Convex approximations for a class of mixed-integer recourse models

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Abstract

We consider mixed-integer recourse (MIR) models with a single recourse constraint. We relate the second-stage value function of such problems to the expected simple integer recourse (SIR) shortage function. This allows to construct convex approximations for MIR problems by the same approach used for SIR models.

Key words: mixed-integer recourse, convex approximation

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1. Introduction

Consider the mixed-integer recourse model

$$\begin{array}{ll} \min_x & cx + \mathcal{Q}(x) \\ \text{s.t.} & x \in X := \{x \in \mathbb{R}_+^{n-n'} \times \mathbb{Z}_+^{n'} : Ax \geq b\} \end{array}$$

where

$$\mathcal{Q}(x) := \mathbb{E}_\omega \left[v(\omega - Tx) \right], \quad x \in \mathbb{R}^n,$$

and, for $s \in \mathbb{R}$,

$$\begin{array}{lll} v(s) := & \min_{y,z} & qy + \bar{q}z \\ & \text{s.t.} & wy + \bar{w}z \geq s \\ & & (y, z) \in C \\ & & y \in \mathbb{R}_+^{m-m'},\ z \in \mathbb{Z}_+^{m'} \end{array}$$

The function v is the second-stage value function, and the function $\mathcal{Q}$ is called the expected value function. These functions model the (expected) costs of recourse actions to compensate for infeasibilities associated with the random goal constraint $Tx \geq \omega$. The right-hand side parameter ω is a random variable with known cumulative distribution function (cdf) F_ω.

This model has only a single recourse constraint, i.e., w and $\bar{w}$ are vectors, and $s \in \mathbb{R}$. In addition, there may be further linear constraints on the second-stage variables y and z, but they do not involve the first-stage decisions x nor the random parameter ω. Such constraints are denoted by $(y, z) \in C$.

Our main motivation to study this model is that it is the simplest extension of pure-integer recourse models, which we studied in a number of papers [4, 5, 8, 18]. In particular, we will see that the approach which we developed to construct convex approximations for the recourse function $\mathcal{Q}$ in the pure-integer case, can be extended to this mixed-integer recourse model. For a general description of this *modification of recourse data* approach, see [17].

In addition, this recourse model can be interpreted as a production planning problem. Using inputs x and given technological constraints $x \in X$, we wish to produce some good Tx to meet uncertain future demand ω, so as to minimize total expected costs $cx + \mathcal{Q}(x)$. In case production falls short of demand, recourse actions y and z can be used to compensate the shortage. The integer variables z represent batches of various sizes $\bar{w}$ (e.g., amounts bought from competitors), whereas the continuous variables y denote 'fractional' but more expensive production. We will return to this interpretation at the end of Section 2.

For a general introduction to recourse models we refer to the textbooks [1, 2, 10], the handbook [11], and the website [14]. Structural properties of mixed-integer recourse models were studied in [12, 13]. Surveys of properties, algorithms, and applications for (mixed-)integer recourse models can be found in [6, 7, 15].

The remainder of this paper is organized as follows. In Section 2 we analyze the second-stage value function v, and show that it can be represented as an *expected integer shortage function.* The latter function was studied extensively in the context of *simple integer recourse* models. In Section 3 we use this knowledge to come up with convex approximations of the recourse function $\mathcal{Q}$, and show that such approximations can be represented by *continuous simple recourse* functions. Section 4 contains a discussion on solution approaches, and a summary and concluding remarks follow in Section 5.

2. Analysis of the second-stage value function

In addition to the standard assumption that the second-stage value function v is finite-valued (i.e., the recourse is complete and sufficiently expensive), we assume

(i) $v(s) = 0$ for $s \leq 0$ and $v(s) > 0$ for $s > 0$.
(ii) For some period $p > 0$, the function v satisfies
$$v(k \cdot p + s) = k \cdot v(p) + v(s), \qquad 0 \leq s < p,\ k \in \mathbb{Z}_+.$$
We say that v is *semi-periodic* on $\mathbb{R}_+$: on this interval v is the sum of a periodic function with period p and a linear function with slope $v(p)$.

Clearly, assumption (ii) means a further restriction of the class of models under study. However, it is satisfied by e.g. simple recourse models (see Figure 2.1), and more generally, if there exists a largest batch of size p which provides the cheapest way of compensating a shortage p.

To simplify the presentation, we assume that the period $p = 1$ and that $v(p) = 1$, which always can be obtained by scaling. Then

$$v(s) = \begin{cases} 0, & s \leq 0 \\ \lfloor s \rfloor + v(\langle s \rangle), & s \geq 0 \end{cases} \tag{1}$$

where $\langle s \rangle := s - \lfloor s \rfloor$ is the fractional part of $s \in \mathbb{R}$.

Observing that $v(0) = 0$, $v(1) = 1$, and that v is non-decreasing on $[0, 1]$ and lower semi-continuous (hence left-continuous), we associate v with the left-continuous cumulative distribution function (cdf) $F_\upsilon(s) := \Pr\{\upsilon < s\}$ of a random variable υ with support in $[0, 1]$,

$$F_\upsilon(s) := \begin{cases} 0, & s \leq 0 \\ v(s), & s \in [0, 1] \\ 1, & s \geq 1 \end{cases} \tag{2}$$

Then $v(s) = \lfloor s \rfloor^+ + F_\upsilon(\langle s \rangle)$, $s \in \mathbb{R}$, with $\lfloor s \rfloor^+ := (\lfloor s \rfloor)^+$ denoting the positive part of $\lfloor s \rfloor$. Analogously, we define $\lfloor s \rfloor^- := (\lfloor s \rfloor)^-$ as the negative part of $\lfloor s \rfloor$, and $\lceil s \rceil^+ := (\lceil s \rceil)^+$.

Example 2.1 Define the (one-dimensional) *mixed-integer simple recourse* (MISR) value function as

$$v(s) = \min\{qy + z : y + z \geq s,\ y \in \mathbb{R}_+,\ z \in \mathbb{Z}_+\}, \quad s \in \mathbb{R},$$

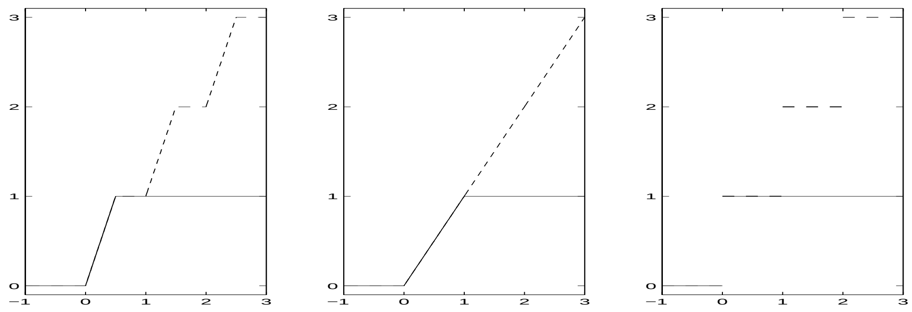

Figure 2.1: The MISR function υ (dashed) and corresponding cdf F_υ of Example 2.1, for $q \in \{2, 1, \infty\}$.

with $q \geq 1$. Then $v(s) = \min\{\lfloor s \rfloor^+ + q\langle s\rangle, \lceil s\rceil^+\}$, so that F_υ is the cdf of the continuous uniform distribution on $[0, 1/q]$ (notation: $\upsilon \sim U(0, 1/q)$).

Continuous simple recourse corresponds to MISR with $q = 1$, and has $v(s) = (s)^+$, $s \in \mathbb{R}$, so that F_υ is the cdf of $\upsilon \sim U(0, 1)$.

With $q = +\infty$ we obtain pure integer simple recourse, with $v(s) = \lceil s\rceil^+$, $s \in \mathbb{R}$, so that F_υ is the cdf of the degenerated random variable υ with $\Pr\{\upsilon = 0\} = 1$.

Figure 2.1 shows these MISR functions v and corresponding F_υ. $\lhd$

Lemma 2.1 *Consider a value function v satisfying (1), and let υ be a random variable with associated cdf F_υ according to (2). Then*

$$v(s) = \mathbb{E}_\upsilon\left[\lfloor \upsilon - s\rfloor^-\right], \quad s \in \mathbb{R}.$$

PROOF. For $s \leq 0$ the result follows trivially.

The random variable υ takes values in $(0, 1]$. Hence, for any fixed $s \in \mathbb{R}$, the random variable $s - \upsilon$ takes values in the interval $[s - 1, s)$, which contains precisely one integer value $\lfloor s\rfloor$. Thus, $\lceil s - \upsilon\rceil$ is a two-valued random variable,

$$\lceil s - \upsilon\rceil = \begin{cases} \lfloor s\rfloor, & \text{if } s - \upsilon \leq \lfloor s\rfloor; \\ \lfloor s\rfloor + 1, & \text{if } s - \upsilon > \lfloor s\rfloor, \end{cases}$$

so that

$$\mathbb{E}_\upsilon\left[\lceil s - \upsilon\rceil\right] = \lfloor s\rfloor + 1 \cdot \Pr\{\upsilon < \langle s\rangle\} = \lfloor s\rfloor + F_\upsilon(\langle s\rangle),$$

which is equal to $v(s)$ for $s > 0$. For such s, observing that $s - \upsilon > -1$ so that $\lceil s - \upsilon\rceil = \lceil s - \upsilon\rceil^+$, the result follows since $\lceil t\rceil^+ = \lfloor -t\rfloor^-$, $t \in \mathbb{R}$. $\square$

We conclude that each value function v under consideration is equivalent to an *integer expected shortage function*

$$H(s) := \mathbb{E}_{\upsilon}\left[\lfloor \upsilon - s \rfloor^{-}\right], \quad s \in \mathbb{R},$$

where the expectation is taken with respect to a random variable υ whose distribution captures the specific properties of v. The integer expected shortage function is well-studied in the context of simple integer recourse models. In the next section we will use its properties to derive convex approximations for the expected recourse function $\mathcal{Q}$.

As mentioned in the Introduction, our recourse model can be seen as a production planning model. In that context, Lemma 2.1 has an interesting interpretation. On the one hand, $v(s)$ represents the minimal costs for satisfying deterministic demand s, using a relatively sophisticated technology with various batch sizes as well as 'fractional' production, all with corresponding unit costs. By Lemma 2.1, the same *expected* production costs can be obtained by using a very simple production technology, which allows only a single unit batch size. Indeed, if one introduces a suitable random disturbance υ, and aims to satisfy the *stochastic* demand $s - \upsilon$, then on average the production costs will be the same for these two models. From the producers perspective, this means that savings on the costs for installing production technology can be obtained. On the other hand, clients may not be happy with the outcomes of such a 'virtual' production technology.

3. The recourse function

We now turn to studying the mixed-integer recourse function $\mathcal{Q}$, which is defined as $\mathcal{Q}(x) = \mathbb{E}_{\omega}\left[v(\omega - Tx)\right]$ for $x \in \mathbb{R}^n$, where the random variable ω represents stochastic demand. To derive properties of this function, we can equivalently study it as a function of the *tender variable* $Tx \in \mathbb{R}$. To this end, we define the one-dimensional function

$$Q(x) := \mathbb{E}_{\omega}\left[v(\omega - x)\right], \qquad x \in \mathbb{R}.$$

3.1 Properties

By Lemma 2.1, we have, for $x \in \mathbb{R}$,

$$\begin{aligned}
Q(x) &= \mathbb{E}_{\omega}\left[\mathbb{E}_{\upsilon}\left[\lfloor \upsilon - (\omega - x) \rfloor^{-}\right]\right] \\
&= \mathbb{E}_{\upsilon}\left[\mathbb{E}_{\omega}\left[\lfloor -\omega + (x + \upsilon) \rfloor^{-}\right]\right] \\
&= \mathbb{E}_{\upsilon}\left[\mathbb{E}_{\omega}\left[\lceil \omega - (x + \upsilon) \rceil^{+}\right]\right] \\
&= \mathbb{E}_{\upsilon}\left[G(x + \upsilon)\right], \qquad (3)
\end{aligned}$$

where $G(t) := \mathbb{E}_{\omega}\left[\lceil \omega - t \rceil^{+}\right]$, $t \in \mathbb{R}$, is the *integer expected surplus function*. This function G is the counterpart of the integer expected shortage function discussed above, and was also studied in the context of simple integer recourse models.

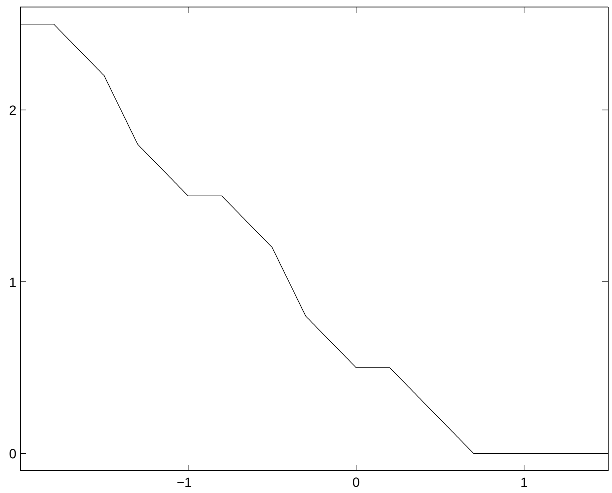

Figure 3.1: The function Q of Example 3.1

Alternatively, defining the random variable $\delta := \omega - \upsilon$, we obtain

$$Q(x) = \mathbb{E}_\delta \left[\lceil \delta - x \rceil^+ \right], \qquad x \in \mathbb{R}, \tag{4}$$

where δ has cdf F_δ,

$$F_\delta(t) = \int_0^1 F_\omega(t+s) dF_\upsilon(s), \qquad t \in \mathbb{R}.$$

The identities (3) and (4) show, that properties of the mixed-integer recourse function Q follow trivially from those of the integer expected surplus function G. Moreover, they provide an easy way to evaluate Q, given the formula

$$G(t) = \sum_{k=0}^{\infty} \left(1 - F_\omega(t+k)\right), \quad t \in \mathbb{R}.$$

For the derivation of this formula and properties of the function G, we refer to [8] and [16].

Example 3.1 Consider the MISR value function $v(s) = \min\{\lfloor s \rfloor^+ + q\langle s \rangle, \lceil s \rceil^+\}$, $s \in \mathbb{R}$. As shown in Example 2.1, for finite $q \geq 1$ the associated cdf F_υ is that of the uniform distribution on $[0, 1/q]$, so that

$$F_\delta(t) = q \int_0^{1/q} F_\omega(t+s) ds, \quad t \in \mathbb{R}.$$

It follows that the random variable δ has a probability density function (pdf) f_δ,

$$f_\delta(t) = q\left(F_\omega(t+1/q) - F_\omega(t)\right), \quad t \in \mathbb{R}. \tag{5}$$

For example, assuming $q = 2$ and that ω is discrete with equally likely realizations 0 and 0.7, it follows by straightforward computation that δ is uniformly distributed on two disjunct intervals:

$$f_\delta(t) = \begin{cases} 1, & t \in [-0.5, 0] \cup [0.2, 0.7] \\ 0, & \text{otherwise.} \end{cases}$$

See Figure 3.1 for the graph of the MISR recourse function $Q(x) = \mathbb{E}_\delta\left[\lceil \delta - x \rceil^+\right]$, $x \in \mathbb{R}$. ◁

We are particularly interested in convexity of the recourse function Q. From (3) and (4) it is clear, that this is directly related to convexity of the function G. In [5] it is shown that G is convex if and only if the underlying random variable ω has a pdf f satisfying

$$f(t) = F(t+1) - F(t), \quad t \in \mathbb{R}, \tag{6}$$

where F is an arbitrary cdf with finite mean value.

Corollary 3.1 *Consider a value function v satisfying (1), and let υ be a random variable with associated cdf F_υ according to (2). Then the recourse function $Q(x) = \mathbb{E}_\omega[v(\omega - x)]$, $x \in \mathbb{R}$, is convex if and only if the random variable $\delta := \omega - \upsilon$ has a pdf satisfying (6).*

In particular, Q is convex if ω has a pdf satisfying (6). □

Remark 3.1 It is well-known that the continuous simple recourse function Q, i.e., the special case of MISR with $q = 1$, is convex for every distribution of ω. This indeed follows trivially from Corollary 3.1 and (5).

Corollary 3.1 shows that the mixed-integer recourse function Q is convex only in exceptional cases. Therefore, we are interested in convex approximations of this function, which is the subject of the next section.

3.2 Convex approximations

Corollary 3.1 suggest that, to obtain convex approximations of a non-convex recourse function Q, we can perturb the distribution of the underlying random variable in such a way that the resulting distribution has a pdf satisfying (6). This approach, which has been applied to pure integer recourse models in [5, 18], will be extended to the current mixed-integer case below.

In principle, any suitable perturbation of the distribution can be used for this purpose, but – as in the pure integer case – we restrict to the following class.

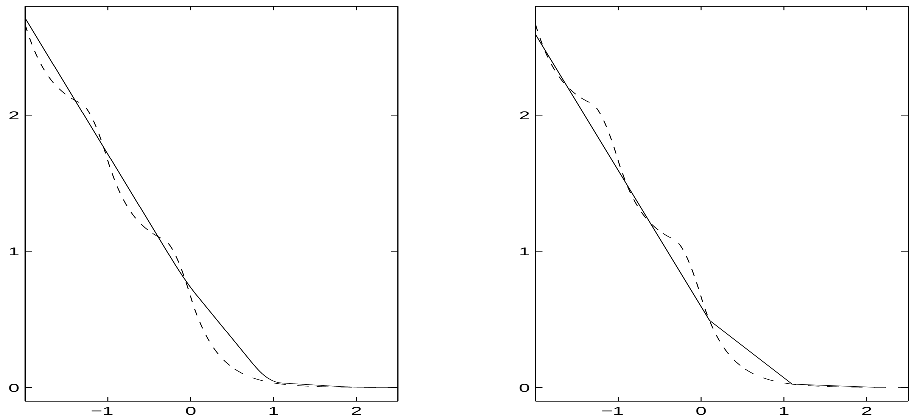

Figure 3.2: The MISR function Q (dashed) and the α-approximations Q^{α} (left) and Q_{α} (right), for $q = 3$, ω exponentially distributed with parameter 3, and $\alpha = 0.1$.

Definition 3.1 For each $\alpha \in [0, 1)$, the *α-approximation* of a random variable φ with cdf F_{φ} is the continuous random variable φ_{α} with pdf $f_{\varphi_{\alpha}}$,

$$f_{\varphi_{\alpha}}(t) := F_{\varphi}(\lfloor t \rfloor_{\alpha} + 1) - F_{\varphi}(\lfloor t \rfloor_{\alpha}), \quad t \in \mathbb{R},$$

where $\lfloor t \rfloor_{\alpha} := \lfloor t - \alpha \rfloor + \alpha$ is the round down of t with respect to $\alpha + \mathbb{Z}$.

Obviously, every pdf $f_{\varphi_{\alpha}}$, $\alpha \in [0, 1)$, satisfies (6). With $F_{\varphi_{\alpha}}$ denoting the cdf of φ_{α}, it is easy to see that $F_{\varphi_{\alpha}}(\alpha + k) = F_{\varphi}(\alpha + k)$, $k \in \mathbb{Z}$. Further properties of α-approximations are discussed in [5].

According to Corollary 3.1, we can choose to either replace ω or δ by an α-approximation. The resulting functions, defined for each $\alpha \in [0, 1)$,

$$Q_{\alpha}(x) := \mathbb{E}_{\delta_{\alpha}}\left[\lceil \delta_{\alpha} - x \rceil^{+}\right], \quad x \in \mathbb{R},$$

and

$$\begin{aligned} Q^{\alpha}(x) &:= \mathbb{E}_{\upsilon}\left[\mathbb{E}_{\omega_{\alpha}}\left[\lceil \omega_{\alpha} - (x + \upsilon) \rceil^{+}\right]\right], \quad x \in \mathbb{R}, \\ &= \mathbb{E}_{\upsilon}\left[G_{\alpha}(x + \upsilon)\right], \end{aligned}$$

are called α-approximations of the recourse function Q; similarly, the function G_{α} is an α-approximation of G. By construction, Q_{α} and Q^{α} are convex approximations of the mixed-integer recourse function Q. See Figure 3.2.

As shown in [5], each α-approximation G_{α}, $\alpha \in [0, 1)$, is piecewise linear and coincides with G on the set $\{\alpha + \mathbb{Z}\}$. Hence, the same is true for the α-approximations Q_{α}, but not for Q^{α}.

Properties of the latter α-approximations depend on the distribution of υ, i.e., on the value function v.

Next we state bounds, uniform in $\alpha \in [0, 1)$, for the respective approximation errors of Q_α and Q^α, for the case that the random variables involved are continuously distributed. If $\delta = \omega - \upsilon$ is not continuously distributed, e.g., if ω is a discrete random variable and the value function v is discontinuous, then we can only prove the trivial upper bound 1 for both approximation errors.

For $t \in \mathbb{R}$, let $\lceil t \rceil_\alpha := \lceil t - \alpha \rceil + \alpha$ denote the round up of t with respect to $\alpha + \mathbb{Z}$.

Theorem 3.1 *If $\delta = \omega - \upsilon$ has a pdf f_δ which is of bounded variation, then, for all $\alpha \in [0, 1)$,*

$$\left|Q(x) - Q_\alpha(x)\right| \le \min\{x - \lfloor x \rfloor_\alpha, \lceil x \rceil_\alpha - x\} \frac{|\Delta| f_\delta}{2} \le \frac{|\Delta| f_\delta}{4}, \quad x \in \mathbb{R},$$

where $|\Delta| f_\delta$ denotes the total variation of f_δ.

If ω has a pdf f_ω which is of bounded variation, then, for all $\alpha \in [0, 1)$,

$$\left|Q(x) - Q^\alpha(x)\right| \le \min\{x - \lfloor x \rfloor_\alpha, \lceil x \rceil_\alpha - x\} \frac{|\Delta| f_\omega}{2} \le \frac{|\Delta| f_\omega}{4}, \quad x \in \mathbb{R}.$$

Moreover, $|\Delta| f_\delta \le |\Delta| f_\omega$ with strict inequality in most cases. Thus, measured by the bounds above, the approximation of Q by Q_α is at least as good as that by Q^α.

PROOF. The bounds on the approximation errors follow immediately from the corresponding result for the function G_α, see [5].

Referring to the definition of total variation, let $U := \{u_0, u_1, \ldots, u_N\} \subset \mathbb{R}$ be such that

$$|\Delta| f_\delta - \varepsilon < \sum_{i=1}^{N} |f_\delta(u_i) - f_\delta(u_{i-1})| \tag{7}$$

for every $\varepsilon > 0$. Then

$$\begin{aligned} |\Delta| f_\delta - \varepsilon &< \sum_{i=1}^{N} \left| \int_0^1 \big(f_\omega(u_i + s) - f_\omega(u_{i-1} + s)\big)\, dF_\upsilon(s) \right| \\ &\le \int_0^1 \sum_{i=1}^{N} \left| f_\omega(u_i + s) - f_\omega(u_{i-1} + s) \right| dF_\upsilon(s) \\ &\le |\Delta| f_\omega, \end{aligned}$$

where the last inequality is strict unless the sets $\{U + s\}$ yield $|\Delta| f_\omega$ (in the sense of (7)) for F_υ-almost all s. □

Remark 3.2 Actually, the bounds derived in Theorem 3.1 apply to α-approximations of the expectation of the scaled value function $v(ps)/v(p)$, $s \in \mathbb{R}$, where p is the period of the semi-periodic function v restricted to $\mathbb{R}_+$. It follows by straightforward calculation that for the general case, the error bounds of Theorem 3.1 are multiplied by a factor $pv(p)$.

For many distributions, the total variation of the pdf decreases as the variance of the distribution increases. For example, the total variation of the normal distribution is proportional to the inverse of its standard deviation. Thus, we would expect that the α-approximations Q_α and Q^α become better as the variance in the respective underlying distributions become larger.

3.3 Continuous simple recourse representation

It follows from the assumed semi-periodicity on $\mathbb{R}_+$ of v, that every convex approximation $Q_\alpha(x)$ or $Q^\alpha(x)$, $\alpha \in [0, 1)$, has an asymptote (with slope -1) for $x \longrightarrow -\infty$. Moreover, both functions decrease to 0 as $x \longrightarrow -\infty$, so that each of them is Lipschitz continuous on $\mathbb{R}$. In [3] we showed that every such function can be represented as a continuous simple recourse function

$$q^+\mathbb{E}_\psi\left[(\psi - x)^+\right] + q^-\mathbb{E}_\psi\left[(\psi - x)^-\right], \quad x \in \mathbb{R},$$

with known distribution of the random variable ψ. Below we apply this result to the α-approximations Q_α and Q^α.

Corollary 3.2 *For $\alpha \in [0, 1)$, consider the α-approximation*

$$Q_\alpha(x) := \mathbb{E}_{\delta_\alpha}\left[\lceil \delta_\alpha - x \rceil^+\right], \quad x \in \mathbb{R},$$

where δ_α is the α-approximation of the random variable $\delta := \omega - \upsilon$, ω has cdf F_ω, and υ has cdf F_υ.

$$Q_\alpha(x) = \mathbb{E}_{\psi_\alpha}\left[(\psi_\alpha - x)^+\right], \quad x \in \mathbb{R},$$

where the random variable ψ_α has cdf Ψ_α,

$$\Psi_\alpha(t) = \int_0^1 F_\omega\left(\lfloor t \rfloor_\alpha + s\right) dF_\upsilon(s), \quad t \in \mathbb{R}.$$

Since Ψ_α is constant on every interval $[\alpha + k, \alpha + k + 1)$, $k \in \mathbb{Z}$, the random variable ψ_α is discrete with support in $\{\alpha + \mathbb{Z}\}$. □

Corollary 3.3 *For $\alpha \in [0, 1)$, consider the α-approximation*

$$Q^\alpha(x) := \mathbb{E}_\upsilon\left[\mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - (x + \upsilon) \rceil^+\right]\right], \quad x \in \mathbb{R},$$

where ω_α is the α-approximation of the random variable ω with cdf F_ω, and υ has cdf F_υ.

$$Q^\alpha(x) = \mathbb{E}_{\varphi_\alpha}\left[(\varphi_\alpha - x)^+\right], \quad x \in \mathbb{R},$$

where the random variable φ_α has cdf Φ_α,

$$\Phi_\alpha(t) = \int_0^1 F_\omega\left(\lfloor t+s \rfloor_\alpha\right) dF_\upsilon(s), \quad t \in \mathbb{R}.$$

□

4. Solving mixed-integer recourse models

Returning to the recourse function $\mathcal{Q}(x) = \mathbb{E}_\omega\left[v(\omega - Tx)\right]$, $x \in \mathbb{R}^n$, we conclude that convex approximations can be obtained by α-approximations. Moreover, each such function $\mathcal{Q}_\alpha(x)$ or $\mathcal{Q}^\alpha(x)$, $\alpha \in [0,1)$, can be represented as a continuous simple recourse function with random right-hand side parameter, whose distribution is known. In particular, for each function $\mathcal{Q}_\alpha$ the corresponding distribution is discrete.

Hence, to approximately solve the mixed-integer recourse problem

$$\min_{x \in X} cx + \mathcal{Q}(x) \tag{8}$$

we can solve instead the continuous simple recourse problem

$$\min_{x \in X} cx + \mathcal{Q}_\alpha(x) \tag{9}$$

or

$$\min_{x \in X} cx + \mathcal{Q}^\alpha(x), \tag{10}$$

for one or more values of the parameter α. If all first-stage variables are continuous, then this can be done by existing algorithms (see e.g. [9]), which are very efficient. If not, then one could apply a branch and bound scheme, and use these algorithms to solve subproblems.

It is easy to see that the bounds presented in Theorem 3.1 also apply to the respective approximation errors in the optimal values of (9) and (10). Depending on the application, this guaranty on the approximation error may be satisfactory or not. If not, then one could solve approximating problems (9) or (10) for a number of parameter values α, yielding respective optimal solutions x_α, and compare them by calculating the true objective values $cx_\alpha + \mathcal{Q}(x_\alpha)$.

As we will show next, the evaluation of $\mathcal{Q}(x_\alpha)$ may not be necessary if x_α is an optimal solution of (9). For each $\alpha \in [0,1)$, the random variable ψ_α underlying the continuous simple recourse function $\mathcal{Q}_\alpha$ is discrete with support in $\alpha + \mathbb{Z}$ by Corollary 3.2. As is well known, the function $\mathcal{Q}_\alpha$ is therefore polyhedral, and it is non-linear at x if and only if Tx belongs to the support of ψ_α. Hence, if x_α is a *free* optimal solution of (9), then $Tx_\alpha \in \alpha + \mathbb{Z}$, so that $\mathcal{Q}_\alpha(x_\alpha) = \mathcal{Q}(x_\alpha)$ as discussed in Section 3.2.

For this reason, and moreover since

(i) solving (9) is in general easier than solving (10), which according to Corollary 3.3 involves a random variable with arbitrary distribution type;

(ii) by Theorem 3.1, the approximation obtained by (9) is at least as good as that of (10),

we conclude that the convex approximations $\mathcal{Q}_\alpha$ are preferable over $\mathcal{Q}^\alpha$ for the purpose of approximately solving the mixed-integer recourse model (8).

5. Summary and concluding remarks

For a restricted class of mixed-integer recourse models, we showed that the second-stage value function v is equivalent to an *expected* integer shortage function H, where the expectation is taken with respect to a distribution which reflects the particular value function. Thus, the mixed-integer recourse function $\mathcal{Q}(x) := \mathbb{E}_\omega[v(\omega - Tx)]$, $x \in \mathbb{R}^n$, can be seen as the expectation of $H(\omega - Tx)$, allowing to derive its properties from those of H, which is well-studied in the context of simple integer recourse models.

In particular, we showed that convex approximations of $\mathcal{Q}$ can be obtained by suitable perturbations of the distributions involved. This approach, first developed for simple integer recourse models [5] and later extended to general pure integer recourse models [18], is thus shown to be applicable to some mixed-integer recourse models too.

Next, we showed that the convex approximations obtained can be represented as recourse functions of continuous simple recourse models. Thus, instead of solving such a mixed-integer recourse model directly, we can

(i) modify the recourse structure (in particular, get rid of integrality constraints in the second-stage problem), and
(ii) perturb the underlying distribution,

to obtain a continuous simple recourse model which is easy to solve, and provides an approximate solution of the original model. The results presented in this paper extend this *modification of recourse data* approach [17], which was previously applied to pure integer (and continuous multiple simple) recourse models, to a class of mixed-integer recourse models.

In future research we hope to further extend this approach to more general mixed-integer recourse models.

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