Realizations of interest rate models

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Abstract

In this paper we comment on a recent paper by Björk and Gombani [1]. In contrast to this paper our starting point is not the Musiela equation but the forward rate dynamics. In our approach we do not need to talk about infinitesimal generators.

As in [1] we consider a bond market living on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ carrying a standard d-dimensional Wiener process W. We assume that the filtration is the usual one generated by W.
Furthermore our setting is that of Björk's paper [2] and Q is assumed to be a martingale measure of

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where f describes the forward rate dynamics.
We have, see [2]

$$\forall T > 0 \quad \forall t \in [0,T] \quad : \quad \alpha(t,T) = \sigma(t,T)\int_0^T \sigma(t,s)ds.$$

Instead of starting with Musiela's parametrization of

$$r(t,x) \quad := \quad f(t,t+x)\ ,\ t \geq 0,\ x \geq 0,$$

we will "derive" it, where the " " marks mean that we assume that everything is such that we may freely apply Fubini and the stochastic version of Fubini.
For this latter theorem we refer to Theorem 44, page 158 in Protter's book [3].
This also means that we do not bother about measurability questions. So in that sense we only consider structural issues. It is far from trivial to fill in the missing details. For readers

willing to do that we refer to Döberlein's thesis [4]. We also will add now and then additional assumptions along the way when considered useful.

As said before we will first of all derive the Musiela parametrization. It really takes some effort and certainly we need more lines than for instance the one-line proof in Björk [2]. We take these efforts because we were not able to understand this very short proof, and we have even doubts whether this one-line proof is correct.

Now the promised derivation.

Step 1:

$$r(t) := f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \alpha(s,t)dW(s).$$

We assume from now on:

$$\begin{aligned} \forall\, 0 \le s \le t \quad &: \\ \alpha(s,t) &= \alpha(s.s) + \int_s^t D_2\alpha(s,u)du \\ \sigma(s,t) &= \sigma(s,s) + \int_s^t D_2\sigma(s,u)du, \end{aligned}$$

where "D_2" means differentiation after the second argument. And furthermore these assumptions hold for every $\omega \in \Omega$. So, for instance, we should read:

$$\forall\, \omega \in \Omega \quad : \quad \alpha(s,t)(\omega) = \alpha(s,s)(\omega) + \int_s^t D_2\alpha(s,u)(\omega)du.$$

In the sequel, however, we omit the dependence on ω.
Now, using these assumptions and applying "Fubini" and "stochastic Fubini" we get:

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t \sigma(s,s)dW(s) \\ &\quad + \int_0^t \int_0^u D_2\alpha(s,u)dsdu + \int_0^t \int_0^u D_2\sigma(s,u)dW(s)du. \end{aligned}$$

Now recall

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s).$$

Hence:

$$D_2 f(u,u) - D_2 f(0,u) = \int_0^u D_2\alpha(s,u)ds + \int_0^u D_2\sigma(s,u)dW(s).$$

Therefore we find:

$$r(t) = r(0) + \int_0^t (\alpha(s,s) + D_2 f(s,s))ds + \int_0^t \sigma(s,s)dW(s). \qquad (*)$$

Step 2: We take an arbitrary $x \geq 0$ and consider now

$$r(t,x) := f(t,t+x).$$

We define

$$\begin{aligned} \widetilde{f}(t,T) &:= f(t,T+x) \\ \widetilde{\alpha}(s,T) &:= \alpha(s,T+x) \\ \widetilde{\sigma}(s,T) &:= \sigma(s,T+x) \end{aligned}$$

Now notice that $r(t,x) = \widetilde{f}(t,t) := \widetilde{r}(t)$.
We now use $(*)$ to find the equation for $\widetilde{r}(t)$, and we find:

$$\begin{aligned} r(t,x) &= r(0,x) + \int_0^t (\alpha(s,s+x) + D_2 r(s,x))ds \\ &\quad + \int_0^t \sigma(s,s+x)dW(s), \end{aligned} \qquad (**)$$

We recall that $\alpha(t,T) = \sigma(t,T)\int_0^T \sigma(t,u)du$, $0 \leq t \leq T$, and we define $\sigma_0(t,x) := \sigma(t,t+x)$.
We further define $D(s,x) := \alpha(s,s+x)$ and we find:

$$r(t,x) = r(0,x) + \int_0^t (D(s,x) + D_2 r(s,x))ds + \int_0^t \sigma_0(s,x)dW(s) \qquad (\mathbf{!})$$

Equation $(\mathbf{!})$ is the parametrization of Musiela.
Given this set-up it is now easy to give a solution of equation $(\mathbf{!})$.
Recall $r(t,x) = f(t,t+x)$. We know

$$f(t,T) = f(0,T) + \int_0^s \alpha(s,T)ds + \int_0^s \sigma(s,T)dW(s)$$

and after some easy calculations we find:

$$r(t,x) \quad = \quad r(0,t+x) + \int_0^t D(s,t-s+x)ds + \int_0^t \sigma_0(s,t-s+x)dW(s) \; \left(*\mathbf{!}\right)$$

So given $x \longrightarrow r(0,x)$ we know $\left(*\mathbf{!}\right)$, **a** solution to Musiela's parametrization.

It is now not difficult to show that given an initial curve $x \longrightarrow r(0,x)$, $x \geq 0$, Musiela's parametrization has a **unique** solution. To that end we consider:

$$r(t,x) \quad = \quad \int_0^t D_2 r(s,x)ds \;,\; \forall\, t \geq 0,\; \forall\, x \geq 0,$$

and ask ourselves whether this equation has solutions and if so what these solutions look like. It is clear that $r(0,x) = 0 \;\forall\, x \geq 0$.
We also know that $\forall\,(t,x) \in \mathbb{R}^2_+$ we have:

$$\triangledown r(t,x) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$$

Now write $\upsilon := \binom{1}{-1}$ and consider $\eta(\lambda) := r((t,x)+\lambda\nu)$. For $\lambda \in \mathbb{R}$ such that $(t,x)+\lambda\nu \in \mathbb{R}^2_+$ we have that $\eta(\lambda)$ is well-defined and then we have $\eta'(\lambda) = 0$, so:

$$\begin{array}{rcl} \forall\,(t,x) \in \mathbb{R}^2_+ & & \exists\, c \in \mathbb{R} \text{ such that} \\ r((t,x)+\lambda\nu) & = & c \text{ whenever } (t,x)+\lambda\nu \in \mathbb{R}^2_+. \end{array}$$

But we know that $r(0,x) = 0$, $\forall\, x \geq 0$, hence we see that $\forall\,(t,x) \in \mathbb{R}^2_+ : r(t,x) = 0$, and this implies that $\left(*\mathbf{!}\right)$ gives the unique solution to Musiela's parametrization given the initial curve $x \longrightarrow r(0,x)$, $x \geq 0$.

From now on we assume that $\sigma_0(t,x)$ does not depend on $\omega \in \Omega$ any longer.
For fixed $x \geq 0$ we now consider the mapping:

$$\mathbb{R}_+ \ni t \longrightarrow \int_0^t \sigma_0(s,t-s+x)dW(s).$$

Definition: We say that this mapping allows for an $(A,B,C(x))$ realization whenever A, B and $C(x)$ are finite-dimensional real-valued matrices such that A and B are constant $\forall\, x \geq 0$ and:

$$\int_0^t \sigma_0(s, t-s+x)dW(s) = C(x)X(t)$$

where $X(\cdot)$ is the unique strong solution of

$$dX(t) \quad = \quad AX(t) + BdW(t),\ X(0) = 0,\ \forall\, t \geq 0 \qquad (S)$$

For instance from Karatzas and Shreve [5] we know that the solution to (S) for given A and B is given by $X(t) = \int_0^t e^{A(t-s)}B(s)dW(s)$.

We hence find the following result.

Theorem: The mapping $t \to \int_0^t \sigma_0(s, t-s+x)dW(s)$ allows for an $(A, B, C(x))$ realization iff

$$\sigma_0(s, t-s+x) \quad = \quad C(x)e^{A(t-s)}B\ ,\ \forall\, t \geq s \geq 0,\ \forall\, x \geq 0.$$

Now recall $\sigma_0(t, x) := \sigma(t, t+x)$.
Hence

$$\sigma_0(s, t-s+x) = C(x)e^{A(t-s)}B \quad \Leftrightarrow \quad \sigma(s, t+x) = C(x)e^{A(t-s)}B.$$

Assume now in addition that $\sigma(s, y) = \widetilde{\sigma}(y),\ \forall\, s \geq 0$, then

$$\begin{aligned} \widetilde{\sigma}(t) &= C(0)e^{At}B = C(t)B, \text{ hence} \\ \widetilde{C}(t) &:= C(0)e^{At} \end{aligned}$$

together with $\widetilde{\sigma}(t) = C(0)e^{At}B$ gives an $(A, B, C(x))$ realization, and we find back results derived by Björk and Gombani in [1] without the use of systems theoretical results. Therefore our paper can be seen as a shortcut to some of the results in [1].

Except for the use of "Fubini" and "stochastic Fubini" our paper can be considered as a rather elementary treatment of Musiela's parametrization and its connection with simple realizations of $r(t, t+x)$.

References

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[4.] Döberlein, F., On term structure models generated by semimartingales, Thesis, Technische Universität Berlin, Berlin, 1999.