

Term Structure Models: A Perspective from the Long Rate

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ABSTRACT

Term structure models resulted from dynamic asset pricing theory are discussed by taking a perspective from the long rate. This paper attempts to answer two questions about the long rate: in frictionless markets having no arbitrage, what *should* the behavior of the long rate be; and, in existing dynamic term structure models, what *can* the behavior of the long rate be.

In frictionless markets having no arbitrage, the yields of all maturities should be positive and the long rate should be finite and non-decreasing. The yield curve should level out as term to maturity increases and slopes with large absolute values occur only in the early maturities. In a continuous-time framework, the longer the maturity of the yield is, the less volatile it shall be. Furthermore, the long rate in continuous-time factor models with a non-singular volatility matrix should be a non-decreasing deterministic function of time.

In the Black-Derman-Toy model and factor models with the short rate having the mean reversion property, the long rate is finite. The long rate in Duffie-Kan models with the mean reversion property is a constant. The long rate in a Heath-Jarrow-Morton model can be infinite or a non-decreasing process. Examples with the long rate being increasing are given in this paper. A model with the long rate and short rate as two state variables is then obtained.

Keywords: asymptotic long rate, term structure of interest rates, factor models, Heath-Jarrow-Morton model.

JEL Classification: E43, G12

1. INTRODUCTION*

There are two approaches to modeling the term structure of interest rates. The first is fitting curves to data from bond markets using statistical techniques. The objective of this empirical estimation of the term structure of interest rates is to find a smooth function of time to maturity that fits the data sufficiently well. Models in literature include polynomial splines of McCulloch (1971), exponential splines of Vasicek and Fong (1982), parsimonious functional form of Nelson and Siegel (1987), and others. See Anderson, et al (1996) for a detailed explanation and comparison of these models. This approach takes a static view and considers solely the shape of the term structure of interest rates.

The second approach takes a dynamic view and considers both shapes of the term structure of interest rates and their evolution. This approach is based on models from recent advances in dynamic asset pricing theory. These models postulate explicit assumptions about the evolution of factors driving interest rates and deduce characterizations of shapes and movements of the term structure of interest rates in a frictionless market having no arbitrage. A partial list of existing models is in Table 1 in the appendix.

The empirical fitting approach and the dynamic asset pricing approach are closely related. Some dynamic term structure models (e.g., Ho and Lee, 1986, Pederson, Shiu, and Thorlacius, 1989, or Heath, Jarrow, and Morton, 1992) take the current term structure, resulting from the empirical fitting approach, as given and then specify the future evolution of the term structure of interest rates. Meanwhile, the functional form for the term structure from the dynamic asset pricing approach

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is also used to fit the data. This gives an alternative method for the empirical estimation of the term structure.

When analyzing fixed-income portfolios, pricing and hedging fixed-income options or other interest rate sensitive products, it is not enough to know where interest rates currently are. One also needs to know where interest rates can be in the future. Thus the dynamic asset pricing approach is the topic of this paper.

Traditionally, the dynamic asset pricing approach to term structure modeling takes a perspective from the short rate. The short rate plays a central role in these models. There are two major approaches to dynamic asset pricing: the equilibrium pricing and pricing by no-arbitrage. Following the seminal paper of Black and Scholes (1973), pricing by no-arbitrage has been the dominating method in the area of dynamic asset pricing. This pricing method requires that one knows the sequence of short rates for each scenario, so a term structure model must provide this information. Meanwhile, as argued empirically by a number of authors (e.g., Litterman and Scheinkman, 1991), most of the variation in returns on all fixed-income securities can be explained in terms of three factors, or attributes, of the term structure. One of the three factors describes a common shift of all interest rates in the same direction. This factor alone explains a large fraction of the overall movement of the term structure. As a result, in many instances, valuation can be reduced to a one factor problem with little loss of accuracy. For these two reasons many term structure models are simply models of the stochastic evolution of the short rate, or models with the short rate as one of the state variables.

There are presently many different term structure models being used in valuation and hedging, but little agreement on any one of them being preferred one. In general, the choice of models depends on the nature of the problem to be solved. See Rogers (1995) for a discussion about the choice of models for the term structure of interest rates.

Pricing long-term bonds is important in managing fixed-income portfolios and determining how to value and hedge a life insurance and pension liabilities. To price

long-term bonds or other fixed-income securities, we need to have models that give appropriate description about the evolution of interest rates of longer maturities. Hogan (1993) identifies internal inconsistencies linked to the specific parameterization and functional form chosen for the short and long rates in Brennan and Schwartz (1979) model. (It is worth noting that, in Brennan and Schwartz (1979), the long rate is the yield on a consol bond that pays coupons continuously and perpetually. In this paper, the long rate is the yield on a zero-coupon bond with an infinite maturity.) Dybvig, Ingersoll, and Ross (1996) show that, in frictionless markets having no arbitrage, the asymptotic long forward and zero-coupon rates never fall. These two results serve as a caution that not every assumption about comovements of discount rates is consistent with a coherent model of the term structure. They also make clear that it is necessary to examine the behavior of yields of the longer maturity in existing term structure models before using them for pricing and hedging long-term securities. Figure 1 and Figure 2 give another reason.

Figure 1 and Figure 2 illustrate the two yield curves from the Merton (1970) model and the Cox-Ingersoll-Ross (1985) model. The horizontal axis is the time to maturity in years. The vertical axis is the yield to maturity in percentage. Figure 1 shows these two curves with the time to maturity being less than 30 years. These two curves are quite similar to each other. However, extending the figure to show the curves with the time to maturity up to 100 years, which is Figure 2, we find the dramatic difference between the parts with time to maturity being larger than 30 years of these two curves.

So it is necessary to pay attention to theoretical implications as well as empirical implications of term structure models. This paper provides one aspect of theoretical implications of term structure models, that is, the dynamic behavior of the asymptotic long rate in existing term structure models.

Figure 1. Yield curves in Merton model (the dashed curve) and Cox-Ingersoll-Ross model (the solid curve) with the time to maturity up to 30 years.

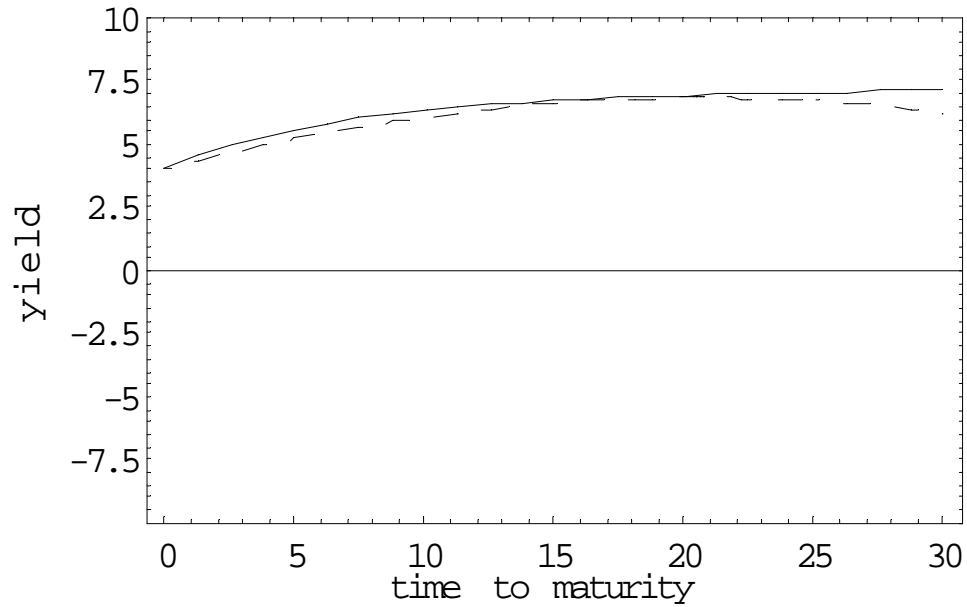
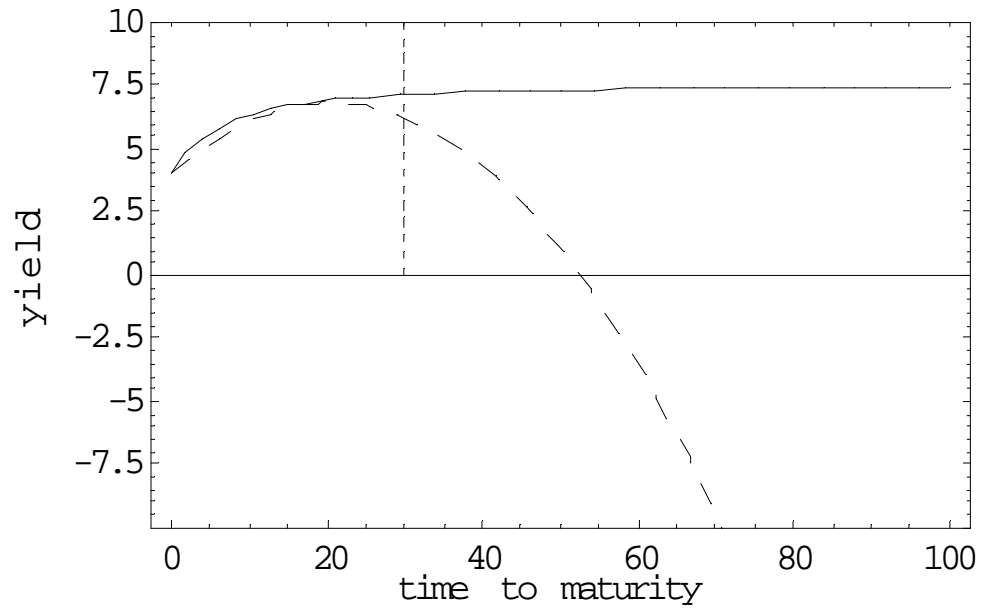


Figure 2. Yield curves in Merton model (the dashed curve) and Cox-Ingersoll-Ross model (the solid curve) with the time to maturity up to 100 years.



This paper attempts to answer partially two questions about the long rate: (1) what should be the behavior of the long rate in frictionless markets having no arbitrage and (2) what can be the behavior of the long rate in existing dynamic term structure models? This paper first discusses the risk neutral approach and argues that, in frictionless markets having no arbitrage, the yield should be positive and finite, and the long rate should be finite and non-decreasing. The yield curve should level out as term to maturity increases and slopes with large absolute values occur only in the early maturities. We then represent the arbitrage-free prices of default-free discount bonds as a stochastic differential system and put the factor models and the Heath-Jarrow-Morton model in this framework. (In general, the dynamic term structure models considered in the literature can be classified as the factor model and the Heath-Jarrow-Morton model. A review of these two classes of models is given in Back, 1996.) A representation for the yield and the long rate is derived from the relationship between the price and the yield. In this setting, we argue that the longer the maturity of the yield, the less volatile it will be and the long rate will be totally determined by the volatility of default-free zero-coupon bond prices. Furthermore, we argue that the long rate in a continuous-time factor model with a non-singular volatility matrix should be a non-decreasing deterministic function of time.

Interest rates have a tendency to be pulled back to some long-run level. This phenomenon is known as mean reversion. We find that explicitly modeling this mean reversion property plays an important role in the boundedness of the long rate in a term structure model. In factor models with the short rate having the mean reversion property, the long rate is bounded above. It is also shown in the paper that the long rate in the Black-Derman-Toy model is finite. Furthermore, the long rate in Duffie-Kan models with the mean reversion property is a constant.

Following Ho and Lee (1986), many researchers and practitioners model the term structure dynamics by taking current yield curve as given (e.g., Pederson, Shiu, and Thorlacius, 1989, or Heath, Jarrow, and Morton, 1992), or making the state variables in factor models being time-inhomogenous and calibrating the model to fit

the current yield curve (e.g., Hull and White, 1990, or Black, Derman, and Toy 1990). This will give a richer behavior of the long rate. However, it also may lead to an infinite long rate, which should be avoided. The long rate from the Heath-Jarrow-Morton model can be infinite or a non-decreasing process. This paper gives examples with increasing long rates and a model with the long rate and short rate as two state variables.

By analyzing the asymptotic behavior of yields on default-free zero-coupon bonds, we also provide a review of dynamic term structure models from the long rate perspective. This can be used to choose among existing models for the purpose of pricing long maturity securities. Meanwhile, this paper attempts to set a foundation for the future research on modeling the yields of longer maturities and pricing and hedging long-maturity bonds and other interest rate sensitive claims.

The outline of the paper is as follows. In Section 2, we define the concepts that describe the term structure of interest rates. To motivate dynamic term structure models, we consider a simplified situation in which all the future movements of the interest rates are known with certainty. In Section 3, we discuss the properties the long rates *should* have in a continuous time framework. In Section 4, the dynamics of the long rates in existing factor models are discussed. Examples from the Heath-Jarrow-Morton framework will be given to explain the dynamics of the long rates. A conclusion is then given in Section 5.

2. DESCRIPTIONS OF THE TERM STRUCTURE OF INTEREST RATES

Much of the difficulty with the term structure of interest rates is caused by cumbersome notation and inconsistent usage of terminology. To minimize this problem, we now define notation used in the paper and keep the symbols to the minimum required.

A default-free discount bond maturing at time T is a security that will pay one unit of currency at time T and nothing at any other time. We denote the price or present value at time t of this bond as $P(t, T)$. At maturity T , we have the

maturation condition $P(T, T) = 1$. The yield (to maturity) is defined as the continuously compounded rate of return that causes the bond price to rise to one at maturity T . The yield at time t is denoted as $y(t, T)$. It is determined by the price-yield relationship:

$$P(t, T) = \exp\{-\tau y(t, T)\}, \quad (2.1)$$

or,

$$y(t, T) = -\frac{\log P(t, T)}{\tau} \quad (2.2)$$

where $\tau = T - t$. As a function of τ , $y(t, T)$ is usually called the (zero coupon) yield curve at time t . The yield curve is also called the term structure of spot interest rates. The yield curve describes the term structure of interest rates by specifying the interest rate of any given maturity.

Two concepts of great importance in this paper are the short rate and the (asymptotic) long rate, which are the two ends of a yield curve. The short rate or instantaneous spot rate is the yield on the currently maturing bond. Hence denoting the short rate at time t by $r(t)$, we have

$$r(t) = y(t, t) \equiv \lim_{T \rightarrow t^+} y(t, T).$$

The (asymptotic) long rate is the yield on the bond with infinite maturity. The long rate prevailing at time t , $\lambda(t)$, is given by

$$\lambda(t) = y(t, \infty) \equiv \lim_{T \rightarrow +\infty} y(t, T).$$

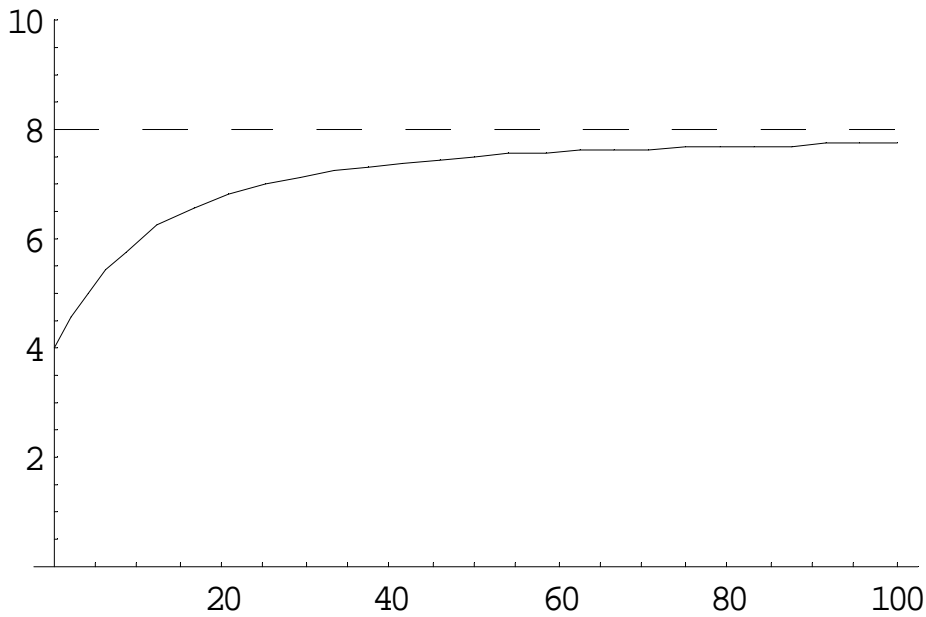
Figure 3 illustrates the short rate and the long rate with an increasing yield curve. The horizontal axis is τ , the time to maturity, in years. The vertical axis is the yield to maturity in percentage. The solid line represents the yield curve. Along the yield curve move left, the limit is the short rate $r(t)$, which is 4% in this example and shown in the figure as the point where the yield curve meets the vertical axis. Along the yield curve move right, the limit is the long rate $\lambda(t)$, which is 8% in this example and at the point where the yield curve meets the dashed line and that can not be shown in the figure.

For one unit of currency invested in a money market account at time $t = 0$, its value at time t will be

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\}.$$

In dynamic asset pricing theory, the money market account is a benchmark for pricing. The last equation partially explains the importance of the short rate concept in dynamic asset pricing theory. The importance of the concept of the long rate is its role in describing the behavior of yields of longer maturity, which is important for pricing and hedging long-maturity bonds or other fixed-income securities.

Figure 3. A yield curve with the short rate at 4% and the long rate at 8%.



Another concept used in this paper is the forward rate. The (instantaneous) forward rate is the instantaneous rate of return that the bond holder can earn by extending his investment for an instant past T . Denoting the forward rate at time t as $f(t, T)$. Then it is given by

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T).$$

It follows from integration of the last equation that,

$$P(t, T) = \exp\left\{-\int_t^T f(t, s) ds\right\}. \quad (2.3)$$

From Equation (2.2) and (2.3), we have the relationship between the yield and the forward rate

$$y(t, T) = \frac{1}{\tau} \left(\int_t^T f(t, s) ds \right). \quad (2.4)$$

As functions of τ , the price $P(t, T)$, the yield $y(t, T)$, and the forward rate $f(t, T)$ can be viewed as equivalent descriptions of the term structure of interest rates at time t before maturity T .

Dynamic term structure models can be better understood by first considering a simplified situation in which all the future movements of the interest rates are **known with certainty**. In this case, the shape and dynamic movement of term structure of interest rates is totally determined by the dynamic movement of future short rates

$$P(t, T) = \exp\left\{-\int_t^T r(s) ds\right\}. \quad (2.5)$$

Differentiating the last equation with respect to t and rearranging

$$dP(t, T) = r(t)P(t, T)dt,$$

which means that the rates of return of all default-free zero-coupon bonds at time t are equated to $r(t)$. This can be used to motivate the term structure models in a risk-neutral framework. From Equation (2.3) and Equation (2.5), we derive

$$\int_t^T r(s) ds = \int_t^T f(t, s) ds,$$

Differentiating the last equation with respect to maturity T and yielding

$$f(t, T) \equiv r(T) \quad \text{for all } t \leq T.$$

Hence

$$f(t, T) \equiv f(0, T) \quad \text{for all } t \leq T,$$

which means that, for each $T > 0$, the forward rate is a constant in a world **without uncertainty**. This can be used to motivate the Heath-Jarrow-Morton model.

3. THE LONG RATE IN ARBITRAGE-FREE ECONOMY

In this section, we discuss what should the behavior of the long rate in frictionless markets having no arbitrage be. To find an answer for this question, we first give a brief description about the risk neutral approach.

A major approach in dynamic asset pricing theory is the risk neutral approach, which is one form of pricing by no-arbitrage. The risk neutral approach to dynamic asset pricing was developed in Ross (1976) and Cox & Ross (1976) and extended in Ross (1978), Harrison & Kreps (1979) and Harrison & Pliska (1981). In frictionless markets having no arbitrage, there exists a risk-neutral measure \mathbf{Q} such that, for each non-dividend paying asset with price at time t denoted as $p(t)$,

$$p(t) = E_t \left[\exp \left(- \exp \int_t^T r(s) ds \right) p(T) \right]. \quad (3.1)$$

where, $E_t[\cdot]$ denotes the conditional expectation given the information available at time t , taken with respect to the measure \mathbf{Q} . One example of this asset is a default-free discount bond.

3.1. The long rate should be finite: an upper bound

Because a default-free discount bond maturing at time T will pay one unit of currency at time T and nothing at any other time, it follows from Equation (3.1) that

$$P(t, T) = E_t \left[\exp \left(- \int_t^T r(s) ds \right) \right]. \quad (3.2)$$

Jensen's Inequality states that, for a random variable X , if $u(X)$ is a strictly convex function, then $E[u(X)] \geq u(E[X])$. See Bowers, et al (1986) for the details. In this inequality, choosing $X = - \int_t^T r(s) ds$, and $u(X) = \exp(X)$ gives

$$P(t, T) \geq \exp\left(-\int_t^T \mathbb{E}_t[r(s)]ds\right).$$

In frictionless markets having no arbitrage, the yield to maturity and the forward rate should be non-negative. Noting Equation (2.2), we obtain

$$0 \leq y(t, T) \leq \frac{1}{T-t} \left(\int_t^T \mathbb{E}_t[r(s)]ds \right),$$

which means that the yield to maturity is less than the average expected short rate, where the expectation is taken with respect to the risk neutral measure \mathcal{Q} . Taking $T \rightarrow +\infty$ in the last equation and Equation (2.4) yields

$$0 \leq \lambda(t) \leq \lim_{T \rightarrow +\infty} \frac{1}{T-t} \left(\int_t^T \mathbb{E}_t[r(s)]ds \right), \quad (3.3a)$$

and

$$\lambda(t) = \lim_{T \rightarrow +\infty} \frac{1}{T-t} \left(\int_t^T f(t, s)ds \right).$$

(3.3b)

(Note that, for a non-decreasing function $g(x)$, $\lim_{x \rightarrow +\infty} \frac{g(x)}{x}$ exists as a finite constant or the infinity. In our case, the short rate and the forward rate are non-negative. For each t , $\int_t^T \mathbb{E}_t[r(s)]ds$ and $\int_t^T f(t, s)ds$ are non-decreasing functions of T . So the limits in Equation (3.3) exist.)

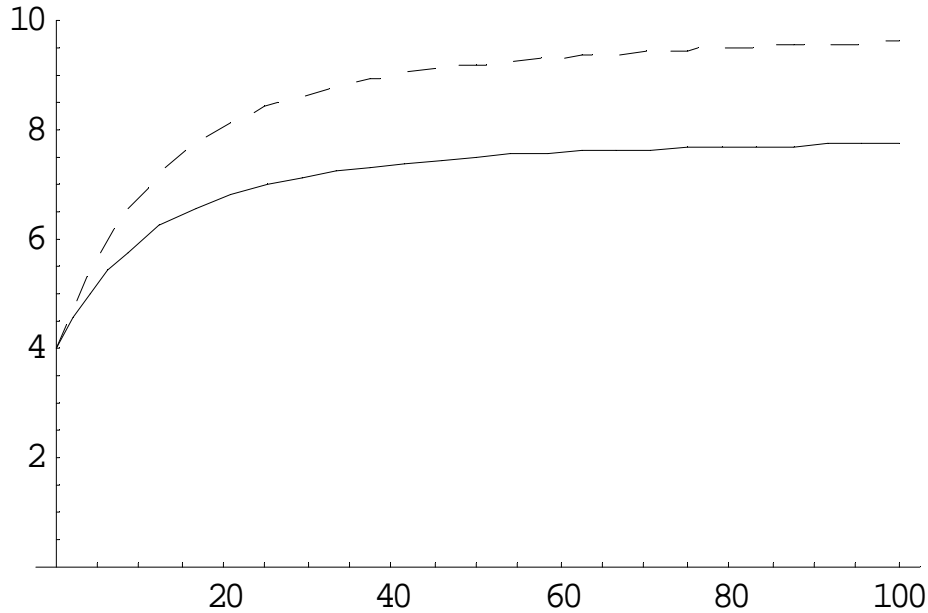
To make economical sense, the long-term average expected short rate should be finite. This means that yields of all maturities as well as the long rate *should* be finite. For a summary as well as applications in Section 4, we give the following proposition:

Proposition 1: In frictionless markets having no arbitrage, the long rate should exist and be finite and non-negative. It is bounded above by the long-term average expected short rate

$$\lambda(t) \leq \lim_{T \rightarrow +\infty} \frac{1}{T-t} \left(\int_t^T \mathbb{E}_t[r(s)]ds \right).$$

Figure 4 illustrates the relationship between the yield to maturity and the expected average short rate. The horizontal axis is the time to maturity in years. The vertical axis is the yield to maturity in percentage. The solid curve is the yield curve. Each point on the dashed curve indicates the expected average short rate.

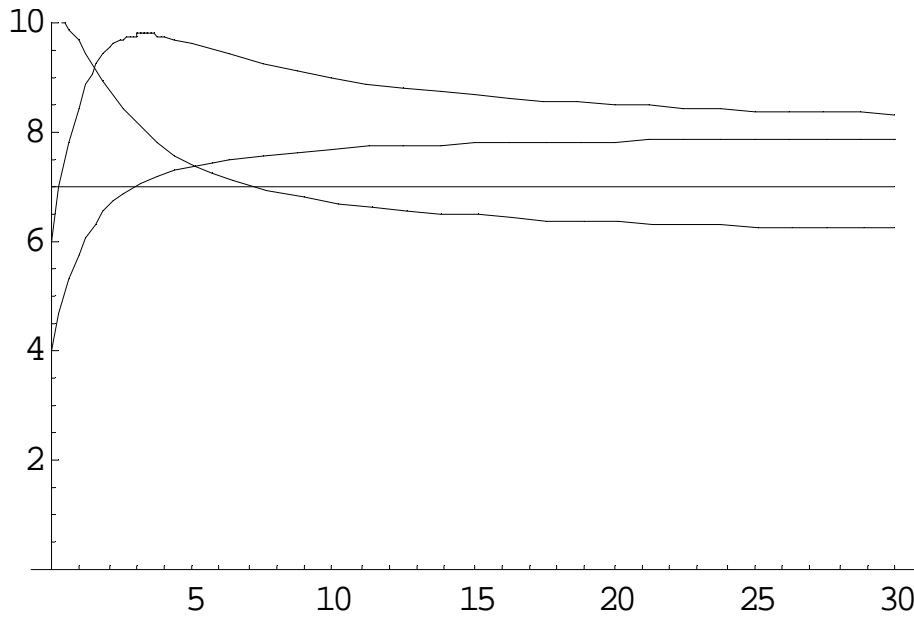
Figure 4. The yield to maturity and the expected average short rate.



3.2. The yield curve should level out as term to maturity increases

Generally, smoothed empirical yield curves have approximated one of four forms: the flat curve, the ascending curve, the decreasing curve, and the humped curve. As pointed out in page 16 of Malkiel (1966), the empirical yield curve has the pervasive tendency to level out as term to maturity increases; and slopes with large absolute values occur only in the early maturities. Figure 5 shows these four types of yield curves. The horizontal axis is the time to maturity in years. The vertical axis is the yield to maturity in percentage.

Figure 5. The four types of yield curves.



In this section, we will show that, in frictionless markets having no arbitrage, the yield curve should level out as term to maturity increases. To do this, we differentiate Equation (2.4.) with respect to T

$$\frac{\partial y(t, T)}{\partial T} = -\frac{1}{(T-t)^2} \left(\int_t^T f(t, s) ds \right) + \frac{f(t, T)}{T-t}.$$

The last equation can be rewritten as

$$\frac{\partial y(t, T)}{\partial T} = \frac{f(t, T) - y(t, T)}{T-t}.$$

From Proposition 1, we know that the long rate is finite. So we have, for each t ,

$\lim_{T \rightarrow +\infty} \frac{y(t, T)}{T-t} = 0$. To make economical sense, it is reasonable to assume that, for

each t , the forward rate should be bounded above. In this case, we have, for each t ,

$\lim_{T \rightarrow +\infty} \frac{f(t, T)}{T-t} = 0$. Thus we have

$$\lim_{T \rightarrow +\infty} \frac{\partial y(t, T)}{\partial T} = 0.$$

which means that the yield curve level out as term to maturity increases and slopes with large absolute values occur only in the early maturities.

3.3. The longer the maturity, the less volatile the yield is: a representation for the long rate

Although various classes of stochastic models are used, the most common language of term structure modellers is that of continuous-time stochastic calculus. In frictionless markets having no arbitrage, for each fixed T , the bond price $P(t, T)$ can be represented as the stochastic differential equation of the form

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma^P(t, T) \cdot dW(t),$$

(3.4)

where $\sigma^P(t, T) = (\sigma_1^P(t, T), \sigma_2^P(t, T), \dots, \sigma_n^P(t, T))$ is a $1 \times n$ vector, and $W(t) = (W_1(t), W_2(t), \dots, W_n(t))^*$ is an n -dimensional standard Brownian motion under the risk-neutral measure \mathcal{Q} . (In this paper, all vectors are column vectors. "*" denotes the vector and matrices transpose operation; and "." denote the inner product of vectors and matrices.) In the appendix, we give a brief discussion of deriving this equation and how to use this equation to derive the partial differential equation of bond prices under the risk-neutral measure in factor models and an arbitrage-free characterization of the term structure in terms of forward rates, which is the Heath-Jarrow-Morton model.

Now we are ready to derive a representation for the long rate in the continuous-time arbitrage-free framework. Using Itô's Lemma, we obtain from the last equation

$$d(\log P(t, T)) = r(t)dt - \frac{1}{2} \sigma^P(t, T) \cdot (\sigma^P(t, T))^* dt - \sigma^P(t, T) \cdot dW(t).$$

Changing t to s in the last equation and integrating from $s = 0$ to $s = t$ yields

$$\log P(t, T) = \log P(0, T) + \int_0^t r(s)ds - \frac{1}{2} \int_0^t \sigma^P(s, T) \cdot (\sigma^P(s, T))^* ds - \int_0^t \sigma^P(s, T) \cdot dW(s)$$

Recalling that $y(t, T) = -\frac{\log P(t, T)}{\tau}$ with $\tau = T - t$, we have

$$y(t, T) = y(0, T) - \frac{1}{\tau} \int_0^t r(s) ds + \frac{1}{2} \int_0^t \frac{\sigma^P(s, T) \cdot (\sigma^P(s, T))^*}{\tau} ds + \int_0^t \frac{\sigma^P(s, T)}{\tau} \cdot dW(s).$$

Let $T \rightarrow +\infty$, and so $\tau \rightarrow +\infty$, we have the following representation for the long rate

$$\lambda(t) = \lambda(0) + \frac{1}{2} \int_0^t \delta(s) ds + \int_0^t \Delta(s) \cdot dW(s),$$

where $\delta(s) = \lim_{T \rightarrow +\infty} \frac{\sigma^P(s, T) \cdot (\sigma^P(s, T))^*}{\tau}$ is a positive scalar, and

$\Delta(s) = \lim_{T \rightarrow +\infty} \frac{\sigma^P(s, T)}{\tau}$ is an $1 \times n$ vector. Because the long rate should exist, $\delta(s)$

and $\Delta(s)$ are well-defined functions. It is easy to see that, if $\delta(s)$ is a well-defined function, then $\Delta(s)$ must be zero. So we have

$$\lambda(t) = \lambda(0) + \frac{1}{2} \int_0^t \delta(s) ds,$$

(3.5)

which is a non-decreasing process. Thus the long rate is determined by the volatility

of default-free zero-coupon bond prices. Also, note that $\frac{\sigma^P(t, T)}{T-t}$ is the

(instantaneous) volatility of the yield on a bond with maturity T . It follows from the

finiteness of $\delta(s)$ that $\lim_{T \rightarrow +\infty} \frac{\sigma^P(t, T)}{T-t} = 0$. Hence, in general, the longer the maturity

of the yield is, the less volatile it is. For a summary as well as applications in

Section 4, we give the following proposition.

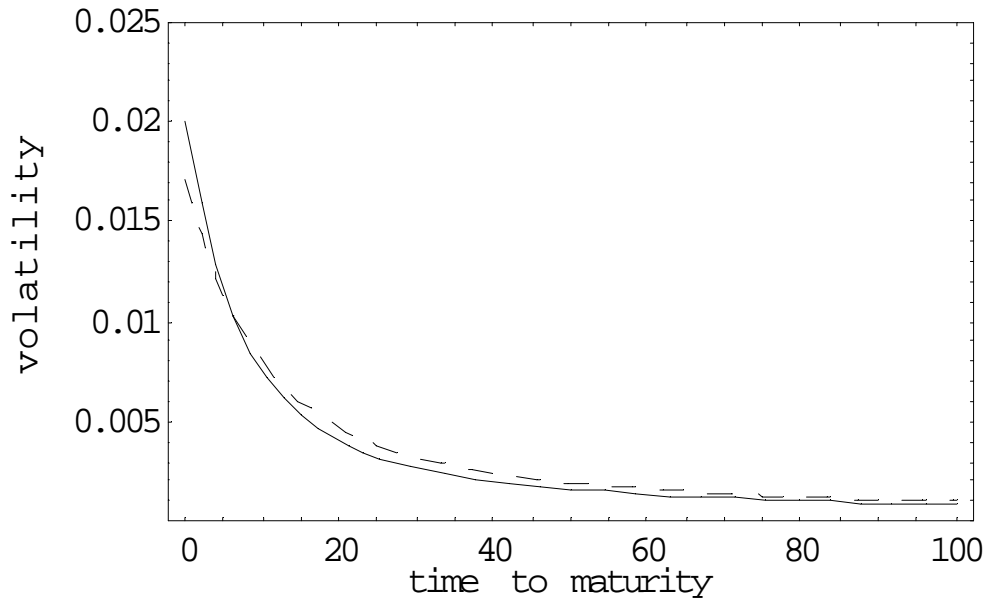
Proposition 2: Assumed that, for each fixed T , the default-free discount bond price $P(t, T)$ can be represented as Equation (3.4) with the volatility vector $\sigma^P(t, T)$.

Then the longer the maturity of the yield is, the less volatile it is; and the long rate is a non-decreasing process with a representation

$$\lambda(t) = \lambda(0) + \frac{1}{2} \int_0^t \delta(s) ds.$$

Figure 6 illustrates how (instantaneous) volatility of the yield to maturity decreases as term to maturity increases in two most popular models. The horizontal axis is the time to maturity in years. The vertical axis is the (instantaneous) volatility of the yield to maturity. The dashed curve represents the volatility of the yield to maturity in the Vasicek (1977) model. The solid curve represents the volatility of the yield to maturity in the Cox-Ingersoll-Ross (1985) model.

Figure 6. The (instantaneous) volatility of the yield to maturity.



3.4. The long rate should be deterministic in factor models: the non-singular volatility matrix case

Most, if not all, of the existing term structure models are factor models. A partial list of existing models is in Table 1 in the appendix. In factor models, the term structure of interest rates is determined by a finite number of state variables $X = (X_1, X_2, \dots, X_n)^*$, which are governed by a stochastic differential system of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t) \cdot dW(t),$$

where $\mu(X, t) = (\mu_1(X, t), \mu_2(X, t), \dots, \mu_n(X, t))^*$, $\sigma(X(t), t)$ is an $n \times n$ matrix, and $W(t)$ is an n -dimensional standard Brownian motion under the risk-neutral measure \mathcal{Q} .

In factor models, the bond price $P(t, T)$ is assumed to be the function of state variables X . From the yield and price relationship, the yields are also the function of state variables X , and so is the asymptotic long rate. Using Itô's lemma, we have

$$d\lambda = \left\{ \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 \lambda}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] \right\} dt + \frac{\partial \lambda}{\partial x} \cdot \sigma \cdot dW(t),$$

(4.1)

where, for notional simplicity, we have omitted variable dependence;

$\frac{\partial \lambda}{\partial x} = \left(\frac{\partial \lambda}{\partial x_1}, \frac{\partial \lambda}{\partial x_2}, \dots, \frac{\partial \lambda}{\partial x_i}, \dots, \frac{\partial \lambda}{\partial x_n} \right)$ is a $1 \times n$ vector of the first-order partial

derivatives, $\frac{\partial^2 \lambda}{\partial x^2} = \left(\frac{\partial^2 \lambda}{\partial x_i \partial x_j} \right)_{n \times n}$ is a $n \times n$ matrix comprising the second-order

partial derivatives, and $\text{tr}[\cdot]$ is the trace of a matrix. Comparing Equation (4.1)

with Equation (3.5), we obtain

$$\frac{\partial \lambda}{\partial x} \cdot \sigma \equiv 0.$$

If the volatility matrix of state variables is non-singular, then

$$\frac{\partial \lambda}{\partial x} \equiv 0.$$

that is, the long rate $\lambda(t)$ in factor models is a non-decreasing function of time t and does depend on the state variables.

In next section, examples from the Heath-Jarrow-Morton framework will be given to show that, if the volatility matrix of state variables is singular, then the long rate $\lambda(t)$ can be an increasing process.

4. THE LONG RATE IN EXISTENCE TERM STRUCTURE MODELS

In this section, we discuss what can the behavior of the long rate in existing dynamic term structure models be.

Many existing factor models explicitly model the mean reversion property of interest rates. Meanwhile, many practitioners tend to use the Heath-Jarrow-Morton model or time-inhomogeneous models of the Hull and White type. We analyze how these two phenomena effect the dynamics of the long rate from term structure models.

4.2. Mean reversion of the short rate and a bound for the long rate

Many one-factor models appearing in the literature explicitly model the mean reversion property of interest rates and are of the form

$$dr(t) = [\alpha_1(t) - \alpha_2(t)r(t)]dt + \sigma(r(t), t)dW(t).$$

Some examples are Merton (1970), Vasicek (1978), Dothon (1978), Courtadon (1982), Cox-Ingersoll-Ross (1985), Ho-Lee (1986), Hull-White (1990), and Chan-Karolyi-Longstaff-Sanders (1992).

Applying the method of integrating factors to the last equation yields

$$r(T) = r(t)e^{-\int_t^T \alpha_2(s)ds} + \int_t^T \alpha_1(S)e^{-\int_S^T \alpha_2(s)ds} dS + \int_t^T \sigma(S)e^{-\int_S^T \alpha_2(s)ds} dW(S).$$

This gives expressions for the expectation of $r(T)$ conditional on the information available at time t

$$E_t[r(T)] = r(t) \exp\left\{-\int_t^T \alpha_2(s)ds\right\} + \int_t^T \alpha_1(S) \exp\left\{-\int_S^T \alpha_2(s)ds\right\} dS.$$

If there exist two positive constants C_1 and C_2 such that $\alpha_1(t) \leq C_1$ and $\alpha_2(t) \geq C_2$ for all t , then we have from the last equation

$$E_t[r(T)] \leq r(t) \exp\{-C_2(T-t)\} + C_1 \int_t^T \exp\{-C_2(T-t)\} dS.$$

Noting that

$$\exp\{-C_2(T-t)\} \leq 1$$

and

$$\int_t^T \exp\{-C_2(T-t)\}dS = \frac{1 - \exp\{-C_2(T-t)\}}{C_2} \leq \frac{1}{C_2},$$

we obtain

$$E_t[r(T)] \leq r(t) + \frac{C_1}{C_2},$$

that is, $E_t[r(T)]$ is bounded from above. From the proposition 1, the long rate is also bounded above.

This condition for $\alpha_1(t)$ and $\alpha_2(t)$ can be easily satisfied by time-homogeneous models, in which the parameters $\alpha_1(t)$ and $\alpha_2(t)$ are constants. A counterexample is the continuous-time equivalent of the Ho-Lee model. In this model, the short rate is given by

$$dr(t) = \left[\frac{d}{dt} f(0, t) + t\sigma^2 \right] dt + \sigma dW(t).$$

In this case,

$$m(T) = m(0) + f(0, T) - r(0) + \frac{T^2\sigma^2}{2}.$$

So $m(T)$ approaches positive infinite as does T . As shown in Dybvig, Ingersoll, and Ross (1996), the long rate is positive infinite, that is, $\lambda(t) = +\infty$.

4.3. The long rate in the Black-Derman-Toy model

In continuous-time equivalent of the Black-Derman-Toy model, the short rate is given by

$$d(\log r(t)) = [\theta(t) - \phi(t)\log r(t)]dt + \sigma(t)dW(t).$$

For simplicity, we denote $R(t) = \log r(t)$. Applying the method of integrating factors yields

$$R(T) = R(t)e^{-\int_t^T \phi(s)ds} + \int_t^T \theta(S)e^{-\int_S^T \phi(s)ds} dS + \int_t^T \sigma(S)e^{-\int_S^T \phi(s)ds} dW(S).$$

This gives expressions for the conditional expectation and variance of $R(T)$

$$E_t[R(T)] = R(t)e^{-\int_t^T \phi(s)ds} + \int_t^T \theta(S)e^{-\int_S^T \phi(s)ds} dS,$$

$$\text{Var}_t[R(T)] = \int_t^T \sigma^2(S)e^{-2\int_S^T \phi(s)ds} dS.$$

Because $R(T)$ is normally distributed, $r(T)$ is lognormally distributed. We have

$$E_t[r(T)] = \exp\left\{E_t[R(T)] + \frac{1}{2}\text{Var}_t[R(T)]\right\}.$$

If $E_t[r(T)]$ is bounded above, then, from the proposition 1, the long rate is also bounded above. One condition that guarantees an upper bound for the long rate is that there exist positive constants C_1 , C_2 and C_3 such that $\phi(t) \geq C_1$, $\theta(t) \leq C_2$ and $\sigma(t) \leq C_3$ for all time t .

4.4. The long rate in the Duffie-Kan affine models

Duffie and Kan (1996) considers a class of term structure models characterized by an affine relation between the drift and diffusion coefficients of the stochastic process describing the evolution of the state variables. It is assumed that the state variables X is a square-root process of form

$$dX(t) = (K \cdot X(t) + K_0)dt + \sqrt{V(X(t))} \cdot \Sigma \cdot dW(t),$$

where K is an $n \times n$ matrix, K_0 is an $1 \times n$ vector, and $\sqrt{V(X(t))}$ is the diagonal matrix

$$\text{diag}\left\{\sqrt{\alpha_1 + \beta_1 \cdot X(t)}, \sqrt{\alpha_2 + \beta_2 \cdot X(t)}, \dots, \sqrt{\alpha_n + \beta_n \cdot X(t)}\right\},$$

where, for each i , α_i is a scalar and β_i is a $n \times 1$ vector, Σ is a $n \times n$ matrix that is positive semi-definite and symmetric. This class of models includes many parametric factor models appearing in the literature or in industry practice as the special cases. Some examples are Merton (1970), Vasicek (1977), Cox, Ingersoll & Ross (1985), Longstaff and Schwartz (1992), Chen and Scott (1992) and Chen (1996).

In this model, the bond prices $P(t,T)$ can be written as the following form:

$$P(t, T) = \exp\{A(\tau) + B(\tau) \cdot X(t)\},$$

where B is a vector-valued function satisfying the following Riccati equation

$$\frac{dB(\tau)}{d\tau} = \frac{1}{2} \text{tr}[B^* \cdot B \cdot \beta] + B \cdot K + \gamma_i ;$$

(4.2a)

and A is a scalar function satisfying the following equation

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2} \text{tr}[B^* \cdot B \cdot \alpha] + B \cdot K_0 + \gamma_0 ;$$

(4.2b)

with $A(0)=0$, $B(0)=0$, and $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$.

From the qualitative theory of differential equation, if all eigenvalues of the matrix K have a negative real part, that is, the state variables have the mean reversion property, then every solution $B(\tau)$ of equation (4.2a) approaches zero as τ approaches infinity (see, for example, Braun (1993) p.386, Theorem 2, and p. 391.), that is, $\lim_{\tau \rightarrow +\infty} B(\tau) = 0$. Then

$$\delta(s) = \lim_{\tau \rightarrow +\infty} \left[\frac{B \cdot \Sigma \cdot V \cdot \Sigma \cdot B^*}{\tau} \right] = 0.$$

From Proposition 2, the long rate in the Duffie-Kan model is a constant.

Now we use several simple examples to explain the technical condition. The simple models are the Merton (1970) model, the Vasicek (1977) model and the Cox-Ingersoll-Ross (1985) model. In these three models, the short rate itself is the state variable. In the Merton model the short rate is given by

$$dr(t) = \theta dt + \sigma dW(t).$$

The yields in this model is given by

$$y(t, T) = r(t) + \frac{1}{2} \theta \tau - \frac{1}{6} \sigma^2 \tau^2.$$

If $\sigma \neq 0$, then the long rate in the Merton model is $\lambda(t) = -\infty$. This example explains that non-singularity of the volatility matrix of state variables is not enough

to guarantee the finiteness of the long rate. In the Vasicek model, the short rate is given by

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t).$$

The yield in this model is given by

$$y(t, T) = \lambda + [r(t) - \lambda] \frac{1 - e^{-kT}}{kT} + \frac{\sigma^2 (1 - e^{-kT})^2}{4k^3 T}.$$

with $\lambda = \theta - \frac{\sigma^2}{2k^2}$. If $k > 0$, then the long rate in the Vasicek model is

$$\lambda(t) \equiv \lambda = \theta - \frac{\sigma^2}{2k^2}.$$

However, if $k \leq 0$, then the long rate in the Vasicek model is $\lambda(t) = -\infty$. In the Cox-Ingersoll-Ross model, the short rate is given by

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t).$$

If $\sigma \neq 0$, then the long rate in the Cox-Ingersoll-Ross model is

$$\lambda(t) = \frac{2k\theta}{k + \sqrt{\sigma^2 + k^2}}.$$

Figure 7. The yield curves for Merton model with $\theta = 0.0055$, $\sigma = 0.02$, and $r(t) = 0.04$.

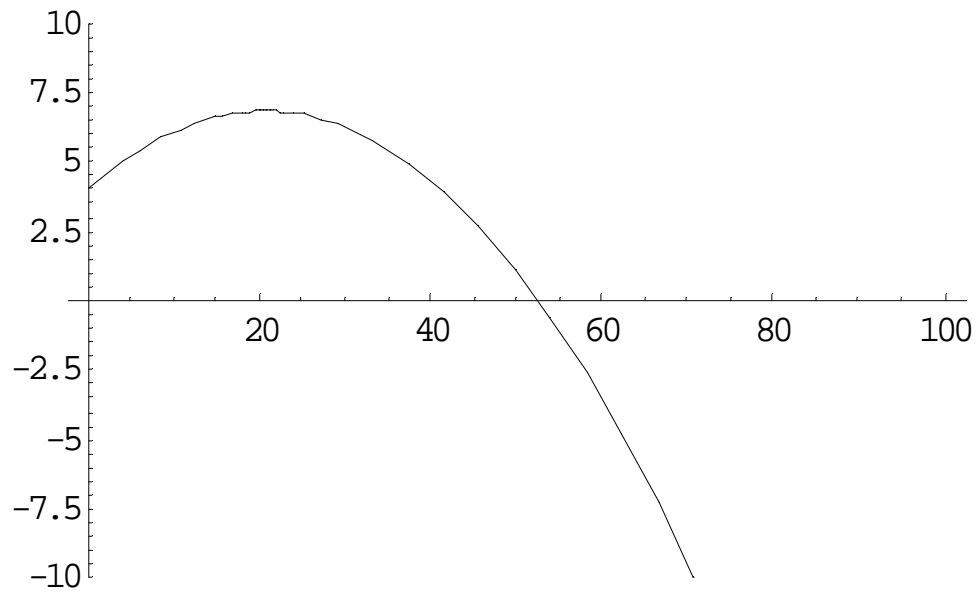
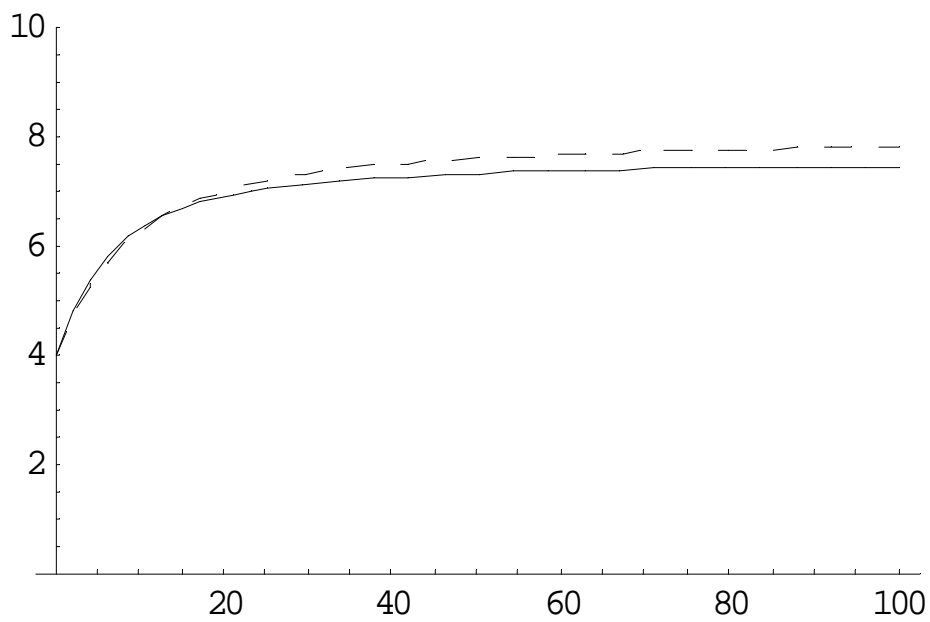


Figure 8. The yield curves for Vasicek model (the dashed curve) with $k = 0.1779$ $\theta = 0.086$, $\sigma = 0.02$, and $r(t) = 0.04$; and Cox-Ingersoll-Ross model (the solid curve) with $k = 0.2339$ $\theta = 0.081$, $\sigma = 0.02$, and $r(t) = 0.04$.



4.5. The long rate in the Heath-Jarrow-Morton model: examples

Heath, Jarrow and Morton (1992) start from modeling the dynamics of the entire forward rate curve. Mathematically, for each fixed T , the forward rate at time t is assumed to satisfy the stochastic differential equation of the form

$$df(t, T) = \sigma_T(t) \cdot \left(\int_t^T \sigma_s(t) dS \right)^* dt + \sigma_T(t) \cdot dW(t), \quad (4.3)$$

where $\sigma_T(f(t, T), t)$ is a $1 \times n$ vector, and $W(t)$ is an n -dimensional standard Brownian motion under the risk-neutral measure Q .

As shown in the appendix, the Heath-Jarrow-Morton model can be put in the risk neutral framework discussed in Section 3. So from Proposition 2, if

$$\delta(t) = \lim_{T \rightarrow +\infty} \frac{\left(\int_t^T \sigma_s(t) dS \right) \cdot \left(\int_t^T \sigma_s(t) dS \right)^*}{\tau}$$

is a finite function, then the long rate exists and is a non-decreasing process with instantaneous volatility being zero. It is easy to find an example in which the long rate being infinite. In Heath-Jarrow-Morton (1992), the authors give a simple model as follow

$$df(t, T) = \left[\sigma_1^2 \tau + \frac{2\sigma_2^2}{\lambda} e^{-(\lambda/2)\tau} (1 - e^{-(\lambda/2)\tau}) \right] dt + \sigma_1 dW_1(t) + \sigma_2 e^{-(\lambda/2)\tau} dW_2(t),$$

where $\tau = T - t$. In this model, $\lambda(t) = +\infty$. In the following, we try to find models in which the long rate $\lambda(t)$ can be an increasing function of time t .

In general, the evolution of the term structure in the Heath-Jarrow-Morton model could depend on the entire path taken by the term structure since it was initialized. Under special volatility restrictions, the path dependence can be completely removed and closed form solutions are available for bond prices. Here we only discuss a one-dimensional case, which has been considered by Cheyette (1992), Jamshidian (1991), and Ritchken and Sankarabramanian (1995). In the one-dimensional case, to remove the path dependence, one can choose

$$\sigma_T(t) = \sigma(t) \exp\left[-\int_t^T k(x) dx\right],$$

where $k(x)$ is a deterministic function of time x . In this case, the price of default-free discount bond is given by

$$P(t, T) = \exp\left\{-\int_t^T f(0, s)ds - B(t, T)[r(t) - f(0, t)] - \frac{1}{2}[B(t, T)]^2 \phi(t)\right\};$$

and the dynamic movement of the short rate is determined by

$$dr(t) = \left\{k(t)[f(0, t) - r(t)] + \phi(t) + \frac{d}{dt}f(0, t)\right\}dt + \sigma(t)dW_1(t);$$

where, $f(0, T)$ is the current forward rate,

$$B(t, T) = \int_t^T \exp\left[-\int_t^U k(x)dx\right]dU,$$

$$\phi(t) = \int_0^t \left\{\sigma(s) \exp\left[-\int_s^t k(x)dx\right]\right\}^2 ds,$$

and $W_1(t)$ is a standard Brownian motion under the risk-neutral measure \mathcal{Q} . This model can be viewed as a factor model, or called a factored Heath-Jarrow-Morton model, by setting $X_1(t) = r(t)$, $X_2(t) = \phi(t)$, and letting $\sigma(t) = \sigma(r(t), \phi(t), t)$ be a function of $r(t)$, $\phi(t)$, and t . In this case

$$dX_1(t) = \left\{k(t)[f(0, t) - X_1(t)] + X_2(t) + \frac{d}{dt}f(0, t)\right\}dt + \sigma(t)dW_1(t)$$

$$dX_2(t) = \left\{\sigma^2(t) - 2k(t)X_2(t)\right\}dt.$$

Obviously, the volatility matrix of these two state variables is singular. It is worth to note that, two state variables are endogenously determined from the model. This will avoid the internal consistent problem cause by arbitrarily choosing state variables in factor models.

To arrive an example in which the long rate $\lambda(t)$ can be an increasing function of time t , we choose

$$k(x) = \frac{1}{2(x+c)},$$

where c is a positive constant. In this case, the volatility of bond price will have the following form

$$\sigma^p(t, T) = 2\sigma(t)\sqrt{t+c}(\sqrt{T+c} - \sqrt{t+c}),$$

and from Proposition 2 of section 3, the long rate is given by

$$\lambda(t) = \lambda(0) + \frac{1}{2} \int_0^t \delta(s) ds = \lambda(0) + 2 \int_0^t [\sigma(s)]^2 (s+c) ds.$$

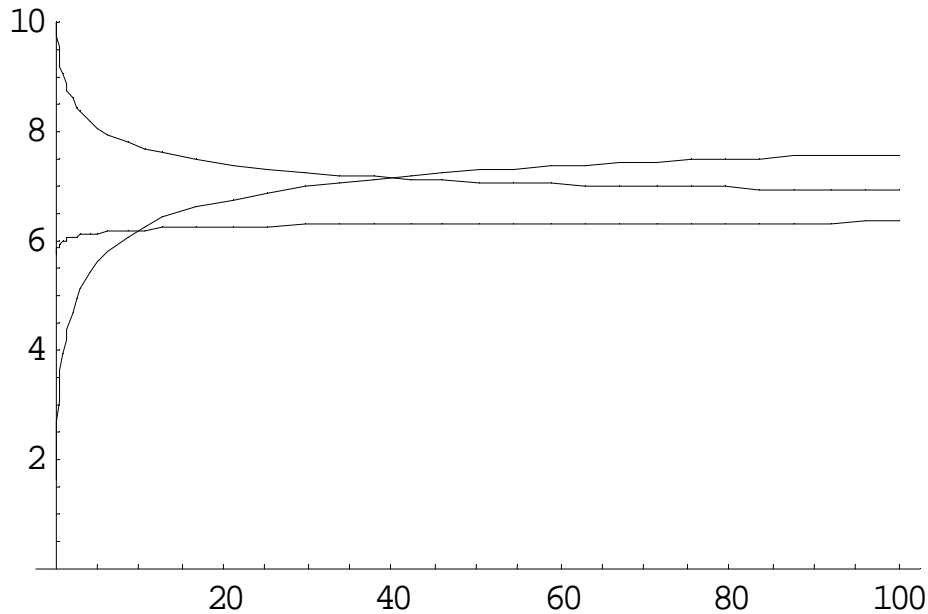
Note that, in this case, $\phi(t) = \frac{1}{t+c} \int_0^t [\sigma(s)]^2 (s+c) ds$,

and

$$\lambda(t) - \lambda(0) = 2(t+c)\phi(t).$$

So we can obtain a two-factor model in which the two state variables are the short rate and the long rate. Figure 9 shows examples of yield curves resulted from this model. The horizontal axis indicates time to maturity from time t in years. The vertical axis is the yield to maturity in percentage. The solid lines represent three types of yield curves from the model.

Figure 9. Examples of yield curves from the model with the short rate and the long rate as two state variables.



By choosing $\sigma(t) = \sigma[r(t)]^\gamma$, we can obtain models in which the long rate is an increasing process of time t with infinite long end. To obtain an example in which

the long rate will be an increasing non-deterministic function of time t with finite long end, we can choose

$$\sigma(t) = \frac{\sigma_1(t)}{\sqrt{t+c}},$$

where $\sigma_1(t)$ is a function with $\int_0^{+\infty} [\sigma_1(s)]^2 ds < \infty$. One example for $\sigma_1(t)$ can be

$$\sigma_1(t) = \exp\{-t - [r(t)]^2\}.$$

5. CONCLUSION

There are presently many different term structure models being used in valuation and hedging interest rate sensitive claims, but little agreement on any one natural one. This is the result of compromise between the properties a term structure model should have and the features a model can have.

In the present paper, we discuss the dynamics of the long zero-coupon rate in the continuous-time arbitrage-free framework. In this setting, the yields of all maturities should be positive and the long rate should be finite and non-decreasing. The yield curve should level out as term to maturity increases and slopes with large absolute values occur only in the early maturities. The longer the maturity of the yield is, the less volatile it is. Furthermore, the long rate in continuous-time factor models with non-singular volatility matrix should be a non-decreasing deterministic function of time.

Many existing factor models explicitly model the mean reversion property of interest rates, this results in the boundedness of the long rate from these term structure models. In many existing factor models, the volatility matrix is non-singular. These two make the long rate in many existing models being a constant or a non-decreasing deterministic function of time.

Many practitioners tend to use the Heath-Jarrow-Morton model or time-inhomogenous factor models of the Hull and White type. These models give a richer behavior of the long rate. The long rate from these models can be infinite or a non-

decreasing process. When using these models in pricing and hedging long-maturity related products, one should be aware of the possibility of an unbounded long rate in these models.

In the Merton (1970) model and the Ho-Lee model, the long rates are infinite. So these two models can not be used to price long-term bonds or other fixed-income securities. In the Vasicek (1970) model and the Langetieg (1980) model, interest rates are normally distributed. There is a positive probability of negative interest rates, which implies arbitrage opportunities. We shall not use these two models. In many applications, A closed-form solution for the default-free discount bond price is a desired property of term structure models. This makes us may choose the Cox-Ingersoll-Ross (1985) model or its generalizations.

Future research should be done with respect to the jump and default effects on the dynamics of the long rates. Empirical testing and comparison of existing term structure models can be also done by taking a perspective from the long rate.

APPENDIX

Table 1: A partial list of existing term structure models

Author(s)	Model Specifications	Long Rate
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Merton (1970)	$dr(t) = \theta dt + \sigma dW_1(t)$	$-\infty$ (if σ is positive)
Vasicek (1977)	$dr(t) = k[\theta - r(t)]dt + \sigma dW_1(t)$	$\theta - \frac{\sigma^2}{2k^2}$ (if k is positive)
Dothon (1978)	$dr(t) = \sigma r(t)dW_1(t)$	a finite non-decreasing deterministic function
Brennan-Schwartz (1979)	$dr(t) = \beta_1(r, \lambda, t)dt + \eta_1(r, \lambda, t)dW_1(t)$ $d\lambda(t) = \beta_2(r, \lambda, t)dt + \eta_2(r, \lambda, t)dW_2(t)$	The long rate is non-decreasing. So it is required that $\eta_2(r, \lambda, t) \equiv 0$ and $\beta_2(r, \lambda, t) \geq 0$
Langtieg (1980)	$dX(t) = (K \cdot X(t) + K_0)dt + \Sigma \cdot dW(t)$ $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$	a finite constant
Courtadon (1982)	$dr(t) = k[\theta - r(t)]dt + \sigma r(t)dW_1(t)$	a finite non-decreasing deterministic function of time
Cox-Ingersoll-Ross (1985)	$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW_1(t)$	$\frac{2k\theta}{k + \sqrt{\sigma^2 + k^2}}$
Ho-Lee (1986)	$dr(t) = \left[\frac{d}{dt}f(0, t) + t\sigma^2\right]dt + \sigma dW(t)$	$+\infty$
Hull-White (1990)	$dr(t) = [\theta(t) + a(t)(b - r(t))]dt + \sigma(t)dW_1(t)$ or $dr(t) = [\theta(t) + a(t)(b - r(t))]dt + \sigma(t)\sqrt{r(t)}dW_1(t)$	can be infinite or a finite non-decreasing function of time
Black-Derman-Toy (1990)	$d(\log r(t)) = [\theta(t) - \phi(t)\log r(t)]dt + \sigma(t)dW_1(t)$	a finite non-decreasing deterministic function of time
Chan-Karolyi-Longstaff-Sanders (1992)	$dr(t) = k[\theta - r(t)]dt + \sigma[r(t)]^\gamma dW_1(t)$	a finite non-decreasing deterministic function of time t (if k is positive)
Longstaff-Schwartz (1992)	$dX(t) = [a - bX(t)]dt + c\sqrt{X(t)}dW_1(t)$	a finite constant (if b and c are positive numbers)
Chen-Scott (1992)	$dY(t) = [d - eY(t)]dt + f\sqrt{Y(t)}dW_2(t)$ $r(t) = \gamma_1 X(t) + \gamma_2 Y(t)$	
Heath-Jarrow-Morton (1992)	$df(t, T) = (\sigma_T(t))^* \cdot \left(\int_t^T \sigma_s(t) dS \right) dt + \sigma_T(t) \cdot dW(t)$ f is the forward rate	can be infinite or a non-decreasing process
Jamshidian (1991) Cheyette (1992) Richken-Sankar (1995)	$dr(t) = \left\{ k(t)[f(0, t) - r(t)] + \phi(t) + \frac{df(0, t)}{dt} \right\} dt + \sigma(t)dW_1(t)$ $d\phi(t) = \left\{ \sigma^2(t) - 2k(t)\phi(t) \right\} dt$	can be infinite or a non-decreasing process
Chen (1996)	$dr(t) = k[\theta(t) - r(t)]dt + \sqrt{V(t)}dW_1(t)$ $d\theta(t) = \nu[\bar{\theta} - \theta(t)]dt + \zeta\sqrt{\theta(t)}dW_2(t)$ $dV(t) = \mu[\bar{V} - V(t)]dt + \eta\sqrt{V(t)}dW_3(t)$	a finite constant (if $k, \nu,$ and μ are positive numbers)
Duffie-Kan (1996)	$dX(t) = (K \cdot X(t) + K_0)dt + \sqrt{V(X(t))} \cdot \Sigma \cdot dW(t)$ $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$	a finite constant (if all the eigenvalues of matrix K have negative real part.)

Factor Models and Heath-Jarrow-Morton Model in risk neutral framework

In Section 2, it is denoted that $B(t) = \exp\left(\int_0^t r(s)ds\right)$. From Equation (3.1), we have

$$\frac{p(t)}{B(t)} = E_t\left[\frac{p(T)}{B(T)}\right], \quad (\text{A.1})$$

that is, the process $\left\{\frac{p(t)}{B(t)}\right\}$ is a martingale with respect to the measure \mathcal{Q} . We will call Equation (A.1) the martingale pricing equation. This equation is valid in a discrete-time framework as well as in a continuous-time framework.

Although various classes of stochastic models are used, the most common language of term structure modellers is that of continuous-time stochastic calculus. Assume that, for each fixed T , the bond price $P(t, T)$ can be represented as the stochastic differential equation of the form

$$\frac{dP(t, T)}{P(t, T)} = \mu^P(t, T)dt - \sigma^P(t, T) \cdot dW(t),$$

where $\mu^P(t, T)$ is a scalar, $\sigma^P(t, T)$ is a $1 \times n$ vector, and $W(t)$ is an n -dimensional standard Brownian motion under the risk-neutral measure \mathcal{Q} . In factor models and the Heath-Jarrow-Morton model discussed in the following, the bond prices have this form. Using Itô's Lemma, we obtain from Equation (A.1)

$$\frac{d[P(t, T) / B(t)]}{P(t, T) / B(t)} = \{\mu^P(t, T) - r(t)\}dt - \sigma^P(t, T) \cdot dW(t).$$

Because the process $\{P(t, T) / B(t)\}$ is a martingale and the drift part of a martingale is zero, we have $\mu^P(t, T) = r(t)$, which means that the rates of return of all default-free zero-coupon bonds at time t are equal to $r(t)$. Therefore we obtain an arbitrage-free characterization of the term structure in terms of bond prices

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma^P(t, T) \cdot dW(t). \quad (\text{A.2})$$

Factor Models

In most factor models, n state variables $X = (X_1, X_2, \dots, X_1, \dots, X_n)^*$ are governed by a stochastic differential equation of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t) \cdot dW(t),$$

where $\mu(X(t), t)$ is a $n \times 1$ vector, $\sigma(X(t), t)$ is a $n \times n$ matrix.

In factor models, the bond price $P(t, T)$ is assumed to be the function of state variables X . Using Itô's lemma, we have

$$dP = \left\{ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] \right\} dt + \frac{\partial P}{\partial x} \cdot \sigma \cdot dW(t)$$

where, for simplicity in symbol, we have omitted variable dependence; $*$ denotes

“transpose;” $\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \dots, \frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_n} \right)$ is a $1 \times n$ vector of the first-order

partial derivatives, and $\frac{\partial^2 P}{\partial x^2} = \left(\frac{\partial^2 P}{\partial x_i \partial x_j} \right)_{n \times n}$ is an $n \times n$ matrix comprising the

second-order partial derivatives. From Equation (A.2), we obtain

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] = r(t)P.$$

Rearranging the last equation yields

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] - r(t)P = 0,$$

which is the partial differential equation for bond prices under the risk-neutral measure.

The Heath-Jarrow-Morton Model

Heath-Jarrow-Morton (1992) start from modeling the dynamics of the entire forward rate curve. Mathematically, for each fixed T , the forward rate at time t , $t \leq T$, is assumed to satisfy the stochastic differential equation of the form

$$df(t, T) = \mu_T(f(t, T), t)dt + \sigma_T(f(t, T), t) \cdot dW(t),$$

(A.3)

where $\mu_T(f(t, T), t)$ is scalar, $\sigma_T(f(t, T), t)$ is a $1 \times n$ vector. For notional simplicity, we will use $\mu_T(t)$ and $\sigma_T(t)$ instead of $\mu_T(f(t, T), t)$ and $\sigma_T(f(t, T), t)$ in the following paragraphs. Recall Equation (2.3), which is $P(t, T) = \exp\{-\int_t^T f(t, s) ds\}$. Using Itô's Lemma, we obtain

$$\frac{dP(t, T)}{P(t, T)} = \left\{ r(t) - \int_t^T \mu_s(t) dS + \frac{1}{2} \left(\int_t^T \sigma_s(t) dS \right) \cdot \left(\int_t^T \sigma_s(t) dS \right)^* \right\} dt - \left(\int_t^T \sigma_s(t) dS \right) \cdot dW(t)$$

Comparing with Equation (A.1), which states that the expected rate of return of each bond at time t is $r(t)$ under the risk-neutral measure, we have

$$r(t) - \int_t^T \mu_s(t) dS + \frac{1}{2} \left(\int_t^T \sigma_s(t) dS \right) \cdot \left(\int_t^T \sigma_s(t) dS \right)^* = r(t).$$

Differentiating the last equation with respect to T and rearranging gives

$$\mu_T(t) = \sigma_T(t) \cdot \left(\int_t^T \sigma_s(t) dS \right)^*.$$

(A.4)

Substituting Equation (A.4) into Equation (A.3), we obtain an arbitrage-free characterization of the term structure in terms of forward rates

$$df(t, T) = \sigma_T(t) \cdot \left(\int_t^T \sigma_s(t) dS \right)^* dt + \sigma_T(t) \cdot dW(t).$$

This specification for forward rates is the essence of the so called the Heath-Jarrow-Morton model.

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