A delegated agent in a winner-take-all contest

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Summary. We consider a winner-take-all contest extended with a principal-agent relationship. One of the two players, say player 1, offers a contract to an agent to act in the contest as a delegate on his behalf. The wage offered to the agent is deliberately chosen by player 1. We characterize the Nash equilibrium of this contest and compare its properties with those of the Nash equilibrium of the corresponding standard contest in which both players compete themselves. We show that the expected utility of player 1 is larger in the contest with a delegate if he is strongly risk averse with respect to his money income and moreover the contested prize is large enough.

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1. Introduction

Consider a situation in which two players are involved in a winner-take-all contest. In the standard approach to this kind of contest, both players compete with each other for a given prize, which is awarded to one of them. Each player exerts effort in order to increase the probability that he will win the prize. A large number of papers have studied these contests and applications to e.g. rent seeking, elections, tournaments, litigation and conflicts. See amongst others Dixit (1987), Hillman and Riley (1989), Nitzan (1994), Rosen (1986) and Tullock (1980). The probabilities of winning are given by a contest success function. Often it is assumed that the probability that a player wins depends on the ratio of the effort of the player himself and the total efforts of both players. See Skaperdas (1996) and Kooreman and Schoonbeek (1997) for a discussion of this kind of success function.

Recently, Baik and Kim (1997) have investigated the situation in which each player has the option to hire a delegated agent who represents him in the contest and expends effort on his behalf. Baik and Kim assume that the wage paid by the player to the delegate consists of two parts: a fixed fee and a contingent fee. The fixed fee is exogenously given, and is paid irrespective of the outcome of the contest. The contingent fee is an exogenously given fraction of the player's valuation of the prize, and is only paid if the player wins the contest. They further assume that a delegate always accepts employment by a player if asked for, and that the players as well as the delegates are risk neutral. Finally, they allow that the players and delegates have different 'abilities', in the sense that the efforts of the delegates might have a different effect on the probability of winning than the efforts of the players. Formally this is accomplished by attaching appropriate weights to the relevant effort levels in the contest success function.

This paper also investigates the situation where a player can hire a delegated agent. In contrast with Baik and Kim (1997), our goal is to analyse this situation explicitly in terms of a principal-agent problem in which a player (principal) offers a contract to a delegate (agent). The delegate can accept or reject the offered contract. In this way we endogenise the wage to be paid by the player, and we take into account the participation constraint of the delegate. We observe that if the contract is accepted, the size of the wage also has an impact on the effort subsequently exerted by the delegate in the contest. Therefore, it is interesting to examine whether a player might have strategic reasons to offer a delegate a higher wage than is strictly necessary to make him accept the contract. Furthermore, we want to analyse the dependency of the outcome of the contest to the extent of risk aversion with respect to the money income of a player who hires a delegate. In order to keep the analysis manageable and to obtain explicit solutions as far as possible, we focus on a situation that includes these elements in the most simple way. That is, we suppose that there is only one player who can hire a delegate, the wage offered to the delegate consists only of a contingent fee, and the players and the delegate all have equal 'abilities'.

To begin with, Section 2 examines briefly the Nash equilibrium of a one-shot standard winner-take-all contest between two players. We suppose that player 1 is either risk averse or risk neutral with respect to his money income, while player 2 is risk neutral with respect to his money income. Next, Section 3 extends this standard contest to a two-stage contest with a principal-agent relationship. In stage 1, player 1 offers a contract to a risk-neutral agent. The contract specifies the wage that will be paid to the agent if he acts in the contest as a delegate of player 1. The wage is contingent on winning the contest, and is equal to a fraction of the contested prize.[1] The size of this fraction is determined by player 1. In stage 2, the agent can accept or reject the contract. If he accepts it, he competes with player 2 on behalf of player 1. If he rejects it, the agent gets a given outside reservation income. We derive the subgame-perfect Nash equilibrium of the extended contest in Section 4.

Section 5 compares the Nash equilibria of the Sections 2 and 4. In particular, we compare the expected utility levels of player 1 and 2 as well as the relevant effort levels in the two equilibria. Doing so, we focus on the two extreme cases in which player 1 is either strongly risk averse or risk neutral with respect to his money income. Section 6 concludes.

2. The contest without a delegate

Let us consider a contest in which two players compete for a prize of money value $v > 0$. We suppose that the utility functions of player 1 and player 2 are given by, respectively, $\nu_1(y_1, e_1) = y_1^{\rho} - e_1$ and $\nu_2(y_2, e_2) = y_2 - e_2$, where y_i denotes the (money) income of player i, and $e_i \geq 0$ denotes the effort level (measured in 'physical' terms, e.g. the amount of time spent on lobbying) that is put forward by player i in order to win the contest, and ρ is a parameter satisfying $0 < \rho \leq 1$ $(i = 1, 2)$. We remark here that utility functions of this form are familiar in the principal-agent literature, see e.g. Rogerson (1985). [2] The size of ρ represents the extent of risk aversion of player 1 with respect to his income (given a fixed value

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1 According to Hay (1996), in many types of lawsuit brought to court by individuals in the U.S.A., this kind of no-cure-no-pay arrangement is offered to the plaintiff's lawyer.

2 In particular, Rogerson (1985) uses utility functions of the form $w(y, e) = z(y) - e$, where y is (money) income, e is effort, and $z(.)$ is a strictly increasing, concave, and twice continuously differentiable function.

of e_1). If $\rho < 1$, then player 1 is risk averse, whereas if $\rho = 1$, he is risk neutral with respect to his income. Notice that player 1 is always risk neutral with respect to his effort level (given a fixed value of y_1). Analogously, player 2 is risk neutral with respect to both his (money) income and effort level.

The probability that player i wins the contest is given by

$$p_i(e_1, e_2) = \frac{e_i}{e_1 + e_2} \tag{1}$$

if e_1 and/or e_2 are positive, while $p_i(0, 0) = \frac{1}{2}$ $(i = 1, 2)$.[3] If player 1 wins, he obtains the prize. Thus, $y_1 = v$ in that case. If player 1 loses, we have $y_1 = 0$. As a result, the expected utility of player 1 equals [4]

$$u_1(e_1, e_2) = p_1(e_1, e_2)\nu_1(v, e_1) + p_2(e_1, e_2)\nu_1(0, e_1) \tag{2}$$

$$= \frac{e_1}{e_1 + e_2} v^\rho - e_1 \tag{3}$$

if e_1 and/or e_2 are positive, while $u_1(0, 0) = \frac{1}{2}v^\rho$. Similarly, the expected utility of player 2 is given by

$$u_2(e_1, e_2) = \frac{e_2}{e_1 + e_2} v - e_2 \tag{4}$$

if e_1 and/or e_2 are positive, while $u_2(0, 0) = \frac{1}{2}v$.

Using Hillman and Riley (1989), it can be shown that the contest has a unique Nash equilibrium with effort levels given by $\tilde{e}_1 = \tilde{s}^2/v$ and $\tilde{e}_2 = \tilde{s}^2/v^\rho$, where $\tilde{s} = \tilde{e}_1 + \tilde{e}_2 = v^{\rho+1}/(v^\rho + v)$. We can rewrite $\tilde{e}_1$ and $\tilde{e}_2$ as

$$\tilde{e}_1(v, \rho) = \frac{v^{2\rho+1}}{(v^\rho + v)^2} \tag{5}$$

$$\tilde{e}_2(v, \rho) = \frac{v^{2+\rho}}{(v^\rho + v)^2} \tag{6}$$

while the corresponding expected utilities of the players can be written as

$$\tilde{u}_1(v, \rho) = \frac{v^{3\rho}}{(v^\rho + v)^2} \tag{7}$$

$$\tilde{u}_2(v, \rho) = \frac{v^3}{(v^\rho + v)^2} \tag{8}$$

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3 Alternatively, one can also suppose that $p_i(0, 0) = 0$ $(i = 1, 2)$. This has no effect on the results derived below.

4 Remark that if the utility function of player 1 would have been $\nu_1(y_1, e_1) = (y_1 - e_1)^\rho$, where e_1 is now measured in terms of money outlays, then $\nu_1(0, e_1)$ is not well defined for all $e_1 > 0$ and $0 < \rho < 1$. Our specification does not have this problem. Note that the problem does not arise if $\rho = 1$.

These expressions will be useful in Section 5.

Finally, we observe that in the risk-neutral case with $\rho = 1$, we have $\tilde{e}_1(v, 1) = \tilde{e}_2(v, 1) = \frac{v}{4}$, and $\tilde{u}_1(v, 1) = \tilde{u}_2(v, 1) = \frac{v}{4}$. The case $\rho = 1$ formally corresponds to the standard rent-seeking model of Tullock (1980). However, we recall that we interpret e_i in terms of physical effort, whereas Tullock interprets e_i in terms of money outlays.

3. The contest with a delegate

Now, let us assume that player 1 will not exert the effort himself in the competition for the prize. Instead, he wants to hire a delegate who represents him in the contest with player 2. We further introduce moral hazard, in the sense that player 1 as a principal is not able to monitor the efforts put forward by his delegate (and by player 2). All other elements of the contest are unchanged.

Let us describe the two-stage game with the principal-agent relationship between player 1 and the delegate in detail. In the first stage, player 1 offers a contract to the (potential) delegate that specifies the (money) wage that will be paid by him to the delegate. Because player 1 cannot observe the effort level of the delegate, $e_d \geq 0$ say, the wage cannot be made dependent on this effort. Instead, we assume that player 1 offers the delegate a simple contingent fee, according to which the delegate gets some fraction $\alpha \in [0, 1]$ of the prize v if player 1 wins the contest, whereas the delegate gets nothing if player 1 loses it. The size of α is determined by player 1. In the second stage, the delegate can accept or reject the contract offer. If the contract is accepted, the delegate competes on behalf of player 1 with player 2 for the prize v. In turn, on the basis of the effort levels chosen, the prize is awarded to either player 1 or player 2. The probabilities that the players 1 and 2 win the prize are given by, respectively,

$$p_1(e_d, e_2) = \frac{e_d}{e_d + e_2}, \qquad p_2(e_d, e_2) = \frac{e_2}{e_d + e_2} \tag{9}$$

if e_d and/or e_2 are positive, while $p_i(0, 0) = \frac{1}{2}$ $(i = 1, 2)$. On the other side, if the contract is rejected, then the delegate obtains some given outside reservation income $\bar{y}_d \geq 0$, player 1 gets nothing and has to pay nothing, and player 2 can obtain the prize by putting forward an infinitesimal positive effort e_2.

The expected utility of player 1 is now given by

$$\begin{aligned} U_1(e_d, e_2, \alpha) &= p_1(e_d, e_2)\nu_1((1-\alpha)v, 0) + p_2(e_d, e_2)\nu_1(0, 0) \\ &= \frac{e_d}{e_d + e_2}\,((1-\alpha)v)^\rho \end{aligned} \tag{10}$$

if e_d and/or e_2 are positive, while $U_1(0, 0, \alpha) = \frac{1}{2}((1-\alpha)v)^\rho$. We use a capital '$U'$' to denote the expected utility in the contest with the delegate. Similarly, as in (4), the expected utility of player 2 equals

$$U_2(e_d, e_2, \alpha) = \frac{e_2}{e_d + e_2} v - e_2 \tag{11}$$

if e_d and/or e_2 are positive, while $U_2(0, 0, \alpha) = \frac{1}{2}v$. We assume that the utility function of the delegate is given by $\nu_d(y_d, e_d) = y_d - e_d$, where y_d is the (money) income of the delegate. Note that the delegate, just like player 2, is risk neutral with respect to both his income and effort level. The expected utility of the delegate if he accepts the contract is

$$\begin{aligned} U_d(e_d, e_2, \alpha) &= p_1(e_d, e_2)\nu_d(\alpha v, e_d) + p_2(e_d, e_2)\nu_d(0, e_d) \\ &= \frac{e_d}{e_d + e_2}(\alpha v) - e_d \end{aligned} \tag{12}$$

if e_d and/or e_2 are positive, while $U_d(0, 0, \alpha) = \frac{1}{2}\alpha v$. The participation constraint of the delegate states that he will only accept the contract if his expected utility derived from participating in the competition is larger than or equal to the expected utility derived from the reservation income, i.e. $\bar{y}_d$.

4. The Nash equilibrium of the contest with the delegate

We will determine the subgame-perfect Nash equilibrium of the contest with the principal-agent relationship between player 1 and the delegate. Using backward induction, we start in stage 2 with a given value of $\alpha \in [0, 1]$. First, let us ignore the participation constraint of the delegate, and suppose that he accepts the contract. Then the delegate and player 2 have to choose their efforts in the competition for the prize. Let $e_d(\alpha)$ and $e_2(\alpha)$ denote these effort levels as a function of α. If $\alpha \in (0, 1]$, then it follows from Hillman and Riley (1989) that there exists at this stage of the game a unique Nash equilibrium in which the efforts are equal to $e_d(\alpha) = s^2(\alpha)/v$ and $e_2(\alpha) = s^2(\alpha)/(\alpha v)$, where $s(\alpha) = (\alpha v)v/(\alpha v + v) = \alpha v/(\alpha + 1)$. Rewriting this, we obtain

$$e_d(\alpha) = \frac{\alpha^2 v}{(\alpha+1)^2} \tag{13}$$

$$e_2(\alpha) = \frac{\alpha v}{(\alpha+1)^2} \tag{14}$$

The corresponding expected utility of the delegate equals

$$U_d(e_d(\alpha), e_2(\alpha), \alpha) = \frac{\alpha^3 v}{(\alpha+1)^2} \tag{15}$$

If $\alpha = 0$, then it follows trivially that the optimal effort level of the delegate is equal to $e_d(0) = 0$. In turn, player 2 can obtain the prize by putting forward an infinitesimal positive effort $e_2(0)$. The corresponding expected utility of the delegate equals $U_d(e_d(0), e_2(0), 0) = 0$. Remark that $U_d(e_d(\alpha), e_2(\alpha), \alpha)$ is an increasing function of α, for all $\alpha \in [0, 1]$.

Taking into account again the participation constraint of the delegate and examining his decision to accept or not the offered contract, it is useful to define r as the ratio of the reservation income of the delegate and the prize, i.e. $r = \bar{y}_d/v$. We observe that if $r > \frac{1}{4}$, then $U_d(e_d(\alpha), e_2(\alpha), \alpha) < \bar{y}_d$ for all $\alpha \in [0, 1]$. Thus, in that case there is no feasible value of α for which the participation constraint holds, which makes this case further uninteresting. Therefore, we make from now on the assumption that $0 \leq r \leq \frac{1}{4}$. Using this, we define the function $\alpha_0(r)$ for $0 \leq r \leq \frac{1}{4}$ as follows. If $0 < r \leq \frac{1}{4}$, then $0 < \alpha_0(r) \leq 1$ is the unique number such that

$$\frac{(\alpha_0(r))^3}{(\alpha_0(r) + 1)^2} = r \tag{16}$$

If $r = 0$, then $\alpha_0(r) = 0$. Observe that $\alpha_0(r)$ is an increasing function of r, taking on values in $[0, 1]$ if r increases over $[0, \frac{1}{4}]$.

We have to distinguish now two cases regarding the given value of α in stage 2, i.e. $\alpha < \alpha_0(r)$ and $\alpha \geq \alpha_0(r)$. First, suppose that $\alpha < \alpha_0(r)$. In that case the delegate will not accept the contract, and as a result his income and expected utility will be equal to $\bar{y}_d$. In turn, player 2 puts forward an infinitesimal positive effort, obtains the prize and gets expected utility equal to (just below) v. Second, suppose that $\alpha \geq \alpha_0(r)$. In that case the delegate will accept the contract. Depending on the sign of α, there are now two subcases regarding the competition between the delegate and player 2, i.e. $\alpha > 0$ and $\alpha = 0$.

Taking the subcase with $\alpha > 0$, we recall that then the delegate and player 2 will choose efforts levels given by, respectively, (13) and (14). Substituting these efforts in (15), we see that the resulting expected utility of the delegate will respectively be equal to $\bar{y}_d$ if $\alpha = \alpha_0(r)$ and greater than $\bar{y}_d$ if $\alpha > \alpha_0(r)$. Thus, the participation constraint of the delegate is binding if $\alpha = \alpha_0(r)$, and it is not binding if $\alpha > \alpha_0(r)$. Substituting (13) and (14) in (11) shows that the expected utility of player 2 always will be positive.

Turning to the subcase with $\alpha = 0$, we recall that then the optimal effort level of the delegate is equal to $e_d(0) = 0$, while player 2 obtains the prize by putting forward an infinitesimal positive effort $e_2(0)$. The resulting expected utility of the delegate is equal to zero, while the expected utility of player 2 equals (just below) v. Notice that $\alpha \geq \alpha_0(r)$ together with $\alpha = 0$ implies that $\alpha_0(r) = 0$. In turn, this means that

$r = 0$, or $\bar{y}_d = 0$. Thus, the second subcase can occur only if $\bar{y}_d = 0$. From this it also follows that in the second subcase the expected utility of the delegate is equal to $\bar{y}_d$ $(= 0)$, i.e. the participation constraint is binding in this subcase.

Next, let us turn to stage 1 of the game. In this stage, player 1 has to choose the optimal value of $\alpha \in [0, 1]$, taking into account the behaviour of the delegate and player 2 in stage 2. In order to solve the problem of player 1, first take $\alpha = 0$. It then follows from the above discussion that irrespective of whether the delegate will accept the contract or not, the resulting expected utility of player 1 will be equal to zero. Second, take $\alpha = 1$. It follows directly that in that case the expected utility of player 1 is always equal to zero as well. Third, consider the following maximization problem:

$$\max_{\alpha \in (0,1)} U_1(e_d(\alpha), e_2(\alpha), \alpha) = \max_{\alpha \in (0,1)} \left(\frac{\alpha}{\alpha + 1} \right) ((1 - \alpha)v)^{\rho} \tag{17}$$

The first-order condition pertaining to this maximization problem equals

$$\rho\alpha^2 = 1 - (1 + \rho)\alpha \tag{18}$$

By making a figure in the (α, ρ)-plane one can verify that given $\rho \in (0, 1]$, there exists a unique $\alpha_1(\rho)$ with $0 < \alpha_1(\rho) < 1$ that satisfies (18) , i.e.

$$\alpha_1(\rho) = \frac{-(1 + \rho) + \sqrt{\rho^2 + 6\rho + 1}}{2\rho} \tag{19}$$

Note that $\alpha_1(\rho)$ does not depend on $\bar{y}_d$ and v, and that $\alpha_1(\rho)$ is a decreasing function for all feasible ρ, i.e. $\alpha_1'(\rho) < 0$ for all $\rho \in (0, 1]$. Moreover, we observe that $\lim_{\rho\downarrow 0} \alpha_1(\rho) = 1$ [5] and $\alpha_1(1) = -1 + \sqrt{2} \approx 0.414$. Next, define for $\rho \in (0, 1]$ the function $f(\rho)$ as

$$f(\rho) = \frac{(\alpha_1(\rho))^3}{(\alpha_1(\rho) + 1)^2} \tag{20}$$

Note that $f'(\rho) < 0$ for all $\rho \in (0, 1]$. Further, $\lim_{\rho\downarrow 0} f(\rho) = \frac{1}{4}$ and $f(1) = \frac{1}{2}(-1 + \sqrt{2})^3 \approx 0.036$.

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5 We have

$$\begin{aligned}
\lim_{\rho\downarrow 0} \alpha_1(\rho) &= \lim_{\rho\downarrow 0} \frac{-(1+\rho)^2 + \rho^2 + 6\rho + 1}{2\rho\left(1 + \rho + \sqrt{\rho^2 + 6\rho + 1}\right)} \\
&= \lim_{\rho\downarrow 0} \frac{2}{1 + \rho + \sqrt{\rho^2 + 6\rho + 1}} \\
&= 1
\end{aligned}$$

Denoting the optimal value of α for player 1 as $\hat{\alpha} = \hat{\alpha}(\rho,\, r)$, we now distinguish three situations depending on the size of r. First, consider the situation with $0 \le r \le \frac{1}{2}(-1+\sqrt{2})^3$. In that case $f(\rho) \ge r$ for all $\rho \in (0, 1]$. As a result, the optimal choice equals $\hat{\alpha}(\rho,\, r) = \alpha_1(\rho) < 1$, giving player 1 a positive expected utility. Second, consider the situation with $\frac{1}{2}(-1+\sqrt{2})^3 < r < \frac{1}{4}$. In that case there exists for each r a unique $\bar{\rho}(r) \in (0, 1)$ such that $f(\bar{\rho}(r)) = r$, i.e. $\bar{\rho}(r)$ is defined by

$$\frac{(\alpha_1(\bar{\rho}(r)))^3}{(\alpha_1(\bar{\rho}(r)) + 1)^2} = r \tag{21}$$

As a result, if $\rho < \bar{\rho}(r)$, then the optimal choice of player 1 is $\hat{\alpha}(\rho,\, r) = \alpha_1(\rho) < 1$, giving him a positive expected utility. On the other side, if $\rho \ge \bar{\rho}(r)$, then $\hat{\alpha}(\rho,\, r) = \alpha_0(r) < 1$, again giving player 1 a positive expected utility. Note that $\hat{\alpha}(\bar{\rho}(r),\, r) = \alpha_1(\bar{\rho}(r)) = \alpha_0(r)$. Third, take the situation with $r = \frac{1}{4}$. Then $f(\rho) < r = \frac{1}{4}$ for all $\rho \in (0, 1]$. As a result, the optimal choice is now $\hat{\alpha}(\rho,\, \frac{1}{4}) = \alpha_0(\frac{1}{4}) = 1$, providing player 1 with an expected utility equal to zero. Notice that in all three situations, player 1 chooses a positive optimal value $\hat{\alpha}(\rho,\, r)$ such that the delegate subsequently accepts the contract in stage 2 and puts forward a positive effort. Summarizing, we obtain the following proposition:

Proposition 4.1. *Let $v > 0$, $\bar{y}_d \ge 0$, and $0 < \rho \le 1$ be given. Let $r = \bar{y}_d/v$ and assume that $r \le \frac{1}{4}$. Let $\alpha_0(r)$ and $\alpha_1(\rho)$ be defined by (16) and (19). Then the contest with player 1, the delegate, and player 2 has a unique subgame-perfect Nash equilibrium, in which:*

(i) Player 1 offers a contract with $\hat{\alpha} = \hat{\alpha}(\rho,\, r)$. Here $\hat{\alpha}(\rho,\, r) > 0$ is defined as follows:
- *In case $0 \le r \le \frac{1}{2}(-1+\sqrt{2})^3$, then $\hat{\alpha}(\rho,\, r) = \alpha_1(\rho) < 1$;*
- *In case $\frac{1}{2}(-1+\sqrt{2})^3 < r < \frac{1}{4}$, then $\hat{\alpha}(\rho,\, r) = \alpha_1(\rho) < 1$ if $\rho < \bar{\rho}(r)$, whereas $\hat{\alpha}(\rho,\, r) = \alpha_0(r) < 1$ if $\rho \ge \bar{\rho}(r)$, where $\bar{\rho}(r)$ is defined in (21);*
- *In case $r = \frac{1}{4}$, then $\hat{\alpha}(\rho,\, r) = \alpha_0(r) = 1$.*

(ii) The delegate accepts this contract;

(iii) The effort levels of the delegate and player 2 are given by

$$\hat{e}_d(v, \rho, r) = \frac{\left(\hat{\alpha}(\rho,\, r)\right)^2 v}{(\hat{\alpha}(\rho,\, r) + 1)^2} > 0 \tag{22}$$

$$\hat{e}_2(v, \rho, r) = \frac{\hat{\alpha}(\rho,\, r) v}{(\hat{\alpha}(\rho,\, r) + 1)^2} > 0 \tag{23}$$

The corresponding expected utility levels of player 1, the delegate, and player 2 are *equal to*

$$U_1(v,\rho,r) \;=\; \left(\frac{\hat{\alpha}(\rho,\ r)}{\hat{\alpha}(\rho,\ r)+1}\right)\big((1-\hat{\alpha}(\rho,\ r))v\big)^{\rho} \geq 0 \tag{24}$$

$$U_d(v,\rho,r) \;=\; \frac{\big(\hat{\alpha}(\rho,\ r)\big)^3 v}{(\hat{\alpha}(\rho,\ r)+1)^2} > 0 \tag{25}$$

$$U_2(v,\rho,r) \;=\; \frac{v}{(\hat{\alpha}(\rho,\ r)+1)^2} > 0 \tag{26}$$

In order to understand (i) of Proposition 4.1 in an intuitive way, take a given value of r and observe the situation faced by player 1. First, suppose that the delegate has no participation constraint. Then, player 1 prefers for strategic reasons to offer a contract with $\hat{\alpha}(\rho,\ r) = \alpha_1(\rho) > 0$, such that the delegate is given the incentive to behave in an optimal way from the standpoint of view of player 1 in the competition with player 2. Further, if player 1 becomes more risk averse with respect to his (money) income (lower values of ρ), then he prefers to offer higher values of α in order to induce larger efforts of the delegate. Similarly, if player 1 becomes less risk averse (higher value of ρ), then he would prefer to offer lower values of α. However, returning now to the participation constraint of the delegate, we see that then at a certain point this constraint might become binding. This means that from such a point on player 1 has to offer a contract with a value of $\hat{\alpha}(\rho,\ r) = \alpha_0(r) > \alpha_1(\rho)$ in order to be sure that the delegate does not refuse the contract and thus puts in no effort at all. Clearly, the question whether, and if so, when the participation constraint becomes binding depends on the size of r and ρ. In particular, in the case where r is relatively small, i.e. $0 \leq r \leq \frac{1}{2}(-1+\sqrt{2})^3$, the constraint is not binding for any ρ. Thus, in that case player 1 has a strategic reason to offer the delegate a higher wage than is strictly necessary to make him accept the contract. In the opposite case where r is relatively large, i.e. $r = \frac{1}{4}$, the participation constraint is binding for each ρ. Now, the above strategic reason is 'outweighed' by the necessity to offer an even higher wage in order to be sure that the delegate accepts the offer at all. In the intermediate case with $\frac{1}{2}(-1+\sqrt{2})^3 < r < \frac{v}{4}$, we see that the question whether the participation constraint is binding or not depends on the degree of risk aversion of player 1 with respect to his money income, i.e. it is not binding if $\rho < \bar{\rho}(r)$ whereas it is binding if $\rho \geq \bar{\rho}(r)$.

Finally, we also see from Proposition 4.1 that if $\hat{\alpha}(\rho,\ r) < 1$, then the effort level and expected utility level of the delegate are less than those of player 2, whereas they are equal to each other if $\hat{\alpha}(\rho,\ r) = 1$. The latter is intuitively clear, because, basically, in that case the delegate and player 2 are involved in a symmetric contest (which is formally identical to the symmetric standard contest of Tullock (1980)).

5. Comparison of the Nash equilibria

Next, let us compare the Nash equilibrium of the contest without the delegate of Section 2 with the Nash equilibrium of the contest with the delegate of Section 4. In particular, we will compare the expected utility levels of player 1 in the two Nash equilibria as well as the effort level chosen by player 1 in the first equilibrium with the effort level chosen by the delegate in the second one. Analogously, we compare the expected utility and effort levels of player 2 in the two Nash equilibria. In our analysis we focus on the impact of the (income) risk-aversion parameter ρ of player 1. More particularly, taking fixed values of $\bar{y}_d \geq 0$ and $v > 0$ which satisfy $r = \bar{y}_d/v \leq \frac{1}{4}$, we examine two opposite extreme situations with respect to ρ. First, we consider the situation in which player 1 is strongly risk averse, as characterized by the limit case where ρ approaches zero from above. Second, we consider the case in which player 1 is risk neutral, i.e. with $\rho = 1$.

Let us start with the situation in which player 1 is strongly (income) risk averse. Using (5) up to and including (8), we can state the following results for the Nash equilibrium of the contest without the delegate in that situation:

$$\tilde{u}_1^*(v) = \lim_{\rho \downarrow 0} \tilde{u}_1(v, \rho) = \frac{1}{(1+v)^2} \tag{27}$$

$$\tilde{u}_2^*(v) = \lim_{\rho \downarrow 0} \tilde{u}_2(v, \rho) = \frac{v^3}{(1+v)^2} \tag{28}$$

$$\tilde{e}_1^*(v) = \lim_{\rho \downarrow 0} \tilde{e}_1(v, \rho) = \frac{v}{(1+v)^2} \tag{29}$$

$$\tilde{e}_2^*(v) = \lim_{\rho \downarrow 0} \tilde{e}_2(v, \rho) = \frac{v^2}{(1+v)^2} \tag{30}$$

Turning to the Nash equilibrium of the contest with the delegate, we consider $\hat{\alpha}(\rho, r)$ of Proposition 4.1, and observe that $\lim_{\rho \downarrow 0} \hat{\alpha}(\rho, r) = 1$ for all $r \in [0, \frac{v}{4}]$. As a result, we can derive that[6]

$$\hat{U}_1^*(v,r) = \lim_{\rho\downarrow 0} \hat{U}_1(v,\rho,r) = \frac{1}{2} \tag{31}$$

$$\hat{U}_2^*(v,r) = \lim_{\rho\downarrow 0} \hat{U}_2(v,\rho,r) = \frac{v}{4} \tag{32}$$

$$\hat{e}_d^*(v,r) = \lim_{\rho\downarrow 0} \hat{e}_d(v,\rho,r) = \frac{v}{4} \tag{33}$$

$$\hat{e}_2^*(v,\bar{y}_d) = \lim_{\rho\downarrow 0} \hat{e}_2(v,r,\rho) = \frac{v}{4} \tag{34}$$

Comparing the two Nash equilibria, we obtain the following four results: (a) $\tilde{u}_1^*(v) < \hat{U}_1^*(v,r)$ if and only if $v > \sqrt{2}-1$; (b) $\tilde{u}_2^*(v) < \hat{U}_2^*(v,r)$ if and only if $v < 1$; (c) $\tilde{e}_1^*(v) < \hat{e}_d^*(v,r)$ if and only if $v > 1$; and (d) $\tilde{e}_2^*(v) < \hat{e}_2^*(v,r)$ if and only if $v < 1$. We make a number of observations here. First, we see from result (a) that there are situations such that for player 1 the expected utility in the contest in which he hires a delegate is larger than in the contest in which there is no delegate and player 1 competes himself. Intuitively speaking we can say that this situation occurs if player 1 is enough risk averse, i.e. ρ is small enough, and moreover the prize is larger than a certain critical value, i.e. $v > \sqrt{2}-1$. Second, result (b) implies that a reverse result holds for player 2. That is, if player 1 is strongly risk averse, and moreover the prize is larger than a certain critical value, i.e. $v > 1$, then the expected utility of player 2 is *smaller* in the contest with a delegate than in the contest without a delegate. Observe that the critical values of v mentioned in (a) and (b) are different. This implies that if player 1 is strongly risk averse and furthermore the size of the prize satisfies $\sqrt{2}-1 < v < 1$, then both player 1 and player 2 have a larger expected utility in the Nash equilibrium of the contest in which player 1 hires a delegate than in the contest in which there is no delegate. The interpretation of results (c) and (d) is straightforward, and can be left to the reader.

Proceeding, let us turn to the situation in which player 1 is (income) risk neutral, i.e. $\rho = 1$. Recall that in that case we have in the Nash equilibrium of the contest without

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6 In order to find $\hat{U}_1^*(v,r)$, we define $\alpha_1(\rho) = x$, and notice that it follows from (19) and the definition of $\alpha_1(\rho)$ that $\rho = (1-x)/(x(1+x))$. Consequently, we can write

$$\begin{aligned}\lim_{\rho\downarrow 0}(1-\alpha_1(\rho))^{\rho} &= \lim_{x\uparrow 1}(1-x)^{\left(\frac{1-x}{x(1+x)}\right)} \\ &= \lim_{x\uparrow 1}\exp\left(\left(\frac{1-x}{x(1+x)}\right)\ln(1-x)\right) \\ &= 1.\end{aligned}$$

where the last equality follows from the well-known fact that $\lim_{x\uparrow 1}(1-x)\ln(1-x) = 0$, see Apostol (1967, p. 302). Using this, the expression for $\hat{U}_1^*(v,r)$ follows directly.

a delegate the following expected utility and effort levels for player 1 and player 2: $\tilde{u}_1(1, v) = \tilde{u}_2(1, v) = \frac{v}{4}$ and $\tilde{e}_1(1, v) = \tilde{e}_2(1, v) = \frac{v}{4}$. Taking the Nash equilibrium of the contest with a delegate, we see that while applying Proposition 4.1, we have to distinguish two cases depending on the size of r. First, take r relatively small, i.e. $0 \leq r < \frac{1}{2}(-1+\sqrt{2})^3$. Recalling that $\alpha_1(1) = -1+\sqrt{2}$ if $\rho = 1$, we then get the following simple expressions for the expected utility and effort levels:

$$\hat{U}_1(v, 1, r) = (3 - 2\sqrt{2})\, v \approx 0.172\, v \tag{35}$$

$$\hat{U}_2(v, 1, r) = \frac{1}{2}\, v \tag{36}$$

$$\hat{e}_d(v, 1, r) = \frac{1}{2}(-1+\sqrt{2})^2\, v \approx 0.086\, v \tag{37}$$

$$\hat{e}_2(v, 1, r) = \frac{1}{2}(-1+\sqrt{2})\, v \approx 0.207\, v \tag{38}$$

Comparing the two Nash equilibria we obtain the following results: (i) $\tilde{u}_1(v, 1) > \hat{U}_1(v, 1, r)$; (ii) $\tilde{u}_2(v, 1) < \hat{U}_2(v, 1, r)$; (iii) $\tilde{e}_1(v, 1) > \hat{e}_d(v, 1, r)$; and (iv) $\tilde{e}_2(v, 1) > \hat{e}_2(v, 1, r)$. Thus, if player 1 is risk neutral and moreover r is relatively small, then we see from result (i) that the expected utility of player 1 is smaller in the Nash equilibrium of the contest with a delegate than in the Nash equilibrium of the contest without a delegate. Result (ii) shows that the opposite conclusion holds for player 2. Results (iii) and (iv) demonstrate that the effort levels in the contest with the delegate are smaller than the corresponding ones in the contest without a delegate.

Second, let us take r such that $\frac{1}{2}(-1+\sqrt{2})^3 \leq r \leq \frac{1}{4}$. We notice that because $\rho = 1$, the critical value $\bar{\rho}(r)$ used in Proposition 4.1 satisfies $\bar{\rho}(r) < \rho = 1$. This implies that $\hat{\alpha}(1, r) = \alpha_0(r)$ for all r with $\frac{1}{2}(-1+\sqrt{2})^3 \leq r \leq \frac{1}{4}$. As a result, if we now compare the expected utility and effort levels in the two Nash equilibria, then the size of $\hat{\alpha}(1, r) = \alpha_0(r)$ depends on the exact value of r. Recall that if r increases over the interval $[\frac{1}{2}(-1+\sqrt{2})^3, \frac{1}{4}]$, then $\hat{\alpha}_0(r)$ increases over the interval $[-1+\sqrt{2}, 1]$. So, if r is close to the lowerbound $\frac{1}{2}(-1+\sqrt{2})^3$, then $\hat{\alpha}(1, r)$ is close to $-1+\sqrt{2}$, and the results of the comparison of the two Nash equilibria are similar to those mentioned above under results (i) up to and including (iv) for the case where r was relatively small. If r increases to the upperbound $\frac{v}{4}$, then $\hat{\alpha}(1, r)$ approaches 1, and starting from the levels given in (35) up to and including (38), the expected utilities of players 1 and 2 decrease in the Nash equilibrium in the contest with the delegate (from $0.172\, v$ and $\frac{1}{2}\, v$) to respectively 0 and $\frac{v}{4}$, whereas the corresponding effort levels of the delegate and player 2 both increase (from $0.086\, v$ and $0.207\, v$) to $\frac{v}{4}$. Concluding, we see that if $\rho = 1$, then not only for all $0 \leq r < \frac{1}{2}(-1+\sqrt{2})^3$ but also for all $\frac{1}{2}(-1+\sqrt{2})^3 \leq r \leq \frac{1}{4}$, player 1 is always worse off in the contest with a delegate than in the contest without a delegate. The reverse conclusion holds for player 2.

6. Conclusion

This paper has extended a standard two-player winner-take-all contest with a principal-agent. First, we discussed the Nash equilibrium of the standard case in which both players exert effort themselves. Player 1 can be either risk averse or risk neutral with respect to his money income, and he is risk neutral with respect to his effort level. Player 2 is risk neutral in both respects. Next, we considered the extended case in which player 1 does not compete himself with player 2, but rather wants to hire a delegate who acts on his behalf in the contest. The delegate is risk neutral with respect to his money income and effort level. Player 1 offers the delegate as a wage a contingent fee consisting of a fraction α of the contested prize. The delegate accepts or rejects this offer depending on the size of his reservation income. We derived the unique subgame-perfect Nash equilibrium of the resulting two-stage contest, characterizing the equilibrium value of α and the corresponding expected utility and effort levels. In particular, we showed that the equilibrium value of α is always positive, and that the delegate always accepts the contract. Moreover, it turned out that there are situations in which player 1 offers the delegate for strategic reasons a higher fraction α of the contested prize than is strictly necessary to make him accept the contract. The reason for doing so is that the higher value of α gives the delegate an incentive to exert a greater effort in the competition with player 2, which is beneficial for player 1.

Finally, we compared the Nash equilibria of the contest with and without a delegate, focusing on the impact of the risk aversion of player 1 with respect to his money income. It turned out that the following holds: (i) if player 1 is strongly risk averse with respect to his money income and if moreover the contested prize is large enough, then the expected utility of player 1 is larger in the contest with a delegate than in the contest without a delegate; (ii) if player 1 is risk neutral with respect to his money income, then the expected utility of player 1 is smaller in the contest with a delegate than in the contest without a delegate. In both cases, an opposite conclusion holds for player 2.

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