Balinski-Tucker Simplex Tableaus: Dimensions, Degeneracy Degrees, and Interior Points of Optimal Faces

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Abstract

This paper shows the relationship between degeneracy degrees and multiplicities in linear programming models. The usual definition of degeneracy is restricted to vertices of a polyhedron. We introduce degeneracy for nonempty subsets of polyhedra and show that for linear programming models for which the feasible region contains at least one vertex holds that the dimension of the optimal face is equal to the degeneracy degree of the optimal face of the corresponding dual model. This result is obtained by means of so-called Balinski-Tucker Simplex Tableaus. Furthermore, we give a strong polynomial algorithm for constructing such a Balinski-Tucker Simplex Tableau when an optimal interior point solution is known.

Keywords: Linear Programming, Degeneracy, Multiple Solutions, Optimal Faces.

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1. Introduction

In this paper the following primal-dual pair of linear programming models is used:
Primal LP-model: $max\{c^T x | Ax \leq b; x \geq 0\}$
Dual LP-model: $min\{b^T y | A^T y \geq c; y \geq 0\}$,
with A, b, c, x and y being matrices and vectors of appropriate sizes. Note that we restrict ourselves to LP-models in which all variables are nonnegative; the so-called canonical LP-models. The theory of linear programming can be found in many textbooks, for instance Nering & Tucker[9]. The definitions of the concepts of polyhedron, face, et cetera, used in this paper, can be found in, for instance, Schrijver[11]. In Section 2 we will generalize the usual definition for degenerate vertices to faces and arbitrary nonempty subsets of polyhedra. In Section 3 we take a closer look at the so-called Balinski-Tucker Simplex Tableaus, introduced in Balinski & Tucker[1] as part of a proof of the Complementary Slackness Theorem. From a Balinski-Tucker Simplex Tableau we will determine the dimensions and degeneracy degrees of the optimal faces of both the primal and the dual LP-models. The theorems, concerning the relationships between dimensions and degeneracy degrees of the optimal faces are given in Section 4. In Section 5 a strong polynomial algorithm is given, that generates a Balinski-Tucker Simplex Tableau when an optimal interior point solution is known.

2. Degeneracy

In this section the definition of degeneracy, which is usually defined for basic feasible solutions, is generalized to faces and subsets of faces of the polyhedron defined by the feasible region of the LP-model.
Let P be a collection of constraints representing a nonempty polyhedron in $\mathbb{R}^n$, consisting of m_1 inequalities and $m - m_1$ equalities in the variables $x_1, \ldots, x_n$; say:

$$\begin{aligned} P = \quad & \{\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m_1; \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = m_1 + 1, \ldots, m\}. \end{aligned} \tag{1}$$

A constraint of a constraint collection P is called a *redundant constraint of* P if its deletion results in a collection of constraints representing the same polyhedron as P. An inequality of a polyhedron-representation P is called an *implied equality of* P if that inequality is satisfied with equality for every point of the polyhedron represented by P. A *minimal representation* is a polyhedron-representation that does neither contain redundant constraints nor implied equalities. For simplicity reasons we will often refer to the 'polyhedron P', instead of 'the polyhedron represented by constraint collection P'.

Let F be a face of the polyhedron P. A polyhedron-representation of F can be obtained from P by replacing appropriate inequalities of P by equalities. A constraint of a polyhedron-representation P is called *binding on* F, if it is satisfied with equality for every point of F.

Denote the number of constraints of P that are binding on F by $b(F, P)$, and the dimension of F (i.e. the dimension of the affine hull of F) by $dim(F)$.

For example, let $P = \{x_1 - x_2 \geq 0; x_1 \geq 0; x_2 \geq 0\}$. The face $F = \{(0, 0)\}$ (with dimension 0) can then be defined in different ways using the constraints of P. For instance, both $\{x_1 - x_2 = 0; x_1 = 0; x_2 \geq 0\}$ and $\{x_1 - x_2 \geq 0; x_1 = 0; x_2 = 0\}$ represent F. All three constraints of P are binding on F. Hence, $dim(F) = 0$ and $b(F, P) = 3$.

Let F be any face of the polyhedron in $\mathbb{R}^n$ represented by the collection of constraints P. The *degeneracy degree* of F with respect to P, denoted by $\sigma(F, P)$, is defined by $\sigma(F, P) = b(F, P) + dim(F) - n$. F is called *degenerate* w.r.t. P iff $\sigma(F, P) > 0$, and F is called *non-degenerate* w.r.t. P iff $\sigma(F, P) = 0$. These definitions are motivated as follows. The number of hyperplanes that determines the affine hull of face F with dimension $dim(F)$ is at least equal to $n - dim(F)$, and this lowerbound is sharp. If the number of constraints from P, that are binding on F, is larger than $n - dim(F)$, then there is a redundancy in the collection of hyperplanes that defines F. Note that the definition of degenerate face generalizes the usual definition of degenerate vertex, because $b(v, P) + dim(v) > n$ reduces for a vertex to $b(v, P) > n$, which is the usual definition for degenerate vertex. With the definition of degenerate face, also "degenerate polyhedron" is defined, since P is a face of P itself. In case of linear programming, this means that also the concept of "degenerate feasible region" is defined by this definition. In the literature of linear programming, degeneracy is usually defined for basic solutions and vertices. However, in Nering & Tucker [9], an LP-model is called degenerate if it has at least one degenerate basic solution (not necessarily feasible). In Güler et al. [6], an LP-model is called degenerate if there exists at least one feasible point that is degenerate.

The set of faces of a polyhedron P, together with the empty set, form a lattice under inclusion. Therefore, for any nonempty subset S of P, there is a unique smallest face F of P with $S \subseteq F$. This allows us to define degeneracy for any nonempty subset of a polyhedron. Let S be a nonempty subset of a polyhedron in $\mathbb{R}^n$ represented by P, and let F be the smallest face of P with $S \subseteq F$. The *degeneracy-degree* of S w.r.t. P, denoted by $\sigma(S, P)$, is defined by $\sigma(S, P) = \sigma(F, P)$. S is called *degenerate* w.r.t. P iff $\sigma(S, P) > 0$, and S is called *non-degenerate* w.r.t. P iff $\sigma(S, P) = 0$. A consequence of the definition of degeneracy degree for faces is the following theorem.

Theorem 2.1 *Let P be a polyhedron-representation in $\mathbb{R}^n$. Then the following assertions hold.*

1. *If F_1 and F_2 are faces of P with $F_2 \subseteq F_1$, then $\sigma(F_2, P) \geq \sigma(F_1, P)$.*
2. *A face of P with dimension at least 1 is degenerate w.r.t. P, if and only if all proper nonempty subsets of F are degenerate w.r.t. P.*
3. *If P degenerate w.r.t. P, then P contains either a redundant constraint or an implied equality.*

Proof.

(1) Let F_1 be a face of P with $\sigma(F_1, P) > 0$. Clearly, $b(F_1, P) = n - dim(F_1) + \sigma(F_1, P)$. Let

F_2 be a subface of F_1. Then $dim(F_2) \leq dim(F_1)$. Hence, the number of binding constraints of P on F_2 is at least $b(F_1, P) + (dim(F_1) - dim(F_2))$, and we have that $\sigma(F_2, P) = b(F_2) + dim(F_2, P) - n \geq b(F_1) + (dim(F_1) - dim(F_2)) + dim(F_2) - n = b(F_1) + dim(F_2) - n = \sigma(F_1, P)$.

(2) Let F be any face with dimension at least 1 of the polyhedron P. We first prove the 'only if' part. Let $\sigma(F, P) > 0$. Then, according to Theorem 2.1(1), all subfaces of F have a positive degeneracy-degree. Hence, all nonempty subsets of F have a positive degeneracy-degree w.r.t. P.
The proof of the 'if' part can be given as follows. If all proper nonempty subsets of F are degenerate w.r.t. P, then also the relative interior of F is degenerate w.r.t. P. Since F has dimension at least 1, the relative interior of F is a proper subset of F. Because F is the smallest face containing the relative interior of F, F is degenerate w.r.t. P.

(3) Let P be degenerate w.r.t P. Then, $\sigma(P, P) > 0$. Let e denote the number of equalities in P. If $e > n - dim(P)$, then P contains at least one redundant equality. If $e \leq n - dim(P)$, then $b(P, P) - e$ inequalities are binding on P. Since $b(P, P) - e = n - dim(P) + \sigma(P, P) - e \geq n - dim(P) + \sigma(P, P) - n + dim(P) = \sigma(P) > 0$, P contains at least one implied equality. □

The following example may illustrate these concepts. Let $P = \{x_1 + x_2 \leq 2; x_1 \leq 1; x_2 \leq 1; x_1, x_2 \geq 0\}$, $F = \{x_1 + x_2 \leq 2; x_1 \leq 1; x_2 = 1; x_1, x_2 \geq 0\}$, and $S = \{(0.2, 1), (0.4, 1)\}$. F is the line segment $[(0, 1), (1, 1)]$. Note that $dim(F) = 1$, and that $x_2 \leq 1$ is the only inequality of P that is binding on F. F is non-degenerate w.r.t P, because $\sigma(F, P) = b(F, P) + dim(F) - n = 1 + 1 - 2 = 0$. Note that the degeneracy-degree w.r.t. P of the face consisting of the single vertex $v = (1, 1)$ satisfies $\sigma(v, P) = b(v, P) + dim(v) - n = 3 + 0 - 2 = 1$. Since the smallest face of P containing S is F, we have that $\sigma(S, P) = \sigma(F, P) = 0$.

In general, it is not true that all subfaces of a non-degenerate face are non-degenerate. In the above example, the vertex (1,1) is degenerate and a subface of the non-degenerate face F. Another example is the regular octahedron in $\mathbb{R}^3$: Every vertex is degenerate, but if this polyhedron is represented by a minimal representation with 8 inequality constraints, then the edges and facets are non-degenerate.

The following example shows how the representation of a polyhedron may influence its degeneracy. Let $P = \{x_1 + x_2 = 1; x_1, x_2 \geq 0\}$ and $P' = \{x_1 + x_2 \leq 1; x_1 + x_2 \geq 1; x_1, x_2 \geq 0\}$. P and P' are two different representations of the same polyhedron in $\mathbb{R}^2$. P is non-degenerate w.r.t. P and P' is degenerate w.r.t P'. P' contains 2 implied equalities. If these inequalities are replaced by equalities, then one of these two equalities is redundant.

The definitions for degeneracy, given above, are dependent on the polyhedron-representation. However, it is possible to define degeneracy of nonempty subsets of a polyhedron independent of the polyhedron-representation. For instance, the degeneracy-degree of a nonempty subset S of a polyhedron Q, denoted by $\sigma(S, Q)$ could be defined as $\sigma(S, Q) = \min_P\{\sigma(S, P) \mid P$ is a representation of Q$\}$.

3. Balinski-Tucker Simplex Tableaus

LP-models can be represented by means of tableaus in many different ways. The tableau representation that we will use is a variation of the tableau introduced in Balinski & Tucker[1]: we place the 'right-hand-side' $a_{00}, a_{10}, \ldots, a_{m0}$ on the left side of the tableau, and call it a *Tucker Tableau*. An example of a Tucker Tableau is shown in Figure 3.1. Let $(p_1, \ldots, p_{n+m})$ be

$$
\begin{array}{c|c|cccc|c}
 & 1 & x_{p_1} & x_{p_2} & \cdots & x_{p_n} & \\
\hline
1 & a_{00} & a_{01} & a_{02} & \cdots & a_{0n} & = -f \\
\hline
y_{p_{n+1}} & a_{10} & a_{11} & a_{12} & \cdots & a_{1n} & = -x_{p_{n+1}} \\
y_{p_{n+2}} & a_{20} & a_{21} & a_{22} & \cdots & a_{2n} & = -x_{p_{n+2}} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
y_{p_{n+m}} & a_{m0} & a_{m1} & a_{m2} & \cdots & a_{mn} & = -x_{p_{n+m}} \\
\hline
 & = -g & = y_{p_1} & = y_{p_2} & \cdots & = y_{p_n} &
\end{array}
$$

Figure 3.1: A Tucker Tableau

a permutation of the integers $1, \ldots, n+m$, with m and n strictly positive integers. The variables $x_{p_1}, \ldots, x_{p_n}$ denote the primal non-basic variables, $x_{p_{n+1}}, \ldots, x_{p_{n+m}}$ the primal basic variables, $y_{p_{n+1}}, \ldots, y_{p_{n+m}}$ the dual non-basic variables, and $y_{p_1}, \ldots, y_{p_n}$ the dual basic variables.
The rows of the tableau of Figure 3.1 are then represented by:

$$
\begin{aligned}
a_{00} + \sum_{j=1}^{n} a_{0j} x_{p_j} &= -f \\
a_{i0} + \sum_{j=1}^{n} a_{ij} x_{p_j} &= -x_{p_{n+i}}, \quad i = 1, \ldots, m.
\end{aligned}
$$

The corresponding primal LP-model is defined as:

$$
\begin{aligned}
\max \quad f = \quad & \sum_{j=1}^{n} -a_{0j} x_{p_j} - a_{00} \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_{p_j} \leq -a_{i0}, \quad i = 1, \ldots, m \\
& x_{p_j} \geq 0, \qquad j = 1, \ldots, n
\end{aligned}
$$

Similarly, the columns of this tableau represent the equations:

$$
\begin{aligned}
a_{00} + \sum_{i=1}^{m} a_{i0} y_{p_{n+i}} &= -g \\
a_{0j} + \sum_{i=1}^{m} a_{ij} y_{p_{n+i}} &= y_{p_j}, \quad j = 1, \ldots, n,
\end{aligned}
$$

and the corresponding dual LP-model reads:

$$\begin{array}{rlll} \min \;\; g = & \displaystyle\sum_{i=1}^{m} -a_{i0} y_{p_{n+i}} - a_{00} & & \\ s.t. & \displaystyle\sum_{i=1}^{m} a_{ij} y_{p_{n+i}} & \geq -a_{0j}, & j = 1, \ldots, n \\ & y_{p_{n+i}} & \geq 0, & i = 1, \ldots, m \end{array}$$

Each row i, with $1 \leq i \leq m$, corresponds to a pair of dual complementary variables; namely, the basic primal variable $x_{p_{n+i}}$ and the non-basic dual variable $y_{p_{n+i}}$. Similarly, each column j, with $1 \leq j \leq n$, corresponds to a pair of dual complementary variables; namely, the non-basic primal variable x_{p_j} and the dual basic variable y_{p_j}. If the row equations are used as column equations and vice versa, the tableau of Figure 3.2 is obtained. Note, that it is equivalent to the tableau in Figure 3.1. The Tucker Tableau in Figure 3.2 is called the *negative transpose* of the tableau in Figure 3.1.

	1	$y_{p_{n+1}}$	$y_{p_{n+2}}$	$\cdots$	$y_{p_{n+m}}$	
1	$-a_{00}$	$-a_{10}$	$-a_{20}$	$\cdots$	$-a_{m0}$	$= g$
x_{p_1}	$-a_{01}$	$-a_{11}$	$-a_{21}$	$\cdots$	$-a_{m1}$	$= -y_{p_1}$
x_{p_2}	$-a_{02}$	$-a_{12}$	$-a_{22}$	$\cdots$	$-a_{m2}$	$= -y_{p_2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{p_n}	$-a_{0n}$	$-a_{1n}$	$-a_{2n}$	$\cdots$	$-a_{mn}$	$= -y_{p_n}$
	$= f$	$= x_{p_{n+1}}$	$= x_{p_{n+2}}$	$\cdots$	$= x_{p_{n+m}}$	

Figure 3.2: The negative transpose of a Tucker Tableau

From the theory of linear programming the following facts are known; see for instance Nering & Tucker[9]. If a Tucker Tableau is given that represents a pair of dual LP-models, then a pivot operation on a non-zero entry a_{ij} ($1 \leq i \leq m$, $1 \leq j \leq n$) transforms it to an equivalent Tucker Tableau that represents the same pair of dual LP-models. A Tucker Tableau is optimal if $a_{i0} \leq 0$ for each i with $1 \leq i \leq m$, and $a_{0j} \geq 0$ for each j with $1 \leq j \leq n$. If both the primal and the dual LP-model have a finite optimal solution, then any Tucker Tableau that represents this pair of dual LP-models can be transformed by means of a finite number of pivot operations into an equivalent optimal Tucker Tableau. For an excellent description of pivot operations, we refer to Nering & Tucker[9]. An optimal Tucker Tableau corresponds to a primal-dual pair of optimal basic feasible solutions. The primal optimal basic feasible solution satisfies $x_{p_j} = 0$ for $j = 1, \ldots, n$, $x_{p_{n+i}} = -a_{i0}$ for $i = 1, \ldots, m$, and $f = -a_{00}$. The corresponding dual optimal basic feasible solution satisfies $y_{p_{n+i}} = 0$ for $i = 1, \ldots, m$, $y_{p_j} = a_{0j}$ for $j = 1, \ldots, n$, and $g = -a_{00}$.

If an LP-model has more than one optimal solution, then it has at least one optimal basic solution. In general such an LP-model has several optimal Tucker Tableaus. The set of all optimal solutions of an LP-model is a face of the feasible region of that LP-model.

A primal-dual pair of optimal solutions (x^*, y^*) is called *strictly complementary* iff $x^* + y^* > 0$. If the primal and dual LP-models both have finite optimal solutions, then there is a pair of strictly complementary optimal solutions; see e.g. Goldman & Tucker[3]. In Balinski & Tucker[1], a constructive proof is given for the existence of a strictly complementary primal-dual pair of optimal solutions. It is shown that in a finite number of pivot operations and rearrangements of rows and columns an optimal Tucker Tableau can be constructed with the structure shown in Figure 3.3. We will call such a tableau a *Balinski-Tucker Tableau*

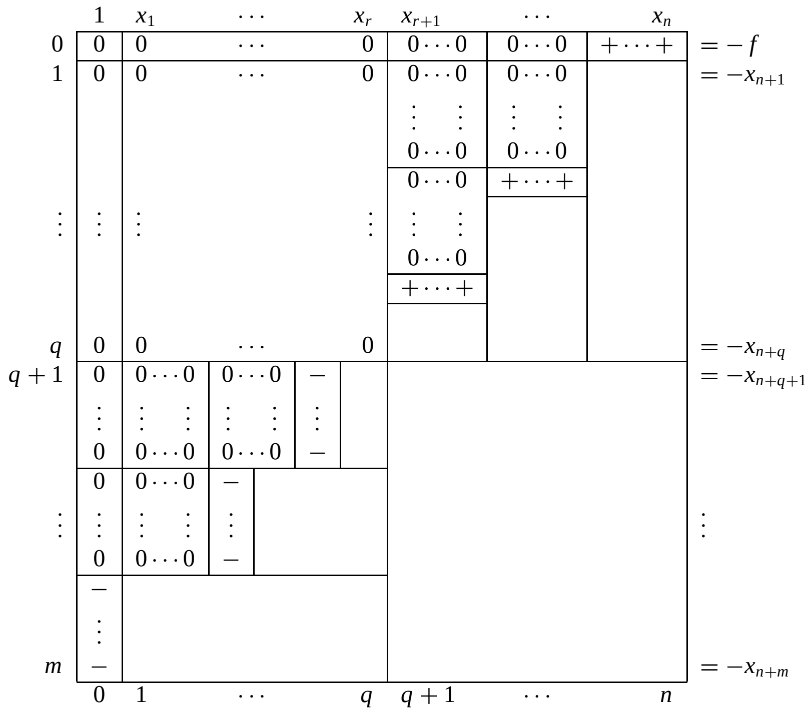

Figure 3.3: A Balinski-Tucker Tableau

(B-T Tableau). In B-T Tableaus it is assumed that the optimal objective values are zero ($f = g = a_{00} = 0$). This can easily be accomplished by giving a_{00} an appropriate value in the primal and dual objective functions. B-T Tableaus have the following characteristics.

1. The first column (with index 0 in Figure 3.3) contains no positive entries. This accounts for the feasibility of the corresponding primal optimal solution and the optimality of the corresponding dual optimal solution.
2. The first row (with index 0 in Figure 3.3) contains no negative entries. This accounts for the feasibility of the corresponding dual optimal solution and the optimality of the

corresponding primal optimal solution.

3. The left upper corner (the matrix consisting of the columns with indices $0, \ldots, r$ and the rows with indices $0, \ldots, q$) is a $(q+1) * (r+1)$ all-zero matrix.
4. The left lower corner (the matrix consisting of the columns with indices $0, \ldots, r$ and the rows with indices $q+1, \ldots, m$) consists of lexicographically negative rows. The rows in this matrix are lexicographically nonincreasing ordered. (if $i < j$, then row i is not lexicographically smaller than row j).
5. The right upper corner (the matrix consisting of the columns with indices $r+1, \ldots, n$ and the rows with indices $0, \ldots, q$) consists of lexicographically positive columns. The columns in this matrix are lexicographically nondecreasing ordered.

From B-T Tableaus of primal-dual pairs of LP-models several properties of the primal an dual optimal faces can be derived.

Theorem 3.1 *For a primal-dual pair of LP-models with finite optimal solutions, represented by a B-T Tableau as shown in Figure 3.3, the following assertions hold.*

1. *For each optimal primal solution (i.e. for each point of the primal optimal face) holds that $x_j = 0$ for $r+1 \leq j \leq n$, and $x_{n+i} = 0$ for $1 \leq i \leq q$.*
2. *For each optimal dual solution (i.e. for each point of the dual optimal face) holds that $y_{n+i} = 0$ for $q+1 \leq i \leq m$, and $y_j = 0$ for $1 \leq j \leq r$.*
3. *A strictly complementary pair of optimal solutions (x^*, y^*) satisfies $x_j^* > 0$ for $1 \leq j \leq r$ and for $n+q+1 \leq j \leq n+m$, and $y_i^* > 0$ for $n+1 \leq i \leq n+q$ and for $r+1 \leq i \leq n$.*

Proof

(1) Let Figure 3.3 be a B-T Tableau of a primal-dual pair of LP-models. The columns corresponding to the primal non-basic variables $x_{r+1}, \ldots, x_n$ are nondecreasingly ordered with respect to the lexicographic ordering. Let β_j, with $r+1 \leq j \leq n$, be the row index with the first positive entry in the column with index j. Let $\beta_r = q+1$ and $\beta_{n+1} = 0$. For $j = r+1, \ldots, n+1$, let $\alpha_i = j$ with $\beta_j \leq i < \beta_{j-1}$. It then follows that $\alpha_q = r+1$.

The first row of this B-T Tableau is $a_{00} + \sum_{j=\alpha_0}^{n} a_{0j} x_{p_j} = -f$. Since in an optimal solution holds that $f = -a_{00}$, we have that $\sum_{j=\alpha_0}^{n} a_{0j} x_j = 0$ with $a_{0\alpha_0}, \ldots, a_{0n} > 0$. Since all variables are nonnegative, we have that $x_{\alpha_0} = \ldots = x_n = 0$ for each primal optimal solution. The row equations for the rows i with $1 \leq i \leq q$ can be written as $\sum_{j=\alpha_i}^{\alpha_{i-1}-1} a_{ij} x_j + \sum_{j=\alpha_{i-1}}^{n} a_{ij} x_j + x_{p+i} = 0$ in which a_{ij} is positive for $\alpha_i \leq j < \alpha_{i-1}$. Suppose that $x_{\alpha_i} = \ldots = x_n = 0$ for each primal optimal solution, and for some i with $0 \leq i < q$. Then the row equation for the row with index $i+1$ reduces to $\sum_{j=\alpha_{i+1}}^{\alpha_i - 1} a_{ij} x_j + x_{p+i} = 0$ in which a_{ij} is positive for $\alpha_{i+1} \leq j < \alpha_i$. Since all variables are nonnegative, we have that $x_{\alpha_{i+1}} = \ldots = x_{\alpha_i} = 0$ for each primal optimal solution. Using mathematical induction, it follows that for each primal optimal solution $x_j = 0$ for each j with $r+1 \leq j \leq n$.

From $\sum_{j=1}^{n} a_{ij} x_j = -x_{n+i}$ for $1 \leq i \leq q$, together with $x_j = 0$ for $r+1 \leq j \leq n$, and $a_{ij} = 0$ for $1 \leq i \leq q$ and $1 \leq j \leq r$, follows that $x_{n+i} = 0$ for $1 \leq i \leq q$ holds for every optimal primal solution.

(2) The proof of this part of the theorem is similar to the proof of the first part if the negative transpose of the B-T Tableau of the first part is used.

(3) See Balinski & Tucker [1] □

The indices $1, \ldots, n+m$ can be partitioned into two sets B and N such that $i \in B$ implies that $y_i = 0$ and $x_i > 0$ for every strictly complementary optimal solution, and $i \in N$ implies that $x_i = 0$ and $y_i > 0$ for every strictly complementary optimal solution. This partition is called the *optimal partition*. Using Theorem 3.1, it follows that $B = \{1, \ldots, r, n+q+1, \ldots, n+m\}$ and $N = \{r+1, \ldots, n+q\}$.

Moreover, a representation of both the primal and the dual optimal face can easily be derived from B-T Tableaus. Let Figure 3.3 be a B-T Tableau of a pair of primal and dual LP- models. The equations that define the primal optimal face are:

$$\sum_{j=1}^{n} a_{ij} x_j = -a_{i0} \qquad i = 1, \ldots, m \tag{2}$$

$$x_j = 0 \qquad j = r+1, \ldots, n \tag{3}$$

$$x_{n+i} = 0 \qquad i = 1, \ldots, q \tag{4}$$

$$x_j \geq 0 \qquad j = 1, \ldots, n \tag{5}$$

$$x_{n+i} \geq 0 \qquad i = 1, \ldots, m \tag{6}$$

which can be reduced to:

$$\sum_{j=1}^{r} a_{ij} x_j = -a_{i0} \qquad i = q+1, \ldots, m \tag{7}$$

$$x_j = 0 \qquad j = r+1, \ldots, n \tag{8}$$

$$x_{n+i} = 0 \qquad i = 1, \ldots, q \tag{9}$$

$$x_j \geq 0 \qquad j = 1, \ldots, r \tag{10}$$

$$x_{n+i} \geq 0 \qquad i = q+1, \ldots, m \tag{11}$$

From the same tableau (or from its negative transpose) a similar representation of the dual optimal face can be derived.

4. Degeneracy degrees and multiple solutions

If the solution of an LP-model is not unique, then the model has multiple solutions and the dimension of its optimal face is larger than zero. In Theorem 4.1 a remarkable dual relationship between the degeneracy degree and the dimension of the optimal faces of a dual pair of LP-models is established.

Theorem 4.1 *Let P be the collection of inequality constraints that represents the feasible region of a primal LP-model, and let D be the collection of inequality constraints that* *represents the feasible region of the corresponding dual LP-model. Let F_P and F_D denote the primal and dual optimal faces and let a B-T Tableau of this primal-dual pair of LP-models (see Figure 3.3) be given. Then the following assertions hold.*

- $dim(F_P) = \sigma(F_D, D) = r.$
- $dim(F_D) = \sigma(F_P, P) = q$

Proof.

We first prove that $dim(F_P) = r$. In Balinski and Tucker[1] it is shown that there exists a strictly complementary optimal solution with $x_j^* > 0$ for $1 \leq j \leq r$, and $x_{n+i}^* > 0$ for $q + 1 \leq i \leq m$. Therefore, the inequalities (10) and (11) are not binding in every point of the primal optimal face, and hence are not implied equalities. The equalities (4) are implied by (2) and (3) (see the proof of Theorem 3.1) and therefore redundant. The dimension of the primal model is $m + n$ (the total number of variables). The dimension of the primal optimal face F_P is determined by the affine independent collection of equalities (2) and (3) and is equal to $(m + n) - (m + (n - r)) = r$.

We now prove that $\sigma(F_P, P) = q$. The binding constraints for the primal optimal face F_P are (2), (3) and (4). Hence, $b(F_P, P) = m + (n - r) + q = m + n - r + q$. Therefore, $\sigma(F_P, P) = b(F_P, P) + dim(F_P) - (m + n) = (m + n - r + q) + r - (m + n) = q$.

The proofs of $dim(F_D) = q$ and $\sigma(F_D, D) = r$ are similar to the above ones; namely, use the negative transpose of the B-T Tableau. □

The symmetry in Theorem 4.1 leads to the following theorem.

Theorem 4.2 *In a primal-dual pair of LP-models with finite optimal solutions, the degeneracy degree of the primal (dual) optimal face is equal to the dimension of the dual (primal) optimal face.*

Proof. Construct a B-T Tableau for the pair of primal and dual LP-models. From Theorem 4.1 the result is obvious. □

Note that the above theorems are restricted to LP-models with nonnegative variables and inequality constraints. Actually, Theorem 4 is valid for general LP-models with nonnegative, nonpositive and free variables, and inequality and equality constraints. In Sierksma & Reay [10] a proof for these general LP-models is given.

The following examples may illustrate above theorems. Consider the following pair of primal and dual LP-models:

$$
\begin{array}{rlll}
\max & -2x_5 & & \\
\text{s.t.} & 3x_3+4x_4-x_5 & \leq 0 & \\
& -x_2+2x_3+3x_5 & \leq 0 & \\
& -2x_1+x_4+5x_5 & \leq 0 & \\
& 3x_1+x_2 & \leq 1 & \\
& x_1,x_2,x_3,x_4,x_5 & \geq 0, &
\end{array}
\qquad
\begin{array}{rll}
\min & y_9 & \\
\text{s.t.} & -2y_8+3y_9 & \geq 0 \\
& -y_7+y_9 & \geq 0 \\
& 3y_6+2y_7 & \geq 0 \\
& 4y_6+y_8 & \geq 0 \\
& -y_6+3y_7+5y_8 & \geq -2 \\
& y_6,y_7,y_8,y_9 & \geq 0.
\end{array}
$$

A Tucker Tableau for these models reads:

	1	x_1	x_2	x_3	x_4	x_5	
1	0	0	0	0	0	2	$=-f$
y_6	0	0	0	3	4	-1	$=-x_6$
y_7	0	0	-1	2	0	3	$=-x_7$
y_8	0	-2	0	0	1	5	$=-x_8$
y_9	-1	3	1	0	0	0	$=-x_9$
	$=-g$	$=y_1$	$=y_2$	$=y_3$	$=y_4$	$=y_5$	

Note that it is a B-T Tableau. The zero matrix in the left upper corner has two rows and three columns. Hence, according to Theorem 4.1, $dim(F_P)=2$, $\sigma(F_P,P)=1$, $dim(F_D,P)=1$ and $\sigma(F_D,P)=2$. From the first three columns of this tableau follows that $F_P=\{-x_2+x_7=0;\ -2x_1+x_8=0;\ 3x_1+x_2+x_9=1;\ x_1,x_2\geq 0;\ x_3=x_4=x_5=x_6=0\}$, and $F_D=\{3y_6-y_3=0;\ 4y_6-y_4=0;\ -y_6-y_5=-2;\ y_1=y_2=y_7=y_8=y_9=0\}$. The corresponding primal and dual optimal basic solutions satisfy $x_1=x_2=x_3=x_4=x_5=x_6=x_7=x_8=0$, $x_9=1$, $f=0$ and $y_1=y_2=y_3=y_4=y_6=y_7=y_8=y_9=0$, $y_5=2$, $g=0$.

The solutions $x_1=1/9$, $x_2=1/3$, $x_3=x_4=x_5=x_6=0$, $x_7=1/3$, $x_8=2/9$, $x_9=1/3$ and $y_1=y_2=0$, $y_3=3$, $y_4=4$, $y_5=1$, $y_6=1$, $y_7=y_8=y_9=0$ form a strictly complementary pair of optimal solutions, and are in the relative interior of the corresponding optimal faces (see Balinski & Tucker [1]).

The following example shows that a small change in an entry of the coefficients matrix may change the optimal faces rigorously. Consider a pair of primal and dual LP-models of which the B-T Tableau can be written as follows:

	1	x_1	x_2	x_3	x_4	x_5	
1	0	0	0	0	0	1	$=-f$
y_6	0	a_{11}	1	1	1	-1	$=-x_6$
y_7	0	-1					$=-x_7$
y_8	0	-1		$\cdots$			$=-x_8$
y_9	0	-1					$=-x_9$
y_{10}	-1	1					$=-x_{10}$
	$=-g$	$=y_1$	$=y_2$	$=y_3$	$=y_4$	$=y_5$	

If $a_{11} > 0$, this tableau is a B-T Tableau of which the zero matrix in the left upper corner consists of one column and five rows. Hence, according to Theorem 4.1, the primal optimal face has dimension equal to zero (the primal optimal face consists of a single vertex) and the dual optimal face has dimension equal to four. If $a_{11} < 0$, this tableau is a B-T Tableau of which the zero matrix in the left upper corner consists of one row and five columns. So, the dimension of the primal optimal face is equal to four and the dimension of the dual optimal face is equal to zero. If $a_{11} = 0$, then the zero matrix in the left upper corner has two rows and two columns. Hence, both primal and dual optimal faces have a dimension equal to one. If the value of a_{11} is close to zero and it is the result of a computer program that uses inaccurate arithmetic, it may be difficult to determine the dimension of the optimal faces.

5. Uniqueness and degeneracy

If an optimal solution of an LP-model is unique, then the optimal face consists of a single vertex which has dimension equal to zero. If an LP-model has multiple optimal solutions, then the optimal face has a positive dimension. Note that, if an LP-model has multiple solutions, this does not necessarily mean that it has more than one basic optimal solution. Theorem 4.2 gives rise to several interesting corollaries about the uniqueness and degeneracy of optimal solutions. See also Sierksma & Reay [10].

Corollary 1.

(a) *A primal LP-model has a unique and degenerate optimal solution, if and only if the corresponding dual LP-model has multiple optimal solutions of which at least one is non-degenerate.*

(b) *A primal LP-model has a unique and non-degenerate optimal solution, if and only if the corresponding dual LP-model has a unique and nondegenerate optimal solution.*

(c) *A primal LP-model has multiple optimal solutions that are all degenerate, if and only if the corresponding dual LP-model has multiple solutions that are all degenerate.*

(d) *A pair of primal and dual LP-models has unique optimal solutions, if and only if their optimal solutions are non-degenerate.*

Proof. See Sierksma & Reay [10].

Corollary 2. *If a primal LP-model has a non-degenerate optimal basic solution and the corresponding dual LP-model has a degenerate basic solution, then the primal LP-model has multiple optimal solutions.*

Proof. The primal optimal face is either a non-degenerate vertex or it contains a non-degenerate vertex as proper subset. Therefore, the degeneracy degree of the primal optimal face is equal to zero (Theorem 2.1(2)). According to Theorem 4.1 the dual optimal solution is unique. If this dual solution is degenerate, then the dual optimal face is degenerate and hence, the primal optimal face has positive dimension. □

Corollary 2 may sometimes help to decide whether or not LP-models have multiple solutions, because computer programs that use simplex methods for solving LP-models give only one optimal primal and dual basic solution.

In the literature several theorems about uniqueness and multiplicity can be found. Here we mention some of them.

In Greenberg[4] the following theorem is proved: A primal-dual pair of optimal solutions is unique if and only if it is a strictly complementary pair of basic solutions.
Note that since a strictly complementary pair of basic solutions is a pair of primal and dual optimal basic feasible solutions that have strictly positive basic variables, this theorem is equivalent to Corollary 1b.

In Mangasarian [7] the following theorem is proved. An optimal solution of a LP-model is unique if and only if it remains an optimal solution when the objective function is changed by an arbitrary but sufficient small perturbation.
This result can be related to our theorems as follows. If an arbitrary, but sufficient small perturbation, has to keep the optimal dual solution non-negative, then the optimal values of the dual basic variables should be strictly positive, which is another way of saying that the optimal dual solution (face) is non-degenerate.

In Nering & Tucker[9] the following is proved. If a pair of primal and dual LP-models has a complementary pair of optimal basic solutions that are both degenerate, then at least one of these two models has multiple optimal solutions. This is proved in Nering & Tucker by showing that there exists a strictly complementary pair of optimal solutions, which differs from the pair of optimal basic solutions. Therefore, at least one of the two models must have multiple solutions.

6. Constructing a B-T Tableau from an interior point solution

In the previous sections it was shown that among all optimal Tucker Tableaus, the B-T Tableaus give additional information. Besides the optimal primal and dual basic solutions, a B-T Tableau provides the optimal faces, the dimensions and the degeneracy degrees of the primal and dual optimal faces. As far as we know, all computer programs that solve LP-models by means of Simplex methods give as solution a more or less arbitrary optimal basic solution. It might be an idea to extend simplex LP-programs in such a way that the optimization algorithm does not stop when an optimal solution is found, but continues until an optimal basis is found that corresponds to a B-T Tableau. Using such a basis it is possible, without too much extra work, to determine the optimal faces, the dimensions and the degeneracy degrees of the optimal faces, the uniqueness of the optimal values of the variables, and the optimal partition of all strictly complementary solutions.

On the other hand, computer programs that use interior point methods, provide primal and dual solutions that form strictly complementary pairs. From these pairs of solutions it is not difficult to find a description of the primal and dual optimal faces, but it is not immediately clear what the dimensions and the degeneracy degrees of these optimal faces are. In general, the optimal solutions found by means of interior point methods are not basic solutions. In Megiddo[8], it is shown how an optimal basic solution can be constructed when an optimal pair of primal and dual solutions is known. But, in general, this optimal basic solution does not correspond to a B-T Tableau. However, it is possible to construct a B-T Tableau in strong polynomial time given a strictly complementary pair of optimal solutions. In Zhang [13] such an algorithm is given, which first uses the algorithm of Megiddo [8] to find an optimal basic solution, and then constructs a B-T Tableau by means of extremal rays of the optimal vertex. In this section we will give an algorithm, called Algorithm Construct-BT, that constructs a B-T Tableau from an interior point solution and an arbitrary Tucker Tableau in strong polynomial time, without using Megiddo's algorithm.

Algorithm Construct-BT

Input: A primal-dual pair of feasible LP-models represented by means of a Tucker Tableau; see Figure 3.1. A strictly complementary pair of optimal solutions (x^*, y^*).

Output: A B-T Tableau.

Step 1. Separate the primal and dual optimal faces and determine the all-zero matrix in the upper-left corner of the B-T Tableau.

Step 2. Make the rows below the zero matrix lexicographically negative, and the columns to the right of the zero matrix lexicographically positive.

Step 3. Transform x^* to a primal optimal basic solution, preserving the zero matrix and the lexicographic properties of the rows and the columns.

- **Step 3.1.** Select a positive non-basic variable x_{p_j}. If no such variable exists, go to Step 3.4.
- **Step 3.2.** Decrease the value of x_{p_j}.
- **Step 3.3.** Pivot if necessary and return to Step 3.1.
- **Step 3.4.** Sort the lexicographically negative rows.

Step 4. Transform y^* to a dual optimal basic solution, preserving the zero matrix and the lexicographic properties of the rows and the columns.

Next, we discuss these steps in more detail.

Ad Step 1. Determining the zero matrix.
Make the optimal value of the objective function equal to zero.

Replace a_{00} in the current tableau by $-\sum_{j=1}^{n} a_{0j} x^*_{p_j}$.

The row equation for the objective row is now $-\sum_{j=1}^{n} a_{0j} x^*_{p_j} + \sum_{j=1}^{n} a_{0j} x_{p_j} = -f$ Substituting $x = x^*$ gives $f^* = 0$. (x^*, y^*) is a strictly complementary pair of optimal solutions. This means that $x^*_i + y^*_i > 0$ and $x^*_i y^*_i = 0$ for $1 \leq i \leq m + n$. The indices $1, \ldots, m + n$ can be partitioned into two sets B and N such that $i \in B$ implies that $x^*_i > 0$, and $i \in N$ implies that $y^*_i > 0$. Let R denote the indices of the primal non-basic variables that belong to B, and let Q denote the indices of the primal basic variables that belong to N.

$$R := B \cap \{p_1, \ldots, p_n\}.$$
$$Q := N \cap \{p_{n+1}, \ldots, p_{n+m}\}.$$

$$
\begin{array}{c|c|ccc|ccc|l}
 & 1 & x_{p_1} & \cdots & x_{p_r} & x_{p_{r+1}} & \cdots & x_{p_n} & \\
\hline
1 & 0 & 0 & \cdots & 0 & a_{0,r+1} & \cdots & a_{0,n} & = -f \\
\hline
y_{p_{n+1}} & 0 & & & & & & & = -x_{p_{n+1}} \\
\vdots & \vdots & & 0 & & & A_1 & & \vdots \\
y_{p_{n+q}} & 0 & & & & & & & = -x_{p_{n+q}} \\
\hline
y_{p_{n+q+1}} & a_{q+1,0} & & & & & & & = -x_{p_{n+q+1}} \\
\vdots & \vdots & & A_2 & & & A_3 & & \vdots \\
y_{p_{n+m}} & a_{m,0} & & & & & & & = -x_{p_{n+m}} \\
\hline
 & = -g & = y_{p_1} & \cdots & = y_{p_r} & = y_{p_{r+1}} & \cdots & = y_{p_n} &
\end{array}
$$

Figure 6.1: Tucker Tableau after Step 1

> As long as the current tableau contains a nonzero entry a_{ij} with $p_{n+i} \in Q$ and $p_j \in R$, perform a pivot operation on a_{ij} and adjust the sets R and Q.

Each time such a pivot operation is performed, the number of elements in both R and Q is decreased by one. If no such pivot operations are possible, then $a_{ij} = 0$ for each $p_{n+i} \in Q$ and $p_j \in R$. The sets Q and R are then minimal.

> Rearrange the rows and the columns of the current tableau, such that the entries a_{ij} with $p_i \in R$ and $p_j \in Q$ form a zero matrix in the left upper corner of the tableau with $r = |R|$ and $q = |Q|$.
> $R = \{p_1, \ldots, p_r\}$.
> $Q = \{p_{n+1}, \ldots, p_{n+q}\}$.

The tableau is now in the form of the tableau in Figure 6.1. Representations of the primal and dual optimal faces can be read from this tableau in the same way as from a B-T Tableau.

Ad Step 2. Lexicographically ordering of the rows and the columns.
The row equations of the current tableau are $a_{i0} + \sum_{j=1}^{n} a_{ij} x_{p_j} = -x_{p_{n+i}}$ for $i = 1, \ldots, m$. Since x^* is a solution of the primal LP-model, $a_{i0} = -x^*_{p_{n+i}} - \sum_{j=1}^{n} a_{ij} x^*_{p_j}$ for $i = 1, \ldots, m$. Combining these equations, the row equations can be written as

$$
-x^*_{p_{n+i}} + \sum_{j=1}^{n} a_{ij}(x_{p_j} - x^*_{p_j}) = -x_{p_{n+i}}, \quad i = 1, \ldots, m. \tag{12}
$$

Similarly, the column equations can be written as

$$
y^*_{p_j} + \sum_{i=1}^{m} a_{ij}(y_{p_{n+i}} - y^*_{p_{n+i}}) = y_{p_j}, \quad j = 1, \ldots, n. \tag{13}
$$

In order to adjust the tableau to these equations, the following operation has to be carried out.

> Replace, in the first column of the current tableau the entries a_{i0} with $i = 1, \dots, m$ by $-x^*_{p_{n+i}}$. Similarly, replace a_{0j} by $y^*_{p_j}$ for $j = 1, \dots, n$.

Since x^* is a point in the relative interior of the primal optimal face, we have that $a_{i0} < 0$ for $i = q+1, \dots, m$. So, the rows with indices $q+1, \dots, m$ are lexicographically negative. Similarly, the columns with indices $r+1, \dots, n$ are lexicographically positive.

Ad Step 3. Determination of a primal optimal vertex.
In the current tableau all non-basic variables $x_{p_1}, \dots, x_{p_r}$ have a positive value. In order to find an optimal basic solution, the values of the non-basic variables have to be lowered to zero; if that is not possible without destroying the feasibility, some non-basic variables have to be replaced by basic variables with a zero value by means of pivot operations.

Ad Step 3.1: Selection of a positive non-basic variable.

> Select a column j with $1 \leq j \leq r$ and $x^*_{p_j} > 0$. If no such column exists, go to Step 3.4.

If more than one non-basic variable has a positive value, it is not relevant which one is chosen.

Ad Step 3.2: Decreasing of the value of the selected variable.
Decrease the value $x^*_{p_j}$ of the variable x_{p_j} as much as possible without loosing feasibility. In order to find the maximal decrease, perform the following ratio test

$$\mu := \sup_{i=1,\dots,m} \{\frac{a_{i0}}{a_{ij}} \mid a_{ij} < 0\}.$$
$$\lambda := \min\{\mu, x^*_{p_j}\}.$$

If column j does not contain any negative entry, the value of x_{p_j} can be decreased to zero and λ will be equal to $x^*_{p_j}$. The value of the variable x_{p_j} is decreased by λ and the first column of the tableau is adjusted in order to keep the row equations (12) valid.

$$x^*_{p_j} := x^*_{p_j} - \lambda.$$
$$a_{i0} := a_{i0} - \lambda * a_{i,j}, \quad x^*_{p_{n+i}} := -a_{i0}, \quad \text{for } i = q+1, \dots, m.$$

Ad Step 3.3: Pivoting.
If, in Step 3.2, some of the rows with indices $q+1, \dots, m$ have become lexicographically positive, a pivot operation on an entry in column j has to be performed in order to make the rows lexicographically negative again. If $x^*_{p_j}$ is still positive after Step 3.2, a pivot operation can replace the non-basic variable x_{p_j} by a basic variable with zero value. In order to determine the pivot row, find a row index k such that, if all rows with $a_{ij} < 0$ are divided by a_{ij}, the row with index k is the lexicographically smallest row among these rows. If k can not be determined uniquely, an arbitrary choice is made.

> If $x^*_{p_j} > 0$, or for some i with $q+1 \leq i \leq m$, row i is lexicographically positive then:

$$k := \underset{i=1,\ldots,m}{\text{arg lexico-min}} \ \{\frac{1}{a_{ij}}(\text{row } i) \mid a_{ij} < 0\}$$

Pivot on a_{kj}.
Set $a_{k0} := -x^*_{p_{n+k}}$.

If a pivot operation is performed in this step, then before the pivot operation, a_{k0} is equal to zero. After the pivot operation the lexicographic negativity of the rows with indices $q+1, \ldots, m$ is restored. Furthermore, the number of primal non-basic variables with a strict positive value is decreased by one.

Return to Step 3.1

Ad Step 3.4. All primal non-basic variables are zero and x^* is a primal basic feasible solution.

Sort the lexicographically negative rows with indices $q+1, \ldots, m$ in lexicographically non-increasing order.

Ad Step 4: Determination of a dual optimal vertex.
This step is similar to Step 3.

Take the negative transpose of the current tableau.
Perform Step 3 to find a primal optimal basic solution.
Take the negative transpose of the current tableau.

y^* is now an optimal dual basic solution.

The ratio test in Step 3.2 is similar to the ratio-test in the simplex algorithm, but here the aim is to decrease the current value of a non-basic variable as much as possible. Note that in Step 3.3 a pivot on an entry in matrix A_2 in Figure 6.1 does not affect the first row and the matrix A_1; the lexicographically positive columns will remain lexicographically positive. At the end of this algorithm the current tableau is a B-T Tableau.

Theorem 6.1 *Given an interior point solution of an LP-model, a corresponding B-T Tableau can be constructed in strong-polynomial time.*

Proof. Perform Algorithm Construct-BT. In Step 1 at most $\min(n-r, m-q)$ pivot operations are performed. In Step 3 at most r and in step 4 at most q. The total number of pivot operations is $\min(n+q, m+r)$ and has $m+n$ as upperbound. Clearly, this algorithm is strong-polynomial, since, apart from the pivot operations, the number of all other operations can be bounded from above by a polynomial in m and n. □

We will illustrate algorithm Construct-BT with the following pair of LP-models. The primal

LP-model

$$\begin{array}{lrrrrcrl}
\max & -4x_1 & +4x_2 & -8x_3 & +4x_4 & & & \\
\text{s.t.} & -x_1 & +x_2 & -2x_3 & +x_4 & \leq & 1 & (\text{slack} : x_5) \\
 & +4x_1 & -4x_2 & +x_3 & -2x_4 & \leq & 0 & (\text{slack} : x_6) \\
 & & & -3x_3 & +x_4 & \leq & 2 & (\text{slack} : x_7) \\
 & -x_1 & +x_2 & -2x_3 & +x_4 & \leq & 1 & (\text{slack} : x_8) \\
 & -2x_1 & +5x_2 & -9x_3 & +3x_4 & \leq & 7 & (\text{slack} : x_9) \\
 & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \geq 0}
\end{array}$$

and the corresponding dual LP-model

$$\begin{array}{lrrrrrcrl}
\min & +y_5 & & +2y_7 & +y_8 & +7y_9 & & & \\
\text{s.t.} & -y_5 & +4y_6 & + & -y_8 & -2y_9 & \geq & -4 & (\text{slack} : y_1) \\
 & +y_5 & -4y_6 & + & +y_8 & +5y_9 & \geq & 4 & (\text{slack} : y_2) \\
 & -2y_5 & +y_6 & -3y_7 & -2y_8 & -9y_9 & \geq & -8 & (\text{slack} : y_3) \\
 & +y_5 & -2y_6 & +y_7 & +y_8 & +3y_9 & \geq & 4 & (\text{slack} : y_4) \\
 & \multicolumn{8}{l}{y_1, y_2, y_3, y_4, y_5 \geq 0.}
\end{array}$$

We first construct a Tucker Tableau for these LP-models.

	1	x_1	x_2	x_3	x_4	
1	0	4	−4	8	−4	$= -f$
y_5	−1	−1	1	−2	1	$= -x_5$
y_6	0	4	−4	1	−2	$= -x_6$
y_7	−2	0	0	−3	1	$= -x_7$
y_8	−1	−1	1	−2	1	$= -x_8$
y_9	−7	−2	5	−9	3	$= -x_9$
	$= -g$	$= y_1$	$= y_2$	$= y_3$	$= y_4$	

A strictly complementary solution for these models is:
$(f^*, x_1^*, \ldots, x_9^*) = (4, 1, 1, 1, 3, 0, 5, 2, 0, 4)$ and
$(g^*, y_1^*, \ldots, y_9^*) = (4, 0, 0, 0, 0, 3, 0, 0, 1, 0)$.
The optimal partition is $B = \{1, 2, 3, 4, 6, 7, 9\}$ and $N = \{5, 8\}$.

Step 1. Replace a_{00} by $-\sum_{j=1}^{4} a_{0j} x_{p_j}^* = 4$. This makes $f^* = g^* = 0$.

	1	x_1	x_2	x_3	x_4	
1	4	4	−4	8	−4	$= -f$
y_5	−1	−1	1	−2	1	$= -x_5$
y_6	0	4	−4	1	−2	$= -x_6$
y_7	−2	0	0	−3	1	$= -x_7$
y_8	−1	−1	1	−2	1	$= -x_8$
y_9	−7	−2	5	−9	3	$= -x_9$
	$= -g$	$= y_1$	$= y_2$	$= y_3$	$= y_4$	

Determine the sets R and Q. $R = \{1, 2, 3, 4\}$ and $Q = \{5, 8\}$.
The non-zero entry $a_{4,4}$ with value 1 corresponds to (x_4, x_8) with $4 \in R$ and $8 \in Q$. Pivoting

on $a_{4,4}$ gives:

	1	x_1	x_2	x_3	x_8	
1	0	0	0	0	4	$= -f$
y_5	0	0	0	0	-1	$= -x_5$
y_6	-2	2	-2	-3	2	$= -x_6$
y_7	-1	1	-1	-1	-1	$= -x_7$
y_4	-1	-1	1	-2	1	$= -x_4$
y_9	-4	1	2	-3	-3	$= -x_9$
	$= -g$	$= y_1$	$= y_2$	$= y_3$	$= y_8$	

Adjust R and Q. $R = \{1, 2, 3\}$ and $Q = \{5\}$. All entries a_{ij} that could be possible pivot candidates are zero now. Rearranging of the rows and columns is not necessary, since the zero matrix is already in the left-upper part of the tableau. ($q = 1$ and $r = 3$). This ends Step 1.

Step 2. Substitute the values of the primal basic variables in the first column, and the values of the dual basic variables in the first row. From now on we will omit the names of the dual variables from the tableau, but instead we will put there the values of the primal optimal solution.

	1	x_1	x_2	x_3	x_8	
	0	0	0	0	1	$= -f$
0	0	0	0	0	-1	$= -x_5$
5	-5	2	-2	-3	2	$= -x_6$
2	-2	1	-1	-1	-1	$= -x_7$
3	-3	-1	1	-2	1	$= -x_4$
4	-4	1	2	-3	-3	$= -x_9$
	0	1	1	1	0	

Step 3.
The primal non-basic variables with positive value are: $x_1^* = 1$, $x_2^* = 1$ and $x_3^* = 1$.
Step 3.1. Select column 3 corresponding to the primal non-basic variable x_3 with $x_3^* = 1$.
Step 3.2. Calculate λ. $\lambda = 1$. Decrease the value of x_3^* with 1 to 0.

	1	x_1	x_2	x_3	x_8	
	0	0	0	0	1	$= -f$
0	0	0	0	0	-1	$= -x_5$
2	-2	2	-2	-3	2	$= -x_6$
1	-1	1	-1	-1	-1	$= -x_7$
1	-1	-1	1	-2	1	$= -x_4$
1	-1	1	2	-3	-3	$= -x_9$
	0	1	1	0	0	

Step 3.3. $x_3^* = 0$ and the rows $2, \ldots, 5$ are lexicographically negative; no pivot is necessary.
Step 3.1. Select column 2 corresponding to the primal non-basic variable x_2 with $x_2^* = 1$.

Step 3.2. Calculate λ. $\lambda = 1$. Decrease the value of x_2^* with 1 to 0.

	1	x_1	x_2	x_3	x_8	
	0	0	0	0	1	$= -f$
0	0	0	0	0	-1	$= -x_5$
0	0	2	-2	-3	2	$= -x_6$
0	0	1	-1	-1	-1	$= -x_7$
2	-2	-1	1	-2	1	$= -x_4$
3	-3	1	2	-3	-3	$= -x_9$
	0	1	0	0	0	

Step 3.3. Row 2 and row 3 are lexicographically positive; a pivot is necessary.
Determine the row index k.
Row 2 divided by -2 gives : (0.0 -1.0 1.0 1.5 -1.0)
Row 3 divided by -1 gives : (0.0 -1.0 1.0 1.0 1.0).
Row 3 is the lexicographically smallest, so a pivot operation has to be performed on $a_{3,2}$.

	1	x_1	x_7	x_3	x_8	
	0	0	0	0	1	$= -f$
0	0	0	0	0	-1	$= -x_5$
0	0	0	-2	-1	4	$= -x_6$
0	0	-1	-1	1	1	$= -x_2$
2	-2	0	1	-3	0	$= -x_4$
3	-3	3	2	-5	-5	$= -x_9$
	0	1	0	0	0	

Step 3.1. Select column 1 corresponding to the primal non-basic variable x_1 with $x_1^* = 1$.
Step 3.2. Calculate λ. $\lambda = 0$. Decreasing the value of x_1^* with 0 does not change the tableau.
Step 3.3. Since $x_1^* > 0$, a pivot is necessary. Pivot on $a_{3,1}$ and let $a_{3,0} := -x_1^* = -1.0$.

	1	x_2	x_7	x_3	x_8	
	0	0	0	0	1	$= -f$
0	0	0	0	0	-1	$= -x_5$
0	0	0	-2	-1	4	$= -x_6$
1	-1	-1	1	-1	-1	$= -x_1$
2	-2	0	1	-3	0	$= -x_4$
3	-3	3	-1	-2	-2	$= -x_9$
	0	0	0	0	0	

Step 3.4. All primal non-basic variables are zero. The lexicographically negative rows $2, \ldots, 5$ are already sorted. This finished Step 3.

Step 4. Take the negative transpose of the current tableau and place the values of the optimal dual basic variables on the left of the tableau, and the values of the non-basic dual variables under the tableau.

	1	y_5	y_6	y_1	y_4	y_9	
	0	0	0	1	2	3	$= g$
0	0	0	0	1	0	-3	$= -y_2$
0	0	0	2	-1	-1	1	$= -y_7$
0	0	0	1	1	3	2	$= -y_3$
1	-1	1	-4	1	0	2	$= -y_8$
	0	3	0	0	0	0	

Step 3.
The only non-basic variable with positive value is y_5 with $y_5^* = 3$.
Step 3.1. Select column 1 corresponding to the dual non-basic variable y_5 with $y_5^* = 3$.
Step 3.2. Calculate λ. $\lambda = 3$. Decrease the value of y_5^* with 3 to 0.

	1	y_5	y_6	y_1	y_4	y_9	
	0	0	0	1	2	3	$= g$
0	0	0	0	1	0	-3	$= -y_2$
0	0	0	2	-1	-1	1	$= -y_7$
0	0	0	1	1	3	2	$= -y_3$
4	-4	1	-4	1	0	2	$= -y_8$
	0	0	0	0	0	0	

Step 3.3. No pivot is necessary.
Step 3.4. No sorting is necessary.
This finishes Step 3.
Take the negative transpose of the current tableau.

	1	x_2	x_7	x_3	x_8	
1	0	0	0	0	4	$= -f$
y_5	0	0	0	0	-1	$= -x_5$
y_6	0	0	-2	-1	4	$= -x_6$
y_1	-1	-1	1	-1	-1	$= -x_1$
y_4	-2	0	1	-3	0	$= -x_4$
y_9	-3	3	-1	-2	-2	$= -x_9$
	$= -g$	$= y_2$	$= y_7$	$= y_3$	$= y_8$	

Step 4 is finished.
The current tableau is a B-T Tableau.
End of algorithm Construct-BT.

7. Conclusions

By introducing a more general definition of degeneracy, we established a remarkable relationship between the degrees of degeneracy and the dimensions of the optimal faces of LP-models. The optimal simplex tableaus as introduced by Balinski & Tucker [1] provide the degrees of degeneracy and the dimensions of the optimal faces. For analyzing the optimal solutions of an LP-model it is worthwhile to know whether the optimal solution is unique or whether there are multiple solutions. Computer programs that solve LP-models by means of Simplex Methods only give one basic optimal solution. We recommend that computer programs are extended in such a way that they calculate B-T Tableaus. Then it is possible to give, besides an optimal solution, also the optimal faces and their dimensions and degeneracy degrees.

In recent years a lot of attention is given to non-simplex methods, such as the interior points methods. These methods provide strictly complementary optimal solutions, which are located in the relative interior of the optimal faces. For LP-models that are relaxations of combinatorial and integer models, usually optimal basic solutions are needed. In Bixby & Saltzman [2] and Megiddo [8], algorithms are given that transform an optimal interior point solution into a basic feasible solution. An optimal interior point solution can be found in polynomial time. In this paper we have presented an algorithm that constructs a B-T Tableau given an optimal interior point solution. Therefore, it is now possible to construct B-T Tableaus of LP-models in polynomial time.

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