

Adding-up constraints and gross substitution in portfolio models

Jan Jacobs and Lambert Schoonbeek*
CCSO and Faculty of Economics, University of Groningen,
The Netherlands

Abstract

We consider a (static) portfolio system that satisfies adding-up constraints and the gross substitution theorem. We show the relationship of the two conditions to the weak dominant diagonal property of the matrix of interest rate elasticities. This enables us to investigate the impact of simultaneous changes in interest rates on the asset demands.

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* Corresponding author: Department of Econometrics, Faculty of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Tel. +31 50 63 3798. Fax: +31 50 63 3720. (From 10 october 1995 onwards these numbers become +31 50 3 63 3798 and +31 50 3 63 3720, respectively.) E-mail: L.Schoonbeek@eco.rug.nl. We thank Elmer Sterken for helpful comments.

1. Introduction

Consider the following static (or long-run equilibrium) portfolio model:

$$x = Ar + Bz + u, \quad (1)$$

where $x = (x_1, \dots, x_n)'$ is an $n \times 1$ vector of asset demands, $A = (a_{ij})$ is an $n \times n$ matrix of parameters, $r = (r_1, \dots, r_n)'$ is an $n \times 1$ vector of interest rates, r_i is the interest rate of the i th asset, B is an $n \times k$ matrix of parameters, z is a $k \times 1$ vector of exogenous variables, and u is an $n \times 1$ vector of disturbances, $u \sim (0, \Sigma)$. The prime attached to a symbol denotes the transpose. Strictly speaking x might contain assets and liabilities, but liabilities are multiplied by -1 and can thus be treated as assets. Therefore, in this paper we will only speak about asset demands.

To ensure consistency it is generally assumed that the following adding-up constraints hold: $\iota_n' A = \iota_n' B = \iota_n' u = 0$, where ι_n is an $n \times 1$ vector of ones. These restrictions guarantee that the portfolio model satisfies the wealth constraint: i.e. the sum of the asset demands equals wealth. Since in our framework all assets are included in the model, we may say that the asset demands sum to zero, i.e. $\iota_n' x = 0$. For a general discussion, see e.g. Brainard and Tobin (1968) or Owen (1986).

In this paper we focus primarily on the first of the adding-up constraints, i.e. on:

Condition 1 $\iota_n' A = 0$, i.e. the column sums of matrix A are all equal to zero.

Another assumption that is often made with respect to model (1) is gross substitution, see e.g. Tobin (1982). Tobin's gross substitution theorem states that the sign of the own interest rate elasticity must be nonnegative while the elasticities of all other interest rates must have nonpositive signs, i.e.:

Condition 2 $a_{ii} \geq 0, a_{ij} \leq 0, i, j = 1, \dots, n, j \neq i$.

The condition implies that if one increases the interest rate of a *single* asset r_k , say, then the quantity demanded of the associated k th asset, x_k , does not decrease whereas the demands for the other assets, x_j ($j \neq k$), will not increase. We observe that using Condition 2 only, in general nothing can be said about the impact on the asset demands of simultaneous changes in more than one interest rate.

The constraint $\iota_n' u = 0$, discussed above, makes that the variance-covariance matrix Σ of the disturbance vector is singular, which complicates the estimation of the matrices A and B . A practical workaround to circumvent the singularity of Σ is to omit one of the asset demand equations in the estimations, see e.g. Owen (1986). Assume

that the equation for the n th asset is deleted. The remaining $n - 1$ asset demand equations are then estimated, e.g. with some system estimation method. Interest rates with the wrong sign, i.e. violating Tobin's gross substitution, are excluded from these $n - 1$ equations. At the end the parameters of the n th equation are calculated as 'residuals' using $l'_n A = l'_n B = 0$. Applying this procedure one should be careful that the parameters of the n th equation satisfy Condition 2 as well.

This paper analyses the implications of the Conditions 1 and 2 by relating them to the concept of a weak dominant diagonal matrix as presented by Schoonbeek (1992). Doing so, we first explore the requirement that the n th 'residual' row of matrix A must satisfy the gross substitution constraint. Next, we characterize matrix A itself as a weak dominant diagonal matrix, which enables us to investigate the effect of simultaneous changes in the interest rates on the asset demands. Finally, we show that the asset demands are invariant under a certain nonnegative (but nonzero) change of the interest rates.

2. The results

Using Condition 1 we write matrix A in the obvious way as

$$A = \begin{pmatrix} A_{n-1,n-1} & a_n \\ -l'_{n-1} A_{n-1,n-1} & -l'_{n-1} a_n \end{pmatrix}, \quad (2)$$

where the elements of the n th row of A amount to $a_{nj} = -\sum_{i=1}^{n-1} a_{ij}$, $j = 1, \dots, n$. Next, applying the gross substitution condition with respect to the first $n - 1$ rows of matrix A only, we obtain

Condition 3 $a_{ii} > 0$, $a_{ij} < 0$, $i = 1, \dots, n - 1$; $j = 1, \dots, n$, $j \neq i$.

Let us now recall the definition of a (column) dominant diagonal matrix, see Takayama (1985), and of a weak (column) dominant diagonal matrix, see Schoonbeek (1992). We state the definitions in terms of matrix A . Clearly, analogous definitions can be given with respect to matrix $A_{n-1,n-1}$. Matrix A has a weak dominant diagonal (wdd) if there exist positive scalars (weights) μ_1, \dots, μ_n such that

$$\mu_j |a_{jj}| > \sum_{i \neq j} \mu_i |a_{ij}|, \quad \text{for all } j = 1, \dots, n. \quad (3)$$

If all weak inequalities ($>$) in (3) are replaced by strict inequalities ($>$), A has a dominant diagonal (dd).

Using the above definitions one can easily verify the following result:

Result 1 *Let Condition 1 and Condition 3 hold. Then matrix A satisfies Condition 2 if and only if matrix $A_{n-1,n-1}$ has a wdd with weights $\mu_1 = \dots = \mu_{n-1} = 1$.*

Thus, assuming that the first $n - 1$ rows of matrix A satisfy the gross substitution constraint, Result 1 gives a necessary and sufficient condition on these $n - 1$ rows such that the n th row – which is obtained from Condition 1 – satisfies the gross substitution constraint as well.

We proceed with the analysis of the properties of matrix A of model (1). Note that if A satisfies Condition 2 and is indecomposable, and, furthermore, has a wdd such that at least one of the weak inequalities of (3) is a strict inequality, then A has also has a dd, see Takayama (1985). Recall that A is indecomposable if there does not exist a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}, \quad (4)$$

where A_{11} and A_{22} are square submatrices and O is a submatrix with all elements equal to zero. In the context of model (1), indecomposability of A is an appealing and natural property. It means that the asset demands do not depend in a block-recursive way on the associated interest rates.

Next, we state the following result:

Result 2 *Let Condition 1 and Condition 2 hold. Then matrix A has a wdd with weights $\mu_1 = \dots = \mu_n = 1$.*

We can make four remarks with respect to Result 2. First, consider model (1) under the Conditions 1 and 2. Suppose that the interest rates are exposed to a change represented by the $n \times 1$ vector Δr , say. This change results in a change in the asset demands equal to $\Delta x \equiv A \Delta r$. Under the assumptions of Result 2 we know that matrix A has a wdd and nonnegative diagonal elements. This implies in turn that for *each* vector $\Delta r \neq 0$ (so, simultaneous changes of more than one interest rate are allowed) there exists an index k such that $\Delta r_k \neq 0$ and $\Delta x_k \Delta r_k > 0$, see Schoonbeek (1992). Notice that this means that if the element Δr_k is positive (negative) then the corresponding element Δx_k is *not* negative (positive). In other words, there is at least one nonzero element of which the sign does not strictly reverse. Note that we say that all nonzero elements strictly reverse sign if $\Delta x_i \Delta r_i = (A \Delta r)_i \Delta r_i < 0$ for all i with $\Delta r_i \neq 0$. As an example, let Δr be a vector of which the signs of the elements alternate, i.e the first element is positive, the second negative, the third positive, and so on. We then

know that there must be at least one interest rate which either (a) increases whereas the demand for the corresponding asset does not decrease, or (b) decreases whereas the demand for the associated asset does not increase. Observe that this conclusion cannot be derived if we only impose Condition 2 with respect to model (1).

Secondly, it is known that a matrix with a dd is nonsingular, see e.g. Takayama (1985). Because Condition 1 implies that matrix A is singular, we directly conclude that A cannot have a dd. So, the wdd-property of A in Result 2 cannot be sharpened in this respect.

Thirdly, under the assumptions of Result 2, matrix A has a wdd with all weights equal to unity. Observe that in this case equation (3) holds with equalities for all $j = 1, \dots, n$. Suppose now that $a_{n,j^*} < 0$ for at least one $j^* \neq n$, and that $A_{n-1,n-1}$ is indecomposable. It then follows that $A_{n-1,n-1}$ must have a dd. We then further obtain the following two properties:

- (i) Because the diagonal (off-diagonal) elements of matrix $A_{n-1,n-1}$ are nonnegative (nonpositive) and $A_{n-1,n-1}$ has a dd, all its diagonal elements must in fact be positive: i.e. all the own interest rate elasticities are positive.
- (ii) Let Δr^{n-1} denote an $(n-1) \times 1$ vector. Because $A_{n-1,n-1}$ has a dd and the diagonal elements of $A_{n-1,n-1}$ are positive, we can conclude that for *each* vector $\Delta r^{n-1} \neq 0$ there is an index k such that $(\Delta r^{n-1})_k (\Delta x^{n-1})_k > 0$, where $\Delta x^{n-1} \equiv A_{n-1,n-1} \Delta r^{n-1}$, see Schoonbeek (1992). Thus, if we arbitrarily change in model (1) one or more of the first $n-1$ interest rates, then there is at least either (a) one interest rate that increases whereas the corresponding asset demand increases as well, or (b) one interest rate that decreases whereas the corresponding asset demand decreases as well. (Compare with our first remark.)

Finally, observing that each arbitrary $n-1 \times n-1$ submatrix of matrix A (obtained by skipping one row and the corresponding column from A) has a wdd if A has a wdd, we conclude from Result 1 and Result 2 that it is not relevant with respect to the gross substitution constraint which row of A is calculated by using Condition 1: instead of considering the n th row of A as the ‘residual’ row, we could have taken equally well any other row of A .

Our next result reads as follows:¹

Result 3 *Let Condition 1 and Condition 2 hold. Then there exists a vector $r^* \geq 0$ such that $Ar^* = 0$. If, in addition, matrix A is indecomposable, then we can take*

¹ With respect to a vector y , say, we use the following notation: $y > 0$ means that all elements of y are positive; $y \geq 0$ means that all elements are nonnegative while at least one element is positive; $y \geq 0$ means that all elements are nonnegative. In an analogous way we use the symbols $<$, \leq and \cdot .

$r^* > 0$, and r^* is unique up to a scalar multiple.

Proof We start with the first statement. From Condition 2 we know that the signs of all diagonal (off-diagonal) elements of A are nonnegative (nonpositive). Therefore, we can write $A = (\rho I - A^+)$, where ρ is a nonnegative real scalar, I is the $n \times n$ identity matrix, and A^+ is a nonnegative matrix. From the Frobenius theorem it follows that A^+ has a nonnegative real eigenvalue τ , say, with corresponding $n \times 1$ left eigenvector and right eigenvector $p \geq 0$ and $q \geq 0$ respectively: i.e. $p'A^+ = \tau p'$ and $A^+q = \tau q$. From Condition 1 we know that $l'_n A = 0$, or $l'_n(\rho I - A^+) = 0$. From the latter it follows that $\rho = \tau$, see Kemp and Kimura (1978, p. 84). We now have to distinguish two cases depending on the magnitude of ρ and τ . Case (i): Suppose $\rho = \tau$. In this case we can simply take $r^* = q$. This completes the proof for this case. Case (ii): Suppose $\rho < \tau$. Because $(\tau I - A^+)q = 0$, we obtain in this case that $(\rho I - A^+)q \leq 0$. Using a result of Kemp and Kimura (1978, p. 3), it then follows that there exists no solution $s > 0$ of the system of equations $s'A = 0$. However, this gives a contradiction, because we know from Condition 1 that $l'_n A = 0$, where $l'_n > 0$. We conclude that $\rho < \tau$ cannot hold. So the first statement is established.

Next, suppose that A is indecomposable. Then A^+ is indecomposable as well, and the second statement follows from Kemp and Kimura (1978, p. 82). \square

Consider model (1) under the Conditions 1 and 2. Suppose again that the interest rates are exposed to a change represented by the vector Δr with a resulting impact of $\Delta x \equiv A\Delta r$ on the asset demands. It then follows from Result 3 that there is a vector $r^* \geq 0$ (or > 0 , if A is indecomposable) such that if we take $\Delta r = r^*$, the change of the interest rates from r to $r + mr^*$, where m is an arbitrary positive real scalar, induces no change in the asset demands. Clearly, this can be interpreted as an invariance property. Alternatively, if the vector of interest rates itself satisfies $r = mr^* \geq 0$, where m is an arbitrary positive real scalar, then in fact the asset demand vector x does not depend on the interest rates.

Finally, recall that we have discussed in the third remark below Result 2 a situation in which matrix $A_{n-1,n-1}$ turned out to have a dd. As a result then $A_{n-1,n-1}$ must be a nonsingular matrix, and so $A_{n-1,n-1}\Delta r^{n-1} = 0$ implies that $\Delta r^{n-1} = 0$. Thus, the invariance property just mentioned with respect to the complete model (1) breaks down if we limit the attention to the first $n - 1$ assets and interest rates only.

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