. For this reason, it is natural to ask first for the conditions under which other

[8] The corresponding result for positive interactions is that $|\sum_{i=1}^N y_i| \nearrow N$ as $\gamma \to \infty$. That is, in the limit all agents conform to $y = 1$ or to $y = -1$ regardless of their private utility such that variation in behavior is minimized.

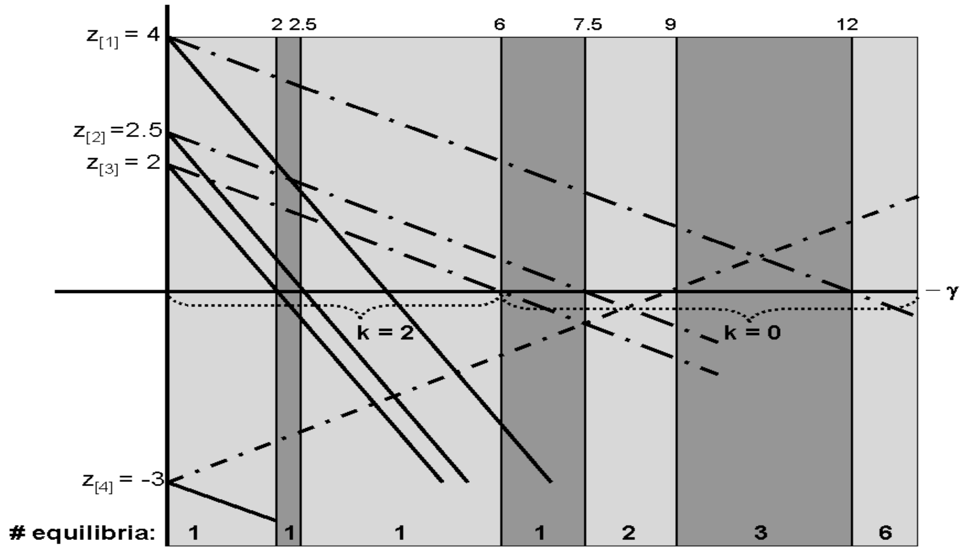

Figure 2: Development of the number of equilibria as γ decreases in a group of four agents, with $z_{[1]} = 4, z_{[2]} = 2.5, z_{[3]} = 2$ and $z_{[4]} = -3$.

equilibria with $k = 2$ are admissible. Consider the outcome in which the y-values of agents 3 and 4 have switched, $\mathbf{y} = (1, 1, -1, 1)$. For this outcome to be an equilibrium, the utility difference of smoking has to be non-positive for agent 3 and positive for agent 4. That is: a) $z_{[3]} + \gamma\frac{k+1}{N-1} = 2 + \gamma \leq 0$, and b) $z_{[4]} + \gamma\frac{k-1}{N-1} = -3 + \gamma/3 > 0$. (See the solid lines in figure 2.) It folllows (from the figure) that the second condition cannot be satisfied for $\gamma < 0$ and for this reason, no other equilibria with $k = 2$ are obtained when γ decreases.

Moreover, one knows (see figure 2) that all equilibria with $k = 2$ become infeasible when, given $k = 2$, the third agent's utility from smoking turns negative, that is $z_{[3]} + \gamma\frac{k-1}{N-1} = 2 + \gamma/3 \leq 0$, which happens at $-\gamma = 6$. Thus, for values of $-\gamma \geq 6$, only those equilibria are admissible in which at least two agents choose $y = -1$. The equilibrium is unique for values of $-\gamma$ slightly larger than 6: agent 3 and 4 choose $y = -1$ and the other two $y = +1$. As $-\gamma$ becomes larger than 7.5, a second equilibrium is possible

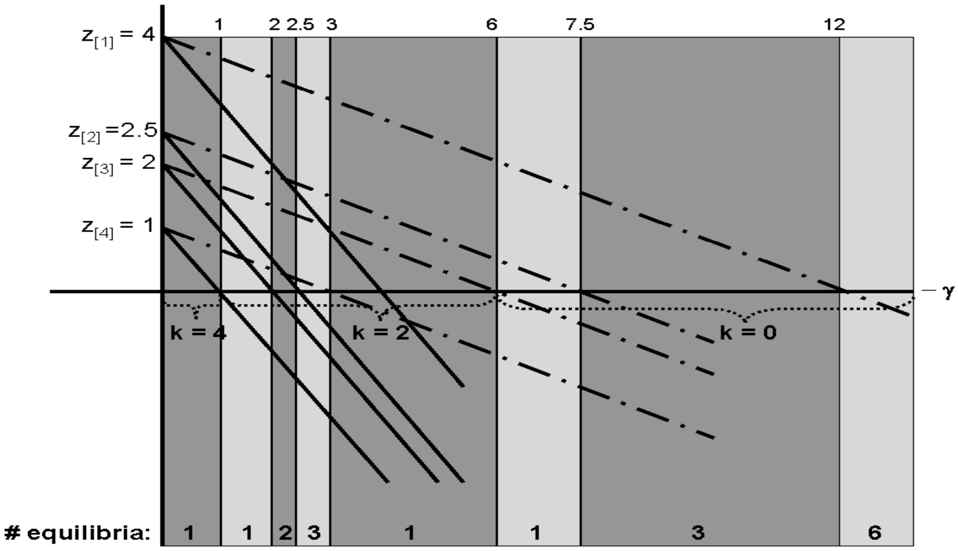

Figure 3: Development of the number of equilibria as γ decreases in a group of four agents, with $z_{[1]} = 4, z_{[2]} = 2.5, z_{[3]} = 2$ and $z_{[4]} = 1$.

($\mathbf{y} = (1, -1, 1, -1)$), since now $z_{[2]} + \gamma\frac{k-1}{N-1} = 2.5 + \gamma/3 \leq 0$. If $-\gamma$ gets lager than 9, again an equilibrium is added ($\mathbf{y} = (1, -1, -1, 1)$) since — given the choice of the other agents — the utility difference to choosing $y = 1$ is now positive for agent 4. Eventually, the number of equilibria in the set doubles when $-\gamma \geq 12$ (and attains it's maximum value $d^e(N) = 6$; see the dashed lines in figure 2).

Example 2 The previous example might lead to the impression that the set of possible equilibria increases as $-\gamma$ increases. In general, this is not the case. Consider the slightly different example with $z_{[1]} = 4, z_{[2]} = 2.5, z_{[3]} = 2$ and $z_{[4]} = 1$. (See figure 3.) For values of $-\gamma$ just below 3, the equilibrium set consists of 3 different equilibria, all having one agent choosing $y = -1$: $\mathbf{y} = (1, 1, 1, -1), \mathbf{y} = (1, 1, -1, 1)$ and $\mathbf{y} = (1, -1, 1, 1)$. For values of $-\gamma$ just above 3, given that exactly one agent chooses $y = -1$, this agent must be agent 4, since then $z_{[4]} + \gamma/3 < 0$.

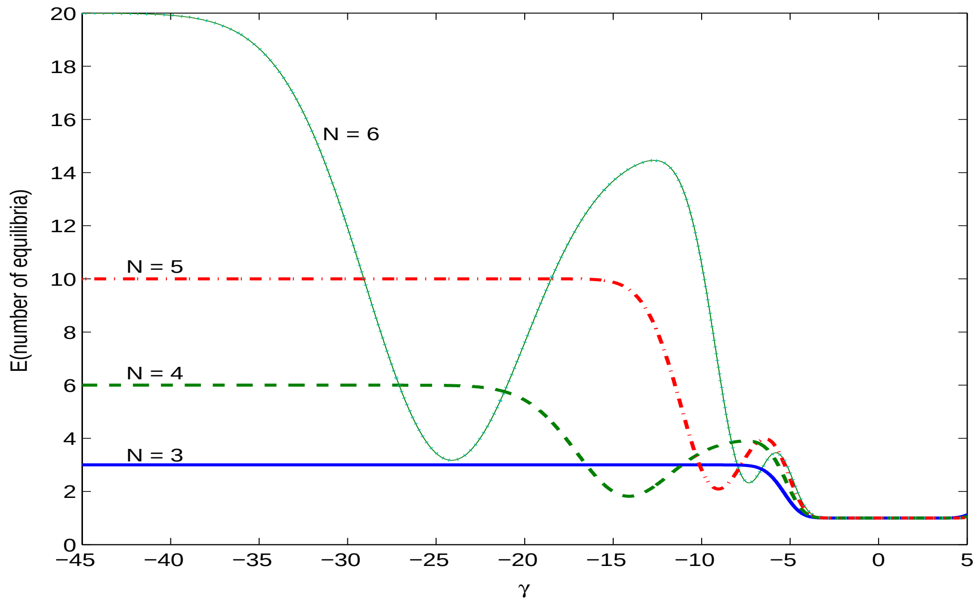

Figure 4: The expected number of equilibria of model (3) for $N \in \{3,4,5,6\}$, $\beta'\mathbf{x_i} = 5$ and $\epsilon_i \sim N(0,1)$, $\forall i$.

3.4 Expected number of equilibria

In section 3.2, upper bounds for the number of admissible equilibria were derived for every combination of $\{\beta, \gamma, \mathbf{x}, \epsilon\}$. In this section, I will focus on the *expected* number of equilibria given a combination $\{\beta, \gamma, \mathbf{x}, N\}$.

With independently and identically distributed disturbances ϵ_i, the expected number of equilibria can be expressed as

$$E[Q(\beta, \gamma, \mathbf{x}, N)] = \int Q(\beta, \gamma, \mathbf{x}, \epsilon, N) dF(\epsilon).$$

Equation (4) for $Q(\cdot)$ implies that $E[Q(\beta, \gamma, \mathbf{x}, N)]$ reduces to

$$\sum_{t=1}^{2^N} \left[\prod_{i=1}^{N} \left(1 - F\left(-\beta'\mathbf{x_i} - \frac{\gamma}{N-1} \sum_{j \neq i} y_{jt} \right) \right)^{\frac{1+y_{it}}{2}} F\left(-\beta'\mathbf{x_i} - \frac{\gamma}{N-1} \sum_{j \neq i} y_{jt} \right)^{\frac{1-y_{it}}{2}} \right],$$

with $F(\cdot)$ the cumulative distribution function of the ϵ_i's.

Using this expression and making some fairly general assumptions on the distribution of the ϵ's, proposition 4 can be derived. This proposition describes how the expected number of equilibria changes with γ for a binary choice game with N agents.

Proposition 4 Expected number of equilibria

Assume that the ϵ_i's in (3) are i.i.d. distributed according to a symmetric p.d.f. $f(\cdot)$.

Then for $N \leq 3$, $\partial E[Q(\beta, \gamma, \mathbf{x}, N)]/\partial\gamma$ is non-positive for $\gamma \in (-\infty, 0)$ and non-negative for $\gamma \in (0, +\infty)$. In particular,

$$\frac{\partial E[Q(\beta, 0, \mathbf{x}, N)]}{\partial\gamma} = 0 \text{ , and,}$$

$$\lim_{\gamma\to-\infty} \frac{\partial E[Q(\beta, \gamma, \mathbf{x}, N)]}{\partial\gamma} = \lim_{\gamma\to+\infty} \frac{\partial E[Q(\beta, \gamma, \mathbf{x}, N)]}{\partial\gamma} = 0.$$

For $N > 3$, $\partial E[Q(\beta, \gamma, \mathbf{x}, N)]/\partial\gamma$ may change sign at points other than $\gamma = 0$.

Proof: See the Appendix.

In figure 4, the expected number of equilibria is plotted for $N = 3, 4, 5$ and 6 for the specific case where agents are homogenous with respect to deterministic private utility ($\beta'\mathbf{x_i} = 5,\ \forall i$) and where the ϵ's are assumed to be i.i.d. $N(0, 1)$ distributed. One observes that $E[Q(\beta, \gamma, \mathbf{x}, 3)]$ (the solid line) is nonincreasing in the domain $\gamma \in (-\infty, 0)$ but that in the same domain, $E[Q(\beta, \gamma, \mathbf{x}, N)]$ has decreasing as well as increasing parts for $N \in \{4, 5, 6\}$. These oscillations are a consequence of changes in the set of equilibria as γ changes value. Example 2 in the previous section gives a particular example of how the number of equilibria may increase as well as decrease as gamma becomes more negative. Note that proposition 4 imposes no assumptions on the value of the $\beta'\mathbf{x_i}$'s.

Figure 4 shows that for a broad range of γ, the expected number of equilibria is well below the upper bound. In particular, the range for which this holds seems to increase with N. Notice that especially for positive values of γ, multiple equilibria do not seem to be an important issue.[9]

4 Extension to more general interactions

The model considered so far only allows for identical interactions between all individuals in the population. One can think of more general interactions, where the degree of interaction between two given individuals depends on e.g. their socio-economic characteristics. In this section, I will briefly discuss the consequences of one particular extension of the model given by (3), in which the degree of interaction is made gender-dependent. This leads to four different interaction parameters: γ_{GB} gives the effect of boys on girls; γ_{BG} from girls on boys, and γ_{GG} and γ_{BB} the intra-gender effects between girls and boys, respectively. Specify

$$\begin{cases} y_i^* = \beta' \mathbf{x_i} + S_i + \epsilon_i & \\ y_i = 1 & \text{if } y_i^* > 0, \\ y_i = -1 & \text{if } y_i^* \leq 0, \end{cases} \tag{8}$$

where

$$S_i = \begin{cases} (\gamma_{GG} \sum_{j=1, j\neq i}^{N} y_j^G \quad + \gamma_{GB} \sum_{j=1}^{N} y_j^B \quad)/(N-1) & \text{if } i \text{ is a girl,} \\ (\gamma_{BG} \sum_{j=1}^{N} y_j^G \quad + \gamma_{BB} \sum_{j=1, j\neq i}^{N} y_j^B \quad)/(N-1) & \text{if } i \text{ is a boy,} \end{cases}$$

with $y_j^G \equiv y_j \cdot I(j \text{ is a girl})$ and $y_j^B \equiv y_j \cdot I(j \text{ is a boy})$, $\forall j$, and $I(\cdot)$ an indicator function. See Kooreman and Soetevent (2003) for an empirical application of this model.

Corollary 2 *For every combination* $\{\beta, \gamma_{BB} \geq 0, \gamma_{GG} \geq 0, \gamma_{GB}, \gamma_{BG}, \mathbf{x}, \epsilon\}$ *there exists at least one vector* $\mathbf{y} \equiv (y_1, y_2, \ldots, y_N)'$ *for which (8) holds.*

Proof: See the Appendix.

The equivalent of proposition 2 for the extended model follows automatically:

[9] This is consonant with the result in section **??**, where in most cases the probability of an unique equilibrium was estimated to be larger than 80 percent.

Corollary 3 *For every combination* $\{\beta, \gamma_{BB} > 0, \gamma_{GG} > 0, \gamma_{GB}, \gamma_{BG}, \mathbf{x}, \epsilon\}$, *the discrete interaction model given by (8) with* N_G *girls and* N_B *boys can have at most* $d^*(N_B, N_G)$ *distinct equilibria, where*

$$d^*(N_B, N_G) \;\; = \;\; \lfloor \frac{N_B}{2} + 1 \rfloor \cdot \lfloor \frac{N_G}{2} + 1 \rfloor.$$

Moreover, for all N_G *and* N_B*, there exists a combination of* $\{\beta, \gamma_{BB} > 0, \gamma_{GG} > 0, \gamma_{GB}, \gamma_{BG}, \mathbf{x}, \epsilon\}$ *for which the maximum number of equilibria is obtained.*

It is noteworthy that the values of the cross-gender interaction parameters γ_{GB} and γ_{BG} do not play a role in determining the maximum number of equilibria.

5 Conclusions

In this paper, a number of equilibrium properties were derived for the binary choice interaction model with a finite number of agents. Both for the case with strategic complements and strategic substitutes, equilibrium existence was proved and tight upper bounds were derived for the number of equilibria, given the number of agents and the degree of interaction between them.

For the case with strategic substitutes, I showed that two outcomes can only be both an equilibrium outcome if for each of them the same number of agents chooses +1. When the number of agents is larger than 3, the expected number of equilibria is non-monotone with respect to changes in the degree of interaction γ. I also briefly discussed the consequences for the equilibrium set when the model is extended to allow for gender-dependent interactions. The main finding here is that the introduction of cross-gender interactions does not affect the upper bounds for the number of equilibria.

One major challenge for future research is to incorporate more general interaction structures in empirical work, by allowing interaction parameters to depend on socio-economic characteristics. Another is to develop an efficient algorithm for finding all equilibria in games with strategic substitutes.

6 Appendix: Proofs

Proof of Proposition 1: Equilibrium existence

The case for $\gamma = 0$ is obvious. I prove proposition 1 for the game with strategic complements ($\gamma > 0$) and the game with strategic substitutes ($\gamma < 0$) separately. For the first case, existence can be readily proved by showing that the game described in section 2 belongs to the class of supermodular games. Existence then immediately follows from using Theorem 5 in Milgrom and Roberts (1990, p. 1265). In this appendix however, I will follow for both cases the alternative route of proving equilibrium existence through finding an explicit equilibrium for all combinations of $\{\beta, \gamma, \mathbf{x}, \epsilon\}$. The main reason is that this procedure gives more insight into some of the peculiarities of the model.

Strategic complements ($\gamma > 0$)

Every possible combination of $\{\beta, \gamma > 0, \mathbf{x}, \epsilon\}$ clearly falls into one of the three following categories

(i) $z_{[1]} \leq 0$;

(ii) $z_{[N]} > 0$;

(iii) $z_{[1]} > 0, z_{[N]} \leq 0$.

I show that for each $\mathbf{z}$ in every category there is an associated $\mathbf{y}$ for which (3) holds, for all values $\gamma > 0$.

(i) $z_{[1]} \leq 0$:
$y_i = -1, i = 1, 2, \ldots, N$ $(k = -N)$ is an equilibrium solution, since $y^*_{[1]} = z_{[1]} - \gamma \frac{N-1}{N-1} \leq 0$. This implies that $y^*_{[i]} \leq 0$ $\forall i$ since $\gamma \frac{N-1}{N-1}$ is a constant and $z_{[i]}$ weakly decreases with i.

(ii) $z_{[N]} > 0$:

$y_i = 1, i = 1, 2, \ldots, N$ $(k = N)$ is an equilibrium solution, since $y^*_{[i]} = z_{[i]} + \gamma \frac{N-1}{N-1} > 0, \forall i$.

(iii) $z_{[1]} > 0, z_{[N]} \leq 0$:

Define $M \equiv 0$ if $z_{[j]} \leq -\gamma \frac{(2j-N-1)}{N-1}, \forall j,\ j \in \{1, 2, \ldots, N\}$ and $M \equiv \arg\max_i \left(z_{[j]} > -\gamma \frac{(2i-N-1)}{N-1}; \forall j \leq i \right)$ otherwise. Five examples of sequences of $z_{[i]}$ with $N = 6$ and $\gamma = 1$ are plotted in figure 5 together with the corresponding values of M. The solid line represents the equation $z_{[i]} = -\gamma \frac{(2i-N-1)}{N-1}$.

If $M = 0$, $y_{[i]} = -1$, $i = 1, 2, \ldots, N$ is an equilibrium solution, since $y^*_{[i]} = z_{[i]} - \gamma \leq z_{[1]} - \gamma \leq 0$, $\forall i$. (See the +-sequence in figure 5.)

If $M > 0$, $y_{[i]} = 1$ for $i = 1, 2 \ldots, M$ and $y_{[i]} = -1$ for $i = M + 1, M + 2, \ldots, N$ $(k = M - [N - M] = 2M - N)$ is an equilibrium solution, since $y^*_{[i]} = z_{[i]} + \gamma \frac{2M-N-1}{N-1} > 0$ for $i = 1, 2, \ldots, M$ and $y^*_{[j]} \leq y^*_{[M+1]} = z_{[M+1]} + \gamma \frac{2M-N+1}{N-1} = z_{[M+1]} + \gamma \frac{2(M+1)-N-1}{N-1} \leq 0$ for all $j = M + 1, M + 2, \ldots, N$.

Note that for sequences of $z_{[i]}$'s for which $M = N$ (like the sequence of circles and x-es in figure 5), $y_{[i]} = -1$, $i = 1, 2, \ldots, N$ is another equilibrium solution if and only if $z_{[1]} \leq \gamma$. In figure 5, this condition holds for the sequence of x-es but not for the sequence of circles. □

Strategic substitutes ($\gamma < 0$)

In this case, I distinguish between the case where the number of subjects N is even and the case where this number is odd.

N even Let $\gamma < 0$. Define $m \equiv \arg\max_i \left(z_{[i]} > 0 \right)$. Suppose that $m > N/2$, that is, the majority of the subjects have a value of z greater than zero. Define the non-overlapping non-empty intervals $I_0 \equiv \left[0, \frac{z_{[m]}(N-1)}{2m-N-1}\right)$;

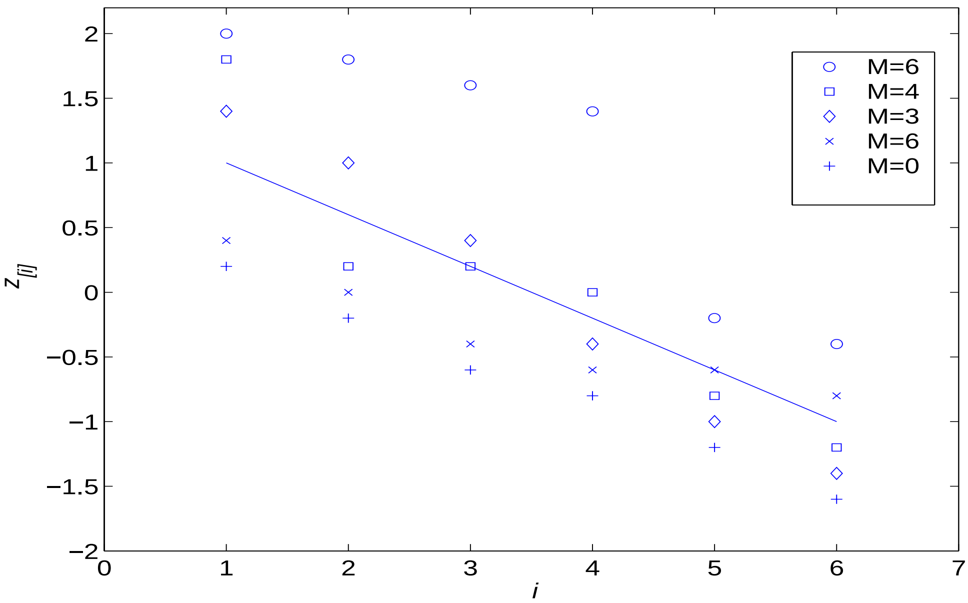

Figure 5: Five examples of $z_{[i]}$-sequences and the corresponding solutions for $M \equiv \arg\max_i \left(z_{[j]} > -\gamma\frac{(2i-N-1)}{N-1}; \forall j \leq i \right)$ for the case with $N = 6$ and $\gamma = 1$.

$I_{m-N/2} \equiv \left[\frac{z_{[N/2+1]}(N-1)}{2(N/2+1)-N-1}, \infty \right) = [z_{[N/2+1]}(N-1), \infty)$ and, if $m > N/2+1$, $I_r \equiv \left[\frac{z_{[m-r+1]}(N-1)}{2(m-r+1)-N-1}, \frac{z_{[m-r]}(N-1)}{2(m-r)-N-1} \right)$, for $r = 1, 2, \ldots, m - N/2 - 1$.

First consider the case $m > N/2 + 1$. Since the intervals are non-overlapping and since $I_0 \cup I_1 \cup \ldots \cup I_{m-N/2} = [0, \infty)$, $-\gamma$ is in one and only one of these intervals. If $-\gamma \in I_0$, $\mathbf{y} = (1, 1, \ldots, 1_m, -1, \ldots, -1)'$, $(k = 2m - N)$ is an equilibrium, since for this solution $y^*_{[1]} \geq \ldots \geq y^*_{[m]} = z_{[m]} + \gamma\frac{2m-N-1}{N-1} > 0$ and $y^*_{[N]} \leq \ldots \leq y^*_{[m+1]} = z_{[m+1]} + \gamma\frac{2m-N+1}{N-1} \leq 0$. If $-\gamma \in I_r$, for $r = 1, 2, \ldots, m-N/2-1$, $\mathbf{y} = (1, 1, \ldots, 1_{m-r}, -1, \ldots, -1)'$ $(k = 2(m-r)-N)$ is an equilibrium, since for this solution $y^*_{[1]} \geq \ldots \geq y^*_{m-r} = z_{[m-r]} + \gamma\frac{2(m-r)-N-1}{N-1} > 0$ and $y^*_{[N]} \leq \ldots \leq y^*_{[m-r+1]} = z_{[m-r+1]} + \gamma\frac{2(m-r)-N+1}{N-1} \leq 0$. If $-\gamma \in I_{m-N/2}$, $\mathbf{y} = (1, 1, \ldots, 1_{N/2}, -1, \ldots, -1)'$ $(k = 0)$ is an equilibrium, since for this solution $y^*_{[1]} \geq \ldots \geq y^*_{[N/2]} = z_{[N/2]} + \gamma\frac{-1}{N-1} > 0$ and $y^*_{[N]} \leq \ldots \leq y^*_{[N/2+1]} = z_{[N/2+1]} + \gamma\frac{1}{N-1} \leq 0$.

If $m = N/2 + 1$, then $I_0 \cup I_{m-N/2} = I_0 \cup I_1 = [0, \infty)$. Applying similar reasoning, one can verify that $\mathbf{y} = (1, 1, \ldots, 1_{N/2+1}, -1, \ldots, -1)'$, $(k = 2)$ is an equilibrium when $-\gamma \in I_0$ and that $\mathbf{y} = (1, 1, \ldots, 1_{N/2}, -1, \ldots, -1)'$ $(k = 0)$ is an equilibrium when $-\gamma \in I_1$.

If $m = N/2$, then $\mathbf{y} = (1, 1, \ldots, 1_{N/2}, -1, \ldots, -1)'$ is an equilibrium for all $-\gamma \in (0, \infty)$, since $y^*_{[1]} \geq \ldots \geq y^*_{[N/2]} = z_{[N/2]} + \gamma \frac{-1}{N-1} > z_{[N/2]} > 0$ and $y^*_{[N]} \leq \ldots \leq y^*_{[N/2+1]} = z_{[N/2+1]} + \gamma \frac{1}{N-1} < z_{[N/2+1]} \leq 0$.

Due to symmetry, the above argument can be applied for $m < N/2$ with m replaced by $\tilde{m} \equiv N - m \geq N/2$ and the roles of the outcomes +1 and -1 interchanged.

N **odd** The above argument can also be applied for odd N. Suppose that $m > (N+1)/2$ and define $I_0 \equiv \left[0, \frac{z_{[m]}(N-1)}{2m-N-1}\right)$, $I_{m-(N+1)/2} \equiv \left[\frac{z_{[(N+1)/2+1]}(N-1)}{2\left(\frac{N+1}{2}+1\right)-N-1}, \infty\right) = \left[\frac{z_{[(N+1)/2+1]}(N-1)}{2}, \infty\right)$ and, if $m > (N+1)/2+1$, $I_r \equiv \left[\frac{z_{[m-r+1]}(N-1)}{2(m-r+1)-N-1}, \frac{z_{[m-r]}(N-1)}{2(m-r)-N-1}\right)$, for $r = 1, 2, \ldots, m - (N+1)/2 - 1$.

Taking the case that $m > (N+1)/2 + 1$, it follows that for $-\gamma \in I_0$, $\mathbf{y} = (1, 1, \ldots, 1_m, -1, \ldots, -1)'$ $(k = 2m - N)$ is an equilibrium; for $-\gamma \in I_r$, $r = 1, 2, \ldots, m - (N+1)/2 - 1$, $\mathbf{y} = (1, 1, \ldots, 1_{m-r}, -1, \ldots, -1)'$ $(k = 2(m-r) - N)$ is an equilibrium; and for $-\gamma \in I_{m-(N+1)/2}$, $\mathbf{y} = (1, 1, \ldots, 1_{(N+1)/2}, -1, \ldots, -1)'$ $(k = 1)$ is an equilibrium.

If $m = (N+1)/2+1$, then $I_0 \cup I_{m-(N+1)/2} = I_0 \cup I_1 = [0, \infty)$. Applying similar reasoning, one can verify that $\mathbf{y} = (1, 1, \ldots, 1_{(N+1)/2+1}, -1, \ldots, -1)'$ $(k = 3)$ is an equilibrium when $-\gamma \in I_0$ and that $\mathbf{y} = (1, 1, \ldots, 1_{(N+1)/2}, -1, \ldots, -1)'$ $(k = 1)$ is an equilibrium when $-\gamma \in I_1$.

If $m = (N+1)/2$, then $\mathbf{y} = (1, 1, \ldots, 1_{(N+1)/2}, -1, \ldots, -1)'$ is an equilibrium for all $-\gamma \in (0, \infty)$, since $y^*_{[1]} \geq \ldots \geq y^*_{[(N+1)/2]} = z_{[(N+1)/2]} + \gamma \cdot 0 > 0$ and $y^*_{[N]} \leq \ldots \leq y^*_{[(N+1)/2+1]} = z_{[(N+1)/2+1]} + \gamma \frac{2}{N-1} < z_{[N/2+1]} \leq 0$. Again, the case with $m < (N+1)/2$ follows from symmetry. $\square$

Proof of Lemma 1

Consider an agent i with $y_i = 1$ and an agent j with $y_j = -1$. Suppose $z_j \geq z_i - \frac{2\gamma}{N-1}$. Then $y_j^* = z_j + \gamma\left(\frac{k+1}{N-1}\right) \geq z_i + \gamma\left(\frac{k-1}{N-1}\right) = y_i^*$. But since $y_i = 1$ and $y_j = -1$ implies $y_i^* > 0 \geq y_j^*$, we have a contradiction. □

Proof of Lemma 2

Suppose that $\mathbf{y}$ with $\sum_{i=1}^N y_i = k$ and $\tilde{\mathbf{y}}$ with $\sum_{i=1}^N \tilde{y}_i = \tilde{k}$ and $\tilde{k} \neq k$ are both equilibria of (3), given a combination $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$. From lemma 3 it follows that this is true only if $\mathbf{y^k} = (1_1, 1_2, \ldots, 1_{\frac{N+k}{2}}, -1_{\frac{N+k+2}{2}}, \ldots, -1_N)$ and $\mathbf{y^{\tilde{k}}} = (1_1, 1_2, \ldots, 1_{\frac{N+\tilde{k}}{2}}, -1_{\frac{N+\tilde{k}+2}{2}}, \ldots, -1_N)$ are both equilibria given $\{\beta, \gamma < 0, \mathbf{x}, \epsilon\}$. Assume without loss of generality that $\tilde{k} > k$, that is: $\tilde{k} - k \geq 2$. Let ν be the first subject whose choice is -1 in equilibrium $\mathbf{y^k}$ and $+1$ in equilibrium $\mathbf{y^{\tilde{k}}}$. Then, for this subject

$$z_{[\nu]} + \gamma\frac{k+1}{N-1} \leq 0 \text{ and } z_{[\nu]} + \gamma\frac{\tilde{k}-1}{N-1} > 0.$$

But also

$$z_{[\nu]} + \gamma\frac{\tilde{k}-1}{N-1} = z_{[\nu]} + \gamma\frac{\tilde{k}-k+k+1-2}{N-1} = \underbrace{z_{[\nu]} + \gamma\frac{k+1}{N-1}}_{\leq 0} + \underbrace{\gamma\frac{(\tilde{k}-k)-2}{N-1}}_{\leq 0} \leq 0,$$

and the contradiction follows. □

Proof of Lemma 3

From the fact that $\mathbf{y}$ is an equilibrium with $y_{[j]} = -1$ and $y_{[j+1]} = 1$, it follows that

$$\begin{aligned} y^*_{[j]} &= z_{[j]} + \gamma\frac{k+1}{N-1} \leq 0 \\ y^*_{[j+1]} &= z_{[j+1]} + \gamma\frac{k-1}{N-1} > 0. \end{aligned}$$

However, since $\gamma < 0$, we have

$$\begin{aligned} \tilde{y}^*_{[j]} &= z_{[j]} + \gamma\frac{k-1}{N-1} \geq z_{[j+1]} + \gamma\frac{k-1}{N-1} > 0 \\ \tilde{y}^*_{[j+1]} &= z_{[j+1]} + \gamma\frac{k+1}{N-1} \leq z_{[j]} + \gamma\frac{k+1}{N-1} \leq 0. \end{aligned}$$

It then follows that $\tilde{\mathbf{y}}$ with $\tilde{y}_{[i]} = y_{[i]}$ for $i \neq j, j+1$ and $\tilde{y}_{[j]} = 1$ and $\tilde{y}_{[j+1]} = -1$ is also an equilibrium. $\square$

Proof of Corollary 2

Define $\forall i$, $z_i^G \equiv \beta'\mathbf{x_i} + \frac{\gamma_{GB} \sum_{j=1}^N y_j^B}{N-1} + \epsilon_i$ if i is a girl and $z_i^B \equiv \beta'\mathbf{x_i} + \frac{\gamma_{BG} \sum_{j=1}^N y_j^G}{N-1} + \epsilon_i$ if i is a boy. Denote the ordered values of z_i^G (z_i^B) as $z_{[i]}^G$ ($z_{[i]}^B$) such that $z_{[1]}^G \geq z_{[2]}^G \geq \ldots \geq z_{[N_G]}$ ($z_{[1]}^B \geq z_{[2]}^B \geq \ldots \geq z_{[N_B]}$), with N_G (N_B) denoting the total number of girls (boys) in the sample.

The line of reasoning used in the proof of proposition 1 now can be applied to the subset of girls (boys), with $z_{[i]}$ replaced by $z_{[i]}^G$ ($z_{[i]}^B$) and γ replaced by γ_{GG} (γ_{BB}). $\square$

Proof of Proposition 4

Suppose the disturbances ϵ_i are i.i.d. according to a symmetric p.d.f. $f(\cdot)$, such that $f(-x) = f(x)$. Define $m_{it} \equiv \sum_{j \neq i} y_{jt}/(N-1)$ for all $i = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, 2^N$. Then

$$E[Q(\beta, \gamma, \mathbf{x}, N)] = \sum_{t=1}^{2^N} \prod_{i=1}^{N} F(\beta'\mathbf{x_i} + \gamma m_{it})^{\frac{1+y_{it}}{2}} F(-\beta'\mathbf{x_i} - \gamma m_{it})^{\frac{1-y_{it}}{2}}.$$

It then follows that

$$\frac{\partial E[Q(\beta, \gamma, \mathbf{x}, N)]}{\partial \gamma} =$$
$$\sum_{t=1}^{2^N} \left\{ \sum_{i=1}^{N} \left[\left(\prod_{j \neq i} F(\beta'\mathbf{x_j} + \gamma m_{jt})^{\frac{1+y_{jt}}{2}} F(-\beta'\mathbf{x_j} - \gamma m_{jt})^{\frac{1-y_{jt}}{2}} \right) \cdot \left(\frac{\partial [F(\beta'\mathbf{x_i} + \gamma m_{it})^{\frac{1+y_{it}}{2}} F(-\beta'\mathbf{x_i} - \gamma m_{it})^{\frac{1-y_{it}}{2}}]}{\partial \gamma} \right) \right] \right\}.$$

In this expression, $\partial [F(\beta'\mathbf{x_i} + \gamma m_{it})^{\frac{1+y_{it}}{2}} F(-\beta'\mathbf{x_i} - \gamma m_{it})^{\frac{1-y_{it}}{2}}]/\partial \gamma$ can be rewritten as

$$\frac{1+y_{it}}{2} m_{it} f(\beta'\mathbf{x_i} + \gamma m_{it}) F(\beta'\mathbf{x_i} + \gamma m_{it})^{\frac{y_{it}-1}{2}} F(-\beta'\mathbf{x_i} - \gamma m_{it})^{\frac{1-y_{it}}{2}} -$$
$$\frac{1-y_{it}}{2} m_{it} f(-\beta'\mathbf{x_i} - \gamma m_{it}) F(-\beta'\mathbf{x_i} - \gamma m_{it})^{-\frac{y_{it}+1}{2}} F(\beta'\mathbf{x_i} + \gamma m_{it})^{\frac{y_{it}+1}{2}}$$

$$= \frac{m_{it}}{2}\left(\sqrt{\frac{F(\beta'\mathbf{x_i}+\gamma m_{it})}{F(-\beta'\mathbf{x_i}-\gamma m_{it})}}\right)^{y_{it}} f(\beta'\mathbf{x_i}+\gamma m_{it})\cdot$$

$$\left[\frac{F(-\beta'\mathbf{x_i}-\gamma m_{it})/F(\beta'\mathbf{x_i}+\gamma m_{it})-1}{\sqrt{F(-\beta'\mathbf{x_i}-\gamma m_{it})/F(\beta'\mathbf{x_i}+\gamma m_{it})}} + y_{it}\left(\frac{F(-\beta'\mathbf{x_i}-\gamma m_{it})/F(\beta'\mathbf{x_i}+\gamma m_{it})+1}{\sqrt{F(-\beta'\mathbf{x_i}-\gamma m_{it})/F(\beta'\mathbf{x_i}+\gamma m_{it})}}\right)\right]$$

$$= y_{it} m_{it} f(\beta'\mathbf{x_i}+\gamma m_{it}).$$

The correctness of the last equation is easily verified by inserting $y_{it} = 1$ and $y_{it} = -1$. As a result

$$\frac{\partial E[Q(\beta,\gamma,\mathbf{x},N)]}{\partial\gamma} =$$

$$\sum_{t=1}^{2^N}\left\{\sum_{i=1}^{N}\left[\left(y_{it} m_{it} f(\beta'\mathbf{x_i}+\gamma m_{it})\cdot\left(\prod_{j\neq i} F(\beta'\mathbf{x_j}+\gamma m_{jt})^{\frac{1+y_{jt}}{2}} F(-\beta'\mathbf{x_j}-\gamma m_{jt})^{\frac{1-y_{jt}}{2}}\right)\right]\right\}.$$

For $N = 2$, $\partial E[Q(\beta,\gamma,\mathbf{x},N)]/\partial\gamma$ reduces to

$$[F(\beta'\mathbf{x_2}+\gamma) - F(\beta'\mathbf{x_2}-\gamma)]f(\beta'\mathbf{x_1}+\gamma)+$$

$$[F(-\beta'\mathbf{x_2}+\gamma) - F(-\beta'\mathbf{x_2}-\gamma)]f(\beta'\mathbf{x_1}-\gamma)+$$

$$[F(\beta'\mathbf{x_1}+\gamma) - F(\beta'\mathbf{x_1}-\gamma)]f(\beta'\mathbf{x_2}+\gamma)+$$

$$[F(-\beta'\mathbf{x_1}+\gamma) - F(-\beta'\mathbf{x_1}-\gamma)]f(\beta'\mathbf{x_2}-\gamma),$$

and for $N = 3$ to

$$[F(\beta'\mathbf{x_2}+\gamma)F(\beta'\mathbf{x_3}+\gamma) - F(\beta'\mathbf{x_2})F(\beta'\mathbf{x_3})]f(\beta'\mathbf{x_1}+\gamma)+$$

$$[F(\beta'\mathbf{x_1}+\gamma)F(\beta'\mathbf{x_3}+\gamma) - F(\beta'\mathbf{x_1})F(\beta'\mathbf{x_3})]f(\beta'\mathbf{x_2}+\gamma)+$$

$$[F(\beta'\mathbf{x_1}+\gamma)F(\beta'\mathbf{x_2}+\gamma) - F(\beta'\mathbf{x_1})F(\beta'\mathbf{x_2})]f(\beta'\mathbf{x_3}+\gamma)+$$

$$[F(-\beta'\mathbf{x_2}+\gamma)F(-\beta'\mathbf{x_3}+\gamma) - F(-\beta'\mathbf{x_2})F(-\beta'\mathbf{x_3})]f(\beta'\mathbf{x_1}-\gamma)+$$

$$[F(-\beta'\mathbf{x_1}+\gamma)F(-\beta'\mathbf{x_3}+\gamma) - F(-\beta'\mathbf{x_1})F(-\beta'\mathbf{x_3})]f(\beta'\mathbf{x_2}-\gamma)+$$

$$[F(-\beta'\mathbf{x_1}+\gamma)F(-\beta'\mathbf{x_2}+\gamma) - F(-\beta'\mathbf{x_1})F(-\beta'\mathbf{x_2})]f(\beta'\mathbf{x_3}-\gamma).$$

One readily observes that for $\gamma < 0$ ($\gamma > 0$) the terms within brackets are all non-positive (non-negative). In the same way, one can show that this is not true for values of $N > 3$. This completes the proof. □

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