# ALM model for pension funds: numerical results for a prototype model

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## Abstract

A multistage mixed-integer stochastic programming model is formulated for an Asset Liability Management problem for pension funds. Since these models are too difficult to solve for realistically sized problems, a heuristic is described. Numerical results for several instances of a prototype model are presented and discussed.

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## 1. Introduction

Stochastic programming for planning under uncertainty provides a versatile tool for dynamic financial analysis, and its use in finance has been gaining popularity since the 1980s [0]. In Drijver et al. [0], we presented a multistage decision model of this type to support the Asset Liability Management (ALM) of a pension fund. Special attention is paid to the incorporation of risk measures that are relevant in the actual practise. Some features of risk measures give rise to the introduction of binary decision variables. Because of that, it is expected that realistically sized models cannot be solved to optimality. However, it is hoped that good feasible solutions can be found by the use of a heuristic.

In this discussion paper, we specify a prototype model of relatively small size. All data are fictitious, but realistic; they are chosen in cooperation with an existing pension fund. The size is such that it is possible to calculate the optimal solution. The results are used to discuss the validity and the usefulness of the model, and to get an idea about suitable parameter values. Moreover, we specify a heuristic, and give a first report on its quality.

The contents of this paper can be summarized as follows. For the readability, in Section 2, we repeat the description of the model in Drijver et al. [0]. Special attention is paid to specification of added features, such as the use of integrated chance constraints with the aim to exclude decisions that give rise to a too large risk of underfunding next year. Section 3 describes the heuristic. In Section 4 the prototype model is specified completely. We choose a planning period of 5 successive years, and restricted the scenario tree to two branches at every node, so that the number of scenarios is only 32. In some detail it is explained which numerical values we selected for these scenarios (in particular, the development of the returns on the 4 asset classes that we distinguish), and how we calculated corresponding values for the discount factors. Moreover, using these discount factors, a valuation of the liabilities in the scenarios is derived. Section 5 contains the numerical results.

## 2. ALM model for pension funds

A pension fund has the task of making benefit payments to participants who have ended their active income earning career. We assume that the pension fund has three sources of funding its liabilities: revenues from its asset portfolio, regular contributions made by the sponsor of the fund and remedial contributions made by the sponsor. The latter payments may be called for if the value of the assets is too low compared to the value of the liabilities. The pension fund has to decide periodically how to distribute the investments over different asset classes and what the contribution rate should be in order to meet all its obligations. This decision process is called Asset Liability Management.

A pension fund has long term obligations, up to decades, and therefore its planning horizon is large, too. The main goal of ALM is to find acceptable investment and contribution policies that guarantee that the *solvency* of the fund is sufficient during the planning horizon. Usually, the solvency at time $t$ is characterized by the *funding ratio* $F_t$ at time $t$, defined by $F_t := A_t/L_t$, where $A_t$ denotes the value of the assets and $L_t$ is the value of the liabilities.

In this section, we give a short description of our stochastic programming model for the ALM process of a pension fund. For more details we refer to Drijver et al. [0]. The model is dynamic, that is, information on the actual value of uncertain parameters is revealed in stages, and the decisions of any stage do depend on the known observations of them at that time, but not on unknown future realizations.

### 2.1 Scenarios and decisions

In this section we discuss basic elements of our ALM model. This is a finite-horizon, discrete time optimization model. It is assumed, that decisions on asset mix, contribution rate and remedial contributions are made once a year. These moments are denoted as $t = 0, 1, \ldots, T-1$, where $t = 0$ is the current decision moment and $T$ is the number of years in the planning horizon. The year $t$ ($t = 1, \ldots, T$) is then the time between decision moment $t-1$ and decision moment $t$.

Uncertainty is modeled through a finite number $S$ of scenarios. Each scenario represents a possible realization of all uncertain parameters in the model. Each scenario $s$ has a probability $p^s$, where $p^s > 0$ and $\sum_{s=1}^{S} p^s = 1$. Since in a dynamic model information on the actual value of the uncertain parameters is revealed in stages, a suitable representation of the set of scenarios is given by a scenario tree.

In our model formulation it is convenient to introduce a complete set of decision variables for each scenario separately. Since the decisions in all scenarios passing one node in the scenario tree should be the same, so-called *nonanticipativity* or *information constraints* have to be added, in order to guarantee that decisions do not depend on values of random parameters that will be revealed in later periods.

Let us now introduce the random parameters and the decision variables of the ALM model. For $t \in \mathcal{T}_1 := \{1, 2, \dots, T\}$, we define the vector of parameters that get their value during year $t$ as

$$\omega_t = (r_{1t}, r_{2t}, \dots, r_{Nt}, W_t, P_t, L_t),$$

where

| | | |
|---|---|---|
| $r_{it}$ | = | return on asset class $i$ in year $t$, $\quad i = 1, \dots, N$, |
| $W_t$ | = | total wages of active participants in year $t$, |
| $P_t$ | = | total benefit payments in year $t$, |
| $L_t$ | = | total value of liabilities after year $t$. |

Their realizations in scenario $s \in \mathcal{S}$ are denoted by

$$\omega_t^s = (r_{1t}^s, r_{2t}^s, \dots, r_{Nt}^s, W_t^s, P_t^s, L_t^s).$$

The decisions at time $t \in \mathcal{T}_0 := \{0, 1, \dots, T-1\}$ are for scenario $s \in \mathcal{S}$:

| | | |
|---|---|---|
| $Z_t^s$ | = | remedial contribution by the sponsor, |
| $\Delta^+ X_{it}^s$ | = | value of assets in class $i$ bought, $i = 1, \dots, N$, |
| $\Delta^- X_{it}^s$ | = | value of assets in class $i$ sold, $i = 1, \dots, N$, |
| $c_{t+1}^s$ | = | contribution rate for year $t+1$. |

At the time horizon $t = T$ only the decisions $Z_T^s$ occur. The following additional variables are important too. For each scenario $s$, and for $t \in \mathcal{T}_1$:

| | | |
|---|---|---|
| $A_t^s$ | = | total asset value at time $t$ |
| $X_{it}^s$ | = | value of investments in asset class $i$, at the beginning of year $t$ (after the adjustment of the portfolio at time $t-1$). |

These are state variables (together with $L_t^s$). They are determined by the parameters and the decision variables, but from an optimization point of view they are decision variables too, if one includes their definitions as constraints in the model, as we shall do. Next, we have to explain in more detail what we mean by 'time $t$' in the definition of $A_t^s$. We assume that at the end of year $t$, i.e., just before decision moment $t$, the realization of $\omega_t$ becomes available, the corresponding contribution $c_t^s W_t^s$ of year $t$ comes in, and the corresponding benefit payments $P_t$ of year $t$ are made. After this, the value of the assets is calculated, based on the revealed returns:

$$A_t^s = \sum_{i=1}^{N} (1 + r_{it}^s) X_{it}^s + c_t^s W_t^s - P_t^s, \quad s \in \mathcal{S}, \quad t \in \mathcal{T}_1.$$

It is compared with the actual value of the liabilities $L_t^s$, and then a possible remedial contribution $Z_t^s$ is added. Finally, all assets are (re)allocated (and the contribution rate $c_{t+1}^s$ of next year is chosen):

$$\sum_{i=1}^{N} X_{i,t+1}^s = A_t^s + Z_t^s, \, s \in \mathcal{S}, t \in \mathcal{T}_0,$$

with

$$X_{i,t+1}^s = (1 + r_{it}^s)X_{it}^s - \Delta^- X_{it}^s + \Delta^+ X_{it}^s - k_i(\Delta^- X_{it}^s + \Delta^+ X_{it}^s), \, s \in \mathcal{S}, t \in \mathcal{T}_0,$$

where $k_i$ denotes the proportional transaction cost for asset class $i$. In the last formula, for $t = 0$ the term $(1 + r_{it}^s)X_{it}^s$ is to be replaced by the parameter $X_{i0}$, denoting the initial position of asset class $i$, just before a possible remedial contribution $Z_0^s$ and possible reallocations $\Delta^+ X_{i0}^s$,$\Delta^- X_{i0}^s$. Similarly, the asset value at the same time is a given parameter $A_0$.

## 2.2 Basic constraints

In the previous subsection we defined the scenarios and the decision variables of the model. Here, we will discuss the constraints on the decision variables. Some constraints have already been mentioned before: the nonanticipativity constraints and the definitions of the state variables. In addition, nonnegativity is required for the value of each asset class $i$, and for restitutions and remedial contributions

$$\Delta^+ X_{it}^s \geq 0, \quad \Delta^- X_{it}^s \geq 0, \quad Z_t^s \geq 0, \quad s \in \mathcal{S}, t \in \mathcal{T}_0, i = 1, \ldots, N.$$

At the horizon, the remedial contributions should also be nonnegative, which is denoted by $Z_T^s \geq 0$. Since the sponsor is not willing to pay extremely large remedial contributions, an upper bound is given on this amount. This upper bound is defined as a fraction $\tau$ of the total level of the wages:

$$Z_t^s \leq \tau W_t^s \quad s \in \mathcal{S}, t = 0, 1, \ldots, T.$$

There are also lower and upper bounds for the asset mix:

$$w_i^l \sum_{j=1}^{N} X_{jt}^s \leq X_{it}^s \leq w_i^u \sum_{j=1}^{N} X_{jt}^s \quad s \in \mathcal{S}, t \in \mathcal{T}_1, i = 1, \ldots, N,$$

where $w_i^l$ and $w_i^u$ are parameters that specify upper and lower bounds on the value of asset class $i$ as a fraction of the total asset portfolio.

In addition, lower and upper bounds on the contribution rate are given by

$$c^l \leq c_t^s \leq c^u, \quad s \in \mathcal{S}, t \in \mathcal{T}_1,$$

where the numbers $c^l$ and $c^u$ have to be decided by the management of the pension fund.

The constraints described up to now are called basic constraints. There are more constraints in our model, however:

- Constraints on one-stage risks of underfunding.
- Constraints on remedial contributions.
- Soft constraints on (too) large changes in the contribution rate.

These constraints are described in Sections 2.3, 2.4 and 2.5, respectively. After that, in Section 2.6 the objective function is specified.

### 2.3 One-stage risk constraints

In order to avoid decisions, that are considered to be too risky, we introduce a set of (one-stage) risk constraints. *One-stage risk* is the risk of underfunding at time $t+1$ as seen at time $t$ (for any $t = 0, \ldots, T-1$), measured in a way to be described. The underlying idea is, that the management of the pension fund specifies a maximum acceptable value for the one-stage risk. The risk constraints enforce, that in the model only solutions are considered where all one-stage risks do not exceed the upper bound.

How to measure such risks? We assume, that the management of the pension fund specifies a value $\alpha$ for the funding ratio, in such a way that values below $\alpha$ are considered to be risky; they should be avoided if possible. The *shortage at level* $\alpha$ at any time is then the minimum amount by which the asset value of that moment must increase in order to have a funding ratio of at least $\alpha$. As risk measure we use the expected shortage at level $\alpha$, at time $t+1$, as calculated with data up to and including time $t$ ($t = 0, \ldots, T-1$). This definition leads to the following one-stage risk constraints:

$$[(A_{t+1} - \alpha L_{t+1})^- | (s, t)] \leq \beta, \quad s \in \mathcal{S}, t \in \mathcal{T}_0,$$

where

| | | |
|---|---|---|
| $A_{t+1}$ | = | asset value at time $t+1$, |
| $L_{t+1}$ | = | liability value at time $t+1$, |
| $(A_{t+1} - \alpha L_{t+1})^-$ | = | $\max(0, \alpha L_{t+1} - A_{t+1})$ |
| | = | shortage at level $\alpha$ at time $t+1$, |
| $\beta$ | = | maximum acceptable value of one-stage risk. |

By $[.|(s,t)]$ we mean that the expected value is calculated at time $t$, based on all information revealed in scenario $s$ from the beginning up to and including $t$. By writing out this expectation we get

$$\sum_{s'=1}^{S} p_{t+1}(s'|s,t)\big(A_{t+1}^{s'} - \alpha L_{t+1}^{s'}\big)^- \leq \beta.$$

Here, $p_{t+1}(s'|s,t)$ denotes the conditional probability that at time $t+1$ scenario $s'$ occurs, given that $(\omega_1, \omega_2, \ldots, \omega_t)$ coincides with scenario $s$; it can be calculated from the data. In this way, the risk constraints fit in a linear programming framework, since the only nonlinearity, caused by taking the negative part, can be eliminated by replacing $A_{t+1}^{s'} - \alpha L_{t+1}^{s'}$ by the difference of two nonnegative variables.

In Subsection  we explain in more detail why our choice of risk measure is suitable.

## 2.4 Constraints on remedial contributions

Our model adopts the following rules for remedial contributions from the sponsor:
- Remedial contributions can only occur at states of the world where the funding ratio is less than $\alpha$.
- If there is a positive remedial contribution, it should cover at least the actual shortage (at level $\alpha$).
- If there is a positive shortage at time $t$ at level $\alpha$, and if also at time $t-1$ there was a positive shortage at that level, then a remedial contribution has to be made.

These rules are formulated as linear constraints in the decision variables, after adding the following binary decision variables (for $s = 1, \ldots, S; \quad t = 0, \ldots, T$).

$$\delta_t^s := \begin{cases} 1 & \text{if } A_t^s < \alpha L_t^s \\ 0 & \text{if not} \end{cases}$$

$$d_t^s := \begin{cases} 1 & \text{if } Z_t^s > 0 \\ 0 & \text{if not.} \end{cases}$$

These binary variables get the correct values, because of the following 'definition inequalities'. For all $t = 0, \ldots, T$ and $s = 1, \ldots, S$, $\delta_t^s \in \{0, 1\}$, $d_t^s \in \{0, 1\}$ and $Z_t^s \geq 0$, and

$$A_t^s - \alpha L_t^s \geq -M\delta_t^s$$

$$A_t^s - \alpha L_t^s \leq M(1 - \delta_t^s) - \frac{1}{M}$$

$$Z_t^s \geq M(d_t^s - 1) - (A_t^s - \alpha L_t^s)$$

$$Z_t^s \leq M d_t^s$$

Here $M$ is a sufficiently large number. The third inequality also forces the remedial contribution to satisfy the second rule. The first and third rule are forced to hold by the conditions

$$d_t^s \leq \delta_t^s$$

$$d_t^s \geq \delta_t^s + \delta_{t-1}^s - 1$$

for $s \in \mathcal{S}$, $t = 0, \ldots, T$. Here $\delta_{-1}^s$ is a given parameter, just as $\delta_0^s$, not depending on $s$.

### 2.5 Soft constraints on changes of contribution rates

Often it is undesirable that the contribution rate changes too fast. Our model gives the possibility to pay attention to this idea, in the following way. If the increase of the contribution rate in two consecutive years is larger then $\rho$, a parameter to be determined by the management of the pension fund, it is penalized in the objective function. Similarly this is done for any decrease larger than $\rho$, but its penalty is different. That is, with $\lambda_{c+}$ ($\lambda_{c-}$) the penalty parameter for a too fast increase (too fast decrease) of the contribution rate, the incurred penalty cost at time $t$ in scenario $s$ are

$$\left(\lambda_{c+}(c_t^s - c_{t-1}^s - \rho)^+ + \lambda_{c-}(c_t^s - c_{t-1}^s + \rho)^+\right) W_t^s,$$

where, as before, the positive (negative) part operator is defined as $(x)^+ = \max(0, x)$, $(x)^- = \max(0, -x)$. By introducing additional decision variables, these nonlinearities can be removed easily from the formulation.

### 2.6 Objective function

Usually, a pension fund has many goals. Moreover, different parties have different interests. In our model, this phenomenon is reflected in the objective function, which is going to be minimized over all strategies for contribution, asset mix, and remedial contribution that satisfy the constraints. It consists of the expected total discounted funding costs, to which penalties are added for undesirable situations (and rewards for desirable situations). The parameters in the penalties do not have a financial meaning necessarily, but one can play with them in order to generate solutions from the model corresponding to various weights on different goals.

The objective function is

$$
\begin{aligned}
&\sum_{s=1}^{S}\sum_{t=1}^{T} p^s\gamma_t^s(c_t^s W_t^s + Z_t^s) \\
&+\sum_{s=1}^{S}\sum_{t=1}^{T} p^s\gamma_t^s\big(\lambda_{c+}(c_t^s - c_{t-1}^s - \rho)^+ + \lambda_{c-}(c_t^s - c_{t-1}^s + \rho)^+\big)W_t^s \\
&+\sum_{s=1}^{S}\sum_{t=0}^{T} p^s\gamma_t^s\big(\lambda_\delta \delta_t^s\big) \\
&+\sum_{s=1}^{S}\sum_{t=0}^{T} p^s\gamma_t^s\big(\lambda_d d_t^s + (\lambda_Z - 1)Z_t^s\big) \\
&+\sum_{s=1}^{S} p^s\gamma_T^s\big(\lambda_\theta(A_T^s - \theta L_T^s)^- + \lambda_\xi(A_T^s - \xi L_T^s)^+\big).
\end{aligned}
$$

Here, $p^s$ is the probability of scenario $s$, $\gamma_t^s$ is the discount factor for a cash flow at time $t$ in scenario $s$. The first term gives the funding cost, and the second one corresponds to the penalties for rapidly changing contribution rates. The third term indicates that at each node in the scenario tree a cost of $\lambda_\delta$ is incurred if the funding ratio is less than $\alpha$. The fourth term gives the expected total discounted penalties for remedial contributions: each time it occurs, the fixed $\lambda_d$ is incurred, together with a variable penalty with rate $(\lambda_Z - 1)$. The last term penalizes a terminal funding ratio less than $\theta$, and rewards a terminal funding ratio larger than $\xi$ (usually, $\theta \leq \alpha \leq \xi$, $\lambda_\theta > 0$, $\lambda_\xi < 0$).

Although transactions costs can also be seen as funding costs, they are not included in the objective function. They appear in the costraints, such that buying and selling leads to a lower asset value.

## 2.7 One-stage risk constraints: background

We now discuss the conceptual and mathematical background of the one-stage risk constraints introduced in Section 2.3. As far as we know, such constraints have not been used before in related ALM models. Dert [0] used one-stage chance constraints: only the probability of shortage next year is restricted in the model. In our specification of one-stage risk, not only the probability, but also the amount of shortage counts: it is a special case of integrated chance constraints, introduced by W.K. Klein Haneveld [0]. In Subsection 2.7.1 we discuss Dert's chance constraints and their relation to our risk constraint. Next, we make clear that integrated chance constraints fit nicely in a linear programming framework.

### 2.7.1 Chance constraints and integrated chance constraints

Underfunding is an undesirable event. However, it cannot always be avoided. Since it is undesirable, we would like that the probability of underfunding in time period $t + 1$, given the state of the world at time $t$, is sufficiently low. That is,

$$P\{A^s_{t+1} < \alpha L^s_{t+1}\} \leq 1 - \phi_t,$$

where $\phi_t$ gives the minimum required reliability.

In stochastic programming, these type of constraints are called chance constraints. Dert [0] used chance constraints in an ALM model. If discrete distributions are considered, one can add chance constraints in a mixed-integer program using binary variables and linear inequalities. At time $t$, we observe the realization of $\omega_t$, and therefore know the actual state of affairs $(t, s)$. Defining $p_{t+1}(s'|(t, s))$ as the conditional probability that scenario $s'$ occurs at time $t + 1$, given the state of affairs $(t, s)$, the chance constraints can be written as

$$M\delta^{s'}_{t+1} \geq \alpha L^{s'}_{t+1} - A^{s'}_{t+1}, \quad s' \in \mathcal{S}, t \in \mathcal{T}_0 \tag{1}$$

$$\sum_{s'=1}^{S} p_{t+1}(s'|(t, s))\delta^{s'}_{t+1} \leq 1 - \phi_t, \quad s' \in \mathcal{S}, t \in \mathcal{T}_0. \tag{2}$$

Here, $M$ is a sufficiently large number, and $\delta^{s'}_{t+1} \in \{0, 1\}$.

From algorithmic point of view, continuous variables are to be prefered to discrete ones. So it is interesting to find out, what the interpretation of the constraints (1) and (2) will be if the binary variables $\delta^{s'}_{t+1} \in \{0, 1\}$ are relaxed to continuous variables $\delta^{s'}_{t+1} \in [0, 1]$. Condition (1), together with nonnegativity, gives the following lower bound for $\delta^{s'}_{t+1}$:

$$\delta^{s'}_{t+1} \geq (A^{s'}_{t+1} - \alpha L^{s'}_{t+1})^{-}/M \quad s' \in \mathcal{S}, t \in \mathcal{T}_0.$$

Assuming that this lower bound is binding (small values for $\delta^{s}_{t+1}$ are prefered in the objective function), we conclude that the relaxation of (1) and (2) becomes

$$\sum_{s'=1}^{S} p_{t+1}(s'|(t, s))\big(\alpha L^{s'}_{t+1} - A^{s'}_{t+1}\big)^{+} \leq M(1 - \phi_t). \tag{3}$$

This last expression is simply

$${}_{t,s}\Big[\big(\alpha L_{t+1} - A_{t+1}\big)^{+}\Big] \leq \beta, \tag{4}$$

for appropriately chosen $\beta$. These constraints are called integrated chance constraints (ICC). We refer to W.K. Klein Haneveld [0] for details on ICC. Note that the right hand side of (4) also has an interpretation: $\beta$ gives the maximum acceptable expected shortage for the next time period.

Constraint (4) can be used in a linear programming framework by introducing additional decision variables. These are denoted by the nonnegative, continuous variables $G^s_t$ and $H^s_t$. These measure the amount of shortage and surplus in state of the world $(t, s)$ respectively. Adding the constraints

$$A^s_t + G^s_t - H^s_t = \alpha L^s_t \quad s \in \mathcal{S}, t = 0, 1, \ldots, T,$$

the integrated chance constraints (4) can be written as

$$\sum_{s'=1}^{S} p_{t+1}(s'|(t, s))G^s_t \leq \beta \quad s \in \mathcal{S}, t = 0, 1, \ldots, T.$$

The inequalities above define convex, polyhedral feasibility sets. These are very attractive from an optimization point of view. Since the constraints defining the integrated chance constraints are all linear, they can be used in a linear programming framework. The properties of integrated chance constraints are to be preferred over those of the chance constraints, not only from a mathematical point of view. Integrated chance constraints are also preferred over chance constraints here, since not only probabilities of underfunding are considered, but also amounts of shortage. Therefore, we will use integrated chance constraints 4 in our ALM model.

## 3. Heuristic

From computational point of view, the multistage stochastic ALM model described in the previous section is just a deterministic mixed-integer linear programming model, basically. This is due to the fact, that uncertainty and time are modeled as a finite number of scenarios and years, respectively. For mixed-integer linear programming, powerful software exists. In Section 5 we will discuss the optimal solution of a prototype model (to be described in Section 4) that we got by using AIMMS with solver XA. However, the size of the deterministic equivalent model is growing extremely fast with the number of scenarios $S$, and the number of years $T$ in the planning horizon. Since we have decisions in each state of the world $(t, s)$, the total number of states is equal to $S(T+1)$. If each state before the horizon has for example ten child nodes, and the horizon is split in 5 time periods, the total number of decision nodes is equal to 600,000.

Therefore, it is to be expected that for realistically sized ALM problems, even the best available software will not provide the optimal solution in a reasonable time. For that reason, we developed a heuristic, that is able to provide a (hopefully good but not necessarily optimal) feasible solution of the model. The heuristic uses linear programming relaxation (that is: replace binary variables by continuous variables in $[0, 1]$); so we assume that the size of the model is such, that the optimal solution of the model can be found if all binary variables are allowed to take fractional values too.

The main ideas of the heuristic will be described first. Then, this heuristic will be presented. Finally, some comments will be made about the heuristic.

### 3.1 Main idea

Suppose that a feasible solution of the ALM model can be found. How to find a better solution, with a lower value of the objective function? Suppose that a remedial contribution has to be made in two consecutive years. The main idea of the heuristic is then, to analyze whether it is advantageous to shift remedial contributions to an earlier time moment. The idea of shifting decisions in a multistage mixed-integer stochastic program is also used by Ahmed and Sahinidis [0] in a capacity expansion model.

Increasing remedial contributions may be advantageous, since this may lead to less states of the world in which these contributions have to be made. As a result, this leads to a lower contribution of the fixed costs to the objective function. On the other hand, such increases lead to earlier payments, which are therefore more expensive. The relationship between the fixed costs and the variable costs play a crucial role in determining whether it is beneficial to increase a remedial contribution or not.

Before we will discuss the consequences of shifting in more detail, we first give a description of the heuristic.

### 3.2 Description of the heuristic

The heuristic consists of five steps. These steps will now be explained.

Step 1

Solve the linear programming relaxation of the multistage mixed-integer stochastic program. Relaxation means that the requirements $\delta_t^s \in \{0, 1\}$ and $d_t^s \in \{0, 1\}$ are replaced by $\delta_t^s \in [0, 1]$ and $d_t^s \in [0, 1]$, respectively.

- If the linear programming relaxation is infeasible: *STOP*. The original ALM model is infeasible, too.
- If the linear programming relaxation has integer optimal values for all $\delta_t^s$ and all $d_t^s$: *STOP*. The solution found is also an optimal solution for the original ALM model.
- If the linear programming relaxation has an optimal solution, but some of the variables $\delta_t^s$ and/or $d_t^s$ are non-integer, go to step 2.

Step 2

In this step, the optimal solution of the linear programming relaxation (having some fractional $\delta_t^s$ and/or $d_t^s$) is transformed into a feasible solution of the original mixed-integer problem by only changing some of the variables $\delta_t^s$, $d_t^s$ and/or $Z_t^s$ as follows.

First, all binary variables $\delta_t^s$ are set to their correct binary value. If $\delta_t^s$ and $\delta_{t-1}^s = 1$, or $Z_t^s > 0$, we set $d_t^s = 1$. Then, the following procedure is applied:

for $t = 0$ to $T$ do
    for $s = 1$ to $S$ do
        if $d_t^s = 1$ then
            $Z_t^s = \alpha L_t^s - A_t^s$
            update asset values
            update $\delta$s and $d$s
        endif
    endfor
endfor

In this forward procedure, the funding ratio is restored to its minimum required level $\alpha$ if the sponsor of the fund has to make a remedial contribution. A cash inflow has consequences for all future states of the world. If the remedial contribution is made in state of the world $(t, s)$, all asset values in states with $(t, s)$ as parent are updated. This is done by keeping contribution rates fixed as found in step 1 of the heuristic. Also the fractions of assets in the asset portfolio are not changed. Due to the increases in the asset values, it may happen that funding ratios increase to values above $\alpha$, while this ratio was strictly less than $\alpha$ before this step. If this is the case, we set the corresponding $\delta$ equal to zero. As a result, it may also happen that a remedial contribution is not necessary any more. Therefore, also the values of $d$ are changed to zero, if this is necessary.

Step 3

Check if we can improve the solution constructed in step 2. This is done by checking whether the value of the objective function will decrease if a remedial contribution will be increased, such that in at least one child node a remedial contribution is not necessary anymore. Formally, we apply the following procedure:

for $t = 0$ to $T - 1$ do
    for $s = 1$ to $S$ do
        if $Z_t^s > 0$ and $Z_{t+1}^s > 0$ then
            if increase in $Z_t^s$ is profitable and possible
                update asset values

update $Z$s accordingly
update $\delta$s and $d$s
endif
endif
endfor
endfor

To check whether the increase in $Z_t^s$, denoted by $\Delta Z_t^s$, is profitable, we compare the costs of this increase with the corresponding returns. The costs are $p^s \gamma_t^s \lambda_Z \Delta Z_t^s$, where, as before, $p^s$ denotes the probability of scenario $s$ and $\gamma_t^s$ is the discount rate for state of the world $(t, s)$. On the other hand, an increase $\Delta Z_t^s$ may lead to a decrease of the objective function value, due to one or more of the following effects:

- In all child nodes with $(t, s)$ as parent, the asset values are increased. This is done in the same way as explained in step 2. As a result, some values of $\delta$ may change from 1 to 0, and the corresponding fixed costs are removed.
- The same argument holds for $d_t^s$, the binary variable which indicates whether a remedial contribution has to be made or not.
- There may be some child nodes in which we still have $Z > 0$, although the asset values in these states are increased. However, the amount of the remedial contribution is decreased. This also lowers the value of the objective function.
- Surplusses at the horizon are rewarded if the funding ratio is above $\xi$. Due to the increases in the asset values, the objective function decreases.
- Shortages at the horizon are penalized if the funding ratio is below $\theta$. Since asset values are increased, this may also lead to lower penalties due to underfunding with respect to the level $\xi$ at the horizon.

Suppose in state $(t, s)$ of the current solution, we observe $Z_t^s > 0$. If there is only one child node in which $Z > 0$, we check whether it is profitable to increase $Z_t^s$ such that the funding ratio is equal to $\alpha$ in this child node. If there are more child nodes in which the sponsor of the fund has to make a remedial contribution, we consider all increases in the remedial contribution in state of the world $(t, s)$ such that the funding ratio is equal to $\alpha$ in one child node. The increase (if any) which leads to the lowest value of the objective function is finally chosen.

In the process of checking whether it is profitable to increase a remedial contribution, we take into account the upper bound on this amount. Since we have the constraint $Z_t^s \leq \tau W_t^s$ in our ALM model, we check whether this constraint is not violated by increasing $Z_t^s$.

If it is profitable to increase the remedial contribution in state of the world $(t, s)$, we update the asset values and also the values of $\delta$ and $d$. This is done in the same way as explained in step 2 of the heuristic.

Step 4

Solve the multistage mixed-integer stochastic program with $\delta_t^s$ and $d_t^s$ fixed, as found in the previous step of the heuristic.

### 3.3 Comments on the heuristic

In this subsection, we will make some additional remarks on the heuristic as described above.

The linear programming relaxation in the first step of the heuristic is a multistage stochastic linear program. For realistically sized ALM problems, solving it may already require a lot of work.

In general, we do not expect to find a feasible solution for the mixed-integer model after the first step. However, if we obtain a feasible solution here, it is also an optimal solution. It may for example be the case that an optimal solution is found if the funding ratio at time $t = 0$ is very high, and, as a result, in the next $T$ years underfunding never occurs.

If no feasible solution is found after this first step, we will try to find one in the second step. The result of this step may be that we cannot find a feasible solution. This may be the case if a remedial contribution $Z$ should be greater than allowed by its upper bound, to restore the funding ratio to its minimum required level $\alpha$.

If we obtain a feasible solution in step 3, we check whether this can be improved by shifting remedial contributions to an earlier time period, as described above. We consider only a finite number of potential increases in the remedial contribution and take the best of these values. Maybe a larger amount leads to an even lower value of the objective function. The increase in the remedial contribution is not optimized, since in that case we have to solve a multistage mixed-integer stochastic program, which is too difficult to solve in general.

In step 4 of the heuristic, again a multistage stochastic linear program is solved. The reason to resolve the model is that the current portfolio decisions an the levels of the contribution rates are presumably non-optimal, given the fixed values of $\delta$ and $d$. For states $(t, s)$ in which we had $\delta_t^s = 0$, we get the hard constraint $A_t^s \geq \alpha L_t^s$, indicating that for these states, underfunding is not allowed.

If we have found a feasible solution after step 2 of the heuristic, we obtain a feasible solution after steps 3 and 4. That is, the result of the heuristic is a feasible solution.

The following example is intended to clarify the central step of the heuristic.

### 3.4 Example step 3 of heuristic

In this example, we will show how step 3 of heuristic works. It is simplified in the sense that we do not take into account discounting or an upper bound on the remedial contributions. Consider the following (small) part of the scenario tree, which is depicted in Figure . There are two scenarios, denoted by 1 and 2. We assume that both scenarios have equal probabilities. Since the nonanticipativity constraints have to be satisfied, the decisions at time 1 coincide in this example.

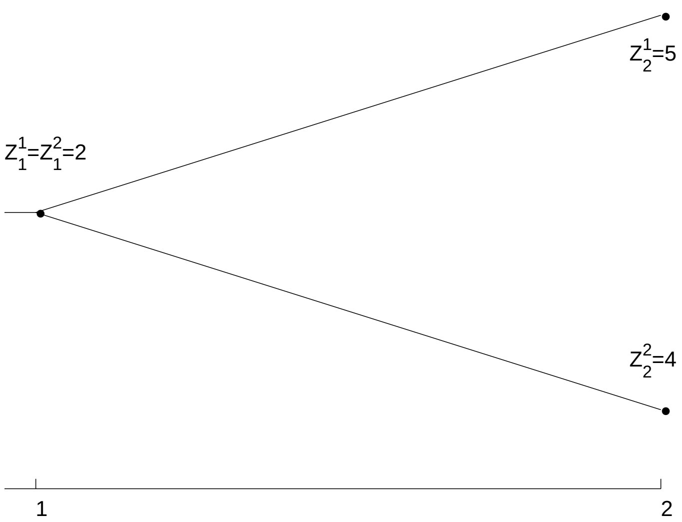


Figure 3.1: Small part of scenario tree to demonstrate idea heuristic.

The sponsor of the fund has to make a remedial contribution at time 1 and in the two states of the world at time 2. This is the solution after step 2 of the heuristic.

We assume in addition that the penalty parameters are $\lambda_Z = 1$, $\lambda_\delta = 2$ and $\lambda_d = 5$.

The relevant returns on the portfolios in the last time period are 3 and 2 percent for scenarios 1 and 2 respectively.

Since we have $Z_t^s > 0$ for all $s = 1, 2$ and $t = 1, 2$, we also have $\delta_t^s = d_t^s = 1$ for $s = 1, 2$ and $t = 1, 2$. The contribution to the objective function due to underfunding and making remedial contributions is the following. At time 1, the contribution is equal to $\lambda_\delta + \lambda_d + \lambda_Z Z = 2 + 5 + 2 = 9$. At time 2 we have in scenario 1 a contribution of $\frac{1}{2}$ (2+5+5)=6 and in scenario 2 of $\frac{1}{2}$ (2+5+4)=5$\frac{1}{2}$. Therefore, the total contribution to the objective function after step 2 of the heuristic is equal to 9+20$\frac{1}{2}$.

In step 3 of the heuristic, two increases in the remedial contribution at time 1 are considered. This contribution should be increased with $\frac{4}{1.02} \approx 3.92$ to remove $Z_2^2$, and it should be increased with $\frac{5}{1.03} \approx 4.85$ to remove $Z_2^1$.

If the remedial contribution is increased with 3.92 at time 1, the contribution to the objective function is equal to 9+3.92=12.92 at time 1 and $\frac{1}{2}$ (2+5+(5-(3.92 $\times$ 1.03)))$\approx$ 3.98 at time 2. The remedial contribution is reduced to approximately 0.96 in scenario 1, due to the increase in the asset value at time 1. The total contribution to the objective function is therefore 12.92+3.98=16.9. Since the value of the objective function is decreased, this shift is advantageous.

We will check whether is advantageous to increase this remedial contribution slightly more, such that we also have $Z_2^1 = 0$. Therefore, we have to increase $Z_1^1$ further with $\frac{0.96}{1.03} \approx 0.93$. As a result, the contribution to the objective function is in this case equal to 9+3.92+0.93=13.85. Since the value of the objective function is decreased again, and since the remedial contributions are equal to 0 in both states at time 2, this is the solution obtained from the heuristic after step 3.

## 4. Prototype model

In this section, we specify a relatively small instance of the ALM model of Section 2. The data are fictitious, but supposed to be semi-realistic; the size is sufficiently small, so that the optimal solution can be found by a mixed-integer linear programming (LP) code.

We use this so-called *prototype* model for various purposes. These will be explained in the next section, where also preliminary numerical results will be presented. The current section deals with the specification of the prototype model.

In the prototype model, we have a planning horizon of $T = 5$ years and $S = 2^T =$

32 scenarios. Each state of the world before the horizon has 2 child nodes, and the conditional probabilities are both $\frac{1}{2}$.

The prototype model has 5,672 constraints, 4,033 decision variables, and 17,962 non-zero coefficients. The number of binary variables is equal to $2(T+1)S = 384$. This follows, since the funding ratio is considered at $(T+1)$ time periods and in all $S$ scenarios, and we have 2 binary variables in each state of the world, the total number of binary variables equals $2(T+1)S$. However, due to the nonanticipativity constraints, only 126 of them are free variables.

## 4.1 Data

In the prototype model, we consider four asset classes: stocks, (10-year) bonds, real estate and cash. These four asset classes are considered by the management of the pension fund we cooperate with. Lower and upper bounds on the fractions of the various asset classes in the asset portfolio, and on the contribution rate, together with the values for the transaction costs for asset class $i$, denoted by $k_i$, are presented in table . The management of a pension fund provided us with these data.

| asset class | $i$ | lower bound | upper bound | $k_i$ |
|---|---|---|---|---|
| stocks | 1 | 0.45 | 0.65 | 0.00425 |
| bonds | 2 | 0.24 | 0.44 | 0.0015 |
| real estate | 3 | 0.06 | 0.16 | 0.00425 |
| cash | 4 | 0 | 0.05 | 0.0005 |
| contribution rate | | 0 | 0.21 | |

Table 4.1: Lower and upper bounds and transaction costs.

In the scenario tree, numerical values for $\omega_t^s = (r_{1t}^s, \ldots, r_{4t}^s, W_t^s, P_t^s, L_t^s)$ are required. In addition, the discount factor $\gamma_t^s$, associated with state $(t, s)$, has to be specified. In the next subsections will be described how the scenarios are generated.

## 4.2 Scenario generation: methods

Ultimately, the goal of scenario generation is to get realizations for the uncertain parameters in each node of the scenario tree. These realizations have to satisfy certain criteria, as explained below.

First, it will be explained how the relationships between the uncertain parameters are defined. Once these relationships are defined, we will use them to get realizations for the parameters in all states of the world in the scenario tree. We will find realizations for the returns on the different asset classes and for the increase in the general wage level. Then, special attention is paid to the valuation of the liabilities. This is necessary to understand how the scenarios are generated for the liabilities. As a result of finding the realizations for the uncertain future value of the liabilities, we also come to the question of how we can get realizations for $P_t^s$, the level of the benefit payments in each state of the world $(t, s)$. Finally, numerical values for $p_t^s$ and $\gamma_t^s$ are found.

### 4.2.1 Vector Autoregressive Model

In this subsection, the relationship between the change in the general wage level, denoted by $v_t^s$, and the returns on stocks, bonds, real estate, and deposits in a bank account will be modeled. These returns are denoted by $r_{it}^s$, where, as before, $i$ denotes the asset class, and $t$ and $s$ are the indices for time and scenario respectively. If the values for $v_t^s$ are found, the numerical values for $W_t^s$, the level of the wages at time $t$ in scenario $s$, is given by

$$W_t^s = W_0 \Pi_{q=1}^t (1 + v_q^s),$$

where $W_0$ is the level of the wages at time 0.

In the literature, it is customary to model the relationships between these returns as a first order vector autoregressive (VAR) model, see for example Sims [0], Boender et al. [0], Dert [0], and Kouwenberg [0]. Kouwenberg [0] uses only the returns which we actually need in each state of the world in our scenario tree, while Boender et al. [0] also use other macroeconomic quantities, like nominal GNP growth, to model these relationships.

Defining $h_t = (h_{1t}, \ldots, h_{N+1,t})$, where

$$h_{it} = \ln(1 + r_{it}) \quad i = 1, \ldots, N, \quad t = 1, \ldots, T$$

and

$$h_{N+1,t} = \ln(1 + v_t) \quad t = 1, \ldots, T,$$

Kouwenbergs VAR model can be written as follows:

$$h_t = a + \Omega h_{t-1} + \epsilon_t \quad \epsilon_t \sim N(0, \Sigma), \quad t = 1, \ldots, T. \tag{5}$$

Here, $h_t$ is a vector of continuously compounded returns and $\epsilon_t$ is an $(N+1)$-vector of independently and identically distributed error terms. It is assumed that these errors are normally distributed with mean zero and covariance matrix $\Sigma$. The periods considered are all one year. The vector $a$ and the matrix $\Omega$ are to be estimated. This can be done using econometric techniques. Lagged variables are only used for the changes in the wage level and the returns on the bank account. This is done, since otherwise returns on stocks, bonds and real estate are predictable, which is in contradiction with what is generally believed, see for example Campbell et al. [0]. To get these numerical values for $a$, $\Omega$ and $\Sigma$, we used the VAR model estimated by Kouwenberg [0]. As starting values for $h_{i0}, i = 1, \ldots, 5$, the mean values of the returns and the mean increase in the general wage level are used.

As we will see in the next subsections, these 5 random variables are used, together with the stream of future benefit payments, to generate scenarios for all random parameters.

### 4.2.2 Generating error terms

In this subsection will be described how the scenarios for the returns on the four asset classes and the change in the general wage level are generated. We assume that the scenario tree is specified in advance, that is, for each state of the world, we know the number of child nodes in advance. An alternative method is described in King and Warden [0]; they convert isolated scenarios into a tree structure.

We assume that the relation between the uncertain parameters can be described by the VAR model (5). We will find future realizations of the parameters by finding appropriate values for these error terms. Once we have found these value for $\epsilon_t^s$, $r_{it}^s$ and $v_t^s$ are found by means of (5).

Moment fitting is used to find numerical values for the disturbance terms in the VAR model (5). To that end, define the following minimization problem:

$$\min_{\epsilon \in \mathbb{R}^5} \|\epsilon\epsilon^T - \hat{\Sigma}\|, \tag{6}$$

with $\|S\| = \sum_{i=1}^5 \sum_{j=i}^5 S_{ij}^2$. Its optimal solution $\epsilon^*$ determines the symmetric two point distribution $P\{\epsilon_t = \epsilon^*\} = P\{\epsilon_t = -\epsilon^*\} = \frac{1}{2}$, with the best fitting second order moment matrix (in the sense of the norm: all variances and covariances are weighted equally).

Mathematical program (6) is solved only once, since each state of the world has two child nodes. This does have consequences, since dependencies between the error terms are introduced. Once we have found numerical values for $(\epsilon_i^s)^*$, $i = 1, \ldots, 5$, $s = 1, 2$, we use them to find returns on the asset classes and the changes in the general wage level for every time period by means of the VAR model (5).

### 4.2.3 Other random parameters

In this subsection will be described how to get values for $L_t^s$ and $P_t^s$, the value of the liabilities and the benefit payments at time $t$ in scenario $s$ respectively. In addition, it is described how we found values for $p_t^s$ and $\gamma_t^s$.

Valuing liabilities is done by finding the present value of all future benefit payments. We used the results of H.A. Klein Haneveld [0] to find appropriate discount factors. Therefore, we have to specify a zero-coupon yield curve and an equity spot curve to find the pension spot curve, the set of discount factors used in finding values for $L_t^s$.

### 4.2.4 Probabilities and discount rates

In the prototype model considered, we assume that the conditional probabilities are a half in each node of the scenario tree. Therefore, probabilities of all scenarios are equal. As a result, we have

$$p^s = \frac{1}{32}, \quad s \in \mathcal{S}.$$

In our ALM model, all cash flows are discounted by the risk-free interest rate, using compounding. That is,

$$\gamma_t^s = \left(\Pi_{q=1}^t (1 + r_{4q}^s)\right)^{-1}.$$

### 4.2.5 Generating scenarios for the liabilities and benefit payments

First the returns for the four asset classes and the increase in the general wage level were generated, using the VAR model (5) in each node of the scenario tree. Therefore, we have already the risk free interest rate, denoted by $r_{4t}^s$, and the return on a 10 year bond, denoted by $r_{2t}^s$. These two data points will be used to estimate the zero coupon yield curve. The following characteristics will be satisfied in finding the zero coupon yield curves:

- The zero coupon yield curve should contain only positive zero coupon yields.

- The forward rates implied by the estimated zero coupon yield curve should all be nonnegative.
- The zero coupon yield curve is monotonically increasing and concave.

This last characteristic is used, since Haugen [0] concludes that upward sloping term structures are far more common than other shapes. In addition, some shapes of the bond spot curve may be a priori implausible. There exists a lot of literature about the shape of term structures and estimation techniques. Famous term structure models are for example those by Vasicek [0], Cox et al. [0], Ho and Lee [0] and Hull and White [0].

We want to satisfy the three characteristics above to find the zero coupon yield curve. We use $r_{4t}^s$ as a point on the yield curve. In general, we cannot use $r_{2t}^s$ as an estimate for the ten year zero coupon yield. For example, if the return on this bond would be negative, this estimate would not make sense. Campbell et al. [0] conclude that the average spread of the ten year zero coupon yield over the one year zero coupon yield is $\mu_b = 1.367$ percent, with a standard deviation of $\sigma_b = 1.237$ percent.

The ten year zero coupon yield used to estimate the yield curve in state $(t, s)$ is denoted by $f_t^s$. We define $f_t^s$ as follows:

$$f_t^s := \begin{cases} r_{2t}^s & \text{if } r_{2t}^s \in [r_2^s, \mu_b + 2\sigma_b]; \\ \max\{r_{4t}^s, r_{4t}^s + \nu_t^s\} & \text{otherwise,} \end{cases}$$

where $\nu_t^s$ is a random drawing from a normal distribution with the mean and standard deviation of the spread as defined above.

The zero coupon yield curve in state of the world $(t, s)$ corresponding to maturity $q$, is denoted by $b_t^s(q)$. We define $b_t^s(q)$ as follows:

$$b_t^s(q) = r_{4t}^s + e_t^s \ln(q + 1),$$

where $r_{4t}^s$ is the risk free interest rate in state $(t, s)$ and $e_t^s$ is a parameter. Since we want to use $f_t^s$ as an estimate for the ten-year yield, we have

$$f_t^s = r_{4t}^s + e_t^s \ln(q + 1),$$

or

$$e_t^s = \frac{f_t^s - r_{4t}^s}{\ln(11)}.$$

Therefore, we will use the following zero coupon yield curve in each state of the world $(t, s)$:

$$b_t^s(q) = r_{4t}^s + \frac{f_t^s - r_{4t}^s}{\ln(11)} ln(1+q). \tag{7}$$

To find the equity spot curve, we add the average risk premium for stocks to (7). This average risk premium is approximately 0.0495, as can be found in Chan et al. [0]. This is consistent with the data used in estimating (5).

The return on a broadly diversified stock portfolio always outperformed a bond portfolio if the considered period was twenty years or longer, see H.A. Klein Haneveld [0]. Therefore, we will use the equity spot curve to discount expected future benefit payments for maturities greater than or equal to twenty years. For earlier time periods, we will use a linear combination of the zero coupon yield curve and the equity spot curve to define the pension spot curve. These ideas are taken from H.A. Klein Haneveld [0].

Defining $D_t^s(q)$ as the discount rate used to discount expected future benefit payments with maturity $q$ years, in state of the world $(t, s)$, we get the following definition for the pension spot curve:

$$D_t^s(q) := \begin{cases} b_t^s(q) + \frac{q}{20} 0.0495 & q = 0, 1, \ldots, 20 \\ b_t^s(q) + 0.0495 & q \geq 20 \end{cases}$$

where, as before, the equity risk premium is 0.0495.

We assume that workers retire at the age of 65 and that they are replaced by workers who are 25. In addition, we assume that the fund guarantees full indexation.

The value of the liabilities in state $(t, s)$ is given by

$$L_t^s := \sum_{q=t}^{\infty} \frac{P_q^s}{1 + O_t^s(q)},$$

where $P_q^s$ is the value of the benefit payments in state of the world $(q, s)$, given by

$$P_q^s = B_q \Pi_{z=1}^q (1 + v_z^s),$$

and $B_q$ is the expected benefit payment for year $q$ as given at time $t = 0$.

The valuation of the liabilities and benefit payments in this way is new insofar as in other ALM models, it is explicitly mentioned how these values are actually calculated.

### 4.3 Scenario generation: numerical results

In this subsection, numerical results from the scenario generation will be described. The numerical values for $a$, $\Sigma$ and $\Omega$ are copied from Kouwenberg [0]. The vector $a$, and the matrix $\Sigma$, introduces in (5) are given by

$$a = \begin{bmatrix} 0.086 \\ 0.058 \\ 0.072 \\ 0.020 \\ 0.018 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 0.0253 & 0.0037 & 0.0526 & -0.0014 & -0.0019 \\ 0.0037 & 0.0036 & 0.0023 & -0.0003 & -0.0003 \\ 0.0526 & 0.0023 & 0.0125 & -0.0003 & -0.0000 \\ -0.0014 & -0.0003 & -0.0003 & 0.0003 & 0.0001 \\ -0.0019 & -0.0003 & -0.0000 & 0.0001 & 0.0009 \end{bmatrix}.$$

The matrix $\Omega$ consists of zeros, except the elements $\Omega_{44} = 0.644$ and $\Omega_{55} = 0.693$.

For the numerical values of the returns on the four asset classes, the value of the liabilities, the level of the wages, the benefit payments, and the discount factors in each node of the scenario tree, we refer to appendix .

## 5. Numerical Results

In this section, we describe numerical results for several instances of the prototype model. The two most important questions we would like to answer are:

- Can we understand the results of the model? For example, do we observe a positive correlation between the level of the funding ratio and the fraction of the assets invested in stocks, as we would expect? Does the level of the contribution rate increase as the funding ratio decreases?
- How good is the solution of the heuristic, compared to the optimal solution? Are there large differences between the optimal and the heuristic solutions? If so, how can we explain these differences?

To be able to answer the questions posed above in detail, we would also answer the following questions:

- How do the solutions change if we replace the lower and upper bounds on the fractions invested in each asset class by zero and one, respectively?
- If the funding ratio at time $t = 0$ is below $\alpha$, are the contribution rates higher than in the basic instance, and are the asset portfolios different?
- What are the consequences if a euro paid by the sponsor is penalized harder than a euro paid by the active participants? Do the contributions by the active participants increase in this case?
- Do the solutions change if all returns on stocks are decreased by, for example, four percent?
- Do the solutions change if there is no upper bound on the remedial contributions?
- Do the solutions change if surplusses and shortages at the horizon are not rewarded or penalized? And how do the solutions change if the rewards at the horizon are increased?
- Are the integrated chance constraints binding? If not, what are the results if they become binding due to a lower maximum expected shortage for the next year?
- Are the contribution rates more volatile if large changes in these rates are not penalized?
- Does the requirement that the sponsor has to make a remedial contribution only after two consecutive years of underfunding lead to lower funding costs, compared to the common restriction which asks for immediate correction?
- If the sponsor has to make a remedial contribution, are these contributions always such that the funding ratio becomes exactly equal to $\alpha$, or are there also situations in which the remedial contribution is higher?

We will answer these questions by one basic instance and eleven other instances. First, the basic instance is analyzed in detail. The other instances differ each slighly from the basic instance, so that we are able to analyze the effect of a single change in the inputs on the restults.

**Instance 1: basic instance**

In this basic instance, the minimum required funding ratio, denoted by $\alpha$, is equal to 1.05. The initial funding ratio, that is, the funding ratio at time $t = 0$, just before changes in the asset portfolio can be made at time $t = 0$, is equal to 1.10. The initial asset value equals 10,394 and the value of the liabilities is equal to 9,449. These numbers are all in million euros.

If the funding ratio is below $\alpha$ in a certain state of the world, fixed penalty costs of $\lambda_\delta = 200$ (million euros) are incurred. In addition, if the sponsor of the fund has to make a remedial contribution, fixed costs of $\lambda_d = 600$ (million euros) are incurred. In this case, also variable costs have to be paid. We set $\lambda_Z = 1$ in this basic instance, indicating that one euro paid by the sponsor is equally expensive as a euro paid by the active participants of the fund. There is an upper bound on the amount the sponsor is able or can be forced to pay in cases of financial distress for the pension fund. As described above, this upper bound is a fraction $\tau$ of the level of the wages. In this basic instance, it is 150 percent of the wages, denoted by $\tau = 1.5$. At time $t = 0$, this implies that the upper bound on a remedial contribution is equal to 366.

The contribution rate in 'year -1' is supposed to be 17 percent. This information is relevant, since changes in the contribution rate from one time period to the next are penalized if they ar greater than $\rho = 0.03$. Larger increases are multiplied by $\lambda_{c+} = 2$ times the level of the wages, and larger decreases are multiplied by $\lambda_{c-} = 1.5$ times the level of the wages. Thus, the management of the fund pursues a relatively stable contribution rate; large increases in the contribution rates are considered to be worse than large decreases.

The maximum allowed expected shortage for the next year is equal to $\beta = 450$ (million euros). This is slightly more than four percent of the initial asset value.

At the horizon, shortages are penalized and surplusses are rewarded. Shortages with respect to the level $\theta = 1.05$ are penalized by $\lambda_\theta = 0.0125$, on top of $\lambda_\delta = 200$. On the other hand, surplusses with respect to the level $\xi = 1.05$ are rewarded by $\lambda_\xi = -0.0045$, thus shortages are penalized harder than surplusses are rewarded at the horizon.

A summary of these parameter values is given in Table .

We solved the basic instance of the prototype model (with parameter settings as presented in Tables  and ) by using AIMMS with solver XA. It took 25.56 seconds to find the optimal solution. We also applied the heuristic, described in Section 3. It took 7.99 seconds to find the heuristic solution. Detailed information on the optimal solution and the heuristic solution can be found in the appendix. The core of the numerical results is described in Table .

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | = | 1.05 | $\lambda_Z$ | = | 1 | $\beta$ | = | 400 |
| $F_0$ | = | 1.10 | $\tau$ | = | 1.5 | $\theta$ | = | 1.05 |
| $A_0$ | = | 10,394 | $c_0$ | = | 0.17 | $\lambda_\theta$ | = | 0.00125 |
| $L_0$ | = | 9,449 | $\rho$ | = | 0.03 | $\xi$ | = | 1.05 |
| $\lambda_\delta$ | = | 200 | $\lambda_{c+}$ | = | 2 | $\lambda_\xi$ | = | -0.00450 |
| $\lambda_d$ | = | 600 | $\lambda_{c-}$ | = | 1.5 | | | |

Table 5.1: Parameter values for model 1.

In Table , it can be seen that the optimal value of the objective function is equal to 337 (million euros), whereas the value of the objective functin for the heuristic solution is 23 percent higher: 414 (million euros). Moreover, in the second and fourth column of Table  it can be seen how these values are composed in funding costs and various penalties. Also, the corresponding values of the decisions are indicated in the first and third column. Basically, they denote 'total expected discounted value'.

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 23 | | 7 |
| Remedial contributions | | 41 | | 40 |
| **Penalties Underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 21 | 31 | 28 | 42 |
| **Penalties Horizon** | | | | |
| Surplus | 6,447 | -29 | 6,383 | -29 |
| Shortage | 132 | 0 | 177 | 0 |
| **Total** | | 337 | | 414 |

Table 5.2: Decisions and contributions to the objective function for model instance 1.

From the numerical results in appendix **??**, we conclude that in the basic instance, there is a positive relationship between the level of the funding ratio and the fraction of stocks in the asset portfolio. There is a negative relationship between the level of the funding ratio and the fraction of the assets invested in bonds and real estate. This is what we expected, since if the funding ratio is relatively high, a little bit more risk may be acceptable to invest in more promising assets. These relations hold for both the optimal solution and the heuristic solution.

We will consider three scenarios (1, 25, and 32) in more detail, since in these scenarios some interesting aspects appear. Although the numerical results for all scenarios for this instance can be found in the appendix, we will present the results for the three scenario mentioned above in Table  for convenience. The return on the asset portfolio from time $t$ to time $t+1$ is denoted by $r_p^t$.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.34 | 0.16 | 0 | 0.132 | 0.03 | 0 | 0 | 0 | 1.099 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.182 |
| | 3 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.343 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.521 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.751 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.160 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.214 |
| | 5 | | | | | | 0 | 0 | 0 | 0 | 1.333 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 5.3: Results for optimal solution model instance 1.

In the initial asset portfolio, 45 percent of the assets is invested in stocks, which is equal to its lower bound. The fraction invested in real estate is equal to its upper bound (16 percent), while the remaining assets are invested in bonds.

The contribution rate at time $t = 0$ is equal to 6 percent. Penalty costs are incurred, since the decrease in the contribution rate is 11 percent, which is more than $\rho$.

In scenario 1, the funding ratio increases from time $t = 1$ on. The fund never has to deal with underfunding, and therefore the sponsor does not have to make a remedial contribution. The contribution rate at time $t = 1$ is lowered to 3 percent, which is the maximum decrease such that no penalties are incurred.

In scenario 25, we see that the sponsor of the pension fund has to make a remedial contribution at time 2, since the funding ratios are less than $\alpha$ at times 1 and 2. Although the funding ratio falls below its minimum required level at time $t = 1$, the contribution rate decreases to 3 percent. This is not very realistic, and most managers would not accept this decision. It would be more realistic if the contribution rate is relatively high in cases of underfunding, so that the remedial contributions are not unnecessarily high.

In scenario 32, we have the same results as in scenario 25 for decision moments up to and including time $t = 2$. This is forced by the nonanticipativity constraints, since scenarios 25-32 coincide before time $t = 3$. In this scenario, the fund also has to deal with underfunding at the horizon. The remedial contribution $Z$ at time $t = 2$ is chosen, such that underfunding is avoided at time $t = 3$.

The asset portfolios is constant through time in this scenario. At each decision moment, assets are bought and sold, in such a way that the fraction of stocks is equal to its lower bound, the fraction of real estate is equal to its upper bound, and the remaining assets are invested in bonds.

In this scenario we have again unrealistic levels of the contribution rate. They are according to us generally too low, and even equal to zero in the last three years, even though the fund has to deal with underfunding at the horizon.

As already mentioned above, the solution of the heuristic differs from the optimal solution. However, the portfolio at time $t = 0$ coincides. The contribution rate at time $t = 0$ is 3 percent in the heuristic solution, which is even lower than the contribution rate in the optimal solution. As a result, a larger penalty due to unstable contribution rates in the heuristic solution is incurred.

The numerical results of the heuristic solution for all scenarios can be found in the appendix. Scenarios 1, 25, and 32 will be discussed in more detail here. The numerical results of these scenarios are presented in Table .

The numerical results for scenario 1 are approximately the same as in the optimal solution. The funding ratio increases over time, the contribution rate decreases towards

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.130 | 0 | 0 | 0 | 0 | 1.097 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.178 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.338 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.180 | 0 | 0 | 0 | 0 | 1.510 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.731 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.150 | 0 | 0 | 0 | 0 | 1.166 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.226 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.380 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 40 | 1.046 |
| | 4 | 0.55 | 0.29 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.077 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.032 |

Table 5.4: Results for heuristic solution model instance 1.

zero, and the fraction of the assets invested in stocks increases to its upper bound.

In scenario 25, there are also similarities between the solutions, since the funding ratio is below $\alpha$ at times $t = 1$ and $t = 2$. The level of the remedial contribution in the heuristic solution is equal to the amount of the shortage. This amount is lower than the level of the remedial contribution made by the sponsor in the optimal solution.

In scenario 32, the sponsor has to make a remedial contribution twice. The first at time $t = 1$ and the second one year later. The reason is that the contribution rate at the beginning of the planning horizon is too low, such that underfunding cannot be avoided at these decision moments. The level of the second remedial contribution is again equal to the amount of the shortage.

We summarize the results of this first instance as follows:

- The portfolio decisions are in agreement with what we expect: the fraction of stocks in the portfolio increases with the funding ratio.

- The contribution rates are low, even if the fund has to deal with underfunding. This is not what most managers would accept.
- The solution of the heuristic differs from the optimal solution. The major differences are the number of remedial contributions to be made by the sponsor, and the level of the contribution rates.

**Instance 2: Other bounds on fractions in asset classes**

In this instance, we will answer the fourth question. We consider a fund which is allowed to invest all assets in one asset class. However, short selling is not allowed, resulting in lower bounds equal to zero and upper bounds equal to one on the fractions of the four asset classes in the portfolio. In the basic instance, the lower and upper bounds on the asset classes were often binding. We expect that in this instance the fractions of stocks in the asset portfolio increases above 65 percent in states of the world where the funding ratio is relatively high, resulting in lower fractions of bonds, real estate and cash in the portfolio. Due to other compositions of the portfolios, also the contribution rates may change.

The objective function value for both the optimal and the heuristic solution are split up in components. The results can be fount in Table  in appendix . The numerical results for scenarios 1, 25 and 32 are presented in Table  in the appendix.

We will describe the results of the optimal solution first. It appears that relaxing the asset mix constraints results in an almost ideal solution for the pension fund: the funding cost reduces from 64 to 8 (million euros), there is no underfunding and no remedial contributions at any time in any scenario, and the levels of the contributions is modest in the first year, and disappears in further years. The fund only invests in two asset classes: stocks and real estate. In some states of the world, all assets are invested in stocks. The reason that investments are only made in these two asset classes, is that the mean returns on these two asset classes are the highest, and that there is a perfect negative correlation between stocks and real estate in our model. The best way to analyze these results is by considering the funding ratios. In many states this ratio is equal to $\alpha$. We conclude that the amounts invested in real estate are such that there is a guarantee that underfunding never occurs. Of course, this in an irrealistic feature of the scenarios. It indicates that arbitrage is possible. By exploiting this property, the model is able to guarantee that underfunding never occurs, by investing at a sufficient level in real estate. The remaining fraction of the assets is invested in stocks. This results in large changes in the composition of the asset portfolios.

At time $t = 0$, the contribution rate is three percent. This is the minimum amount, necessary to have a guarantee that underfunding at time $t = 1$ does not occur in the two states of the world which are possible. At later time moments, this guarantee also holds for a contribution rate equal to zero. Due to this way of investing, the funding ratios increase considerably in some scenarios.

The heuristic solution coincides with the optimal solution. This follows, since the linear programming relaxation already gives the optimal solution.

**Instance 3: Underfunding at time** $t = 0$

In this instance, we will answer question five. We consider a pension fund which starts at time $t = 0$ with a funding ratio of 1, which is less than $\alpha$. We expect higher funding costs in this case, due to both higher contributions by the active participants and higher remedial contributions. If only the initial funding ratio is decreased to 1, i.e. $A_0$ is set equal to $L_0$, and all other parameter values are as specified in Table , no feasible solution exits. In order to be able to obtain a feasible solution, the maximum expected shortage in the next year, denoted by $\beta$, is increased to 1,250 (million euros). Also the upper bound on a remedial contribution has to be increased The value of $\tau$ has to be increased to 6, given the value of $\beta = 1,250$.

From the numerical results, which are presented in Table  in the appendix, we conclude that the funding costs increase dramatically, from 64 to 545 (million euros). There are five states of the world in which the funding ratio is below $\alpha$ (one of them is the state at time $t = 0$). In both states at time $t = 1$, the funding ratio is too low, indicating that the sponsor has to make a remedial contribution in these states. In the optimal solution, these amounts are such that no remedial contribution has to be made at later decision moments.

Also the contribution rates are higher than in the basic instance in many states of the world. The level of the initial contribution rate is surprising, however. Although the sponsor has to make a remedial contribution at the end of the first year, the contribution rate is lowered to 14 percent in the first year. The contribution rate gradually decreases to zero at later decision moments.

The fraction of the assets invested in stocks at time $t = 0$ is 4% higher than in the basic instance, and the fraction of the assets invested in bonds is lowered by this percentage. The amounts invested in real estate and cash at time $t = 0$ are the same as in the basic instance.

In the heuristic solution, the funding ratio is below $\alpha$ in only four states of the world. This is the result of an even higher remedial contribution at time $t = 1$. The sponsor has to make these contributions in the same states of the world.

The contribution rates are always zero, except in two states of the world, in which this rate is equal to three percent. This is not what most managers will consider as reasonable solutions, given the low initial funding ratio.

The portfolios in the optimal solutions closely resemble those in the heuristic solution. The largest difference in the optimal and heuristic solutions between the fractions of the assets invested in one asset class is 5%.

**Instance 4: Increase variable costs sponsor**

Here, we will answer question 6. In this instance, each euro the sponsor has to pay is more expensive than each euro which is paid by the active participants. This may be reasonable, since the sponsor may have to lend the money, in which case interest costs also have to be paid. We represent these higher costs by increasing $\lambda_Z$ to 1.1. That is, each euro paid by the sponsor is ten percent more expensive than a euro paid by the participants.

In the optimal solution, we obtained the same portfolios and also the same levels of the contribution rates as in the basic instance. The contribution rate at time $t = 0$ is 6 percent. This is the minimum level such that the funding ratio in scenario 32 at time $t = 3$ is equal to $\alpha$, and such that no penalty costs are incurred after time $t = 0$ due to large increases in the contribution rate. The results of this scenario, together with those of scenarios 1 and 25, are presented in the appendix.

There are three states of the world in which the funding ratio is below its minimum required level. The sponsor has to make a remedial contribution in one of them.

The heuristic solution coincides with the corresponding solution in the basic instace. The only difference between the solutions, is that in instance 4 there are also penalties incurred for the remedial contributions.

**Instance 5: Higher penalty variable costs remedial contributions**

In the previous instance, each euro to be paid by the sponsor is ten percent more expensive than a euro to be paid by the active participants. However, the contribution rates are still very low in that instance. This can be explained by the fact that a positive contribution rate implies costs in both child nodes, while a remedial contribution is only made in the child node in which underfunding occurs. Since the cash flows are multiplied by the probabilities, making remedial contributions is more advantageous in cases where only in one of the two child nodes the sponsor has to make a remedial contribution if $\lambda_Z$ is not much higher than 1. Therefore, we increase the variable costs for each euro paid by the sponsor to $\lambda_Z = 2.5$. We expect that the contribution rates are higher in this case.

The numerical results of this instance are presented in Table  in the appendix. From this table we conclude that the contribution rate is indeed higher than in the basic instance at time $t = 0$. It equals 14 percent at this decision moment.

As a result of the higher contribution rates, it suffices to pay a lower remedial contribution at time $t = 2$. The asset portfolio is approximately equal to those in the basic instance.

The results of the heuristic solution are presented in Table . In this heuristic solution, the funding ratio is below its minimum required level in 4 states of the world, and in two of them the sponsor has to make a remedial contribution. As already mentioned in a few instances described above, the contribution rates are very low. In fact, most managers will consider them to be too low. The asset portfolios in the heuristic solution are close to the portfolios in the optimal solution.

**Instance 6: Lower returns on stocks**

In the sixth instance we consider, the returns on stocks are lowered by four percent in all time periods in all scenarios. This is done, since the average return of more than ten percent per year, is considered as too optimistic by some managers. We expect that the fraction of assets invested in stocks will decrease.

This is indeed the case. The fraction of assets invested in real estate is always equal to its upper bound. The fund invests also more in cash, and, to a lesser degree, it increases in bonds. The initial portfolio is approximately the same as in the first instance.

The contribution rate is in our opinion more realistic than in the instances considered before. The initial contribution rate is almost 20 percent. However, even if the funding ratio drops to a level below $\alpha$ at time $t = 1$, the contribution rate is decreased considerably. This is not very realistic.

The value of the objective function equals 937. This is 178 % higher than the corresponding value in the basic instance. The funding costs are 436 % higher than in the basic instance. In the appendix, the results for scenarios 1, 13, 25 and 32 are presented.

In scenario 13, the funding ratio is below $\alpha$ at time $t = 3$. However, the sponsor does not have to make a remedial contribution. In scenario 32, the funding ratio is below its minimum required level at three decision moments. However, the sponsor has to make a remedial contribution only once.

The composition of the asset portfolios in the heuristic solution closely resembles those in the optimal solution. However, the differences in the levels of the contribution rates in these two solutions are dramatically. The contribution rates are all zero in the heuristic solution, although there are four states of the world in which the funding ratio is below $\alpha$, and the sponsor has to make a remedial contribution in three of them.

#### Instance 7: No upper bound on remedial contribution

Instead of requiring an upper bound on the amount the sponsor is prepared to (or can be forced to) pay in case of financial distress, we consider an instance in which no upper bound on this amount is given. We obtained the same solution as in the basic instance, both for the optimal and the heuristic solution.

We considered also the parameter settings as described in instance 3, except that the fixed costs associated with underfunding is increased to $\lambda_\delta = 1500$. The funding ratio at time $t = 0$ is 1, which is strictly below its minimum required level. We expect that higher fixed costs associated with underfunding will lead to discrepancies with the basic instance.

The numerical results of this instance are presented in Table **??** in appendix . The sponsor of the fund has to make a remedial contribution immediately in this case. The level of this contribution is such that in all future states considered, the funding ratio is sufficiently high, so that fixed costs due to underfunding are avoided in the future. The contribution rates are in all time periods equal to zero.

The heuristic solution coincides coincides with the heuristic solution of instance 3. The contribution rates are lower than in the optimal solution. This results in underfunding at the horizon in scenario 32.

#### Instance 8: No rewards and penalties horizon

In all instances described above, surplusses at the horizon were rewarded, and shortages penalized. In this instance of the prototype model, we remove these rewards and penalties. We do not expect major changes in the decisions.

The initial portfolio in the optimal solution is the same as in the basic instance. However, there are significant differences in the composition of the portfolios as we approach the horizon. If surplusses and shortages at the horizon are rewarded or penalized, the fraction of assets invested in stocks is often equal to its upper bound. The reason is that the fund invests rather aggressively in order to reap the profits in case of high funding ratios. Other decisions are more or less the same: low contribution rates, and the sponsor has to make a remedial contribution only once.

At times $t = 3$ and $t = 4$, there are large differences in the compositions of the asset portfolios in the optimal and heuristic solution. In the heuristic solution, the fraction of assets invested in stocks is higher than in the optimal solution in most states of the world at times $t = 3$ and $t = 4$.

The average funding ratio is one percent higher if surplusses are rewarded and shortages are penalized.

**Instance 9: Higher rewards horizon**

In the first eight instances of the prototype model, shortages at the horizon were penalized harder than surplusses rewarded. In this instance, we do the opposite. This is done by changing the value of $\lambda_\xi$ to -0.01. The numerical results of this change can be found in Table  in the appendix.

In the optimal solution, all decisions regarding contribution rates and remedial contributions are exactly the same as in the basic instance. The asset portfolios are also the same up to and including time $t = 2$. In decision nodes at times time $t = 3$ and $t = 4$, a higher fraction of the assets is invested in stocks.

The same conclusion can be drawn from the heuristic solution. The portfolios are the same in the first three decision periods. In the last two years, there are differences: more is invested in stocks.

**Instance 10: Integrated chance constraints binding**

In the previously described instances, the integrated chance constraints were not binding. We consider here an instance, in which the maximum expected shortage in the next year is decreased to 200. Question ten will be answered here.

The asset portfolios do not change much, as compared to the basic instance. In particular, the initial portfolios are equal. In scenario 32 at time $t = 1$, the integrated chance constraints are binding. However, the same portfolio is selected as in the basic instance at this decision moment.

The contribution rate is 11.5 percent at time $t = 0$, while this rate was 0 in the basic instance. This level is necessary to obtain an asset value at time $t = 1$, such that the integrated chance constraints can be satisfied.

In the heuristic solution, we obtain the same initial portfolio, and also the same initial contribution rate as in the optimal solution. Of course, the reason is the same as described previously. For the remaining decision moments, the heuristic solution closely resembles the solutions of many other instances described above. Although the funding ratio is below its minimum required level in some states of the world, the contribution rates are considered as too low by most managers.

**Instance 11: No penalties large changes contribution rate**

In the first ten instances, changes (either increases or decreases) in the contribution rates in two consecutive time periods greater than three percent were penalized. In this instance we do not penalize large deviations. Given the results of the previously described instances, we expect that the contribution rates decrease faster or become even zero in all periods, since at time $t = 0$ a contribution rate equal to zero will not be penalized.

The numerical results can be found in Table  in the appendix. The contribution rates in the optimal solution are zero in all states of the world. The initial portfolio is equal to the portfolio in the basic instance. Also at later decision moment, the portfolios closely resemble.

The absence of the contributions by the active participants results in a higher remedial contribution in scenario 25 at time $t = 2$. This is the only state of the world in which the sponsor has to make a remedial contribution in this instance.

In the heuristic solution, the contribution rates are always zero too. As in the basic instance, the sponsor of the fund has to make a remedial contribution in two states of the wold. The asset portfolios are approximately the same as in the optimal solution, although the fund invests more in cash in the heuristic solution when the horizon is approached.

### Instance 12: Remedial contribution as soon as funding ratio is too low

In this instance, the sponsor of the fund has to make a remedial contribution as soon as the funding ratio is below $\alpha$. The results of this instance are presented in Table .

The funding costs increase in this case to 131. This is an increase of 105% compared with the funding costs in the basic instance. These costs increase due to both higher contribution rates and a higher level of the remedial contribution in scenario 17 at time $t = 1$. The higher funding costs result in only one state of the world in which the funding ratio is too low.

The contribution rates are more in accordance with what we expect. They increase in scenario 32, so that the funding ratio is sufficiently high at the horizon.

In the solution of the heurisitic, the sponsor has to make a remedial contribution twice. The total amount the sponsor has to pay is larger, due to lower levels of the contribution rate.

### Answers to the questions

We started this section with a number of questions. Now we will give provisional answers to them, based on the numerical results of the instances described.

1. The portfolios are in accordance with what we expected. There is a positive correlation between the level of the contribution rate and the fraction of assets invested in stocks.
   The contribution rates are more puzzling. They are generally very low, and in most states of the world even equal to zero, although the sponsor has to make remedial contributions in some states of the world.
2. In the optimal solutions, the sponsor has to make a remedial contribution only if the funding ratio is below its minimum required level in two consecutive years. The funding costs are in this case lower than if the sponsor has to restore the funding ratio as soon as it drops below $\alpha$. In the optimal solutions, the levels of the remedial contributions are such that in the next time period(s) the funding ratio is greater than or equal to $\alpha$.
3. By and large, the portfolios in the heuristic solution resemble those in the optimal solution. However, in most instances the contribution rates are equal to zero in all states of the world, whereas they are strictly positive in some states of the world in the optimal solutions. This results in higher remedial contributions. To compensate, the remedial contributions are larger and more frequent

in the heuristic solution.

4. The portfolios change substantially if the lower and upper bounds on the fractions of the asset classes are replaced by zero and one respectively. The pension fund only invests in stocks and real estate in this case, since there is a perfect negative correlation between the returns of these asset classes in our model. This results in a sufficiently high funding ratio in all 32 scenarios in all time periods.
5. If the initial funding ratio is below $\alpha$ at time $t = 0$, the contribution rates are higher than in the basic instance. This results in lower remedial contributions at time $t = 1$.
6. If each euro paid by the sponsor is ten percent more expensive than a euro paid by active participant, the contribution rates are still very low in both the optimal and the heuristic solution. However, as the penalty costs associated with payments by the sponsor increases to 150 percent ($\lambda_Z = 2.5$), the contributions rates do increase, leading to lower remedial contributions. This conforms to our expectations.
7. If all returns on stocks are lowered by four percent, the fund invests less in this asset class in most states of the world. The contribution rates increase in the first years. However, there are large discrepancies between the optimal and the heuristic solution, both in asset allocations and contribution rates.
8. The solutions do not change if there are no upper bounds on the remedial contributions in the eleven instances described above. However, for sufficiently large fixed costs, a remedial contribution has to be made only once, if the funding ratio at time $t = 0$ is below its minimum required level $\alpha$.
9. The optimal and heuristic solution change only marginally from those of the basic instance in the first three years if surplusses and shortages at the horizon are not rewarded and penalized anymore. Only in the last two years, the composition of the portfolios differ much from those in the basic instance.
10. Instance 10 is the only instance in which the integrated chance constraints are binding. In this instance, the contribution rates are higher than in the basic instance, because the asset value in one state of the world has to increase in order to be able to meet this constraint. The composition of the asset portfolio are not much different from those in the basic instance.
11. If there are no penalties on the contribution rate increases or decreases substantially in two consecutive years, the contributions become zero in all states of the world.

### Future research

The numerical results of the instances of the prototype model described in this paper, are used to make decisions regarding future research. First of all, we would like to improve the heuristic. In some instances, the solution of the heuristic resembles the optimal solution, but in some cases, the results differ too much. Especially the levels of the contribution rates are of much concern.

A second improvement can be made in the way the scenarios are generated. In particular the dependencies between the returns on stocks, bonds, and real estate are not realistic. In addition, the number of scenarios should be increased, to make decision making more realistic.

In the future, we will also add other decision variables to the ALM model. For example, we would like to add indexations as decisions. Also, a restitution to the sponsor in case of high funding ratios can be considered.

## 6. Summary and conclusions

We developed an Asset Liability Management (ALM) model which contains some new and important aspects which we did not encounter in other ALM models, leading to more realistic ALM models for pension funds. The first new aspect is the flexible modeling of remedial contributions after several periods of underfunding. This new modeling feature is important, since as described in H.A. Klein Haneveld [0] the supervisor of pension funds in The Netherlands uses this criterion to judge the solvency position of a pension fund.

Special attention is also paid to the incorporation of risk constraints. We have seen that there is a close relationship between chance constraints and integrated chance constraints. The latter are to be preferred, from a mathematical point of view, and also by the fact that not only probabilities are taken into account, but also amounts of shortage.

We also presented a heuristic to find a feasible solution for our ALM model, since for realistically sized ALM models, it is in general impossible to find an optimal solution, due to the introduction of binary variables.

The most remarkable results are those for the contribution rates. They are in general very low, and in most states of the world even equal to zero. This is explained by the relatively low variable costs if a remedial contribution has to be made. If these costs are increased, contribution rates increase in the years before the sponsor has to restore the funding ratio. If all returns on stocks are lowered by four percent, or in cases where the integrated chance constraints are binding, the initial contribution rates are more on a level that we expect. In these instances, high contribution rates are necessary to obtain a feasible solution. In the solution of the heuristic, the contribution rates generally are even lower than those in the optimal solution.

At time $t = 0$, the asset portfolios of the different instances are very similar. At later decision moments, however, there are substantial differences between these portfolios. The results of instance 8 stand out in particular. In this instance, we also saw the largest differences between the portfolios in the heuristic solution, and the portfolios in the optimal solution.

In many instances, the funding ratio is below $\alpha$ in 3 states of the world in the optimal solution, but the sponsor has to make a remedial contribution only once. In the solution obtained with the heuristic, in almost all instances the funding ratio is below its minimum required level in 4 states, while the sponsor has to make a remedial contribution to the fund twice.

If the sponsor of the fund has to make a remedial contribution as soon as the funding ratio is below its minimum required level, funding costs increase. The reason is that the sponsor has to make these payments more often, and the total amount the sponsor has to pay increases significantly in this case. If a remedial contribution has to be made after two consecutive periods of underfunding, there is a possibility of a recovery of financial markets, which may cause funding ratios to increase.

]

Appendix

## Appendix A:

## Realizations uncertain parameters

| *s* | Returns stocks | | | | | Returns bonds | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.278 | 0.278 | 0.278 | 0.278 | 0.278 | -0.002 | -0.002 | -0.002 | -0.002 | -0.002 |
| 2 | | | | | -0.070 | | | | | 0.125 |
| 3 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 4 | | | | | -0.070 | | | | | 0.125 |
| 5 | | | -0.070 | 0.278 | 0.278 | | | 0.125 | -0.002 | -0.002 |
| 6 | | | | | -0.070 | | | | | 0.125 |
| 7 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 8 | | | | | -0.070 | | | | | 0.125 |
| 9 | | -0.070 | 0.278 | 0.278 | 0.278 | | 0.125 | -0.002 | -0.002 | -0.002 |
| 10 | | | | | -0.070 | | | | | 0.125 |
| 11 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 12 | | | | | -0.070 | | | | | 0.125 |
| 13 | | | -0.070 | 0.278 | 0.278 | | | 0.125 | -0.002 | -0.002 |
| 14 | | | | | -0.070 | | | | | 0.125 |
| 15 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 16 | | | | | -0.070 | | | | | 0.125 |
| 17 | -0.070 | 0.278 | 0.278 | 0.278 | 0.278 | 0.125 | -0.002 | -0.002 | -0.002 | -0.002 |
| 18 | | | | | -0.070 | | | | | 0.125 |
| 19 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 20 | | | | | -0.070 | | | | | 0.125 |
| 21 | | | -0.070 | 0.278 | 0.278 | | | 0.125 | -0.002 | -0.002 |
| 22 | | | | | -0.070 | | | | | 0.125 |
| 23 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 24 | | | | | -0.070 | | | | | 0.125 |
| 25 | | -0.070 | 0.278 | 0.278 | 0.278 | | 0.125 | -0.002 | -0.002 | -0.002 |
| 26 | | | | | -0.070 | | | | | 0.125 |
| 27 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 28 | | | | | -0.070 | | | | | 0.125 |
| 29 | | | -0.07 | 0.278 | 0.278 | | | 0.125 | -0.002 | -0.002 |
| 30 | | | | | -0.070 | | | | | 0.125 |
| 31 | | | | -0.070 | 0.278 | | | | 0.125 | -0.002 |
| 32 | | | | | -0.070 | | | | | 0.125 |

| *s* | Returns real estate | | | | | Returns cash | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | -0.039 | -0.039 | -0.039 | -0.039 | -0.039 | 0.054 | 0.050 | 0.048 | 0.046 | 0.045 |
| 2 | | | | | 0.202 | | | | | 0.061 |
| 3 | | | | 0.202 | -0.039 | | | | 0.062 | 0.055 |
| 4 | | | | | 0.202 | | | | | 0.071 |
| 5 | | | 0.202 | -0.039 | -0.039 | | | 0.063 | 0.056 | 0.051 |
| 6 | | | | | 0.202 | | | | | 0.067 |
| 7 | | | | 0.202 | -0.039 | | | | 0.072 | 0.062 |
| 8 | | | | | 0.202 | | | | | 0.078 |
| 9 | | 0.202 | -0.039 | -0.039 | -0.039 | | 0.066 | 0.058 | 0.052 | 0.049 |
| 10 | | | | | 0.202 | | | | | 0.065 |
| 11 | | | | 0.202 | -0.039 | | | | 0.068 | 0.059 |
| 12 | | | | | 0.202 | | | | | 0.075 |
| 13 | | | 0.202 | -0.039 | -0.039 | | | 0.074 | 0.063 | 0.056 |
| 14 | | | | | 0.202 | | | | | 0.072 |
| 15 | | | | 0.202 | -0.039 | | | | 0.079 | 0.066 |
| 16 | | | | | 0.202 | | | | | 0.082 |
| 17 | 0.202 | -0.039 | -0.039 | -0.039 | -0.039 | 0.070 | 0.060 | 0.054 | 0.050 | 0.048 |
| 18 | | | | | 0.202 | | | | | 0.063 |
| 19 | | | | 0.202 | -0.039 | | | | 0.066 | 0.058 |
| 20 | | | | | 0.202 | | | | | 0.074 |
| 21 | | | 0.202 | -0.039 | -0.039 | | | 0.070 | 0.060 | 0.054 |
| 22 | | | | | 0.202 | | | | | 0.070 |
| 23 | | | | 0.202 | -0.039 | | | | 0.076 | 0.064 |
| 24 | | | | | 0.202 | | | | | 0.081 |
| 25 | | 0.202 | -0.039 | -0.039 | -0.039 | | 0.076 | 0.064 | 0.057 | 0.052 |
| 26 | | | | | 0.202 | | | | | 0.068 |
| 27 | | | | 0.202 | -0.039 | | | | 0.073 | 0.062 |
| 28 | | | | | 0.202 | | | | | 0.078 |
| 29 | | | -0.07 | -0.039 | -0.039 | | | 0.08 | 0.067 | 0.058 |
| 30 | | | | | 0.202 | | | | | 0.074 |
| 31 | | | | 0.202 | -0.039 | | | | 0.083 | 0.069 |
| 32 | | | | | 0.202 | | | | | 0.085 |

| *s* | Liabilities | | | | | | Wages | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9449 | 10118 | 10198 | 10139 | 10177 | 10086 | 244 | 257 | 256 | 255 | 254 | 254 |
| 2 | | | | | | | 10127 | | | | | 259 |
| 3 | | | | | 10073 | 10157 | | | | | 260 | 258 |
| 4 | | | | | | 9970 | | | | | | 263 |
| 5 | | | | 10179 | 9897 | 9808 | | | | 260 | 258 | 256 |
| 6 | | | | | | 10150 | | | | | | 262 |
| 7 | | | | | 9970 | 9654 | | | | | 264 | 260 |
| 8 | | | | | | 9520 | | | | | | 266 |
| 9 | | | 10321 | 9904 | 10063 | 10067 | | | 261 | 259 | 257 | 256 |
| 10 | | | | | | 10086 | | | | | | 261 |
| 11 | | | | | | 9977 | 9684 | | | | 262 | 259 |
| 12 | | | | | | 9935 | | | | | | 265 |
| 13 | | | | 10403 | 9646 | 10148 | | | | 264 | 261 | 258 |
| 14 | | | | | | 10159 | | | | | | 264 |
| 15 | | | | | 9522 | 9336 | | | | | 266 | 262 |
| 16 | | | | | | 9381 | | | | | | 267 |
| 17 | | 10104 | 9941 | 9920 | 10021 | 10269 | | 262 | 259 | 257 | 256 | 255 |
| 18 | | | | | | 10274 | | | | | | 260 |
| 19 | | | | | 10291 | 9846 | | | | | 261 | 259 |
| 20 | | | | | | 9893 | | | | | | 264 |
| 21 | | | | 9879 | 9560 | 10064 | | | | 263 | 260 | 258 |
| 22 | | | | | | 10016 | | | | | | 263 |
| 23 | | | | | 9525 | 9382 | | | | | 265 | 261 |
| 24 | | | | | | 9193 | | | | | | 267 |
| 25 | | | 9999 | 9549 | 10143 | 10102 | | | 265 | 261 | 259 | 257 |
| 26 | | | | | | 10023 | | | | | | 262 |
| 27 | | | | | 10048 | 9470 | | | | | 264 | 261 |
| 28 | | | | | | 9606 | | | | | | 266 |
| 29 | | | | 9941 | 9400 | 9942 | | | | 267 | 262 | 259 |
| 30 | | | | | | 9782 | | | | | | 265 |
| 31 | | | | | 9500 | 9014 | | | | | 268 | 263 |
| 32 | | | | | | 9676 | | | | | | 269 |

| $s$ | Benefit payments | | | | | Discount factors | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 514 | 538 | 562 | 586 | 609 | 0.949 | 0.904 | 0.863 | 0.825 | 0.789 |
| 2 | | | | | 622 | | | | | 0.778 |
| 3 | | | | 598 | 631 | | | | 0.813 | 0.770 |
| 4 | | | | | 645 | | | | | 0.759 |
| 5 | | | 574 | 607 | 638 | | | 0.850 | 0.805 | 0.765 |
| 6 | | | | | 651 | | | | | 0.754 |
| 7 | | | | 620 | 661 | | | | 0.793 | 0.747 |
| 8 | | | | | 675 | | | | | 0.736 |
| 5 | | 550 | 582 | 613 | 642 | | 0.890 | 0.842 | 0.800 | 0.762 |
| 10 | | | | | 656 | | | | | 0.751 |
| 11 | | | | | 658 | 665 | | | 0.788 | 0.744 |
| 12 | | | | | 684 | | | | | 0.733 |
| 13 | | | 642 | 635 | 672 | | | 0.829 | 0.780 | 0.739 |
| 14 | | | | | 686 | | | | | 0.728 |
| 15 | | | | 649 | 696 | | | | 0.769 | 0.721 |
| 16 | | | | | 711 | | | | | 0.710 |
| 17 | 524 | 558 | 588 | 617 | 645 | 0.935 | 0.882 | 0.836 | 0.796 | 0.760 |
| 18 | | | | | 659 | | | | | 0.749 |
| 15 | | | | 630 | 669 | | | | 0.785 | 0.742 |
| 20 | | | | | 683 | | | | | 0.731 |
| 21 | | | 601 | 640 | 675 | | | 0.824 | 0.777 | 0.737 |
| 22 | | | | | 690 | | | | | 0.726 |
| 23 | | | | 653 | 700 | | | | 0.765 | 0.719 |
| 24 | | | | | 715 | | | | | 0.708 |
| 25 | | 570 | 601 | 646 | 680 | | 0.868 | 0.816 | 0.772 | 0.734 |
| 26 | | | | | 694 | | | | | 0.723 |
| 27 | | | | 660 | 705 | | | | 0.761 | 0.716 |
| 28 | | | | | 720 | | | | | 0.706 |
| 25 | | | 623 | 669 | 712 | | | 0.804 | 0.753 | 0.712 |
| 30 | | | | | 727 | | | | | 0.701 |
| 31 | | | | 684 | 738 | | | | 0.742 | 0.694 |
| 32 | | | | | 753 | | | | | 0.684 |

**Appendix B:**

**Basic instance: optimal solution**

| $s$ | Fractions stocks | | | | | Fractions bonds | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.45 | 0.50 | 0.65 | 0.65 | 0.65 | 0.39 | 0.34 | 0.24 | 0.24 | 0.24 |
| 2 | | | | | | | | | | |
| 3 | | | | | 0.55 | | | | | 0.24 |
| 4 | | | | | | | | | | |
| 5 | | | | 0.65 | 0.65 | | | | 0.25 | 0.24 |
| 6 | | | | | | | | | | |
| 7 | | | | | 0.65 | | | | | 0.24 |
| 8 | | | | | | | | | | |
| 9 | | | 0.45 | 0.65 | 0.55 | | | 0.39 | 0.24 | 0.24 |
| 10 | | | | | | | | | | |
| 11 | | | | | 0.65 | | | | | 0.24 |
| 12 | | | | | | | | | | |
| 13 | | | | 0.65 | 0.65 | | | | 0.24 | 0.25 |
| 14 | | | | | | | | | | |
| 15 | | | | | 0.55 | | | | | 0.24 |
| 16 | | | | | | | | | | |
| 17 | | 0.45 | 0.65 | 0.65 | 0.55 | | 0.39 | 0.24 | 0.24 | 0.24 |
| 18 | | | | | | | | | | |
| 19 | | | | | 0.65 | | | | | 0.29 |
| 20 | | | | | | | | | | |
| 21 | | | | 0.60 | 0.65 | | | | 0.24 | 0.24 |
| 22 | | | | | | | | | | |
| 23 | | | | | 0.56 | | | | | 0.24 |
| 24 | | | | | | | | | | |
| 25 | | | 0.45 | 0.60 | 0.55 | | | 0.39 | 0.24 | 0.24 |
| 26 | | | | | | | | | | |
| 27 | | | | | 0.62 | | | | | 0.27 |
| 28 | | | | | | | | | | |
| 29 | | | | 0.45 | 0.55 | | | | 0.39 | 0.24 |
| 30 | | | | | | | | | | |
| 31 | | | | | 0.45 | | | | | 0.39 |
| 32 | | | | | | | | | | |

| $s$ | Fractions real estate | | | | | Fractions cash | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.16 | 0.16 | 0.11 | 0.06 | 0.06 | 0 | 0 | 0 | 0.05 | 0.05 |
| 2 | | | | | | | | | | |
| 3 | | | | | 0.16 | | | | | 0.05 |
| 4 | | | | | | | | | | |
| 5 | | | | 0.06 | 0.06 | | | | 0.04 | 0.05 |
| 6 | | | | | | | | | | |
| 7 | | | | | 0.06 | | | | | 0.05 |
| 8 | | | | | | | | | | |
| 9 | | | 0.16 | 0.11 | 0.16 | | | 0 | 0 | 0.05 |
| 10 | | | | | | | | | | |
| 11 | | | | | 0.06 | | | | | 0.05 |
| 12 | | | | | | | | | | |
| 13 | | | | 0.06 | 0.06 | | | | 0.05 | 0.04 |
| 14 | | | | | | | | | | |
| 15 | | | | | 0.16 | | | | | 0.05 |
| 16 | | | | | | | | | | |
| 17 | | 0.16 | 0.11 | 0.06 | 0.16 | | 0 | 0 | 0.05 | 0.05 |
| 18 | | | | | | | | | | |
| 19 | | | | | 0.06 | | | | | 0 |
| 20 | | | | | | | | | | |
| 21 | | | | 0.16 | 0.11 | | | | 0 | 0 |
| 22 | | | | | | | | | | |
| 23 | | | | | 0.16 | | | | | 0.04 |
| 24 | | | | | | | | | | |
| 25 | | | 0.16 | 0.16 | 0.16 | | | 0 | 0 | 0.05 |
| 26 | | | | | | | | | | |
| 27 | | | | | 0.06 | | | | | 0.05 |
| 28 | | | | | | | | | | |
| 29 | | | | 0.16 | 0.16 | | | | 0 | 0.05 |
| 30 | | | | | | | | | | |
| 31 | | | | | 0.16 | | | | | 0 |
| 32 | | | | | | | | | | |

| *s* | Portfolio returns | | | | | Contribution rates | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.118 | 0.132 | 0.176 | 0.180 | 0.180 | 0.06 | 0.03 | 0 | 0 | 0 |
| 2 | | | | | 0.000 | | | | | |
| 3 | | | | -0.000 | 0.149 | | | | | 0 |
| 4 | | | | | 0.027 | | | | | |
| 5 | | | 0.007 | 0.180 | 0.180 | | | | 0 | 0 |
| 6 | | | | | 0.000 | | | | | |
| 7 | | | | 0.000 | 0.181 | | | | | 0 |
| 8 | | | | | 0.001 | | | | | |
| 9 | | 0.040 | 0.118 | 0.176 | 0.149 | | | 0 | 0 | 0 |
| 10 | | | | | 0.027 | | | | | |
| 11 | | | | 0.007 | 0.181 | | | | | 0 |
| 12 | | | | | 0.000 | | | | | |
| 13 | | | 0.050 | 0.181 | 0.180 | | | | 0 | 0 |
| 14 | | | | | 0.001 | | | | | |
| 15 | | | | 0.001 | 0.149 | | | | | 0 |
| 16 | | | | | 0.028 | | | | | |
| 17 | 0.050 | 0.118 | 0.176 | 0.180 | 0.149 | | 0.03 | 0 | 0 | 0 |
| 18 | | | | | 0.027 | | | | | |
| 19 | | | | 0.000 | 0.178 | | | | | 0 |
| 20 | | | | | 0.003 | | | | | |
| 21 | | | 0.007 | 0.161 | 0.176 | | | | 0 | 0 |
| 22 | | | | | 0.007 | | | | | |
| 23 | | | | 0.020 | 0.152 | | | | | 0 |
| 24 | | | | | 0.026 | | | | | |
| 25 | | 0.050 | 0.118 | 0.160 | 0.149 | | | 0 | 0 | 0 |
| 26 | | | | | 0.027 | | | | | |
| 27 | | | | 0.020 | 0.173 | | | | | 0 |
| 28 | | | | | 0.006 | | | | | |
| 29 | | | 0.050 | 0.118 | 0.149 | | | | 0 | 0 |
| 30 | | | | | 0.028 | | | | | |
| 31 | | | | 0.050 | 0.118 | | | | | 0 |
| 32 | | | | | 0.005 | | | | | |

| $s$ | $\delta$ | | | | | | $d$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | | | | | 0 | | | | | | 0 |
| 3 | | | | | 0 | 0 | | | | | 0 | 0 |
| 4 | | | | | | 0 | | | | | | 0 |
| 5 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 6 | | | | | | 0 | | | | | | 0 |
| 7 | | | | | 0 | 0 | | | | | 0 | 0 |
| 8 | | | | | | 0 | | | | | | 0 |
| 9 | | | 0 | 0 | 0 | 0 | | | 0 | 0 | 0 | 0 |
| 10 | | | | | | 0 | | | | | | 0 |
| 11 | | | | | | 0 | 0 | | | | 0 | 0 |
| 12 | | | | | | 0 | | | | | | 0 |
| 13 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 14 | | | | | | 0 | | | | | | 0 |
| 15 | | | | | 0 | 0 | | | | | 0 | 0 |
| 16 | | | | | | 0 | | | | | | 0 |
| 17 | | 1 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| 18 | | | | | | 0 | | | | | | 0 |
| 19 | | | | | 0 | 0 | | | | | 0 | 0 |
| 20 | | | | | | 0 | | | | | | 0 |
| 21 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 22 | | | | | | 0 | | | | | | 0 |
| 23 | | | | | 0 | 0 | | | | | 0 | 0 |
| 24 | | | | | | 0 | | | | | | 0 |
| 25 | | | 1 | 0 | 0 | 0 | | | 1 | 0 | 0 | 0 |
| 26 | | | | | | 0 | | | | | | 0 |
| 27 | | | | | 0 | 0 | | | | | 0 | 0 |
| 28 | | | | | | 0 | | | | | | 0 |
| 29 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 30 | | | | | | 0 | | | | | | 0 |
| 31 | | | | | 0 | 0 | | | | | 0 | 0 |
| 32 | | | | | | 1 | | | | | | 0 |

| $s$ | Remedial contributions | | | | | | Funding ratios | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1.100 | 1.099 | 1.182 | 1.343 | 1.521 | 1.751 |
| 2 | | | | | | 0 | | | | | | 1.467 |
| 3 | | | | | 0 | 0 | | | | | 1.292 | 1.410 |
| 4 | | | | | | 0 | | | | | | 1.276 |
| 5 | | | | 0 | 0 | 0 | | | | 1.136 | 1.318 | 1.505 |
| 6 | | | | | | 0 | | | | | | 1.221 |
| 7 | | | | | 0 | 0 | | | | | 1.098 | 1.271 |
| 8 | | | | | | 0 | | | | | | 1.080 |
| 9 | | | 0 | 0 | 0 | 0 | | | 1.068 | 1.185 | 1.311 | 1.441 |
| 10 | | | | | | 0 | | | | | | 1.278 |
| 11 | | | | | | 0 | 0 | | | | 1.118 | 1.292 |
| 12 | | | | | | 0 | | | | | | 1.055 |
| 13 | | | | 0 | 0 | 0 | | | | 1.050 | 1.272 | 1.360 |
| 14 | | | | | | 0 | | | | | | 1.140 |
| 15 | | | | | 0 | 0 | | | | | 1.080 | 1.191 |
| 16 | | | | | | 0 | | | | | | 1.051 |
| 17 | | 1 | 0 | 0 | 0 | 0 | | 1.029 | 1.114 | 1.253 | 1.403 | 1.510 |
| 18 | | | | | | 0 | | | | | | 1.341 |
| 19 | | | | | 0 | 0 | | | | | 1.147 | 1.344 |
| 20 | | | | | | 0 | | | | | | 1.127 |
| 21 | | | | 0 | 0 | 0 | | | | 1.068 | 1.213 | 1.288 |
| 22 | | | | | | 0 | | | | | | 1.097 |
| 23 | | | | | 0 | 0 | | | | | 1.061 | 1.167 |
| 24 | | | | | | 0 | | | | | | 1.050 |
| 25 | | | 190 | 0 | 0 | 0 | | | 1.035 | 1.170 | 1.214 | 1.333 |
| 26 | | | | | | 0 | | | | | | 1.193 |
| 27 | | | | | 0 | 0 | | | | | 1.069 | 1.256 |
| 28 | | | | | | 0 | | | | | | 1.050 |
| 29 | | | | 0 | 0 | 0 | | | | 1.050 | 1.170 | 1.200 |
| 30 | | | | | | 0 | | | | | | 1.081 |
| 31 | | | | | 0 | 0 | | | | | 1.081 | 1.192 |
| 32 | | | | | | 1 | | | | | | 1.036 |

**Appendix C:**

**Basic instance: heuristic solution**

| *s* | Fractions stocks | | | | | Fractions bonds | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.45 | 0.49 | 0.65 | 0.65 | 0.65 | 0.39 | 0.35 | 0.24 | 0.24 | 0.24 |
| 2 | | | | | | | | | | |
| 3 | | | | | 0.60 | | | | | 0.24 |
| 4 | | | | | | | | | | |
| 5 | | | | 0.65 | 0.60 | | | | 0.24 | 0.24 |
| 6 | | | | | | | | | | |
| 7 | | | | | 0.65 | | | | | 0.24 |
| 8 | | | | | | | | | | |
| 9 | | | 0.45 | 0.65 | 0.60 | | | 0.39 | 0.24 | 0.24 |
| 10 | | | | | | | | | | |
| 11 | | | | | 0.65 | | | | | 0.24 |
| 12 | | | | | | | | | | |
| 13 | | | | 0.60 | 0.55 | | | | 0.24 | 0.24 |
| 14 | | | | | | | | | | |
| 15 | | | | | 0.63 | | | | | 0.24 |
| 16 | | | | | | | | | | |
| 17 | | 0.45 | 0.65 | 0.65 | 0.65 | | 0.39 | 0.24 | 0.24 | 0.24 |
| 18 | | | | | | | | | | |
| 19 | | | | | 0.55 | | | | | 0.24 |
| 20 | | | | | | | | | | |
| 21 | | | | 0.58 | 0.55 | | | | 0.24 | 0.24 |
| 22 | | | | | | | | | | |
| 23 | | | | | 0.55 | | | | | 0.24 |
| 24 | | | | | | | | | | |
| 25 | | | 0.45 | 0.65 | 0.65 | | | 0.39 | 0.24 | 0.24 |
| 26 | | | | | | | | | | |
| 27 | | | | | 0.56 | | | | | 0.24 |
| 28 | | | | | | | | | | |
| 29 | | | | 0.45 | 0.65 | | | | 0.39 | 0.29 |
| 30 | | | | | | | | | | |
| 31 | | | | | 0.45 | | | | | 0.39 |
| 32 | | | | | | | | | | |

| *s* | Fractions real estate | | | | | Fractions cash | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.16 | 0.16 | 0.11 | 0.11 | 0.11 | 0 | 0 | 0 | 0 | 0 |
| 2 | | | | | | | | | | |
| 3 | | | | | 0.16 | | | | | 0 |
| 4 | | | | | | | | | | |
| 5 | | | | 0.11 | 0.16 | | | | 0 | 0 |
| 6 | | | | | | | | | | |
| 7 | | | | | 0.11 | | | | | 0 |
| 8 | | | | | | | | | | |
| 9 | | | 0.16 | 0.11 | 0.16 | | | 0 | 0 | 0 |
| 10 | | | | | | | | | | |
| 11 | | | | | 0.06 | | | | | 0.05 |
| 12 | | | | | | | | | | |
| 13 | | | | 0.16 | 0.16 | | | | 0 | 0.05 |
| 14 | | | | | | | | | | |
| 15 | | | | | 0.08 | | | | | 0.05 |
| 16 | | | | | | | | | | |
| 17 | | 0.16 | 0.11 | 0.06 | 0.06 | | 0 | 0 | 0.05 | 0.05 |
| 18 | | | | | | | | | | |
| 19 | | | | | 0.16 | | | | | 0.05 |
| 20 | | | | | | | | | | |
| 21 | | | | 0.13 | 0.16 | | | | 0.05 | 0.05 |
| 22 | | | | | | | | | | |
| 23 | | | | | 0.16 | | | | | 0.05 |
| 24 | | | | | | | | | | |
| 25 | | | 0.16 | 0.11 | 0.11 | | | 0 | 0 | 0 |
| 26 | | | | | | | | | | |
| 27 | | | | | 0.15 | | | | | 0.05 |
| 28 | | | | | | | | | | |
| 29 | | | | 0.16 | 0.06 | | | | 0 | 0 |
| 30 | | | | | | | | | | |
| 31 | | | | | 0.16 | | | | | 0 |
| 32 | | | | | | | | | | |

| $s$ | Portfolio returns | | | | | Contribution rates | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.118 | 0.130 | 0.176 | 0.176 | 0.176 | 0.03 | 0 | 0 | 0 | 0 |
| 2 | | | | | 0.000 | | | | | |
| 3 | | | | 0.007 | 0.160 | | | | | 0 |
| 4 | | | | | 0.020 | | | | | |
| 5 | | | 0.007 | 0.176 | 0.160 | | | | 0 | 0 |
| 6 | | | | | 0.000 | | | | | |
| 7 | | | | 0.007 | 0.176 | | | | | 0 |
| 8 | | | | | 0.007 | | | | | |
| 9 | | 0.041 | 0.118 | 0.176 | 0.160 | | | 0 | 0 | 0 |
| 10 | | | | | 0.020 | | | | | |
| 11 | | | | 0.007 | 0.181 | | | | | 0 |
| 12 | | | | | 0.000 | | | | | |
| 13 | | | 0.050 | 0.160 | 0.149 | | | | 0 | 0 |
| 14 | | | | | 0.027 | | | | | |
| 15 | | | | 0.020 | 0.175 | | | | | 0 |
| 16 | | | | | 0.006 | | | | | |
| 17 | 0.050 | 0.118 | 0.176 | 0.180 | 0.180 | | 0 | 0 | 0 | 0 |
| 18 | | | | | -0.000 | | | | | |
| 19 | | | | 0.000 | 0.149 | | | | | 0 |
| 20 | | | | | 0.028 | | | | | |
| 21 | | | 0.007 | 0.158 | 0.149 | | | | 0 | 0 |
| 22 | | | | | 0.027 | | | | | |
| 23 | | | | 0.020 | 0.149 | | | | | 0 |
| 24 | | | | | 0.028 | | | | | |
| 25 | | 0.050 | 0.118 | 0.154 | 0.176 | | | 0 | 0 | 0 |
| 26 | | | | | 0.007 | | | | | |
| 27 | | | | 0.024 | 0.153 | | | | | 0 |
| 28 | | | | | 0.024 | | | | | |
| 29 | | | 0.050 | 0.118 | 0.150 | | | | 0 | 0 |
| 30 | | | | | 0.022 | | | | | |
| 31 | | | | 0.050 | 0.118 | | | | | 0 |
| 32 | | | | | 0.005 | | | | | |

| $s$ | $\delta$ | | | | | | $d$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | | | | | | 0 | | | | | | 0 |
| 3 | | | | | 0 | 0 | | | | | 0 | 0 |
| 4 | | | | | | 0 | | | | | | 0 |
| 5 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 6 | | | | | | 0 | | | | | | 0 |
| 7 | | | | | 0 | 0 | | | | | 0 | 0 |
| 8 | | | | | | 0 | | | | | | 0 |
| 9 | | | 0 | 0 | 0 | 0 | | | 0 | 0 | 0 | 0 |
| 10 | | | | | | 0 | | | | | | 0 |
| 11 | | | | | | 0 | 0 | | | | 0 | 0 |
| 12 | | | | | | 0 | | | | | | 0 |
| 13 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 14 | | | | | | 0 | | | | | | 0 |
| 15 | | | | | 0 | 0 | | | | | 0 | 0 |
| 16 | | | | | | 0 | | | | | | 0 |
| 17 | | 1 | 0 | 0 | 0 | 0 | | 0 | 0 | 0 | 0 | 0 |
| 18 | | | | | | 0 | | | | | | 0 |
| 19 | | | | | 0 | 0 | | | | | 0 | 0 |
| 20 | | | | | | 0 | | | | | | 0 |
| 21 | | | | 0 | 0 | 0 | | | | 0 | 0 | 0 |
| 22 | | | | | | 0 | | | | | | 0 |
| 23 | | | | | 0 | 0 | | | | | 0 | 0 |
| 24 | | | | | | 0 | | | | | | 0 |
| 25 | | | 1 | 0 | 0 | 0 | | | 1 | 0 | 0 | 0 |
| 26 | | | | | | 0 | | | | | | 0 |
| 27 | | | | | 0 | 0 | | | | | 0 | 0 |
| 28 | | | | | | 0 | | | | | | 0 |
| 29 | | | | 1 | 0 | 0 | | | | 1 | 0 | 0 |
| 30 | | | | | | 0 | | | | | | 0 |
| 31 | | | | | 0 | 0 | | | | | 0 | 0 |
| 32 | | | | | | 1 | | | | | | 0 |

| $s$ | Remedial contributions | | | | | | Funding ratios | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1.100 | 1.098 | 1.178 | 1.338 | 1.510 | 1.731 |
| 2 | | | | | | 0 | | | | | | 1.466 |
| 3 | | | | | 0 | 0 | | | | | 1.297 | 1.430 |
| 4 | | | | | | 0 | | | | | | 1.272 |
| 5 | | | | 0 | 0 | 0 | | | | 1.132 | 1.308 | 1.466 |
| 6 | | | | | | 0 | | | | | | 1.237 |
| 7 | | | | | 0 | 0 | | | | | 1.101 | 1.269 |
| 8 | | | | | | 0 | | | | | | 1.090 |
| 9 | | | 0 | 0 | 0 | 0 | | | 1.068 | 1.185 | 1.311 | 1.456 |
| 10 | | | | | | 0 | | | | | | 1.269 |
| 11 | | | | | | 0 | 0 | | | | 1.118 | 1.292 |
| 12 | | | | | | 0 | | | | | | 1.055 |
| 13 | | | | 0 | 0 | 0 | | | | 1.050 | 1.248 | 1.297 |
| 14 | | | | | | 0 | | | | | | 1.150 |
| 15 | | | | | 0 | 0 | | | | | 1.102 | 1.246 |
| 16 | | | | | | 0 | | | | | | 1.050 |
| 17 | | 1 | 0 | 0 | 0 | 0 | | 1.028 | 1.112 | 1.251 | 1.401 | 1.550 |
| 18 | | | | | | 0 | | | | | | 1.302 |
| 19 | | | | | 0 | 0 | | | | | 1.145 | 1.307 |
| 20 | | | | | | 0 | | | | | | 1.155 |
| 21 | | | | 0 | 0 | 0 | | | | 1.066 | 1.209 | 1.252 |
| 22 | | | | | | 0 | | | | | | 1.116 |
| 23 | | | | | 0 | 0 | | | | | 1.059 | 1.161 |
| 24 | | | | | | 0 | | | | | | 1.050 |
| 25 | | | 167 | 0 | 0 | 0 | | | 1.033 | 1.166 | 1.226 | 1.380 |
| 26 | | | | | | 0 | | | | | | 1.180 |
| 27 | | | | | 0 | 0 | | | | | 1.050 | 1.211 |
| 28 | | | | | | 0 | | | | | | 1.050 |
| 29 | | | | 40 | 0 | 0 | | | | 1.046 | 1.166 | 1.225 |
| 30 | | | | | | 0 | | | | | | 1.050 |
| 31 | | | | | 0 | 0 | | | | | 1.077 | 1.187 |
| 32 | | | | | | 1 | | | | | | 1.032 |

## Appendix D:

## Instance 2

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 8 | | 8 |
| Remedial contributions | 0 | 0 | 0 | 0 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0 | 0 | 0 | 0 |
| Fixed costs $Z > 0$ | 0 | 0 | 0 | 0 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 28 | 42 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 8200 | -36.9 | 8200 | -36.9 |
| Shortage | 0 | 0 | 0 | 0 |
| **Total** | | 13 | | 13 |

Table 4.1: Decisions and contributions to the objective function for model instance 2.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.47 | 0 | 0.53 | 0 | 0.111 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.62 | 0 | 0.38 | 0 | 0.157 | 0 | 0 | 0 | 0 | 1.089 |
| | 2 | 0.97 | 0 | 0.03 | 0 | 0.269 | 0 | 0 | 0 | 0 | 1.197 |
| | 3 | 1.00 | 0 | 0.00 | 0 | 0.278 | 0 | 0 | 0 | 0 | 1.472 |
| | 4 | 0.59 | 0 | 0.41 | 0 | 0.147 | 0 | 0 | 0 | 0 | 1.817 |
| | 5 | | | | | | | 0 | 0 | 0 | 2.041 |
| 25 | 0 | 0.47 | 0 | 0.53 | 0 | 0.073 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.58 | 0 | 0.41 | 0 | 0.043 | 0 | 0 | 0 | 0 | 1.050 |
| | 2 | 0.55 | 0 | 0.45 | 0 | 0.134 | 0 | 0 | 0 | 0 | 1.050 |
| | 3 | 0.77 | 0 | 0.23 | 0 | 0.206 | 0 | 0 | 0 | 0 | 1.183 |
| | 4 | 1.00 | 0 | 0.00 | 0 | 0.278 | 0 | 0 | 0 | 0 | 1.279 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.574 |
| 32 | 0 | 0.47 | 0 | 0.53 | 0 | 0.073 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.58 | 0 | 0.41 | 0 | 0.043 | 0 | 0 | 0 | 0 | 1.050 |
| | 2 | 0.55 | 0 | 0.45 | 0 | 0.053 | 0 | 0 | 0 | 0 | 1.050 |
| | 3 | 0.67 | 0 | 0.33 | 0 | 0.021 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.40 | 0 | 0.60 | 0 | 0.094 | 0 | 0 | 0 | 0 | 1.050 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.050 |

Table 4.2: Results optimal solution for model instance 2.

**Appendix E:**

**Instance 3**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 56 | | 9 |
| Remedial contributions | | 889 | | 1,102 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 2.1 | 413 | 2 | 393 |
| Fixed costs $Z > 0$ | 0.9 | 565 | 0.9 | 565 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 15 | 23 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,214 | -28 | 6,497 | -29 |
| Shortage | 132 | 0 | 100 | 0 |
| **Total** | | 1,920 | | 2,082 |

Table 5.1: Decisions and contributions to the objective function for model instance 3.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.128 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.11 | 1 | 1 | 643 | 1.006 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0.03 | 0 | 0 | 0 | 1.136 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.289 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.453 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.641 |
| 25 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.043 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.47 | 0.37 | 0.16 | 0 | 0.046 | 0.03 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.054 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.231 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.387 |
| 32 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.043 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.47 | 0.37 | 0.16 | 0 | 0.046 | 0.03 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.054 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 5.2: Results optimal solution for model instance 3.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.118 | 0.03 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.130 | 0 | 1 | 1 | 1,198 | 0.999 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.155 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.310 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.180 | 0 | 0 | 0 | 0 | 1.478 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.693 |
| 25 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.057 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.150 | 0 | 0 | 0 | 0 | 1.173 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.235 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.371 |
| 32 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.057 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.053 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.084 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.040 |

Table 5.3: Results heuristic solution for model instance 3.

**Appendix F:**

**Instance 4**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 23 | | 7 |
| Remedial contributions | | 41 | | 40 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | 41 | 4 | 40 | 4 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 21 | 31 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,444 | -29 | 6,422 | -29 |
| Shortage | 132 | 0 | 177 | 0 |
| **Total** | | 341 | | 418 |

Table 6.1: Decisions and contributions to the objective function for model instance 4.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.50 | 0.34 | 0.16 | 0 | 0.132 | 0.03 | 0 | 0 | 0 | 1.099 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.182 |
| | 3 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.343 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.521 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.751 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.160 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.214 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.333 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 6.2: Results optimal solution for model instance 4.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.130 | 0 | 0 | 0 | 0 | 1.097 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.178 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.338 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.180 | 0 | 0 | 0 | 0 | 1.510 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.731 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.150 | 0 | 0 | 0 | 0 | 1.166 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.226 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.380 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 40 | 1.046 |
| | 4 | 0.55 | 0.29 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.077 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.032 |

Table 6.3: Results heuristic solution for model instance 4.

**Instance 5**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 64 | | 9 |
| Remedial contributions | | 60 | | 143 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 17 | 25 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,582 | -30 | 6,353 | -29 |
| Shortage | 132 | 0 | 231 | 0 |
| **Total** | | 391 | | 517 |

Table 6.4: Decisions and contributions to the objective function for model instance 5.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.14 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.51 | 0.33 | 0.16 | 0 | 0.135 | 0.03 | 0 | 0 | 0 | 1.101 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.187 |
| | 3 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.349 |
| | 4 | 0.65 | 0.29 | 0.06 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.529 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.756 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.14 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.11 | 1 | 0 | 0 | 1.031 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 1 | 1 | 111 | 1.039 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.160 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.215 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.333 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.14 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.11 | 1 | 0 | 0 | 1.031 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.14 | 1 | 1 | 111 | 1.039 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.082 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.037 |

Table 6.5: Results optimal solution for model instance 5.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.130 | 0 | 0 | 0 | 0 | 1.098 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.178 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.338 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.510 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.731 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.58 | 0.26 | 0.16 | 0 | 0.154 | 0 | 0 | 0 | 0 | 1.147 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.184 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.330 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 208 | 1.029 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 6.6: Results heuristic solution for model instance 5.

**Instance 6**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 89 | | 0 |
| Remedial contributions | | 254 | | 334 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.8 | 162 | 0.9 | 186 |
| Fixed costs $Z > 0$ | 0.7 | 411 | 0.9 | 544 |
| Penalties variable costs $Z$ | -0 | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 24 | 35 | 36 | 54 |
| **Penalties horizon** | | | | |
| Surplus | 3,422 | -15 | 3,622 | -16 |
| Shortage | 132 | 0 | 32 | 0 |
| **Total** | | 937 | | 1,053 |

Table 6.7: Decisions and contributions to the objective function for model instance 6.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.46 | 0.38 | 0.16 | 0 | 0.103 | 0.20 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0.17 | 0 | 0 | 0 | 1.087 |
| | 2 | 0.60 | 0.24 | 0.16 | 0 | 0.136 | 0.03 | 0 | 0 | 0 | 1.138 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.136 | 0 | 0 | 0 | 0 | 1.245 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.136 | 0 | 0 | 0 | 0 | 1.352 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.489 |
| 13 | 0 | 0.46 | 0.38 | 0.16 | 0 | 0.103 | 0.20 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0.17 | 0 | 0 | 0 | 1.087 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0.07 | 0 | 0 | 0 | 1.050 |
| | 3 | 0.45 | 0.36 | 0.16 | 0.03 | 0.102 | 0.04 | 1 | 0 | 0 | 1.015 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0.01 | 0 | 0 | 0 | 1.141 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.127 |
| 25 | 0 | 0.46 | 0.38 | 0.16 | 0 | 0.029 | 0.20 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0.06 | 1 | 0 | 0 | 1.011 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0.03 | 1 | 1 | 393 | 1.039 |
| | 3 | 0.55 | 0.24 | 0.16 | 0.05 | 0.127 | 0 | 0 | 0 | 0 | 1.171 |
| | 4 | 0.45 | 0.35 | 0.16 | 0.05 | 0.103 | 0 | 0 | 0 | 0 | 1.179 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.238 |
| 32 | 0 | 0.46 | 0.38 | 0.16 | 0 | 0.029 | 0.20 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0.06 | 1 | 0 | 0 | 1.011 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0.03 | 1 | 1 | 393 | 1.039 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.062 |
| | 5 | | | | | | | 1 | 0 | 0 | 0.997 |

Table 6.8: Results optimal solution for model instance 6.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.39 | 0.16 | 0 | 0.100 | 0 | 0 | 0 | 0 | 1.079 |
| | 2 | 0.60 | 0.24 | 0.16 | 0 | 0.127 | 0 | 0 | 0 | 0 | 1.125 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.136 | 0 | 0 | 0 | 0 | 1.230 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.126 | 0 | 0 | 0 | 0 | 1.334 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.456 |
| 13 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.079 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 1 | 1 | 501 | 1.038 |
| | 3 | 0.45 | 0.34 | 0.16 | 0.05 | 0.112 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0 | 0 | 0 | 0 | 1.184 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.171 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 1 | 1 | 417 | 1.009 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.100 | 0 | 1 | 1 | 125 | 1.038 |
| | 3 | 0.56 | 0.28 | 0.16 | 0 | 0.100 | 0 | 0 | 0 | 0 | 1.171 |
| | 4 | 0.64 | 0.30 | 0.06 | 0.05 | 0.120 | 0 | 0 | 0 | 0 | 1.177 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.293 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 1 | 1 | 417 | 1.009 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 1 | 1 | 125 | 1.038 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.032 | 0 | 0 | 0 | 0 | 1.062 |
| | 5 | | | | | | | 1 | 0 | 0 | 0.998 |

Table 6.9: Results heuristic solution for model instance 6.

## Appendix G:

## Instance 7

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 72 | | 9 |
| Remedial contributions | | 962 | | 1,102 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 1.9 | 1,457 | 2 | 1474 |
| Fixed costs $Z > 0$ | 0.9 | 565 | 0.9 | 565 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 12 | 18 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,561 | -30 | 6,497 | -29 |
| Shortage | 0 | 0 | 100 | 0 |
| **Total** | | 3,045 | | 3,163 |

Table 7.1: Decisions and contributions to the objective function for model instance 7.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.128 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.11 | 1 | 1 | 796 | 1.006 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0.03 | 0 | 0 | 0 | 1.153 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.309 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.477 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.670 |
| 25 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.043 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.046 | 0.06 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.058 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.238 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.395 |
| 32 | 0 | 0.49 | 0.35 | 0.16 | 0 | 0.043 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.046 | 0.06 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.058 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.056 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.21 | 0 | 0 | 0 | 1.089 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.050 |

Table 7.2: Results optimal solution for model instance 7.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.118 | 0.03 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 1,198 | 0.999 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.155 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.310 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.478 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.693 |
| 25 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 1,250 | 0.926 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.057 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.150 | 0 | 0 | 0 | 0 | 1.173 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.235 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.371 |
| 32 | 0 | 0.47 | 0.37 | 0.16 | 0 | 0.050 | 0.14 | 1 | 0 | 0 | 1.000 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 1 | 1,250 | 0.932 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.063 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.053 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.084 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.040 |

Table 7.3: Results heuristic solution for model instance 7.

**Instance 8**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 22.7 | | 0 |
| Remedial contributions | | 41 | | 42 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 21 | 31 | 36 | 53 |
| **Penalties horizon** | | | | |
| Surplus | | 0 | | 0 |
| Shortage | 0 | 0 | 0 | 0 |
| **Total** | | 366 | | 399 |

Table 7.4: Decisions and contributions to the objective function for model instance 8.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.099 |
| | 2 | 0.58 | 0.36 | 0.06 | 0 | 0.157 | 0 | 0 | 0 | 0 | 1.167 |
| | 3 | 0.56 | 0.33 | 0.06 | 0.05 | 0.156 | 0 | 0 | 0 | 0 | 1.303 |
| | 4 | 0.65 | 0.25 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.443 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.657 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.56 | 0.38 | 0.06 | 0 | 0.154 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.207 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.325 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 7.5: Results optimal solution for model instance 8.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.130 | 0 | 0 | 0 | 0 | 1.097 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.338 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.507 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.735 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 176 | 1.032 |
| | 3 | 0.55 | 0.24 | 0.16 | 0.05 | 0.150 | 0 | 0 | 0 | 0 | 1.166 |
| | 4 | 0.65 | 0.29 | 0.06 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.198 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.350 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 176 | 1.032 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 40 | 1.046 |
| | 4 | 0.55 | 0.29 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.077 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.012 |

Table 7.6: Results in heuristic solution for model instance 8.

**Instance 9**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 23 | | 7 |
| Remedial contributions | | 41 | | 40 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 21 | 31 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,619 | -66 | 6,460 | -65 |
| Shortage | 132 | 0 | 177 | 0 |
| **Total** | | 301 | | 376 |

Table 7.7: Decisions and contributions to the objective function for model instance 9.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.50 | 0.34 | 0.16 | 0 | 0.132 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.182 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.343 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.516 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.738 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.232 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.387 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 190 | 1.035 |
| | 3 | 0.59 | 0.24 | 0.16 | 0.01 | 0.021 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.020 | 0 | 0 | 0 | 0 | 1.050 |
| | 5 | | | | | | | 1 | 0 | 0 | 0.974 |

Table 7.8: Results optimal solution for model instance 9.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.130 | 0 | 0 | 0 | 0 | 1.098 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.178 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.338 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.510 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.731 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.58 | 0.26 | 0.16 | 0 | 0.154 | 0 | 0 | 0 | 0 | 1.146 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.182 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.328 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.028 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 167 | 1.033 |
| | 3 | 0.59 | 0.24 | 0.16 | 0.01 | 0.050 | 0 | 1 | 1 | 40 | 1.028 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 5 | | | | | | | 1 | 0 | 0 | 0.974 |

Table 7.9: Results heuristic solution for model instance 9.

**Instance 10**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 44 | | 30 |
| Remedial contributions | | 34 | | 35 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 16 | 24 | 32 | 48 |
| **Penalties horizon** | | | | |
| Surplus | 6,556 | -30 | 6,489 | -29 |
| Shortage | 16 | 0 | 16 | 0 |
| **Total** | | 345 | | 436 |

Table 7.10: Decisions and contributions to the objective function for model instance 10.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.51 | 0.33 | 0.16 | 0 | 0.134 | 0.03 | 0 | 0 | 0 | 1.099 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.167 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.303 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.443 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.657 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 1 | 1 | 159 | 1.035 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.207 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.325 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 1 | 0 | 0 | 1.029 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 1 | 159 | 1.035 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 7.11: Results optimal solution for model instance 10.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.50 | 0.34 | 0.16 | 0 | 0.133 | 0 | 0 | 0 | 0 | 1.100 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.184 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.345 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.518 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.747 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.030 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 144 | 1.036 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.158 | 0 | 0 | 0 | 0 | 1.166 |
| | 4 | 0.65 | 0.29 | 0.06 | 0 | 0.178 | 0 | 0 | 0 | 0 | 1.226 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.383 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.12 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.030 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 144 | 1.036 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 40 | 1.046 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.077 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.032 |

Table 7.12: Results heuristic solution for model instance 10.

**Instance 11**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 0 | | 0 |
| Remedial contributions | | 46 | | 47 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.7 | 141 | 0.8 | 161 |
| Fixed costs $Z > 0$ | 0.2 | 130 | 0.3 | 191 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 0 | 0 | 0 | 0 |
| **Penalties horizon** | | | | |
| Surplus | 6,355 | -29 | 6,365 | -29 |
| Shortage | 132 | 0 | 231 | 0 |
| **Total** | | 289 | | 370 |

Table 7.13: Decisions and contributions to the objective function for model instance 11.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.129 | 0 | 0 | 0 | 0 | 1.100 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 3 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.336 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.180 | 0 | 0 | 0 | 0 | 1.513 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.741 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 214 | 1.032 |
| | 3 | 0.60 | 0.24 | 0.16 | 0 | 0.160 | 0 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.55 | 0.24 | 0.16 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.214 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.333 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 214 | 1.032 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.036 |

Table 7.14: Results optimal solution for model instance 11.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.129 | 0 | 0 | 0 | 0 | 1.097 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.336 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.507 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.728 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.47 | 0.37 | 0.16 | 0 | 0.118 | 0 | 1 | 1 | 176 | 1.032 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.153 | 0 | 0 | 0 | 0 | 1.173 |
| | 4 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.235 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.391 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 0 | 0 | 1.027 |
| | 2 | 0.47 | 0.37 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 176 | 1.032 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 90 | 1.041 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.071 |
| | 5 | | | | | | | 1 | 0 | 0 | 1.026 |

Table 7.15: Results heuristic solution for model instance 11.

**Instance 12**

| | Optimal | | Heuristic | |
|---|---|---|---|---|
| | decisions | objective | decisions | objective |
| **Funding costs** | | | | |
| Contributions participants | | 31 | | 8 |
| Remedial contributions | | 100 | | 109 |
| **Penalties underfunding** | | | | |
| Fixed costs underfunding | 0.5 | 94 | 0.5 | 98 |
| Fixed costs $Z > 0$ | 0.5 | 281 | 0.5 | 293 |
| Penalties variable costs $Z$ | - | 0 | - | 0 |
| **Penalties contribution** | | | | |
| Large increases and decreases | 21 | 33 | 28 | 42 |
| **Penalties horizon** | | | | |
| Surplus | 6,455 | -30 | 6,401 | -30 |
| Shortage | 0 | 0 | 231 | 0 |
| **Total** | | 508 | | 520 |

Table 7.16: Decisions and contributions to the objective function for model instance 12.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.50 | 0.34 | 0.16 | 0 | 0.129 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.180 | 0 | 0 | 0 | 0 | 1.336 |
| | 4 | 0.60 | 0.24 | 0.16 | 0 | 0.180 | 0 | 0 | 0 | 0 | 1.513 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.741 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 1 | 215 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.05 | 0 | 0 | 0 | 1.032 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.160 | 0.02 | 0 | 0 | 0 | 1.170 |
| | 4 | 0.65 | 0.24 | 0.06 | 0.05 | 0.149 | 0 | 0 | 0 | 0 | 1.214 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.333 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.06 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 1 | 0 | 215 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.05 | 0 | 0 | 0 | 1.032 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.08 | 0 | 0 | 0 | 1.050 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.21 | 0 | 0 | 0 | 1.081 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.036 |

Table 7.17: Results optimal solution for model instance 12.

| $s$ | $t$ | stocks | bonds | real estate | cash | $r_p$ | c | $\delta$ | d | Z | F |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.49 | 0.35 | 0.16 | 0 | 0.129 | 0 | 0 | 0 | 0 | 1.097 |
| | 2 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.176 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.176 | 0 | 0 | 0 | 0 | 1.336 |
| | 4 | 0.65 | 0.24 | 0.16 | 0.05 | 0.176 | 0 | 0 | 0 | 0 | 1.507 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.728 |
| 25 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 222 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.118 | 0 | 0 | 0 | 0 | 1.032 |
| | 3 | 0.65 | 0.24 | 0.11 | 0 | 0.153 | 0 | 0 | 0 | 0 | 1.173 |
| | 4 | 0.65 | 0.24 | 0.16 | 0.04 | 0.176 | 0 | 0 | 0 | 0 | 1.235 |
| | 5 | | | | | | | 0 | 0 | 0 | 1.391 |
| 32 | 0 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0.03 | 0 | 0 | 0 | 1.100 |
| | 1 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 1 | 1 | 222 | 1.027 |
| | 2 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.032 |
| | 3 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.041 |
| | 4 | 0.45 | 0.39 | 0.16 | 0 | 0.050 | 0 | 0 | 0 | 0 | 1.071 |
| | 5 | | | | | | | 1 | 1 | 232 | 1.026 |

Table 7.18: Results heuristic solution for model instance 12.

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