Degeneracy Degrees of Constraint Collections

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Abstract

This paper presents an unifying approach to the theory of degeneracy of basic feasible solutions, vertices, faces, and all subsets of polyhedra. We use the concept of degeneracy degree for arbitrary subsets of $\mathbb{R}^n$ with respect to linear constraint collections. We discuss the connection with the usual definitions, and establish the relationship between minimal representations of polyhedra and the degeneracy of their faces. We also consider a number of complexity aspects of the problem of determining degeneracy degrees.

Keywords: Linear Programming, Linear Optimization, Degeneracy, Complexity.

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1. Introduction

For a long time, degeneracy was considered something of theoretical value, that appeared only very seldom in practice. This situation changed since the time it occurs more frequently, among that in many combinatorial optimization problems such as crew scheduling. A recent survey about degeneracy in optimization problems can be found in Gal[3]. However, the theory on degeneracy shows not much agreement about the definitions and starting points. In case of linear programming, degeneracy is usually only defined for basic feasible solutions and vertices. A basic feasible solution is then called degenerate if at least one of the basic variables has a zero value. However, in Nering & Tucker[5], an LP-model is called degenerate if it has at least one degenerate basic solution (not necessarily feasible). In Güler et al.[4], an LP-model is called degenerate if there is at least one feasible point that has less than m positive coordinate entries, with m the number of equality constraints in the primal standard model. These definitions are all based on the existence of a degenerate point. In Roos et al.[6], an LP-model is called degenerate if either the primal problem or its dual has multiple optimal solutions. This definition relates the degeneracy of an LP-model to the degeneracy of the optimal faces. Degeneracy sometimes plays an important role in the proofs of the convergence of algorithms; for instance in the convergence of the affine scaling methods; for an overview see e.g. Güler et al.[4]. On the other hand, degeneracy may cause numerical problems in interior point methods, by making the linear systems, that are solved close to the optimum, ill-conditioned; see e.g. Güler et al.[4].

In the underlying paper we provide a unifying approach, in which we define the degree of degeneracy of arbitrary subsets of $\mathbb{R}^n$ with respect to a given constraint collection that defines a polyhedron.

2. Degeneracy of sets

Let P be a collection of m linear constraints in $\mathbb{R}^n$, called a *constraint collection*, consisting of m_1 equalities and $m - m_1$ inequalities in the variables $x_1, \ldots, x_n$; say:

$$P = \{\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, \ldots, m_1; \ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = m_1 + 1, \ldots, m\}. \quad (1)$$

We denote by $pol(P)$ the set of points in $\mathbb{R}^n$ for which all constraints of P are satisfied, i.e. $pol(P)$ is the polyhedron represented by the constraint collection P.

Let S be a subset of $\mathbb{R}^n$. A constraint of P is called *binding* on S, if it is satisfied with equality for every point of S. Note that the empty set is binding on all constraints of P, since the empty set is contained in the intersection of any collection of equality constraints. Denote the number of constraints of P that are binding on S by $bnd(S, P)$, and the dimension of the polyhedron represented by the binding constraints on S by $dimbnd(S, P)$. The *degeneracy degree* of a subset $S \subseteq \mathbb{R}^n$ w.r.t. P, is denoted and defined by

$$\sigma(S, P) = bnd(S, P) + dimbnd(S, P) - n;$$

see Tijssen & Sierksma[9]. The definition can be motivated as follows. The number of hyperplanes that determine the intersection of the binding constraints on S is at least equal to $n - dimbnd(S, P)$, and this lower bound is sharp. If the number of constraints of P that are binding on S is larger than $n - dimbnd(S, P)$, then there is 'redundancy' in the collection of hyperplanes that defines the affine hull of S. Therefore, $\sigma(S, P) \geq 0$ for every S and P. S is called *degenerate* w.r.t. P iff $\sigma(S, P) > 0$, and S is called *non-degenerate* w.r.t. P iff $\sigma(S, P) = 0$. In Gal et al.[2] a definition for "degeneracy degree" is introduced for vertices, in which case $dimbnd(S, P) = 0$.

The degeneracy degree of the empty set is well defined, and depends on the constraint collection in the following way. Let P be a collection of m constraints in R^n. Since the empty set belongs to all m affine subspaces that are the boundaries of the m constraints, it follows that $bnd(\emptyset, P) = m$. The dimension of this intersection is at least equal to the dimension of the empty set, which is defined to be -1. Therefore, $\sigma(\emptyset, P) \geq m + (-1) - n = m - n - 1$. If, with an arbitrary $S \in \mathbb{R}^n$ no constraint of P is binding on S, then $\sigma(S, P) = 0 + n - n = 0$.

Theorem 2.1 *Let P be a constraint collection on $\mathbb{R}^n$, and let S_1 and S_2 be subsets of $\mathbb{R}^n$ with $S_1 \subseteq S_2$, then $\sigma(S_1, P) \geq \sigma(S_2, P)$.*

(1) The degeneracy degree of S_2 satisfies $\sigma(S_2, P) = bnd(S_2, P)+$ $dimbnd(F_2, P) - n$. Hence, $bnd(S_2, P) = n - dimbnd(S_2, P) + \sigma(S_2, P)$. Let S_1 be a subset of S_2. Then, $dimbnd(S_1, P) \leq dimbnd(S_2, P)$. The number of binding constraints of P on S_1 is at least $bnd(S_2, P) + (dimbnd(S_2, P) - dimbnd(S_1, P))$, and we have that $\sigma(S_1, P) = bnd(S_1, P) + dimbnd(S_1, P) - n \geq bnd(S_2, P) + (dimbnd(S_2, P) - dimbnd(S_1, P)) + dimbnd(S_1, P) - n = bnd(S_2, P)+$ $dimbnd(S_2, P) - n = \sigma(S_2, P)$. □

If for two sets S_1 and S_2 the same constraints in P are binding, then $\sigma(S_1, P) = \sigma(S_2, P)$. The polyhedron Q represented by these binding constraints is the largest polyhedron for which $S_1 \in Q$, and $\sigma(S_1, P) = \sigma(Q, P)$.

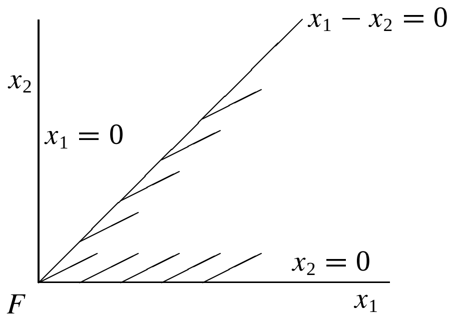

Figure 3.1: Example of a face.

3. Degeneracy on polyhedra

In this section we assume that the constraint collection P represents a nonempty polyhedron, i.e. $pol(P) \neq \emptyset$. A constraint H of a constraint collection P is called *redundant* if its deletion results in a constraint collection representing the same polyhedron as P, i.e.

$$pol(P \setminus \{H\}) = pol(P).$$

Note that the deletion of one redundant constraint may change another redundant constraint into a non-redundant one. An inequality of a constraint collection P is called an *implicit equality of* P if that inequality is satisfied with equality for every point of $pol(P)$. A *minimal representation* of a polyhedron is a constraint collection with a minimal number of constraints; i.e. the deletion of any constraint results in a different polyhedron. A thorough survey of the properties of redundant constraints, implicit equalities, and minimal representations can be found in Telgen[7], where it is shown among others that a minimal representation contains neither redundant constraints nor implicit equalities. Let F be a face of the polyhedron $pol(P)$. A constraint collection that represents F can be obtained from P by replacing an appropriate collection of inequalities of P by equalities. However, such representations are not unique in general; they may contain redundant constraints. This may be clear from the following example.

Let $P = \{x_1 - x_2 \geq 0;\ x_1 \geq 0;\ x_2 \geq 0\}$. The polyhedron represented by P is depicted as the shaded area of Figure 3.1. The face $F = \{(0, 0)\}$ (with dimension 0) can be represented in different ways using the constraints of P by changing a number of inequalities into equalities. For instance, both $\{x_1 - x_2 = 0;\ x_1 = 0;\ x_2 \geq 0\}$ and

$\{x_1 - x_2 \geq 0;\ x_1 = 0;\ x_2 = 0\}$ represent F. All three constraints of P are binding on F. Clearly, $bnd(F, P) = 3$ and $dimbnd(F) = 0$.
Even if P is a minimal representation, the representation of a face F of $pol(P)$ need not be unique; if, for example, the octahedron of Figure 3.3 is represented by a minimal representation, then any vertex can be represented by replacing three of the four binding inequalities by equalities.

The definition of the concepts "degenerate face" and "degenerate vertex" of a polyhedron represented by a constraint collection P can be obtained from the definition by letting S being a face or a vertex of $pol(P)$, respectively. Since the intersection of the constraints that are binding on the face F is the affine hull of that face, it follows that $dimbnd(F, P) = dim(F)$. Note that the definition of "degenerate face" generalizes the usual definition of "degenerate vertex", because $bnd(v, P) + dim(v) > n$ reduces in case of a vertex to $bnd(v, P) > n$, which is in fact the usual definition of "degenerate vertex". The definition of "degenerate face" includes the definition of "degenerate polyhedron", since $pol(P)$ is a face of $pol(P)$ itself. In terms of linear programming, this means that the concept of "degenerate feasible region" is now well defined as well. In the following theorem we collect a number of properties of degeneracy degrees of faces.

Theorem 3.1 *Let P be a constraint collection representing a nonempty polyhedron in $\mathbb{R}^n$. Then the following assertions hold.*

1. *If F_1 and F_2 are faces of $pol(P)$ with $F_1 \subseteq F_2$, then $\sigma(F_1, P) \geq \sigma(F_2, P)$.*
2. *A face F of $pol(P)$ with dimension at least 1 is degenerate with respect to P iff all proper nonempty subsets of F are degenerate w.r.t. P.*
3. *If $pol(P)$ degenerate w.r.t. P, then P contains either a redundant constraint or an implicit equality.*
4. *A face F of $pol(P)$ with dimension at least 1 is non-degenerate w.r.t. P iff F contains a proper nonempty subset that is non-degenerate w.r.t. P.*

(1) This proof is equivalent to the proof of Theorem 2.1 by taking F_1 and F_2 for S_1 and S_2, respectively.

(2) Let F be a face of $pol(P)$ with dimension at least 1. We first prove the 'only if' part. Let $\sigma(F, P) > 0$. Then, according to Theorem 3.1(1), all subfaces of F have a positive degeneracy degree. Hence, all nonempty subsets of F have a positive degeneracy degree w.r.t. P. The proof of the 'if' part can be given as follows. If all proper nonempty subsets of F are degenerate w.r.t. P, then also the relative interior of F is degenerate w.r.t. P. Since F has dimension at least 1, the relative interior of F is a proper subset of F. Because F is the smallest face containing the relative

interior of F, F is degenerate with respect to P.

(3) Let $pol(P)$ be degenerate w.r.t P. Then, $\sigma(pol(P), P) > 0$. Let e denote the number of equalities in P. If $e > n - dim(pol(P))$, then P contains at least one redundant equality. If $e \leq n - dim(pol(P))$, then $bnd(pol(P), P) - e$ inequalities are binding on $pol(P)$. Since $bnd(pol(P), P) - e = n - dim(pol(P)) + \sigma(pol(P), P) - e \geq n - dim(pol(P)) + \sigma(pol(P), P) - n + dim(pol(P)) = \sigma(pol(P), P) > 0$, P contains at least one implicit equality.

(4) This is the logical reversal of (2). □

The following examples illustrate the degeneracy degree of a set S that is not a face of $pol(P)$.

Let $P = \{x_1 + x_2 \leq 2;\ x_1 \leq 1;\ x_2 \leq 1;\ x_1, x_2 \geq 0\}$, $F = pol(\{x_1 + x_2 \leq 2;\ x_1 \leq 1;\ x_2 = 1;\ x_1, x_2 \geq 0\})$, and $S = \{(0.2, 1), (0.4, 1)\}$; see Figure 3.2. F is the line segment $[(0, 1), (1, 1)]$. Note that $dim(F) = 1$, and that $x_2 \leq 1$ is the only inequality of P that is binding on F. F is non-degenerate w.r.t P, because $\sigma(F, P) = bnd(F, P) + dim(F) - n = 1 + 1 - 2 = 0$. The degeneracy degree w.r.t. P of the face consisting of the single vertex $v = (1, 1)$ satisfies $\sigma(v, P) = bnd(v, P) + dim(v) - n = 3 + 0 - 2 = 1$. The only binding constraint on S is the constraint $x_2 \leq 1$. The dimension of $x_2 = 1$ is 1. Therefore, $bnd(S) = 1$, $dimbnd(S) = 1$, and $\sigma(S, P) = 1 + 1 - 2 = 0$.

Corollary 3.1 *The degeneracy degree of a nonempty subset S of a polyhedron Q represented by the constraint collection P is equal to the degeneracy degree of the smallest face F of Q that contains S.*

The faces of a polyhedron together with the empty set form a lattice under inclusion. Therefore, there exists a unique smallest face F with $S \subseteq F$. The constraints that are binding on F are also binding on S. If there is a constraint that is binding on S that is not binding on F, then F is not the smallest face of Q that contains S. Therefore, the same collection of constraints is binding both on F and S. Hence, $bnd(F) = bnd(S)$, $dimbnd(F) = dimbnd(S)$, and $\sigma(S, P) = \sigma(F, P)$. □

As an example, let $P = \{0x \leq 0 \mid x \in \mathbb{R}^n\}$. For any $S \in \mathbb{R}^n$ there is exactly one binding constraint, and the smallest face containing S is $\mathbb{R}^n$ itself. Therefore, $\sigma(S, P) = \sigma(\mathbb{R}^n, P) = bnd(\mathbb{R}^n, P) + dim(\mathbb{R}^n) - n = 1 + n - n = 1$.

In general, it is not true that all subfaces of a non-degenerate face are non-degenerate. In the example preceding Corollary 3.1, the vertex (1,1) is a degenerate subface of the non-degenerate face F. Another example is the regular octahedron in $\mathbb{R}^3$

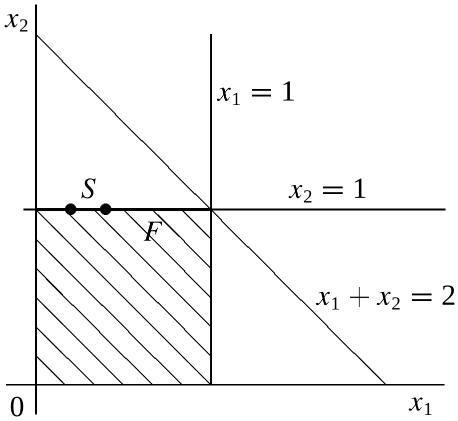

Figure 3.2: Degeneracy degree of a subset.

depicted in Figure 3.3. Each vertex of this octahedron is degenerate, since each vertex has four binding facets. This fact is independent of the representation of this octahedron by a constraint collection. If this octahedron is represented by a minimal representation with 8 inequality constraints (without redundant constraints or implicit equalities), then the edges, the facets, and the polyhedron itself are non-degenerate. In fact this is the smallest example of a polytope that has only degenerate vertices. The following example shows how representations of polyhedra may influence its degeneracy degrees.

Let $P = \{x_1+x_2 = 1;\ x_1, x_2 \geq 0\}$ and $P' = \{x_1+x_2 \leq 1;\ x_1+x_2 \geq 1;\ x_1, x_2 \geq 0\}$. P and P' are two different representations of the same polyhedron in $\mathbb{R}^2$ namely, the line segment between (0,1) and (1,0). $pol(P)$ is non-degenerate with respect to P, since every point in the relative interior of $pol(P)$ is binding on one constraint, and $dim(pol(P)) = 1$, so that $\sigma(pol(P), P) = 1 + 1 - 2 = 0$. However, $pol(P')$ is degenerate with respect to P', since every point of $pol(P')$ is binding on at least two constraints. P' contains two implicit equalities. If these inequalities are replaced by its two corresponding equalities they become redundant constraints, and one of the two can be removed.

The definitions of degeneracy given above are dependent on the way the polyhedron is represented by a constraint collection. However, it is possible to define degeneracy degrees of subsets of polyhedra independent of constraint collection, namely in the following way.

The *degeneracy degree of a subset* S of a polyhedron Q in $\mathbb{R}^n$, denoted by $\sigma(S, Q)$

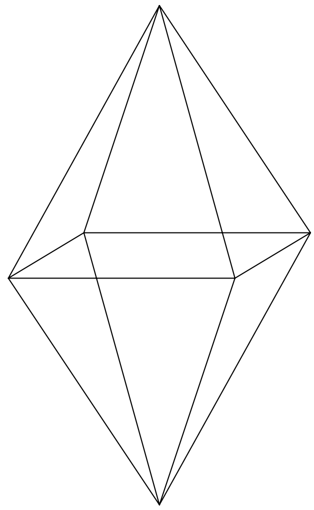

Figure 3.3: Octahedron; all vertices degenerate.

is defined as $\sigma(S, Q) = \sigma(S, P)$, where P is a minimal representation of Q.

Theorem 3.2 *The degeneracy degree of a subset S of a polyhedron Q is minimal if the degeneracy degree is determined with respect to a constraint collection P that is a minimal representation of Q.*

Every minimal representation P of Q contains the same number of equalities $n - dim(Q)$, and precisely one inequality for every facet. Let k be the number of facets from Q that are binding on S, and let F be the smallest face of Q that contains S. Then, $bnd(S, P) = n - dim(Q) + k$, and $\sigma(S, P) = \sigma(F, P) = bnd(F, P) + dim(F) - n = dim(F) - dim(Q) + k$. This degeneracy degree is minimal, since the dimensions of F and Q are not dependent of the constraint collection P that represents Q, and none of the k inequality constraints is redundant. □

If a polyhedron is represented by a constraint collection that is a minimal representation, we can be more precise about the degeneracy of the faces of that polyhedron.

Theorem 3.3 *Let the constraint collection P be a minimal representation of an n-dimensional polyhedron Q. Then Q, and its $(n-1)$- and $(n-2)$-faces are non-degenerate. All other faces of Q are not necessarily non-degenerate.*

Take any constraint collection P, and let $Q = pol(P)$. Let $dim(Q) = n$. If P contains equality constraints, each of them can be eliminated by solving one variable from it and substituting this into the other constraints. This results in an equivalent minimal representation. So, we may assume that Q is full dimensional, i.e. the dimension of the underlying space is n.

Q is non-degenerate. See Theorem 3.1(3).

The $(n-1)$-faces of Q are non-degenerate. Every point in the relative interior of a $(n-1)$-face(facet) F is binding on exactly one inequality constraint; see Telgen[8], Lemma 4.4.1. Hence, $\sigma(F, P) = bnd(F, P) + dimbnd(F, P) - n = 1 + (n-1) - n = 0$

The $(n-2)$-faces of Q are non-degenerate. We will show that if a $(n-2)$-face is degenerate, then P is not a minimal representation, in which case it contains implicit equalities or redundant constraints. Let F be a degenerate $(n-2)$-face of Q, and let v be a point in the relative interior of F. Then $\sigma(v, P) = bnd(v, P) + dimbnd(v, P) - n = bnd(v, P) + dim(F) - n = bnd(v, P) + n - 2 - n > 0$. Hence, $bnd(v, P) > 2$. Since Q is full dimensional, P does not contain an equality constraint and can therefore be written as $P = \{Ax \leq b\}$. Add slack variables to P, and write P as an extended Simplex tableau, i.e. $\{Ax + s = b;\ \ s \geq 0\}$, with s_i the basic variables and x_i the nonbasic variables. Note that the nonbasic variables are not necessarily nonnegative. Let s_1 and s_2 be two slack variables corresponding to constraints that are binding on F and v. Perform some pivots in order to make s_1 and s_2 nonbasic variables. First find a nonzero coefficient in the s_1 row. If this is not possible then this row has the form $0 + s_1 = b_1$. Since v is a feasible point, b_1 must be equal to zero. But $s_1 = 0$ is an implicit equality which contradicts the assumption that P is a minimal representation. Therefore there is a nonzero coefficient in the s_1 row. Perform a pivot on this element. Now consider the s_2 row. If this row has a nonzero coefficient in a nonbasic column different from the one of s_2, then pivot on this coefficient in order to make both s_1 and s_2 nonbasic variables. If, on the other hand, this row has not such nonzero coefficient, then this row has the form $as_1 + s_2 = 0$. The right hand side is equal to zero, because $s_1 = 0,\ s_2 = 0$ has to be feasible. Clearly, $a \geq 0$ implies that $s_1 = s_2 = 0$ for all feasible points. Hence, the constraint of s_2 in P is an implicit equality. Moreover, $a < 0$ implies that the constraint of s_2 is a positive multiple of the constraint of s_1, and therefore redundant in P.

Since $bnd(v, P) > 2$, there must be a third constraint binding on v and F. Let s_3 be the slack variable of this constraint. This constraint has the form $a_1 s_1 + a_2 s_2 + a_p x_p \cdots a_q x_q + s_3 = b_3$, in which $x_p, \ldots, x_q$ denote the $n-2$ nonbasic variables. Since F is $(n-2)$-dimensional, we can find $n-1$ affine independent points in F for which $s_1 = s_2 = s_3 = 0$. Denote these points by y_i, $i = 1, \ldots, n-1$. Substituting the coordinates of the y_i into the row of s_3 gives

$$\sum_{j=p}^{q} y_{ij}a_j = b_3, i = 1, \ldots, n-1.$$

Subtracting the first equation from the other $n-2$ equations gives

$$\sum_{j=p}^{q} y_{1j}a_j = b_3$$

$$\sum_{j=p}^{q} (y_{ij} - y_{1j})a_j = 0, i = 1, \ldots, n-1.$$

Since the $n-2$ vectors $(y_2 - y_1, \ldots, y_{n-1} - y_1$ are linear independent, it follows that $a_j = 0$ for $j = p, \ldots, q$, and from the first equation follows that $b_3 = 0$. Therefore, the row of s_3 has the form $a_1s_1 + a_2s_2 + s_3 = 0$. Now we have to consider several cases. (a). a_1 and a_2 are both non negative. Then the constraint of s_3 is an implicit equality which contradicts the assumptions.
(b). a_1 and a_2 are both non positive. Then the constraint of s_3 is redundant which contradicts the assumptions. (c). $a_1 < 0$ and $a_2 > 0$. Perform a pivot on a_1 which results in a row with slack variable a_1 and two negative coefficients in the columns of s_2 and s_3. Similar as in (2), it can be shown that the constraint corresponding to s_1 is redundant. (d). $a_1 > 0$ and $a_2 < 0$. Perform a pivot on a_2 which results in a row with slack variable s_2 and two negative coefficients in the columns of s_1 and s_3. Similar as in (2), it can be shown that the constraint corresponding to s_2 is redundant.

All other faces may be degenerate or non-degenerate. If P is a minimal presentation of a simplex in $\mathbb{R}^n$, then all faces of $pol(P)$ are non-degenerate. Let $P = \{x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0; x_1 + x_3 \leq 1;\ x_2 + x_3 \leq 1\}$. Then $pol(P)$ is a pyramid in $\mathbb{R}^3$ with top $t = (x_1 = x_2 = 0,\ x_3 = 1)$, that is degenerate; four inequality constraints are binding at the top. For $n > 3$, let $Q_n = \{x_i \geq 0,\ i = 4, \ldots, n\}$, and consider the constraint collection $P \cup Q_n$. Then $pol(P \cup Q_n) = pol(P) + pol(Q_n)$. The face $\{t\} + pol(Q_n)$ has dimension $n-3$ and has a degeneracy degree equal to one. □

In linear programming the optimal solutions form a face of the polyhedron represented by the constraints of the LP-model. By using the definition of degeneracy degree for the optimal faces of the primal LP-model and its dual, the following theorem holds.

Theorem 3.4 *The degeneracy degree of the optimal face of a primal LP-model is equal to the dimension of the optimal face of the corresponding dual LP-model.*

The proof of this theorem can be found in Tijssen & Sierksma[9].

4. Determining degeneracy degrees

Since degenerate basic solutions may cause cycling in Simplex algorithms that are not equipped with special anti-cycling pivot selection rules, LP-models that have at least one degenerate basic solution are called degenerate. On the other hand, interior point algorithms may become numerically unstable in the neighborhood of a degenerate face, and the convergence proofs of a number of interior point algorithms are dependent of the non-degeneracy of the optimal solution; see e.g. Güler et al.[4]. If a degenerate basic solution is encountered during the execution of a Simplex algorithm, it is clear that the LP-model is degenerate (has a degenerate basis). On th other hand, it is difficult to check whether an LP-model has a degenerate basic solution without checking all basic solutions. Actually, in Chandrasekaran et al.[1] it is shown that the problem of checking whether an LP-model is degenerate is NP-complete. This is done by proving that determining whether a transportation problem, formulated as an LP-model, has a degenerate feasible basic solution is as difficult as solving the well known 'subset-sum problem', which is NP-complete.

Theorem 4.1 *The problem of deciding whether a nonempty polyhedron defined by a constraint collection P has a degenerate face is NP-complete.*

If a polyhedron has a degenerate face, then all subfaces of that face are degenerate as well (Theorem 2.1(1)). Therefore, it suffices to decide whether one of the minimal faces is degenerate. But even in the case that the minimal faces are vertices, this problem is already NP-complete (see Chandrasekaran et al.[1]). □

If a constraint collection P is given, together with a point $p \in pol(P)$, it is easy to determine the degeneracy degree of the smallest face of the polyhedron that contains p. This can be done as follows. First, the constraints that are binding on p are determined by substituting the values of the coordinate entries of p into the constraints of the polyhedron representation, and checking which constraints are binding. The intersection of the binding constraints form a representation of the affine hull of the smallest face that contains p. The dimension of this face can be determined by calculating the rank of the matrix formed by the coefficients of the binding constraints. This rank can be calculated in the usual way using Gaussian elimination. All these calculations can be done in polynomial time.

Calculating the degeneracy degree of a nonempty polyhedron when only a representation of the polyhedron in the form of a constraint collection P is given is more difficult. If a point q in the relative interior of the polyhedron is given (together with a proof that it is indeed a point in the relative interior) the method outlined above can be used, since the smallest face that contains q is $pol(P)$ itself. Therefore, the degeneracy degree of q is equal to the degeneracy degree of $pol(P)$. If a point $q \in pol(P)$ is known, then the smallest face that contains q does not have to be P itself. It is possible that the degeneracy degree of that point is larger than the degeneracy degree of $pol(P)$. Therefore, a feasible point does not provide sufficient information to determine the degeneracy degree of $pol(P)$.

In order to calculate the degeneracy degree of P it is necessary to know the number of constraints that are binding on $pol(P)$. Therefore, it is necessary to find all implicit equalities of the constraint collection. This can be done by solving the LP-model shown in the proof of the following theorem. If all binding constraints are determined, the dimension of $pol(P)$ is calculated in the usual way.

Theorem 4.2 *Let P be a constraint collection, representing a nonempty polyhedron. All implicit equalities in P can be found in polynomial time by solving one linear programming problem.*

Let $P = \{A_1 x_1 = b_1;\, A_2 x_1 \leq b_2\}$ with $pol(P) \neq \emptyset$. Consider the primal LP-model

$$(P): \quad \max \{\, 0x_1 \mid A_1 x_1 = b_1;\;\; A_2 x_1 \leq b_2 \,\}$$

and its dual

$$(D): \quad \min \{\, b_1^T y_1 + b_2^T y_2 \mid A_1^T y_1 + A_2^T y_2 = 0;\;\; y_2 \geq 0 \,\}.$$

Since the objective function in the primal model is the zero function, any feasible point is optimal. Moreover, since $pol(P) \neq \emptyset$, we know that both (P) and (D) have finite optimal solutions. For any optimal solution it holds that $0x_1 = 0 = b_1^T y_1 + b_2^T y_2$. Furthermore, we know that both models also have one or more solutions that are strictly complementary, and are therefore located in the relative interior of the optimal faces (see Roos et al.[6]). After including slack variables into the inequality constraints of the primal model, we obtain

$$(P'): \quad \max \{\, 0x_1 \mid A_1 x_1 = b_1;\;\; A_2 x_1 + x_2 = b_2;\;\; x_2 \geq 0 \,\},$$

and the strictly complementarity condition can be written as $x_2 + y_2 > 0$. In order to ensure that the sums of all the coordinate entries of x_2 and y_2 are positive, we change this condition into $x_2 + y_2 \geq \alpha \mathbf{1}$, where α is strictly positive number, and $\mathbf{1}$ an all-unit vector. A strictly complementary pair of optimal solutions can be found by solving the following LP-model:

$$
\begin{array}{llllllll}
\max & \alpha & & & & & & \\
s.t. & A_1 x_1 & & & & & & = b_1 \\
 & A_2 x_1 & + & x_2 & & & & = b_2 \\
 & & & & A_1^T y_1 & + & A_2^T y_2 & = 0 \\
 & & & & b_1^T y_1 & + & b_2^T y_2 & = 0 \\
 & & & x_2 & & + & y_2 \quad -\alpha\mathbf{1} & \geq 0 \\
 & & & & & & \alpha & \leq 1 \\
 & x_2, y_2 \geq 0 & & & & & &
\end{array}
$$

The constraint $\alpha \leq 1$ is used for excluding unbounded solutions. By means of interior point methods we can solve this model in polynomial time. For any optimal solution α has a positive value. The implicit equalities of P can easily be determined from the optimal values of x_2. Every entry of x_2 that is zero corresponds to a slack variable of an inequality constraint in P that has a zero value for every point of $pol(P)$. □

Theorem 4.3 *Let P be a constraint collection in $\mathbb{R}^n$. The degeneracy degree of $pol(P)$ can be calculated in polynomial time.*

First, all implicit equalities of P are determined using the method described in Theorem 4.2. These calculations take polynomial time. The number of equality constraints together with the implicit equalities in P is now equal to the number of binding constraints $bnd(P, P)$. Secondly, $dim(P)$ is calculated by determining the rank of the coefficient matrix corresponding to the equality constraints together with the implicit equalities. This can be done in polynomial time with Gaussian elimination. Hence, $\sigma(P, P) = bnd(P, P) + dim(P) - n$ can be calculated in polynomial time. □

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