Auctions with Rent Seeking

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Abstract

We present a model which combines elements of an auction and a rent-seeking contest. Players compete for a prize. Apart from exerting lobbying efforts, they also have to submit a bid which is payable only if they win the prize. First, we analyze the model if the returns-to-scale parameters of both bids and efforts are unity. We present a necessary and sufficient condition for the existence of a unique Nash equilibrium. In the equilibrium each player submits the same bid, while the sum of all efforts equals that bid. Second, we analyze the case in which the returns-to-scale parameters may differ from unity, and derive the implications of that specification.

Keywords: auctions; lobbying; rent seeking

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1. Introduction

In many economic situations, a number of contestants try to obtain some prize or rent. Several mechanisms can be used to assign a prize to one of the competitors. One obvious way to do so is through a regular auction. Then, all contestants submit a bid and, as a rule, the one submitting the highest bid obtains the prize, and pays an amount that depends in some pre-described way on the total vector of bids. In the simplest case, the highest bidder pays his own bid, whereas the other bidders pay nothing. For recent surveys of this literature, see e.g. Wolfstetter (1996) or Klemperer (1999). Another possible mechanism is the following. In the case of policy decisions, the parties involved often exert effort in an attempt to influence the decision process. This effort can take the form of lobbying, but can also consist of bribes. Such a process can be modelled as an all-pay auction or a rent-seeking contest. In an all-pay auction (see e.g. Baye, Kovenock, and de Vries, 1993), all contestants have to pay for their effort, and the one with the highest effort wins the auction. In a rent-seeking contest, all players also exert some effort, but the outcome of the process is stochastic: each contestant wins with a probability that is increasing in his own effort, but decreasing in that of his competitors. The extensive literature on such contests started with Tullock (1980). See further e.g. Dixit (1987), Hillman and Riley (1989), and for a comprehensive survey, Nitzan (1994).

Yet, in practice, we often have situations that lie somewhere between the two extremes of auctions and rent seeking. Often, when an auction is held, the outcome is not solely determined by the height of the bid. In most cases, other aspects of the competing offers also play a role. In public procurement, the quality of the offers made is also taken into account, usually by some predefined rule that weighs different quantifiable quality criteria of the offers made. Another example is the procedure by which major sports events, such as the Olympic Games, are assigned to cities or countries. On the one hand, this decision is determined by bids the contestants submit, which come in the form of e.g. the quality or quantity of new stadiums and infrastructure. Yet, there is probably also room for some lobbying or bribing of the decision makers. A final example is a takeover battle. Suppose two firms try to take over a third firm. Both firms submit a bid. Shareholders decide whom to tender their shares to. Yet, they will usually base their decisions not only on the bids submitted, but also on the extent to which they feel each firm contributes to the long-term prospects of the firm being taken over.[1] Thus, often, even if an auction is held, there is still room for lobbying or rent seeking to try to influence the outcome of the auction.

In this paper, we try to model this notion. We build on the rent-seeking literature, but assume that the probability of winning not only depends on the effort exerted, but also on the bid made. In section 2, we describe our general framework, and show that it can be seen as an extension of the standard rent-seeking game. In section 3, we consider the simplest possible version of our model in which returns-to-scale parameters of both bids

1 A related example: in a recent hostile takeover battle, the British telephone company Vodafone bid some 132 billion euro to obtain control of its German rival Mannesmann. Reportedly, both firms set aside a total amount of 850 million euro for this fight, trying to influence the voting behavior of shareholders. From this amount, 150 million was reserved for advertising. See The Economist (2000).

and efforts are equal to unity. For a given number of players, we present a necessary and sufficient condition for the existence of a unique (Nash) equilibrium in which all players participate in the contest. We show that in that equilibrium (a) each player submits the same bid, (b) the sum of all outlays equals that bid, and (c) there is underdissipation of rent. Furthermore, we give explicit equilibrium solutions for the case of equal valuations, and for the case in which there are only two contestants. Section 4 uses a more general model, in which the returns-to-scale parameters of bids and efforts may differ from unity, and derives the implications of that specification. If an equilibrium in which all players participate exists, it now has that the sum of all individual ratios of the effort and bid, equals the ratio of the returns-to-scale parameters associated with efforts and bids. We further present a sufficient condition for the existence of an equilibrium of this model for the case of equal valuations. Section 5 concludes.

2. The general model

Our basic model is the following. There are n players trying to obtain some prize. Player i values the prize at $v_i > 0$. We thus allow for asymmetric valuations. Each player can submit a bid $b_i \geq 0$, and spend effort $e_i \geq 0$. The bid b_i only has to be paid if i wins the prize. However, outlays e_i are sunk. A player cannot retrieve these, regardless of whether or not he wins the prize. In general, we assume that the probability p_i that i wins is given by the logit form contest success function

$$p_i(b_1, \ldots, b_n, e_1, \ldots, e_n) = \frac{f(b_i, e_i)}{\sum_{j=1}^{n} f(b_j, e_j)}, \qquad i = 1, \ldots, n, \tag{1}$$

if $b_j > 0$ and $e_j > 0$ for at least one j, and $p_i = 0$ if that is not the case. Here, $f(b_i, e_i)$ is non-negative, and $\partial f/\partial b_i$, $\partial f/\partial e_i \geq 0$. This implies $\partial p_i/\partial b_i$, $\partial p_i/\partial e_i \geq 0$, and $\partial p_i/\partial b_j$, $\partial p_i/\partial e_j \leq 0$ $(j \neq i)$. Thus, based on the bid b_i and the outlays e_i, a 'score' $f(b_i, e_i)$ is computed for each player. The probability that a certain player wins this contest, is equal to the share of his score in the total sum of scores. Note that these probabilities sum to unity.[2] Given (1), player i wants to maximize his expected payoff, which is given by

$$\Pi_i = p_i \, (v_i - b_i) - e_i. \tag{2}$$

This expression reflects that the bid only has to be paid if the player wins the prize, whereas the outlays are non-refundable.

A natural assumption is that the score $f(b_i, e_i)$ links b_i and e_i in some multiplicative fashion. In that way, we capture the idea that there is a trade-off between increasing bid b_i and increasing effort e_i. In section 3, we simply assume $f(b_i, e_i) = b_i e_i$, which we loosely denote as a constant-returns-to-scale score (note that if the size of either b_i or e_i is increased with a certain multiplicative factor, then the score is increased with this same

2 As long as at least one player both submits a positive bid and exerts a positive effort. We assume that the contest is cancelled, i.e. the prize is not awarded at all, if none of the players both submits a positive bid and exerts a positive effort.

factor as well). Note also that in that case, the probability that player i wins the prize is equal to zero if he submits a zero bid or exerts no effort. In section 4, we use a more general Cobb-Douglas score function $f(b_i, e_i) = b_i^\alpha e_i^\beta$, with $\alpha, \beta > 0$ returns-to-scale parameters of, respectively, the bids and efforts. Such a more general function, however, leads to a less tractable model.

In a standard rent-seeking model, only some effort e_i is exerted. Expected payoffs then equal

$$\pi_i = \frac{g(e_i)}{\sum_j g(e_j)} v_i - e_i. \tag{3}$$

Many papers in this literature assume $g(e_i) = e_i$. Hillman and Riley (1989) analyze this model, allowing for n contestants and asymmetric valuations. Ellingsen (1991) gives an application. Our model in section 3 can be seen as a generalization of this approach. Some papers, including Tullock (1980), use a more general contest success function $g(e_i) = e_i^r$, with $r > 0$. Nti (1999) analyzes this model, allowing for asymmetric valuations, but restricting attention to the case $n = 2$. Our model in section 4 generalizes this approach. Finally, we refer to Skaperdas (1996) and Kooreman and Schoonbeek (1997) for a general discussion of the foundations of logit form contest success functions in rent-seeking models.

3. A constant-returns-to-scale score

In this section we use (1), with the constant-returns-to-scale score $f(b_i, e_i) = b_i e_i$. We therefore have

$$\Pi_i = \left(\frac{b_i e_i}{\sum_j b_j e_j}\right)(v_i - b_i) - e_i \tag{4}$$

if $b_j > 0$ and $e_j > 0$ for at least one j, and $\Pi_i = 0$ otherwise. We want to investigate the (Nash) equilibria of the resulting model.

Without loss of generality, we first order the valuations such that $v_1 \geq v_2 \geq \ldots \geq v_n$. Further, we define the following continuous auxiliary function

$$h_n(b) = \sum_{j=1}^{n} \left(\frac{1}{v_j - b}\right) - \frac{n-1}{b}, \tag{5}$$

for $0 < b < v_n$. Observe that $h_n(b)$ is strictly increasing in b. Moreover, $\lim_{b \downarrow 0} h_n(b) = -\infty$, and $\lim_{b \uparrow v_n} = \infty$. This implies that $h_n(b)$ has a unique root, $b(n)$ say, on $(0, v_n)$, i.e. $h_n(b(n)) = 0$. Using this, we present the following theorem which provides a necessary and sufficient condition for the existence of a unique equilibrium in which all n players participate, and which, moreover, gives general characteristics of such an equilibrium.

Theorem 3.1 *Let the valuations be $v_1 \geq v_2 \geq \ldots \geq v_n$. There exists an equilibrium $(\hat{b}_1, \ldots, \hat{b}_n, \hat{e}_1, \ldots, \hat{e}_n)$ in which all n players participate, i.e. with $\hat{b}_i > 0$ and $\hat{e}_i > 0$,*

$\forall i$, if and only if the unique root $b(n)$ of the function $h_n(b)$ as defined in (5) satisfies $b(n) < v_n/2$. If such an equilibrium exists, it is unique and the bids and efforts satisfy:

(i) $\hat{b}_i = \hat{b} = b(n), \forall i,$

(ii) $\hat{e}_i = \frac{\hat{b}(v_i - 2\hat{b})}{(v_i - \hat{b})}, \forall i,$

(iii) $\sum_{i=1}^{n} \hat{e}_i = \hat{b}.$

PROOF. See the Appendix. □

Thus, in this model, where not only efforts but also bids determine the probability of winning the prize, all players submit the same bid in the equilibrium, regardless of their valuation. This implies that in equilibrium, the fact that bids are submitted does not play a role, i.e. differences in the success probabilities are solely determined by differences in the outlays $\hat{e}_i$.

We also have that in equilibrium the bid every participant submits, equals the sum of total outlays. The equilibrium bid is less than one half of the smallest valuation, v_n. Furthermore, using (i) of Theorem 3.1 and (5), we see that the equilibrium bid is strictly increasing in the size of the valuations of the players, i.e. $\partial\hat{b}/\partial v_i > 0$, $\forall i$ (note that for fixed b, the first term on the RHS of (5) strictly decreases if one marginally increases the valuation v_i, whereas the second term remains constant). It also follows that the equilibrium bid and efforts are linear homogeneous in the valuations, in the sense that if all valuations increase with a same factor, then the equilibrium bid and efforts all increase with this factor as well.

Using Theorem 3.1, we further see that $\hat{e}_1 \geq \hat{e}_2 \geq \ldots \geq \hat{e}_n$. Thus, the higher the valuation of a player, the greater the effort he exerts in the equilibrium. It can also be verified that in the equilibrium the probability that player i wins the prize equals $\hat{p}_i = \hat{e}_i/\hat{b}$. This implies that $\hat{p}_1 \geq \hat{p}_2 \geq \ldots \geq \hat{p}_n > 0$. As a result, the player with the highest valuation also has the highest probability to win the prize. The expected profit of player i corresponding to the equilibrium can be expressed as

$$\widehat{\Pi}_i = \frac{(v_i - 2\hat{b})^2}{(v_i - \hat{b})}. \tag{6}$$

Consequently, we obtain that $\widehat{\Pi}_1 \geq \widehat{\Pi}_2 \geq \ldots \geq \widehat{\Pi}_n > 0$.

Theorem 3.1 considers equilibria in which all players participate in the contest. However, for arbitrary valuations, the root $b(n) \in (0, v_n)$ of $h_n(b)$ of (5) does not always satisfy $b(n) < v_n/2$. [3] In that case, an equilibrium with all n participating players does not exist. The same kind of problem appears in the standard rent-seeking model with unequal valuations. To handle this problem, Hillman and Riley (1989) propose an intuitively appealing procedure in which only players with the highest valuations decide to participate in the contest. Applying a similar procedure to our model, we can state that agents n, $n-1, \ldots$ will sequentially drop out of the contest until, for some agent k, we have $b(k) < v_k/2$,

3 Take e.g. $v_1 = 5$, $v_2 = 4$ and $v_3 = 2$. Then $b(3) \approx 1.134 > v_3/2$.

with $b(k)$ the unique root of $h_k(b)$, where $h_k(b)$ is defined as

$$h_k(b) = \sum_{j=1}^{k} \left(\frac{1}{v_j - b} \right) - \frac{k-1}{b}, \tag{7}$$

for $0 < b < v_k$. In that case the equilibrium bid with the k players $1, \ldots, k$ is given by this root.

Admittedly, this procedure, although appealing, has a drawback.[4] Suppose for example that $n = 5$, but there is no equilibrium with all five players participating. Then, rather for the player with the lowest valuation to drop out, it may also be an equilibrium for the player with the second-lowest valuation to drop out. If he does, the condition just given may be satisfied for the player with the lowest valuation.[5]

In the next two subsections, we demonstrate that the condition $b(n) < v_n/2$ holds — and thus that there exists a unique equilibrium in which all players participate — if either all valuations are equal or $n = 2$. Note that the latter implies that if we have a case in which there is no equilibrium with $n > 2$ players participating, then the above procedure in which players with the lowest valuations sequentially drop out, certainly provides us with an equilibrium.

Concluding this section, we discuss the extent of rent dissipation that occurs in the equilibrium of Theorem 3.1. First note that, in order to study rent dissipation, we need a definition for that magnitude in the context of our model. In the rent-seeking literature, the extent of rent dissipation is defined as the total sum of outlays of the contestants trying to obtain the prize. Yet, in our model, there is also a bid $\hat{b}$ paid by the winner. Arguably, this should not be counted as rent dissipation, since it merely consists of a transfer from the winner of the prize to the authority selling the prize. On the other hand, one can argue that, when $\hat{e}_i$ consists of bribes rather than efforts, then these bribes are also merely transfers. We therefore consider both possibilities. First, suppose that the winning bid is considered as dissipated rent. Total rent dissipation then equals $D = \sum_i \hat{e}_i + \hat{b}$. Using Theorem 3.1, it follows that $D = 2\hat{b} < v_n$. Thus, in this case there is always underdissipation of rent, in the sense that total rent dissipation is less than the size of (even) the smallest valuation of the prize. Second, if we suppose that the winning bid is not considered as dissipated rent, then total rent dissipation, D' say, satisfies $D' = \frac{1}{2}D < \frac{1}{2}v_n$. Obviously, again there is always underdissipation of rent.

3.1 The case of equal valuations

Let us now consider the case in which all players have the same valuation. We then obtain the following corollary of Theorem 3.1.

4 Note that this proviso also holds for Hillman and Riley (1989) proposition 5, even though they fail to point this out.

5 To illustrate this possibility for the case $n = 3$, take again the valuations of footnote 3. There is no equilibrium in which all three players participate. However, for *each* combination of two players there exists a well-defined equilibrium. See Corollary 3.2 below.

Corollary 3.1 *If $v_i = v, \forall i$, then a unique equilibrium exists. The equilibrium bids and efforts are given by:*

(i) $\hat{b}_i = \hat{b} = \frac{(n-1)v}{(2n-1)}, \forall i,$

(ii) $\hat{e}_i = \hat{e} = \frac{(n-1)v}{(2n-1)n}, \forall i.$

PROOF. Using $v_i = v, \forall i$, it follows that the root $b(n)$ of the function $h_n(b)$, defined in (5), is equal to $b(n) = (n-1)v/(2n-1)$, thus $b(n) < v/2$. From Theorem 3.1, there is a unique equilibrium. Moreover, from part (i) of Theorem 3.1, the equilibrium bids equal $\hat{b}_i = b(n), \forall i$, hence part (i) of the corollary. Finally, invoking symmetry, i.e. $\hat{e}_i = \hat{e}, \forall i$, part (ii) of the corollary follows from part (iii) of Theorem 3.1. □

Since, for this case, we do have explicit solutions for $\hat{b}$ and $\hat{e}_i$, we can also explicitly characterize the extent of rent dissipation that occurs in the equilibrium. From Corollary 3.1, if the winning bid is considered as dissipated rent, then total rent dissipation is $\frac{2}{3}v$ with $n = 2$, and it strictly increases to v as n goes to infinity. If the winning bid is not considered as dissipated rent, then total rent dissipation is $\frac{1}{3}v$ with $n = 2$, and it strictly increases to $\frac{1}{2}v$ as n goes to infinity.

Next, we recall that total rent dissipation equals $(n-1)v/n$ in the standard rent-seeking model, see e.g. Hillman and Riley (1989). Thus, in our model, total rent dissipation is lower than in the standard rent-seeking model when $\hat{b}$ is not considered as dissipated rent, but higher when $\hat{b}$ is considered as dissipated rent.

For the standard rent-seeking model, in equilibrium it can be shown that $e_i^* = e^* = (n-1)v/n^2, \forall i$, see again Hillman and Riley (1989). The expected profit of contestant i then equals $\pi_i^* = v/n^2$. In our model, using Corollary 3.1, we have

$$\widehat{\Pi}_i = \frac{v}{n}\left(\frac{1}{2n-1}\right). \tag{8}$$

In a regular auction, it is easy to see that each player would bid the common valuation of the prize (v), leaving expected profits equal to zero. Therefore, in our auction with rent seeking, expected profits for contestants are higher than in a regular auction, but lower than in a standard rent-seeking contest.

3.2 The case of two players

Next, we return to the general model in which valuations are allowed to differ, but restrict attention to the case of two contestants, thus $n = 2$. We then have the following corollary of Theorem 3.1.

Corollary 3.2 *If $n = 2$, then a unique equilibrium exists. The equilibrium bids and efforts are given by:*

(i) $\hat{b}_i = \hat{b} = \frac{v_1+v_2}{3} - \frac{1}{3}\sqrt{(v_1+v_2)^2 - 3v_1v_2},$

(ii) $\hat{e}_i = \frac{\hat{b}(v_i - 2\hat{b})}{(v_i - \hat{b})},$

for $i = 1, 2$. Substituting $\hat{b}$ into (ii), we have an explicit solution for $\hat{e}_i$.

PROOF. Taking $n = 2$, it can be verified that the root $b(2)$ of the function $h_2(b)$ defined in (5) is given by

$$b(2) = \frac{v_1 + v_2}{3} - \frac{1}{3}\sqrt{(v_1 + v_2)^2 - 3v_1 v_2}.$$

Again, without loss of generality, assume that $v_1 \geq v_2$. We then have to show that $b(2) < v_2/2$, i.e. $v_2 - 2b(2) > 0$. Now,

$$v_2 - 2b(2) = \frac{v_2 - 2v_1}{3} + \frac{2}{3}\sqrt{(v_1 + v_2)^2 - 3v_1 v_2}. \tag{9}$$

For this expression to be positive, we need

$$2\sqrt{(v_1 + v_2)^2 - 3v_1 v_2} > 2v_1 - v_2. \tag{10}$$

With $v_1 \geq v_2$, the RHS of this expression is positive. Taking squares on both sides and rearranging, (10) simplifies to $3v_2^2 > 0$, which is always satisfied. Using Theorem 3.1, a unique equilibrium exists. Parts (i) and (ii) of the corollary follow directly. □

Suppose we consider the winning bid $\hat{b}$ as dissipated rent. Total rent dissipation then equals $D = \hat{e}_1 + \hat{e}_2 + \hat{b} = 2\hat{b}$. Nti (1999) proposes the following way to study how the extent of asymmetry in valuation influences total rent dissipation. Without loss of generality, assume again that $v_1 \geq v_2$, and write $v_2 = \lambda v_1$, with $\lambda \leq 1$. We then have

$$D = \frac{2}{3}v_1\left(1 + \lambda - \sqrt{1 - \lambda + \lambda^2}\right). \tag{11}$$

Observe that $\partial D/\partial\lambda > 0$. Thus, the more equal valuations are (i.e. the higher λ is), the higher total rent dissipation. Yet, this analysis is in terms of a fixed v_1. More equal valuations then imply a higher v_2, while keeping v_1 fixed. In this analysis, increased rent dissipation is not so much due to lower asymmetry, but rather to a higher v_2. This can be seen as follows. Rather than writing $v_2 = \lambda v_1$, we can also write $v_1 = \mu v_2$, with $\mu \geq 1$. We then have

$$D = \frac{2}{3}v_2\left(1 + \mu - \sqrt{1 - \mu + \mu^2}\right). \tag{12}$$

Now, $\partial D/\partial\mu > 0$. Thus, this suggests that having more equal valuations (i.e. lower μ) leads to *lower* dissipation, since we now do the analysis in terms of a fixed v_2 rather than a fixed v_1.

A better way to study the effect of a decrease in asymmetry is the following. Suppose the sum of valuations of both contestants is fixed: $v_1 + v_2 = V$. Using $v_1 \geq v_2$, we may write $v_1 = \rho V$ and $v_2 = (1 - \rho)V$, with $\rho \in [\frac{1}{2}, 1)$. We can study the effect of decreased asymmetry as a decrease in ρ, without the problem of scale effects that affect the analyses above.

Total rent dissipation now equals

$$D = \frac{2}{3}V\left(1 - \sqrt{1 - 3\rho(1 - \rho)}\right). \tag{13}$$

Observe that

$$\frac{\partial D}{\partial \rho} = \frac{1 - 2\rho}{\sqrt{1 - 3\rho(1 - \rho)}} V. \tag{14}$$

Thus, rent dissipation is maximized when $\rho = \frac{1}{2}$, i.e. when the two valuations are equal. Further, $\partial D / \partial \rho < 0$ for all $\rho \in (\frac{1}{2}, 1)$. Therefore, with two players, we unambiguously have that more equal valuations lead to higher total rent dissipation. Remark that this result does not hinge on the definition of rent dissipation. It does not matter whether or not we count $\hat{b}$ as dissipated rent. If we do not, total rent dissipation simply equals $D' = \frac{1}{2} D$.

4. A general Cobb-Douglas score

In the previous section, we analyzed a model where the returns-to-scale parameters associated with both bidding and rent seeking equal unity. In this section, we use the more general Cobb-Douglas score function $f(b_i, e_i) = b_i^\alpha e_i^\beta$. The returns-to-scale parameters satisfy $\alpha, \beta > 0$. Hence, the model analyzed in the previous section is a special case of this model, with $\alpha = \beta = 1$. Expected profits of player i now equal

$$\Pi_i = \left(\frac{b_i^\alpha e_i^\beta}{\sum_j b_j^\alpha e_j^\beta} \right) (v_i - b_i) - e_i \tag{15}$$

if $b_j > 0$ and $e_j > 0$ for at least one j, and $\Pi_i = 0$ otherwise.

For this model we have the following general result.

Theorem 4.2 *Consider an equilibrium* $(\hat{b}_1, \ldots, \hat{b}_n, \hat{e}_1, \ldots, \hat{e}_n)$ *in which all* n *players participate, i.e. with* $\hat{b}_i > 0$ *and* $\hat{e}_i > 0$, $\forall i$. *We then have:*

$$\left(\sum_{j=1}^{n} \frac{\hat{e}_j}{\hat{b}_j} \right) = \frac{\beta}{\alpha}. \tag{16}$$

PROOF. See the Appendix. □

Thus, if we have an equilibrium in which all players participate, then the sum of all individual ratios of the equilibrium effort and equilibrium bid, equals the ratio of the returns-to-scale parameters associated with efforts and bids. This theorem has a natural interpretation. As β, the parameter that reflects returns to scale with respect to the efforts increases, then efforts become more important, in the sense that the sum of the individual ratios of the equilibrium effort and equilibrium bid increases. Also, as α, the parameter that reflects returns to scale with respect to bids increases, then bids become more important, in the sense that the sum of the individual ratios of the equilibrium effort and equilibrium bid decreases.

In order to analyse this model further, we make in the next subsection the simplifying assumption that all contestants have equal valuations. We remark that (even) for the case of two players, it is in general not possible to find the equilibrium in closed form.[6]

4.1 Equal valuations

Suppose that, in the model described above, all players have equal valuations. We then have the following result.

Theorem 4.3 *Suppose that $v_i = v, \forall i$, and $\beta \leq 1$. Then there exists a unique equilibrium. In particular, the equilibrium bids and efforts are given by:*

(i) $\hat{b}_i = \hat{b} = \frac{\alpha(n-1)v}{\alpha(n-1)+n}, \forall i,$

(ii) $\hat{e}_i = \hat{e} = \frac{1}{n}\frac{\beta(n-1)v}{\alpha(n-1)+n}, \forall i.$

PROOF. See the Appendix. □

In other words, when valuations are equal, $\beta \leq 1$ is a sufficient condition for the existence of a unique equilibrium, which is given by parts (i) and (ii) of Theorem 4.3.

We make the following remarks with respect to the equilibrium bids and efforts of Theorem 4.3. First, the bids and efforts are linear homogeneous in the valuation v. Second, if the returns-to-scale parameter of bids, α, increases, then the equilibrium bids strictly increase as well, whereas the equilibrium efforts strictly decrease. Third, if the returns-to-scale parameter of efforts, β, increases, then the equilibrium efforts strictly increase; however, there is no effect on the equilibrium bids. Fourth, in the equilibrium the probability that player i wins the prize equals $\hat{p}_i = 1/n$, whereas his expected profit equals

$$\widehat{\Pi}_i = \frac{1}{n}(v - \hat{b}) - \hat{e} = \frac{1}{n}\left(\frac{nv - \beta(n-1)v}{\alpha(n-1)+n}\right), \tag{17}$$

which is positive, because we assumed that $\beta \leq 1$.

Using Theorem 4.3, we can again study the extent to which rent is dissipated. To begin with, suppose that the winning bid is considered as dissipated rent. We then have from Theorem 4.3 that

$$D = n\hat{e} + \hat{b} = \frac{(\alpha+\beta)(n-1)v}{\alpha(n-1)+n}. \tag{18}$$

Consequently, with two contestants, total rent dissipation is $(\alpha+\beta)v/(\alpha+2)$. The extent of rent dissipation strictly increases to $(\alpha+\beta)\,v/(\alpha+1)$ as n goes to infinity. Rent dissipation strictly increases in α and β. In order to see that dissipation is strictly increasing in α, note that

$$\frac{\partial D}{\partial \alpha} = \frac{n - \beta(n-1)}{(\alpha n - \alpha + n)^2}\,(n-1)\,v, \tag{19}$$

which is positive, since by assumption $\beta \leq 1$.

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6 In the special case where $\beta = 1$, bids are again equal among agents, regardless of the size of α. This follows from (A.19) of the Appendix. The results given in section 3 can easily be generalized to this special case. However, if $\beta \neq 1$, bids are no longer equal among agents.

Next, suppose we do not count the winning bid as dissipated rent. From Theorem 4.3 we then obtain

$$D' = n\hat{e} = \frac{\beta(n-1)v}{\alpha(n-1)+n}. \tag{20}$$

With two contestants, total rent dissipation now equals $\beta v/(\alpha + 2)$. The extent of rent dissipation strictly increases to $\beta v/(\alpha + 1)$ as n goes to infinity. Rent dissipation strictly decreases in α, but strictly increases in β.

We conclude with two remarks. First, we see that total rent dissipation strictly increases in α if the winning bid is considered as dissipated rent, whereas total rent dissipation strictly decreases in α if the winning bid is not regarded as dissipated rent. Second, it is easy to verify that if the winning bid is considered as dissipated rent, then in equilibrium there is underdissipation of rent. Obviously, this conclusion then also holds if the winning bid is not considered as dissipated rent.

5. Conclusion

In this paper, we presented a model which combines elements of an auction and a rent-seeking contest. The model considers a situation in which players compete for a prize. The probability that a player wins the prize depends not only on the amount of effort exerted, but also on the bid submitted. The bid only has to be paid if the player wins the prize, the effort outlays are sunk.

First, we discussed the model with constant returns to scale in both bids and outlays. We presented a necessary and sufficient condition for the existence of a unique (Nash) equilibrium in which all players participate. We found that in the equilibrium all players will submit the same bid, regardless of their valuations, and that total outlays equal that bid. Moreover, we found underdissipation of rent, even if the winning bid is also considered as dissipated rent. For the two player case, we showed that the extent of total rent dissipation is strictly decreasing in the extent of asymmetry in valuations.

Second, we studied a more general model, in which the probability of success depends on a general Cobb-Douglas function in bids and efforts. For that model, we demonstrated that the sum of the individual ratios of the equilibrium effort and equilibrium bid is equal to the ratio of their respective returns-to-scale parameters. Focusing on the case of equal valuations, we showed that the model has an equilibrium if the returns-to-scale parameter of efforts is not greater than unity. We showed that in the equilibrium there is underdissipation of rent, even if the winning bid is also considered as dissipated rent. Total rent dissipation strictly increases in the returns-to-scale parameter of efforts. Finally, if the winning bid is considered as dissipated rent, then total rent dissipation strictly increases in the returns-to-scale parameter of bids, whereas if the winning bid is not considered as dissipated rent, total rent dissipation is strictly decreasing in this parameter.

Appendix: Proofs of Theorems 3.1, 4.2 and 4.3

Proof of Theorem 3.1

To begin with, we state the first-order conditions for an interior solution of the expected profit maximization problem of player i, given the bids b_j and efforts e_j $(j \neq i)$ of his rivals, i.e.

$$\frac{\partial \Pi_i}{\partial b_i} = \frac{\left(\sum_j b_j e_j\right) e_i \left(v_i - 2b_i\right) - b_i e_i^2 \left(v_i - b_i\right)}{\left(\sum_j b_j e_j\right)^2} = 0 \tag{A.1}$$

and

$$\frac{\partial \Pi_i}{\partial e_i} = \frac{\left(b_i \left(\sum_j b_j e_j\right) - b_i^2 e_i\right)\left(v_i - b_i\right)}{\left(\sum_j b_j e_j\right)^2} - 1 = 0. \tag{A.2}$$

Note that while stating these first-order conditions, we assume that $b_j > 0$ and $e_j > 0$ for at least one $j \neq i$.

Now, assume that $(\hat{b}_1, \ldots, \hat{b}_n, \hat{e}_1, \ldots, \hat{e}_n)$ is an equilibrium with $\hat{b}_i > 0$ and $\hat{e}_i > 0$, $\forall i$. We then have to show that $b(n) < v_n/2$ — where $b(n)$ is the root of $h_n(b)$ as defined in (5) — and that equilibrium bids and efforts satisfy parts (i), (ii) and (iii) of the theorem. Using the first-order conditions (A.1) and (A.2), in the equilibrium we must have $\hat{b}_i < v_i/2$, $\forall i$. Further, evaluated in the equilibrium, (A.1) implies that

$$\left(\sum_{j \neq i} \hat{b}_j \hat{e}_j\right)\left(v_i - \hat{b}_i\right) = \hat{b}_i \left(\sum_j \hat{b}_j \hat{e}_j\right), \tag{A.3}$$

whereas (A.2) yields

$$\left(\sum_{j \neq i} \hat{b}_j \hat{e}_j\right) \hat{b}_i \left(v_i - \hat{b}_i\right) = \left(\sum_j \hat{b}_j \hat{e}_j\right)^2. \tag{A.4}$$

Substituting (A.3) into (A.4) yields $\hat{b}_i^2 \left(\sum_j \hat{b}_j \hat{e}_j\right) = \left(\sum_j \hat{b}_j \hat{e}_j\right)^2$, thus

$$\hat{b}_i^2 = \sum_j \hat{b}_j \hat{e}_j. \tag{A.5}$$

The RHS of (A.5) is a constant, independent of i. Using this we can write $\hat{b}$ for the bid of each player. Hence, the condition $\hat{b}_i < v_i/2$, $\forall i$, reduces to $\hat{b} < v_n/2$. Further, from (A.5) we immediately have

$$\hat{b} = \sum_j \hat{e}_j. \tag{A.6}$$

In turn, using $\hat{b}_i = \hat{b}$, $\forall i$, and (A.6), (A.3) implies $\left(\hat{b}^2 - \hat{b}\hat{e}_i\right)(v_i - \hat{b}) = \hat{b}^3$, so we can solve $\hat{e}_i$, $\forall i$, as a function of $\hat{b}$,

$$\hat{e}_i = \frac{\hat{b}(v_i - 2\hat{b})}{(v_i - \hat{b})}. \tag{A.7}$$

Substituting (A.7) into (A.6) yields

$$\sum_j \left(\frac{v_j - 2\hat{b}}{v_j - \hat{b}}\right) = 1, \tag{A.8}$$

thus $\sum_j \left(1 - \frac{\hat{b}}{v_j - \hat{b}}\right) = 1$, or

$$\hat{b} \sum_j \left(\frac{1}{v_j - \hat{b}}\right) = n - 1. \tag{A.9}$$

From (A.9), $\hat{b}$ is a root of $h_n(b)$ of (5). Since $h_n(b)$ has a unique root, $\hat{b} = b(n)$, and we must have $b(n) < v_n/2$. Part (i) of the theorem is now obvious, and parts (ii) and (iii) follow from, respectively, (A.7) and (A.6).

Next, assume that $b(n) < v_n/2$. We then have to prove that there exists an equilibrium in which all players participate. We will show that such an equilibrium is given by $(\hat{b}_1, \ldots, \hat{b}_n, \hat{e}_1, \ldots, \hat{e}_n)$, where $\hat{b}_i = \hat{b} = b(n)$ and $\hat{e}_i = \hat{b}(v_i - 2\hat{b})/(v_i - \hat{b})$, $\forall i$. Remark that these bids and efforts satisfy $\hat{b}_i > 0$ and $\hat{e}_i > 0$, $\forall i$. It remains to be shown that each player i maximizes his expected profit by chosing $b_i = \hat{b}$ and $e_i = \hat{e}_i$, given the choices $\hat{b}_j$ and $\hat{e}_j$ $(j \neq i)$ of his rivals.

Consider the maximization problem faced by player i, given these choices of his rivals. First, notice that if player i chooses $b_i = 0$, then his corresponding optimal effort is equal to zero, and his expected profit amounts to zero. Second, if player i chooses $e_i = 0$, then his expected profit equals zero irrespective of the size of his bid. Third, examine positive bids b_i and positive efforts e_i of player i, which satisfy the first-order conditions (A.1) and (A.2). It is convenient to write $c_i = \sum_{j \neq i} \hat{b}_j \hat{e}_j$. Note that $c_i > 0$. Using (A.1) and (A.2), it follows that for such bids and efforts of player i we must have

$$c_i(v_i - b_i) = b_i(b_i e_i + c_i) \tag{A.10}$$

and

$$c_i b_i (v_i - b_i) = (b_i e_i + c_i)^2. \tag{A.11}$$

From (A.10) we directly obtain that $b_i < v_i/2$. Further, (A.10) and (A.11) imply that

$$b_i^3 = c_i(v_i - b_i) \tag{A.12}$$

and

$$e_i = \frac{c_i(v_i - 2b_i)}{b_i^2}. \tag{A.13}$$

Define the continuous auxiliary function $k_i(b) = b^3 - c_i(v_i - b)$ for $0 < b < v_i$. By assumption, $0 < \hat{b} < v_i/2$ and $\hat{e}_i > 0$. Further, note that the first-order conditions of player i are satisfied if he chooses $b_i = \hat{b}$ and $e_i = \hat{e}_i$. As a result, $b = \hat{b}$ must be a root of $k_i(b)$, i.e. $k_i(\hat{b}) = 0$. Moreover, one can easily verify that $k_i(b)$ has no other roots. By implication, besides $b_i = \hat{b}$ and $e_i = \hat{e}_i$, there exist no other positive bid b_i and positive effort e_i which satisfy the first-order conditions of player i. Finally, we observe that the expected profit of player i corresponding to $b_i = \hat{b}$ and $e_i = \hat{e}_i$ is equal to

$$\widehat{\Pi}_i = \left(\frac{\hat{e}_i}{\hat{b}}\right)\left(v_i - \hat{b}\right) - \hat{e}_i = \frac{(v_i - 2\hat{b})^2}{(v_i - \hat{b})} \tag{A.14}$$

which is clearly positive. As a result, player i indeed globally maximizes his expected profit by choosing $b_i = \hat{b}$ and $e_i = \hat{e}_i$. □

Next, we present the proofs of Theorems 4.2 and 4.3. We first make some preliminary remarks. The first-order conditions of an interior solution of the expected profit maximization problem of player i in the model of section 4, given the bids b_j and efforts e_j $(j \neq i)$ of his rivals, are given by

$$\frac{\partial \Pi_i}{\partial b_i} = \frac{\left(\sum_j b_j^{\alpha} e_j^{\beta}\right)\left(\alpha b_i^{\alpha-1} e_i^{\beta}(v_i - b_i) - b_i^{\alpha} e_i^{\beta}\right) - \alpha b_i^{2\alpha-1} e_i^{2\beta}(v_i - b_i)}{\left(\sum_j b_j^{\alpha} e_j^{\beta}\right)^2} = 0 \tag{A.15}$$

and

$$\frac{\partial \Pi_i}{\partial e_i} = \frac{\beta b_i^{\alpha} e_i^{\beta-1}\left(\sum_j b_j^{\alpha} e_j^{\beta}\right) - \beta b_i^{2\alpha} e_i^{2\beta-1}}{\left(\sum_j b_j^{\alpha} e_j^{\beta}\right)^2}(v_i - b_i) - 1 = 0, \tag{A.16}$$

where we assume that $b_j > 0$ and $e_j > 0$ for at least one $j \neq i$. We observe that (A.15) reduces to

$$\alpha\left(\sum_{j \neq i} b_j^{\alpha} e_j^{\beta}\right)(v_i - b_i) = b_i\left(\sum_j b_j^{\alpha} e_j^{\beta}\right), \tag{A.17}$$

whereas (A.16) reduces to

$$(v_i - b_i)\beta b_i^{\alpha} e_i^{\beta-1}\left(\sum_{j \neq i} b_j^{\alpha} e_j^{\beta}\right) = \left(\sum_j b_j^{\alpha} e_j^{\beta}\right)^2. \tag{A.18}$$

Using these conditions we present the proofs of Theorems 4.2 and 4.3.

Proof of Theorem 4.2

Suppose that $(\hat{b}_1, \ldots, \hat{b}_n, \hat{e}_1, \ldots, \hat{e}_n)$ is an equilibrium in which all n players participate. In the equilibrium each player's expected profit must be nonnegative, for otherwise the player will not participate in the contest. This implies that we must have $\hat{b}_i < v_i$, $\forall i$. We further know that the equilibrium must satisfy the first-order conditions (A.17) and (A.18).

Using this, we obtain

$$\hat{b}_i^{\alpha+1}\hat{e}_i^{\beta-1} = \frac{\alpha}{\beta}\left(\sum_{j=1}^{n}\hat{b}_j^{\alpha}\hat{e}_j^{\beta}\right). \tag{A.19}$$

Note that the RHS of this equality is a constant, independent of i. Thus, the products $\hat{b}_i^{\alpha+1}\hat{e}_i^{\beta-1}$ are a constant. As a result, (A.19) yields

$$\left(\sum_{j=1}^{n}\frac{\hat{e}_j}{\hat{b}_j}\right) = \frac{\beta}{\alpha}, \tag{A.20}$$

which completes the proof. □

Proof of Theorem 4.3

Assume that $v_i = v$, $\forall i$, and $\beta \leq 1$. We will show that there exists a unique equilibrium, which is given by parts (i) and (ii) of the theorem.

To begin with, we remark that the equilibrium must be symmetric, i.e. $b_i = b$ and $e_i = e$, $\forall i$. It is obvious that the situation with $b_i = b = 0$ and/or $e_i = e = 0$, $\forall i$, is not an equilibrium. Further, substituting $e_i = e > 0$ and $b_i = b > 0$, $\forall i$, into the n first-order conditions (A.17) and (A.18), it follows directly that $b = \hat{b}$ and $e = \hat{e}$, where

$$\hat{b} = \frac{\alpha(n-1)v}{\alpha(n-1)+n} \tag{A.21}$$

and

$$\hat{e} = \frac{1}{n}\frac{\beta(n-1)v}{\alpha(n-1)+n}. \tag{A.22}$$

This implies that there is only one possible equilibrium, i.e. $b_i = \hat{b}_i = \hat{b}$ and $e_i = \hat{e}_i = \hat{e}$, $\forall i$. In order to demonstrate that this indeed constitutes an equilibrium, we have to prove that player i maximizes his expected profit by choosing $b_i = \hat{b}$ and $e_i = \hat{e}$, given the choices $\hat{b}_j = \hat{b}$ and $\hat{e}_j = \hat{e}$ $(j \neq i)$ of his rivals.

Take the maximization problem faced by player i, given these choices of his rivals. First, we see that if player i chooses $b_i = 0$, then his corresponding optimal effort is zero, and hence his expected profit equals zero. Second, if player i chooses $e_i = 0$, then his expected profit is zero independent of the size of his bid. Third, let us examine positive bids b_i and positive efforts e_i of player i which satisfy the first-order conditions (A.17) and (A.18). In that case, (A.17) and (A.18) reduce to

$$\alpha d_i(v - b_i) = b_i(b_i^{\alpha}e_i^{\beta} + d_i) \tag{A.23}$$

and

$$d_i\beta b_i^{\alpha}e_i^{\beta-1}(v - b_i) = (b_i^{\alpha}e_i^{\beta} + d_i)^2, \tag{A.24}$$

where for notational convenience we have defined $d_i = \sum_{j\neq i}\hat{b}^{\alpha}\hat{e}^{\beta}$. Note that $d_i > 0$. It follows directly from (A.23) that we must have $b_i < \alpha v/(1+\alpha)$.

Observe that (A.23) and (A.24) are satisfied if player i chooses $b_i = \hat{b}$ and $e_i = \hat{e}$ (note that $0 < \hat{b} < \alpha v/(1+\alpha)$ and $\hat{e}_i > 0$). We further remark that the expected profit of player i corresponding to these choices equals

$$\widehat{\Pi}_i = \frac{1}{n}(v - \hat{b}) - \hat{e} = \frac{1}{n}\left(\frac{nv - \beta(n-1)v}{\alpha(n-1)+n}\right), \tag{A.25}$$

which is positive, since $\beta \leq 1$. The proof is completed if we show that besides $b_i = \hat{b}_i$ and $e_i = \hat{e}_i$, there exist for player i no other bid b_i with $0 < b_i < \alpha v/(1+\alpha)$ and effort $e_i > 0$, which satisfy (A.23) and (A.24). In order to show that, we distinguish two cases, i.e. $\beta = 1$ and $\beta < 1$.

First, take the case $\beta < 1$. From (A.23), we obtain that $e_i = s_i(b_i)$, where the continuous auxiliary function $s_i(b)$ is defined as

$$s_i(b) = \left(d_i(\alpha v - (\alpha+1)b)b^{-(\alpha+1)}\right)^{\frac{1}{\beta}}, \tag{A.26}$$

for $0 < b < \alpha v/(1+\alpha)$. Observe that $s_i(b)$ is strictly decreasing in b, and, moreover, that $\lim_{b\downarrow 0} s_i(b) = \infty$ and $\lim_{b\uparrow \frac{\alpha v}{(1+\alpha)}} s_i(b) = 0$.

Next, substitution of (A.23) into (A.24) yields

$$\beta b_i^{\alpha+1} e_i^{\beta-1} = \alpha(b_i^{\alpha} e_i^{\beta} + d_i), \tag{A.27}$$

which, in turn, with (A.23) implies that

$$\alpha^2 d_i(v - b_i) = \beta b_i^{\alpha+2} e_i^{\beta-1}. \tag{A.28}$$

The latter gives that $e_i = t_i(b_i)$, where the continuous auxiliary function $t_i(b)$ is defined as

$$t_i(b) = \left(\frac{\alpha^2 d_i}{\beta}(v-b)b^{-(\alpha+2)}\right)^{\frac{1}{\beta-1}}, \tag{A.29}$$

for $0 < b < \alpha v/(1+\alpha)$. Since in this case we have $\beta < 1$, $t_i(b)$ is strictly increasing in b, and $\lim_{b\downarrow 0} t_i(b) = 0$. As a result, the functions $s_i(b)$ and $t_i(b)$ have a unique point of intersection. By implication, this unique point of intersection is given by $b = \hat{b}$. It follows that for player i there exist a unique bid $0 < b_i < \alpha v/(1+\alpha)$ and a unique effort $e_i > 0$ which satisfy (A.23) and (A.24), i.e. $b_i = \hat{b}$ and $e_i = \hat{e}_i$.

Second, take the case $\beta = 1$. It then follows from (A.23) and (A.24) that

$$e_i = \frac{d_i(\alpha v - (\alpha+1)b_i)}{b_i^{\alpha+1}} \tag{A.30}$$

and

$$\alpha^2 d_i(v - b_i) = b_i^{\alpha+2}. \tag{A.31}$$

It is easy to verify that $b_i = \hat{b}$ is the unique solution of (A.31). In turn, we can conclude that for player i there exist a unique $0 < b_i < \alpha v/(1+\alpha)$ and a unique effort $e_i > 0$ which satisfy (A.23) and (A.24), i.e. $b_i = \hat{b}$ and $e_i = \hat{e}_i$. □

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