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Inequality and Network Structure*

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Abstract

This paper explores the manner in which the structure of a social network constrains the level of inequality that can be sustained among its members. We assume that any distribution of value across the network must be stable with respect to coalitional deviations, and that players can form a deviating coalition only if they constitute a clique in the network. We show that if the network is bipartite, there is a unique stable payoff distribution that is maximally unequal in that it does not Lorenz dominate any other stable distribution. We obtain a complete ordering of the class of bipartite networks and show that those with larger maximum independent sets can sustain greater levels of inequality. The intuition behind this result is that networks with larger maximum independent sets are more sparse and hence offer fewer opportunities for coalitional deviations. We also demonstrate that standard centrality measures do not consistently predict inequality. We extend our framework by allowing a group of players to deviate if they are all within distance k of each other, and show that the ranking of networks by the extent of extremal inequality is not invariant in k.

JEL classification: C71, D30, D85.

Keywords: inequality, networks, coalitional deviations, power, centrality.

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1 Introduction

This paper explores the manner in which the structure of a social network constrains the level of inequality that can be sustained among its members. The key idea is that any distribution of value must be stable with respect to coalitional deviations, where the set of feasible coalitions is itself constrained by the requirement that only groups of players that are mutually connected can jointly deviate. That is, we allow for deviations only by *cliques*. A payoff distribution is said to be stable if there is no clique in the network that can profitably deviate. The main research question is the following: What is the relationship between the structure of the network and the maximum level of inequality that can be sustained among its members?

To compare payoff distributions in terms of their level of inequality, we adopt the standard criterion of Lorenz dominance. Lorenz dominance only provides a partial ordering of value distributions so the maximum level of inequality may not be well defined in general. We show, however, that when the network is bipartite and the value of a network is a strictly convex function of the number of players, there is a *unique* stable value distribution that does not Lorenz dominate any other distribution. Hence the most unequal value distribution is well defined. We refer to this distribution as the extremal distribution of this network. Given that the extremal distribution is well defined, we obtain a complete ordering of the class of bipartite networks with respect to the level of extremal inequality that they can sustain. The ordering is based on the cardinality of maximum independent sets: bipartite networks which have a larger maximum independent sets can sustain greater levels of extremal inequality.[1] We then extend this framework to include the case in which players can jointly deviate if they are all within distance k of each other, and explore the manner in which extremal inequality changes as k is varied. Although inequality declines as k increases, it can do so at different rates in different networks. As a result, the ranking of networks by the extent of extremal inequality is not invariant in k.

The idea that network structure influences the allocation of value was initially proposed in a seminal paper by Myerson (1977), who assumed that a coalition of individuals could generate value if and only if they were all connected to each other along some path that did not involve anyone outside the coalition. Such paths could be of arbitrary length, which entails the implicit assumption that communication through intermediaries is as effective as direct communication in the process of coalition formation. This assumption has been maintained in the significant literature on communication games that has followed the work of Myerson; see Slikker and van den Nouweland (2001) for a survey. In contrast, we assume that deviating coalitions require direct communication (or at least sufficiently short paths) between members. Additionally, while Myerson's objective was an axiomatic characterization of a particular value distribution, our concern here is simply with the extent of inequality that is consistent with stability.

A number of writers have previously explored determinants of the degree of inequality in

[1] An independent set in a network is a set of vertices such that no pair of vertices in the set are connected to each other. An independent set is maximum if there is no independent set with greater cardinality. See Section 2 for formal definitions.

equilibrium networks. Goyal and Vega-Redondo (2007) propose an allocation rule whereby connections produce a surplus that is shared with essential intermediaries in the network (see also Hojman and Szeidl, 2008). This model captures the intuition of Burt (2005, p. 4) that "people who do better are somehow better connected," the underlying idea being that centrally located individuals may hold up players that are not directly connected, or for other reasons secure a large share of the goods or services that flow through the network. A number of centrality measures have been proposed, including the number of neighbors of a player, his closeness (mean shortest path to other players) and his betweenness (the fraction of shortest paths between all pairs of players in a network that include the player); see, for instance, Jackson (2008). Inequality in these indices of centrality are thought to induce corresponding levels of inequality in the allocation of value in the network, a supposition for which there is some empirical evidence (see, for instance, Brass, 1984; Podolny and Baron, 1997).

Our approach is different in two important respects. First, in these papers, an agent's central position confers the ability to gain larger shares of the surplus, the intuition being that essential intermediaries can extract rents through their control of flows between players that are not otherwise connected. These "middleman" models are implicitly based on the idea that competition reduces inequality, and monopoly increases it. The centrality measures thus explain distributional advantage by analyzing how well connected the rich are. While these intuitions are undoubtedly correct in many settings, our model stresses another dimension that determines inequality: how isolated the poor are. Intuitively, if the network is dense, inequality will be hard to sustain as disadvantaged players can jointly deviate. Conversely, if the network is sparse, peripheral players can more readily be exploited.[2]

A second important difference is that the papers cited above employ an exogenously given profile of payoff functions that determines for each network the allocation of value between players. The focus is accordingly on the level of inequality that arises in equilibria of the network formation model. This contrasts with the work of Myerson (1977) and this paper, in which networks are given exogenously, and the inequality supportable on that network is investigated in light of the posited rules on coalitional deviation.

Dutta and Ray (1989) also study the interaction between stability and equality. They propose a solution concept that selects among a set of allocations that satisfy core-like participation constraints the one that is most egalitarian in terms of Lorenz dominance. An important difference between their work and ours is that Dutta and Ray do not restrict the set of coalitions that can form and do not therefore explore the manner in which inequality varies with network structure. Focusing on hierarchies, Demange (2004) restricts coalitional deviations to teams and shows that this leads to a unique stable outcome for a range of games. In contrast, we explore deviations by cliques rather than teams. Stable allocations are typically not unique in our framework, and our focus is on allocations that are extremal in a well-defined sense. Finally, Bramoullé and Kranton (2007) find that independent sets play a central role in an entirely different context: the private provision of local public

[2] Even in the original framework of Myerson (1977), the standard intuition need not apply. Kalai, Postlewaite, and Roberts (1978) show that the central player in a star network can be worse off at a core allocation than he would be at any core allocation in a complete network.

goods (see also Corbo, Calvó-Armengol, and Parkes, 2007). They show that there is a class of equilibria that can be characterized in terms of the behavior of players belonging to a maximal independent set.[3] Their model is one of a noncooperative game played on a network, and differs in fundamental respects from our own. It is interesting, therefore, to note that independent sets play a critical role in both solutions.

While our model is too abstract to be directly applicable to empirical cases, we think that the approach captures an important aspect of real world conflict over the joint gains to cooperation. It suggests, for example, that geographically dispersed outsourcing may be profitable for a firm as it limits the opportunities for suppliers to communicate and hence reduces the likelihood that they could jointly deviate. It also provides a possible explanation of the contrast, noted by historians and archaeologists, between the stability of high levels of inequality characteristic of relationships between a landed class and dependent farmers in ancient societies and the frequent challenges to unequal distribution of the surplus in industrial production during the modern era (Hobsbawm, 1964; Trigger, 2003). An explanation consistent with our model is that agrarian inequality is based on the infrequent delivery of crops by otherwise isolated farmers, while inequality between industrial employers and workers is based on the daily delivery of the worker's own labor to a common site (the factory). As a result employees have direct links based on their common place of employment, while share croppers and other agrarian producers do not. These differences may also help explain why one half is the most common crop share, while the labor share in industrial production is commonly much higher.

2 Distributions on networks

2.1 Networks

Players are located on a network. A *network* is a pair (N, g), where $N = \{1, \ldots, n\}$ is a set of *vertices* and g is an $n \times n$ matrix, with $g_{ij} = 1$ denoting that there is a *link* or edge between two vertices i and j, and $g_{ij} = 0$ meaning that there is no link between i and j. A link between i and j is denoted by $\{i, j\}$. We focus on undirected networks, so $g_{ij} = g_{ji}$. Moreover, we set $g_{ii} = 0$ for all i. In the following, we fix the vertex set N and denote a network by the matrix g. If $g_{ij} = 1$, that is, if there is a link between i and j, we say that i and j are *neighbors* or, alternatively, that they are *adjacent* in g. A *clique* is a set of pairwise adjacent vertices. The number of neighbors of a vertex is termed its *degree*. The *degree distribution* of a network is a vector $d = (d_0, \ldots, d_{n-1})$, with d_m the number of vertices with degree m.

A *path* between two vertices i and j in a network g is a list of vertices $i_1, i_2, \ldots, i_K$ such that $i_1 = i$ and $i_K = j$, and $g_{i_t i_{t+1}} = 1$. If $i = j$, the path $i_1, i_2, \ldots, i_K$ is called a *cycle*. If there is a path between any two vertices in the network, we say that the network is *connected*. The *length* of a path $i_1, i_2, \ldots, i_K$ is $K - 1$. The *distance* $d_{ij}(g)$ between two vertices i and j

[3] A *maximal* independent set is an independent set that is not properly contained in another independent set. Note that a *maximum* independent set (one that has maximum cardinality among independent sets) is maximal, while the converse need not hold. Our results pertain to maximum independent sets.

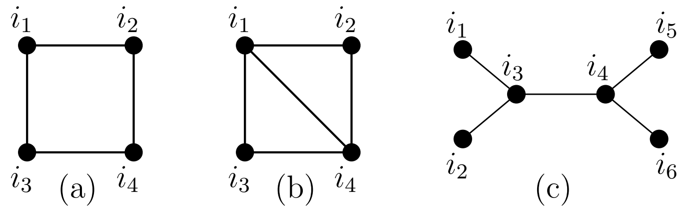


Figure 2.1: (a) A bipartite network, with independent sets $\{i_1, i_4\}$ and $\{i_2, i_3\}$. (b) A network that is not bipartite: there do not exist two independent sets that partition the set of vertices. (c) A bipartite network that is a tree.

in network g is defined as follows. If $i = j$, then $d_{ij}(g) = 0$. If $i \neq j$, then $d_{ij}(g)$ is equal to the length of the shortest path between i and j in g, if such a path exists, and ∞ otherwise.

An *independent set* in a network is a set of vertices that are pairwise nonadjacent. A set of vertices forms a *maximum independent set* in g if it is an independent set and there is no independent set in g with a strictly higher cardinality. Note that while a network may have multiple (maximum) independent sets, the cardinality of a maximum independent set is unique.

We derive several results for bipartite networks. A network is *bipartite* if its vertex set is the union of two disjoint (possibly empty) independent sets; see Figure 2.1. It can be shown that a network is bipartite if and only if it does not have a cycle of odd length. The class of bipartite networks contains the set of trees, which are networks without a cycle (see Figure 2.1(c)).

2.2 Stable allocations

Consider a set of players located on a network. The players jointly generate a surplus, which is an increasing and strictly convex function of the network size. The surplus is divided among the players in the network, in such a way that no coalition of players that form a clique in the network can profitably deviate.

More precisely, consider a set of players $N = \{1, \ldots, n\}$, and a network g with vertex set N. Hence, each player is associated with a vertex. As in Myerson (1977), players generate value if they are connected by some path. The value generated by a set of players S is given by $f(|S|)$, where f is a strictly convex and increasing function with $f(0) = 0$. We assume that f is continuous and twice differentiable. Without loss of generality, we assume that the network is connected, that is, there is a path between each pair of players. Hence, the value of g is $f(n)$.

This surplus is divided among the players. The distribution of the surplus is determined by the deviating coalitions that can form. We assume that only cliques can jointly deviate. Formally, an *allocation* is any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$. An allocation x is *feasible* if $x_i \geq 0$ for all $i \in N$, and

$$\sum_{i \in N} x_i \leq f(n). \tag{2.1}$$

We say that an allocation x is *stable* on g if no clique can gain by deviating: for each clique

C in g,

$$\sum_{i \in C} x_i \geq f(|C|). \tag{2.2}$$

That is, for an allocation to be stable, the members of each clique have to get at least as much collectively under the allocation as they would if they were to deviate collectively and form their own network. In Section 6, we allow for players to coordinate deviations over larger distances.

Since f is a (strictly) convex function, the egalitarian allocation given by $x_i = f(n)/n$ is always stable, so that the set of feasible and stable allocations is nonempty. It is immediate that the set of feasible and stable allocations, being a set of vectors satisfying a set of weak inequalities (2.1) and (2.2), is closed and convex. The definition of the set of feasible and stable allocations is reminiscent of the definition of the core in transferable-utility games (TU-games). The difference is that while inequality (2.2) needs to hold for *all* coalitions for x to be in the core, we only require the inequality to hold for subsets of players that are sufficiently close in the network. Hence, the set of feasible and stable allocations is a superset of the core of an appropriately defined TU-game where the value function is extended to all coalitions. It can be shown that even though the function f is strictly convex, the TU-game with the value function extended to all coalitions will typically not be a convex game in the sense of Shapley (1971); see Van den Nouweland and Borm (1991).

2.3 Inequality

We want to compare allocations in terms of the inequality they generate. Corresponding to any allocation x is a *distribution* $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$. The distribution $\bar{x}$ is simply a permutation of the elements of x that places them in (weakly) increasing order: $\bar{x}_1 \leq \bar{x}_2 \leq ... \leq \bar{x}_n$. We say that a distribution $\bar{x}$ is feasible and stable on g if there exists a corresponding allocation that is feasible and stable on g. While the egalitarian distribution is always stable, there may be multiple stable distributions in general, some of which may be characterized by high levels of inequality.

To compare distributions in terms of the level of inequality, we use the criterion of Lorenz dominance. Consider two distributions $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, $\bar{y} = (\bar{y}_1, ..., \bar{y}_n) \in \mathbb{R}^n_+$ such that

$$\sum_{i \in N} \bar{x}_i = \sum_{i \in N} \bar{y}_i = f(n).$$

Then, we say that $\bar{x}$ *Lorenz dominates* $\bar{y}$ if, for each $m = 1, ..., n$,

$$\sum_{i=1}^{m} \bar{x}_i \geq \sum_{i=1}^{m} \bar{y}_i,$$

with strict inequality for some m. If $\bar{x}$ Lorenz dominates $\bar{y}$, we say that $\bar{x}$ is a *more equal* distribution than $\bar{y}$.

We call a stable distribution $\bar{x}$ on g which is feasible *extremal* if there is no distribution $\bar{y}$ that is stable and feasible such that $\bar{x}$ Lorenz dominates $\bar{y}$. Since the Lorenz dominance

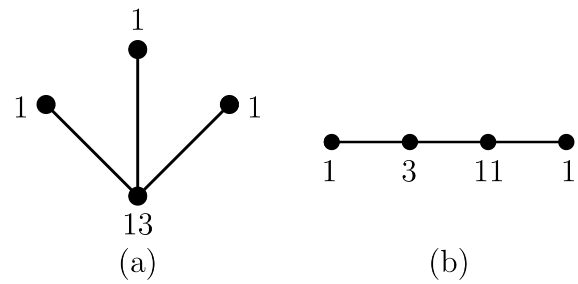


Figure 3.1: (a) The network g of Example 3.1. The numbers represent the unique allocation consistent with the extremal distribution for g. (b) The network g' of Example 3.1. The numbers represent one of the allocations that is consistent with the extremal distribution for g'.

criterion only provides a partial order on the set of feasible and stable distributions, there may be multiple extremal distributions for a given network. We say that a network g has a *unique extremal distribution* if the set of extremal distributions on g is a singleton.

3 Examples

The concepts of stability and extremal distributions may be illustrated with a few examples.

Example 3.1 Suppose $f(n) = n^2$ and consider the networks g and g' depicted in Figure 3.1(a) and (b), respectively. The value of both networks is $f(4) = 16$. The conditions for stability require that each individual is assigned at least $f(1) = 1$, and each pair of neighbors is assigned at least $f(2) = 4$. Both networks have a unique extremal distribution, given by $\bar{x} = (1, 1, 1, 13)$ and $\bar{x}' = (1, 1, 3, 11)$, respectively. Hence, $\bar{x}'$ dominates $\bar{x}$. The extremal distribution for g' corresponds to a unique allocation, as depicted in Figure 3.1: The players with three neighbors receives 13 while the other players each get 1. By contrast, the extremal distribution for g is consistent with a many different allocations to nodes. Any allocation such that two unconnected nodes receive 1 and the other two are assigned 3 and 11 is stable. An example of such an allocation is shown in Figure 3.1(b). $\triangleleft$

In Example 3.1, what properties of network g allow it to support a more unequal distribution than g'? One possibility is the fact that the distribution of the number of neighbors that each player has in g is itself more unequal than that in g'. Could the fact that the degree distribution d' for g' Lorenz dominates the degree distribution d for g be related to the fact that $\bar{x}'$ Lorenz dominates $\bar{x}$? As the following example shows, the answer is negative.

Example 3.2 Suppose $f(1) = 1$, $f(2) = 3$, and $f(10) = 20$. Consider the networks h and h' in Figure 3.2(a) and (b), respectively. In both cases, the value generated by the network is equal to 20. The stability conditions require that each individual be assigned at least

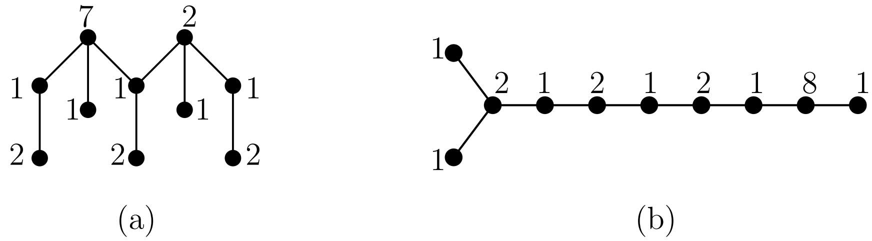


Figure 3.2: (a) The network h of Example 3.2. (b) The network h' of Example 3.2. The numbers represent represent one of the allocations consistent with the unique extremal distribution in each case.

$f(1) = 1$, and each pair of neighbors be assigned at least $f(2) = 3$. Both networks have a unique extremal distribution, given by

$$\bar{x} = (1, 1, 1, 1, 1, 2, 2, 2, 2, 7),$$
$$\bar{x}' = (1, 1, 1, 1, 1, 1, 2, 2, 2, 8).$$

Hence, $\bar{x}$ Lorenz dominates $\bar{x}'$. ◁

In Example 3.2, h' can sustain greater inequality than h. This is the opposite of what one would predict based on inequality in the degree distributions of h and h', which are given by:

$$d = (1, 1, 1, 1, 1, 2, 2, 3, 3, 3)$$
$$d' = (1, 1, 1, 2, 2, 2, 2, 2, 2, 3),$$

respectively. Clearly d' Lorenz dominates d, even though $\bar{x}$ Lorenz dominates $\bar{x}'$. The level of extremal inequality sustainable in a network therefore does not depend in a straightforward manner on inequality of the degree distribution.

Like a player's degree, his betweenness is often taken as a measure of a player's prominence and as a determinant of a player's payoffs. The betweenness of a player i in a network is the number of shortest paths between v and w player i belongs to over the total number of all shortest paths between v and w, averaged over all v and w (see, for example, Jackson, 2008). A player's betweenness may be interpreted as a measure of how essential he is in information transmission between other players. However, inequality in betweenness fares no better in explaining extremal inequality, as the next example demonstrates.

Example 3.3 Suppose $f(1) = 1$, $f(2) = 3$, and $f(7) = 12$. Consider the network in Figure 3.3. The value generated by the network is 12. The stability conditions require that each individual is assigned at least $f(1) = 1$, and that each pair of neighbors is assigned at least $f(2) = 3$. The network has a unique extremal distribution, given by

$$\bar{x} = (1, 1, 1, 1, 2, 2, 4).$$

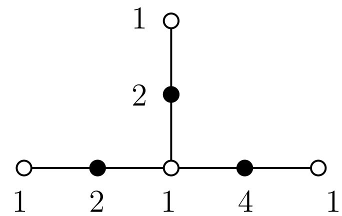


Figure 3.3: The network of Example 3.3. The player with the greatest degree, betweenness and closeness gets the lowest payoff in any extremal allocation.

This distribution is consistent with different allocations to the players, but in any such allocation, each player represented by an open circle ($\circ$) is assigned $f(1) = 1$. This includes the player with the highest degree. This player also has the highest betweenness (0.43), more than double than that of his neighbors, both of whom receive higher payoffs. $\triangleleft$

Taken together Examples 3.2 and 3.3 reveal that a focus on inequality in the degree or betweenness in attempting to understand the extent of inequality in social networks is misleading in two respects. First, networks with more equal degree or betweenness distributions may be capable of sustaining greater inequality than those with more unequal distributions. And second, by either measure, well-connected players can do substantially worse than less well-connected players in a given network. Inspection of Figure 3.3 also shows that another important centrality measure, closeness, also fails to predict high payoffs.[4] In the next section, we show that rather than the degree or betweenness distribution, it is the cardinality of the largest independent sets in a network that is most informative about the extent to which inequality can be sustained in the special case of bipartite networks.

4 Stable inequality in bipartite networks

In this section, we first show that any bipartite network has a unique extremal distribution. We then investigate how the unique extremal distribution changes for bipartite networks when the network structure is varied. The class of bipartite networks is an important one in the network literature in economics, as it it covers the extensively studied buyer-seller networks and it contains the class of trees. Trees play an important role in the network formation literature as in many cases, equilibrium networks are minimally connected.

We first consider uniqueness of the extremal distribution. Let A be a maximum independent set in g. Let $\ell \in N \setminus A$ be an arbitrary player not belonging to A. Define the allocation x^* by

$$x_i^* = \begin{cases} f(1) & \text{if } i \in A, \\ f(2) - f(1) & \text{if } i \in N \setminus (A \cup \{\ell\}), \\ f(n) - |A| f(1) + (n - |A| - 1) f(2) & \text{if } i = \ell. \end{cases} \tag{4.1}$$

The corresponding distribution is denoted by $\bar{x}^*$.

[4]The closeness of a player in the network is the average length of the shortest paths to other players (Jackson, 2008).

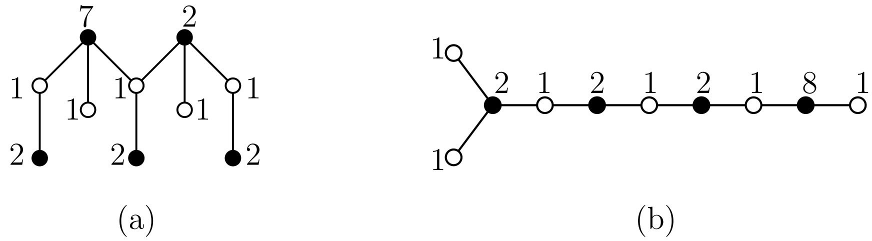


Figure 4.1: (a) The network h of Example 3.2. (b) The network h' of Example 3.2. The numbers represent one of the allocations consistent with the extremal distribution in each case. In both (a) and (b), vertices belonging to a maximum independent set are marked by white circles ($\circ$)

Theorem 4.1 *If g is a bipartite network, then $\bar{x}^*$ is its unique extremal distribution.*

The proof can be found in Appendix A. The idea behind the proof of Theorem 4.1 is simple. The allocation x^* assigns $f(1)$ to each player in a maximum independent set, $f(2) - f(1)$ to all players not in the maximum independent set except ℓ, and the remainder to ℓ. Under any other stable and feasible allocation, the total value allocated to the t players with the smallest assignment must always be as least as large as this sum under x^* for any t, so that any stable and feasible distribution that is not equal to $\bar{x}^*$ must Lorenz dominate it. We show this by dividing the set of players into pairs of neighbors (which together need to get at least $f(2)$ if the allocation is to be stable) and "unmatched" players (who need to get at least $f(1)$ under any stable allocation). Using this, we show that the proposed allocation x^* satisfies all the constraints implied by stability in such a way as to minimize the cumulative sum of the t smallest assignments, making it the unique extremal distribution.

As a corollary of Theorem 4.1, we find that bipartite networks can be ranked in terms of extremal inequality by the cardinality of their maximum independent sets. Hence, even though the Lorenz dominance relation is not a complete order, we obtain a complete order on the set of bipartite networks.

Corollary 4.2 *Consider any two bipartite networks g, g' with vertex set N. Let A and A' denote any maximum independent sets, and $\bar{x}$ and $\bar{x}'$ the unique extremal distributions in g and g' respectively. Then, $\bar{x} = \bar{x}'$ if and only if $|A| = |A'|$. If $|A| \neq |A'|$, then $\bar{x}$ Lorenz dominates $\bar{x}'$ if and only if $|A| < |A'|$.*

We can now return to Example 3.2 to apply this result. Although h has for instance a more unequal degree distribution then h',[5] the cardinality of its maximum independent set is 5, as compared to 6 for h'. A direct application of Theorem 4.1 implies that h' can sustain a more unequal payoff distribution; see Figure 4.1.

[5]It can easily be checked that the same holds for betweenness, while h' has a more unequal distribution of closeness than h.

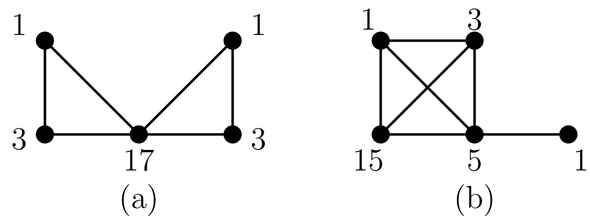


Figure 5.1: (a) The network q of Example 5.1. The numbers represent one of the allocations consistent with the unique extremal distribution for q. (b) The network q' of Example 5.1. The numbers represent represent one of the allocations consistent with the unique extremal distribution for q'.

5 General networks

What can one say about more general networks? There are two issues to consider: the uniqueness of the extremal distribution for a given network, and the ordering of networks with respect to their extremal distributions. Unfortunately, the proof of Theorem 4.1 cannot be easily extended to more general networks. The reason is that in the case of bipartite networks, only deviations by individual players or by neighbors are allowed, so that the unique extremal distribution is easy to characterize. By contrast, general networks can contain cliques with three or more players, so that deviations by larger groups are allowed. In that case, candidate extremal distributions are harder to characterize.

Even if a uniqueness result could be obtained, it would not be as straightforward to rank networks in terms of the inequality of their extremal distribution, as the following example shows. This example demonstrates that two networks that are in the same equivalence class with respect to the cardinality of their maximum independent sets may nevertheless be unambiguously ranked with respect to their extremal distributions.

Example 5.1 Suppose $f(n) = n^2$, and consider the networks q and q' in Figure 5.1(a) and (b), respectively. Both networks have unique extremal distributions, given by

$$\bar{x} = (1, 1, 3, 3, 17),$$
$$\bar{x}' = (1, 1, 3, 5, 15),$$

so $\bar{x}'$ Lorenz dominates $\bar{x}$. $\triangleleft$

The previous example shows that two networks with the *same* cardinality of their maximum independent sets can be unambiguously ranked with respect to their extremal distributions. In contrast, the following example shows that two networks that *differ* in the cardinality of their maximum independent set cannot necessarily be ranked with respect to their extremal distributions.

Example 5.2 Suppose $f(n) = n^2$, and consider the networks r and r' in Figure 5.2(a) and (b), respectively. Both networks have unique extremal distributions, given by

$$\bar{x} = (1, 1, 1, 1, 1, 3, 3, 3, 3, 83),$$
$$\bar{x}' = (1, 1, 1, 1, 1, 1, 3, 5, 7, 79).$$

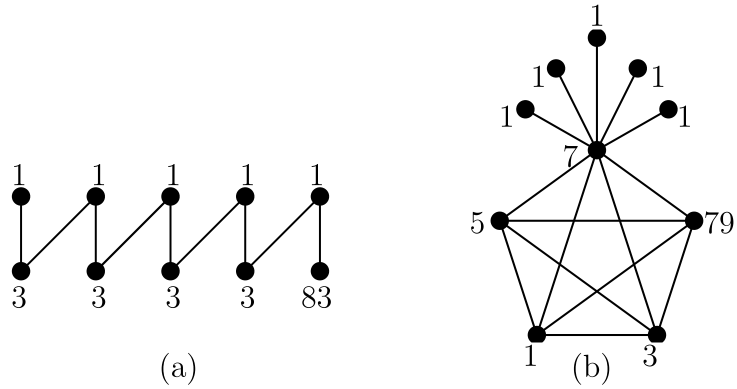


Figure 5.2: (a) The network r of Example 5.2. The numbers represent one of the allocations consistent with the unique extremal distribution for r. (b) The network r' of Example 5.2. The numbers represent represent one of the allocations consistent with the unique extremal distribution for r'.

These two distributions are not comparable based on the Lorenz criterion. $\triangleleft$

6 Broader coalitions

So far, we have only allowed for deviations of cliques. This presumes that players can coordinate on a deviation only if they are all directly connected, that is, if the distance between each pair of players in the coalition is equal to one. What happens if we allow for deviations by coalitions of players that are all within distance k of each other in the network?

Given a network, define a *k-coalition* to be a set of players that are all within distance k of each other. As before, the value that a k-coalition C can obtain on its own is $f(|C|)$. We say that an allocation x is *k-stable* on g if, for each k-coalition C in g,

$$\sum_{i \in C} x_i \geq f(|C|).$$

Hence, no k-coalition can profitably deviate from a k-stable allocation. Stability, as defined in Section 2, corresponds to k-stability for $k = 1$. A k-stable distribution $\bar{x}$ on g which is feasible is called *k-extremal* if there is no distribution $\bar{y}$ that is k-stable and feasible such that $\bar{x}$ Lorenz dominates $\bar{y}$.

To analyze this case, it is useful to define the k-power of a network. A k-power g^k of a connected network g is the network with the same vertex set as g, with $g^k_{ij} = 1$ if and only if the distance between i and j in g is at most k (see, for example, Gross and Yellen, 2003). A set of players is a k-coalition in a connected network g if and only if it is a 1-coalition in the k-power g^k of g. Hence, an allocation is k-stable in g if and only if it is stable in g^k.

For $k > 1$, we do not have results such as Theorem 4.1 and Corollary 4.2, showing that there is a unique extremal distribution for bipartite networks and providing a characterization of how the degree of inequality depends on the network structure. However, the effect of increasing k for given network is straightforward: A group of players that forms a k-coalition

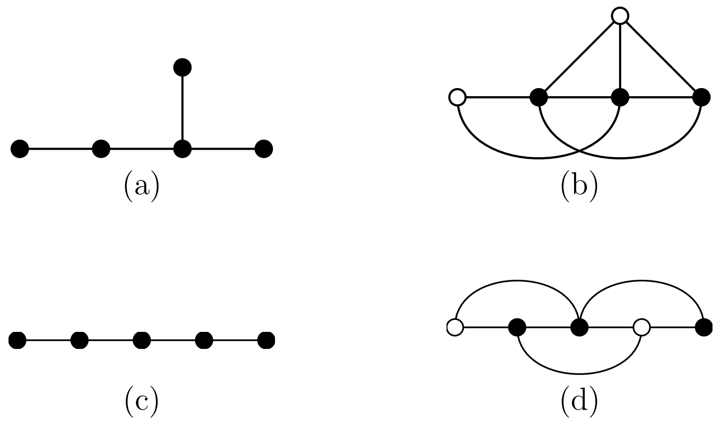


Figure 6.1: (a) The network s of Example 6.2. (b) The 2-power s^2 of s. (c) The network $\tilde{s}$ of Example 6.2. (b) The 2-power $\tilde{s}^2$ of $\tilde{s}$. In both (a) and (b), the vertices in a maximum independent set are marked with an open circle ($\circ$).

in a network g is a k'-coalition in g for $k' > k$. The following result states that the degree of inequality that can be sustained in a network weakly decreases when we increase k:

Proposition 6.1 *For any network g and k, k' such that $k' > k$, if $\bar{x}', \bar{x}$ are extremal distributions in g for k and k', respectively, then $\bar{x}' = \bar{x}$, $\bar{x}'$ Lorenz dominates $\bar{x}$, or $\bar{x}$ and $\bar{x}'$ cannot be compared with respect to Lorenz dominance.*

This states that if we weakly increase the set of possible coalitions by increasing k, the extremal distribution cannot become more unequal. However, it allows for extremal distributions to be noncomparable for different values of k. The reason we cannot rule this out is twofold. First, there may be multiple extremal distributions for some values of k. Second, even if all extremal distributions are unique, the restricted core may change in a nontrivial and unexpected way (cf. Kalai et al., 1978). We do not, however, have an example where the extremal distributions under different values of k are noncomparable.

How does the degree of inequality that can be sustained in a network depend on the network structure in this more general setting? Not surprisingly, a direct extension of Corollary 4.2 does not hold, as the following example illustrates: ranking networks in terms of the cardinality of their maximum independent sets does not provide a ranking in terms of the inequality they can sustain.

Example 6.2 Suppose $f(n) = n^2$ and $k = 2$. Consider the networks s and $\tilde{s}$ in Figure 6.1(a) and (c), respectively. The 2-extremal distributions of s and $\tilde{s}$ are the extremal distributions of s^2 and $\tilde{s}^2$, respectively. It can be easily verified that for both networks there is a unique extremal distribution. While for both networks, the cardinality of the maximum independent set of the 2-powers is equal to 2 (see Figure 6.1(b) and (d), respectively), the unique 2-extremal distribution $(1, 1, 3, 5, 15)$ for s Lorenz dominates the unique 2-extremal distribution $(1, 1, 3, 3, 17)$ for $\tilde{s}$. $\triangleleft$

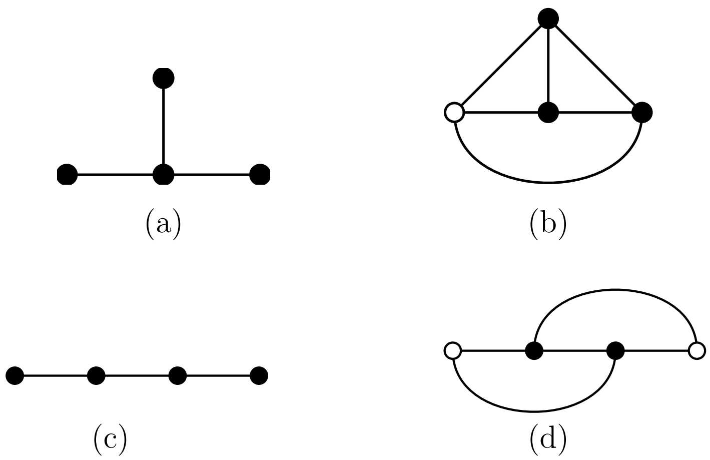


Figure 6.2: (a) The star network g_{star} of Example 6.3. (b) The 2-power g_{star}^2 of g_{star}. (c) The network g_{line} of Example 6.3. (d) The 2-power g_{line}^2 of $\tilde{s}$. In (b) and (d) the vertices in a maximum independent set are marked with an open circle ($\circ$)

Interestingly, while the degree of inequality that can be sustained in a network weakly decreases for any network if k increases (Proposition 6.1), this decrease occurs at very different rates for different networks, as the following example shows.

Example 6.3 Consider the star network g_{star} and the line network g_{line} depicted in Figure 6.2(a) and (c), respectively, and suppose $f(n) = n^2$. When $k = 1$, Theorem 4.1 shows that there is a unique extremal distribution; by Corollary 4.2, the unique extremal distribution $\bar{x}_{line}^1$ for the line Lorenz dominates the unique extremal distribution $\bar{x}_{star}^1$ for the star.

However, when $k = 2$, the situation is reversed. In the case of the star, all players can now form deviating coalitions, while for the line, the two players at the end of the star can still not coordinate a joint deviation. This is most easily seen by considering the 1-coalitions in the 2-powers of the line and the star, shown in Figure 6.2(b) and (d), respectively. This has implications for the degree of inequality that can be sustained. Also for $k = 2$, the extremal distributions for the line and star are unique; however, the unique extremal distribution $\bar{x}_{star}^2$ for the star now Lorenz dominates the unique extremal distribution for the line $\bar{x}_{star}^2$. $\triangleleft$

Intuitively, one might think that the diameter or the characteristic path length of a network determines the rate at which inequality decreases when k increases. The diameter is the maximum distance between any two players in a connected network, while the characteristic path length is the average distance between any two players. Indeed, in Example 6.3, the line network both has a smaller diameter and a smaller characteristic path length than the star network. The increase in k from $k = 1$ to $k = 2$ has a larger impact on the set of feasible coalitions in the star network than in the line, because the star network directly becomes fully connected. However, this intuition is incorrect, as the following example shows.

Example 6.4 Again, assume $f(n) = n^2$, and take $k = 2$. Consider the networks t, t' and t'' in Figure 6.3(a), (c), and (e), respectively. The diameter of t is 3, the diameter of t' is 4, and

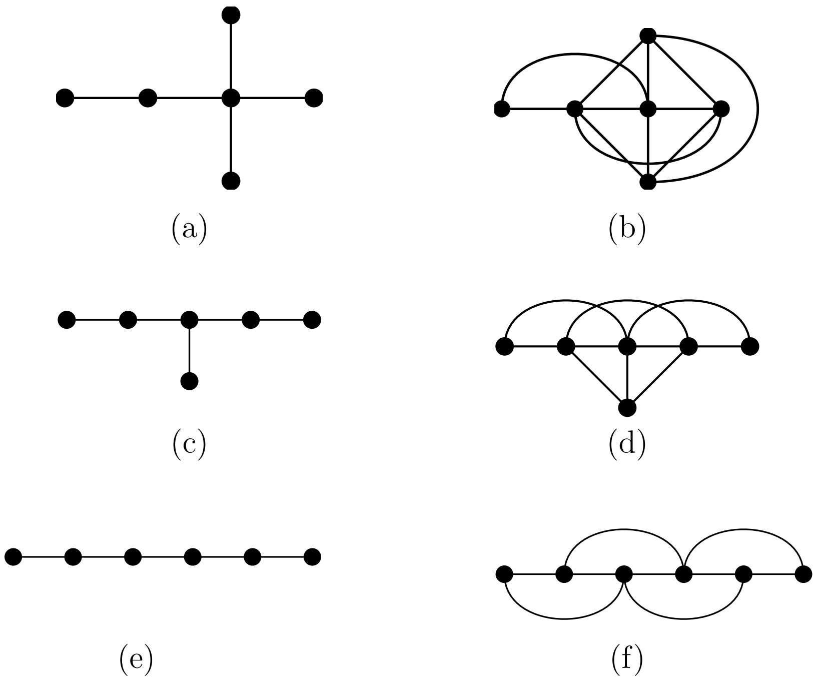


Figure 6.3: (a) The network t of Example 6.4. (b) The 2-power of t. (c) The network t' of Example 6.4. (d) The 2-power of t'. (e) The network t'' of Example 6.4. (f) The 2-power of t''.

the diameter of t'' is 5. The ordering in terms of the characteristic path lengths is the same: t has a characteristic path length of 1.87, t' has a characteristic path length of 2.07, and t'' has a characteristic path length of 2.33. The 2-powers of t, t' and t'' are shown in Figure 6.3(b), (d) and (f), respectively. Then, the unique 2-extremal distribution for t is (1,1,3,5,7,19). For t', the unique 2-extremal distribution is (1,1,1,3,5,25), and for t'' it is (1,1,3,3,5,23). That is, for $k = 2$, the unique k-extremal distributions for t and t'' Lorenz dominate the unique extremal distribution for t': the degree of inequality that can be sustained changes nonmonotonically with the diameter and the characteristic path length. $\triangleleft$

Hence the manner in which the degree of inequality that can be sustained in a network depends on its structure remains an open question in this more general setting. While it may be possible to show that there exists a unique extremal distribution in this case, we conjecture that there is no network property that will give a complete ranking of networks in terms of the inequality they can sustain when deviations by coalitions of size larger than two are allowed.

7 Conclusions

In this paper, we have studied how the degree of inequality that can be sustained on a network depends on its structure. The starting point of our analysis is the intuitive idea that players can only jointly deviate if they form a clique in the network. Our main result gives a complete ordering of the class of bipartite networks in terms of the degree of inequality they can sustain. The key network property is the cardinality of the maximum independent sets. Specifically, we have shown that for each bipartite network, there exists a unique payoff distribution which is more unequal than all other distributions (in terms of Lorenz dominance), called the extremal distribution. We then showed that the unique extremal distribution of a bipartite network Lorenz dominates that of another bipartite network if and only if the cardinality of its maximum independent set is smaller than that of the second network.

We also extended our framework to allow for deviations of players that are within distance k of each other in the network, and provided some examples to show how the degree of inequality changes when k is varied, depending on the network structure. These examples show that inequality in payoffs does not vary monotonically with inequality in the distribution of players' degree, betweenness, or closeness. Global properties of the network matter in determining local outcomes.

There are two obstacles to providing a complete characterization of the relation between inequality and network structure in more general networks or for the case $k > 1$. First, the question of whether or not the set of extremal distributions is a singleton remains open in the general case. And second, as we have illustrated with several examples, even when one compares two networks with a unique extremal distribution, it is not clear which network properties determine the level of inequality that can be sustained. For instance, it is not the case that the cardinality of a network's maximum independent set is the sole determinant of payoff inequality, as in the case of bipartite networks and neighbor deviations.

When one considers general networks or allows players to coordinate over larger distances in the network, the extremal distributions seem to depend on the interaction of different global properties of the network. In the more general setting, the level of inequality that can be sustained will not just depend on the possible pairs that can jointly deviate, as for bipartite networks, but also on deviation opportunities for larger cliques. These considerations suggest that it may not be possible to obtain a complete ordering of networks in terms of the extremal inequality they sustain, as the full network structure comes into play in determining inequality. Nevertheless, some progress may be made using concepts and techniques from the literature on cores of restricted games (Bilbao, 2000). The relationship between inequality and network structure is an economically interesting one, and this seems to be a promising area for future research.

Appendix A Proof of Theorem 4.1

We first derive some preliminary results. Lemma A.1 shows that the set of vertices of any network can be partitioned into a maximum independent set and a set of vertices that are connected to at least one vertex in the maximum independent set.

Lemma A.1 *Consider a network g with at least two vertices, and let A be a maximum independent set in g. Define*

$$B := \{j \in N \mid \exists i \in A \text{ such that } g_{ij} = 1\}$$

to be the set of vertices that have at least one neighbor in A. Then the sets A and B form a partition of the vertex set N.

Proof. First we show that $A \cap B = \emptyset$. Suppose that there is a vertex $i \in A \cap B$. As $i \in A$ and since A is an independent set, there is no $j \in A$ such that $g_{ij} = 1$. However, we also have $i \in B$. By the definition of B, there exists $m \in A$ such that $g_{im} = 1$, a contradiction.

We now establish that $N = A \cup B$. Suppose there exists $i \in N$ that does not belong to $A \cup B$. Then, by the definition of B, there exists no $j \in A$ such that $g_{ij} = 1$. But then $A \cup \{i\}$ is an independent set, contradicting that A is a maximum independent set. □

Lemma A.2 is a technical result on bipartite networks, which allows us to derive Corollary A.3. Corollary A.3 states that for bipartite networks, there exists an injective mapping from vertices not belonging to a maximum independent set to the vertices in the maximum independent set, in such a way that the vertices that are matched in this way are neighbors in the network.

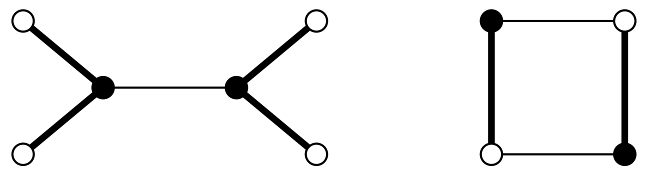
Before we can derive these results, we need some more definitions. The *endpoints* of an edge $\{i, j\}$ are the vertices i and j. A vertex is *incident* to an edge if it is one of the endpoints of that edge. A vertex without any neighbors is called an *isolated vertex*. An *edge cover* of a network with no isolated vertices is a set of edges L such that every vertex of the network is incident to some edge of L. A *minimum edge cover* of a network without isolated vertices is an edge cover of the network such that there is no edge cover with strictly smaller cardinality, see Figure A.1. Note that while a network can have multiple (minimum) edge covers, the cardinality of a minimum edge cover is well defined. A *subgraph* of a network (N, g) is a network (N', g') such that

(i) the vertex set of $(N'g')$ is a subset of that of (N, g), that is, $N' \subseteq N$;

(ii) the edge set of (N', g') is a subset of (N, g), that is, $g'_{ij} = 1$ implies $g_{ij} = 1$ for all vertices i and j.

An *induced subgraph* is a subgraph obtained by deleting a set of vertices. A *component* of a network (N, g) is a maximal connected subgraph, that is, a subgraph $(N'g')$ that is connected and is not contained in another connected subgraph of (N, g). Given a graph (N, g), the subgraph induced by the set non-isolated vertices is referred to as the *core subgraph* of (N, g).[6] Finally, a *star* is a tree consisting of one vertex adjacent to all other vertices. We refer to this vertex as the *center* of the star.

[6]Of course, if a network does not have isolated vertices, the core subgraph is just the network itself.



(a) (b)

Figure A.1: Two bipartite networks; in each network, a minimum edge cover is indicated with bold lines, and vertices belonging to one of the maximum independent sets are marked by white circles (∘). Note that while in the network in (a) the minimum edge cover and the maximum independent set are unique, there two maximum independent sets and two minimum edge covers for the network in (b).

Lemma A.2 *Let (M, h) be a bipartite network, and let (M', h') be an induced subgraph of (M, h). For any maximum independent set of the core subgraph (N, g) of (M', h'), there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ of (N, g) such that*

$$\{i_1, \ldots, i_m\} = A, \qquad \{j_1, \ldots, j_m\} = N \setminus A,$$

and there exists no j_m, j_k, $j_\ell \neq j_k$ such that $i_m = i_k$.

Proof. First note that every induced subgraph of a bipartite network is again a bipartite network (that is, the class of bipartite networks is hereditary). Therefore, we can prove the statement in the lemma by proving that for any bipartite network (N, g) and any maximum independent set A of the core subgraph of (N, g), there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ of the core subgraph with the desired properties (cf. West, 2001, Remark 5.3.20). Without loss of generality, we can restrict attention to a bipartite network (N, g) without isolated vertices. As before, we fix the vertex set N and denote the network (N, g) by g.

Let A be a maximum independent set in g. We will construct a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ with the desired properties. First note that for any minimum edge cover L' of g, for any vertex i belonging to A, there exists an edge e in L' such that i is an endpoint of e, as otherwise L' would not cover all vertices. Moreover, as A is an independent set, there is no edge in L' with two vertices from A as its endpoints. Hence, without loss of generality, we can take $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$, with

$$\{i_1, \ldots, i_m\} \supseteq A.$$

By the Kőnig-Rado edge covering theorem (e.g. Schrijver, 2003, p. 317), the cardinality of a maximum independent set is equal to the cardinality of a minimum edge cover, so that

$$\{i_1, \ldots, i_m\} = A.$$

Since $\{i_1, \ldots, i_m\} = A$, for the vertices of $N \setminus A$ to be covered by L, we need

$$\{j_1, \ldots, j_m\} \supseteq N \setminus A.$$

As A is an independent set, we have

$$\{j_1, \ldots, j_m\} = N \setminus A.$$

Finally, suppose that there exist distinct $j_m l, j_k$ such that $i_m = i_k =: i$. First note that for any minimum edge cover Λ the following holds. If both endpoints of an edge e belong to edges in Λ other than e, then $e \notin \Lambda$, because otherwise $\Lambda \setminus \{e\}$ would also be an edge cover of the network, contradicting that Λ is a minimum edge cover. Hence, each component formed by edges of L has at most one vertex with more than one neighbor and is a star. By assumption, j_m and j_k belong to the same component in L; the center of this component is i. Since each vertex in A is associated with at least 1 edge in L, this means that $|L| > |A|$, which cannot hold by the Kőnig-Rado edge covering theorem. □

Remark 1 In Lemma A.2, we show that for each maximum independent set in the core subgraph of an induced subgraph of a bipartite network, there exists a minimum edge cover such that each vertex i in the core subgraph not belonging to the maximum independent set is matched to a vertex j in the maximum independent set to which it is connected in the network, and there is no other vertex i' in the core subgraph that is matched to j. Note that vertices not belonging to the maximum independent set will typically be connected to multiple vertices in the maximum independent set, see e.g. the network in Figure A.1(a). ◁

Corollary A.3 *Let (M, h) be a bipartite network, and let (M', h') be an induced subgraph of (M, h). For any maximum independent set A of (M', h'), there exists an injective mapping π from $M' \setminus A$ to A such that $h'_{i\pi(i)} = 1$ for all $i \in M' \setminus A$.*

Proof. Denote the set of isolated vertices in (M', h') by B. By Lemma A.2, there exists a minimum edge cover $L = \{\{i_1, j_1\}, \ldots, \{i_m, j_m\}\}$ for the core subgraph (N, g) of (M', h') such that

$$\{i_1, \ldots, i_m\} = A \setminus B, \qquad \{j_1, \ldots, j_m\} = M' \setminus (A \cup B),$$

and there exists no j_m, j_k, $j_m \neq j_k$ such that $i_m = i_k$. Moreover, $B \subseteq A$. Hence, the mapping $\pi : \{j_1, \ldots, j_m\} \to \{i_1, \ldots, i_m\} \cup B$ defined by

$$\pi(j_t) = i_t$$

for $t = 1, \ldots, m$ satisfies the desired properties. □

Finally, Lemma A.4 establishes that the allocation x^* (Equation 4.1) is stable for a bipartite network.

Lemma A.4 *Consider a bipartite network g with at least two vertices. Let A be a maximum independent set in g, and let ℓ be an arbitrary player in $N \setminus A$. Then, the allocation x^* is stable.*

Proof. To show that the allocation x is stable, we need to establish the following:

(i) The total value $\sum_{i \in N} x_i$ allocated to the players does not exceed $f(n)$.

(ii) Each player gets at least $f(1)$, that is, $x_i \geq f(1)$ for each $i \in N$.

(iii) Each pair of neighbors gets at least $f(2)$, that is, for each $i, j \in N$ such that $g_{ij} = 1$, $x_i + x_j \geq f(2)$.

It is easy to see that (i) is satisfied by definition:

$$\sum_{i \in N} x_i^* = f(n).$$

To show (ii) and (iii), first note that by the strict convexity of f,

$$f(2) - f(1) > f(1). \tag{A.1}$$

Hence, each player $i \neq \ell$ gets at least $f(1)$. By (A.1) and Lemma A.1, each pair of neighbors $i, j \in N \setminus \{\ell\}$ gets at least $f(2) - f(1) + f(1) = f(2)$.

Hence, it remains to show that $x_\ell \geq f(2) - f(1)$. First note that

$$\sum_{j \neq \ell} x_j = (n - |A|)(f(2) - f(1)) - (f(2) - f(1)).$$

Hence, it suffices to show that

$$(n - |A|)(f(2) - f(1)) \leq f(n).$$

By Corollary A.3, there exists an injective mapping from $N \setminus A$ to A, so that $n - |A| \leq n/2$. Moreover, $f(2) - f(1) \leq f(2)$. We thus need to show that

$$\frac{f(2)}{2} \leq \frac{f(n)}{n}.$$

This follows if $\frac{d}{dx}(f(x)/x) \geq 0$ for all $x \geq 0$. First consider the case $x = 0$. Using L'Hôpital's rule, we have

$$\begin{aligned} \lim_{x \downarrow 0} \frac{d}{dx}\left(\frac{f(x)}{x}\right) &= \lim_{x \downarrow 0} \frac{1}{x^2}\left(x\tfrac{df}{dx} - f(x)\right) \\ &= \lim_{x \downarrow 0} \frac{1}{2x}\left(\tfrac{df}{dx} + x\tfrac{d^2 f}{dx^2} - \tfrac{df}{dx}\right) \\ &= \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right) > 0. \end{aligned}$$

For $x > 0$, $\frac{d}{dx}\left(\frac{f(x)}{x}\right) \geq 0$ if and only if $x\frac{df}{dx} - f(x) \geq 0$. As $f(0) = 0$, we can rewrite this latter condition as

$$\frac{d}{dx}\left(\frac{f(x)}{x}\right) \geq \frac{f(x) - f(0)}{x - 0},$$

which holds by the convexity of f. □

We are now ready to prove Theorem 4.1. Consider a bipartite network (N, g). As before, we fix N and denote the network by g. When $|N| = 1$, it is easy to see that the set of feasible and stable allocations is the singleton $\{x^*\}$, so that trivially $\bar{x}^*$ is the unique extremal distribution.

Hence, consider the case $|N| \geq 2$. Let A be a maximum independent set of A, and for each t, define

$$S_t := \sum_{i=1} \bar{x}^*$$

to be the sum of the t smallest assignments under $\bar{x}^*$, and note that

$$S_t^* = \begin{cases} t\, f(1) & \text{if } t \leq |A|; \\ |A|\, f(1) + (t - |A|)\,(f(2) - f(1)) & \text{if } |A| < t \leq n-1; \\ f(n) & \text{if } t = n. \end{cases} \tag{A.2}$$

By Lemma A.4, x^* is stable. It remains to show that for any distribution $\bar{y}$ on g that is stable and feasible, either $\bar{y} = \bar{x}^*$ or $\bar{y}$ Lorenz dominates $\bar{x}^*$. Suppose not. Then there exists t such that

$$S_t^* > S_t,$$

where we have defined $S_t := \sum_{i=1}^{t} \bar{y}_i$ to be the sum of the t smallest assignments under $\bar{y}$. Let C_t be any subset of vertices of cardinality t such that

$$\sum_{i \in C_t} y_i = S_t,$$

and let $A_t \subseteq C_t$ be a maximum independent set in the subgraph induced by C_t. Clearly, $|A_t| \leq |A|$.

By Lemma A.1, the set C_t can be partitioned into A_t and the set B_t that have at least one neighbor in A_t. By Corollary A.3, there is an injective mapping π from B_t to A_t such that for each $i \in B_t$, $\{i, \pi(i)\}$ is an edge in the subgraph induced by C_t. Define

$$U_t := \{i \in A_t \mid i = \pi(j) \text{ for some } j \in B_t\}$$

to be the set of players in A_t that are matched with a player in B_t by the mapping π.

In a bipartite network, only singleton coalitions or coalitions consisting of pairs of neighbors can form. Hence, by stability of $\bar{y}$, each individual player needs to be assigned at least $f(1)$ under $\bar{y}$. By the strict convexity of f, it holds that $2f(1) < f(2)$. Hence, under a stable allocation, two neighboring players cannot both be assigned $f(1)$.

Combining these results gives

$$\begin{aligned} S_t &= \sum_{i \in C_t} y_i \\ &= \sum_{i \in B_t} (y_i + y_{\pi(i)} + \sum_{i \in A_t \setminus U_t} y_i \\ &\geq \sum_{i \in B_t} f(2) + \sum_{i \in A_t \setminus U_t} f(1) \\ &= \big(t - |A_t|\big) f(2) + \big(|A_t| - (t - |A_t|)\big) f(1) \\ &= t\,(f(2) - f(1)) + |A_t| \,\big(2f(1) - f(2)\big) \\ &\geq t\,(f(2) - f(1)) + |A| \,\big(2f(1) - f(2)\big), \end{aligned} \tag{A.3}$$

where the last inequality follows from $|A_t| \leq |A|$ and $2f(1) - f(2) < 0$ (by strict convexity of f). We need to consider three cases. Firstly, if $t \leq |A|$, then $S_t^* = t\,f(1)$. Since by stability, $y_i \geq f(1)$ for all i, it follows that $S_t^* \leq S_t$. Secondly, suppose $|A| < t \leq n - 1$. Then it follows from (A.2) and (A.3) that

$$S_t^* = t\,(f(2) - f(1)) + |A|\,(2f(1) - f(2) \leq S_t.$$

Finally, if $t = n$, then $S_t^* = S_t = f(n)$. Hence, for all t, $S_t^* \leq S_t$, a contradiction. □

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