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Weighted average lexicographic values for share sets and balanced cooperative games

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Abstract

Inspired by Kalai-Samet [4] and Tijs [11], weighted average lexicographic values are introduced for share sets and for cores of cooperative games using induction arguments. Continuity properties and monotonicity properties of these weighted lexicographic values are studied. For subclasses of games (convex games, simplex games, big boss games) relations are established with weighted (exact) Shapley values.

KEYWORDS: Cooperative games, average lexicographic value, weighted Shapley value.

JEL code C71

1 Introduction

The average lexicographic value (AL value) is introduced in [11] for balanced games. It is, in an n-player situation, the average of the $n!$ lexicographic maxima of the core corresponding to the $n!$ orderings of the players. The idea was extended in ([2]) for share opportunity sets. Much emphasis is there on the continuity properties of the AL-value on compact convex share sets and especially for perfect share sets. Inspired by the literature on weighted Shapley values ([9],[10],[4]) we became interested in the existence of weighted AL-values. At first sight, there are two approaches to define weighted lexicographic values. On one hand, one on can put weights on orderings of the players leading to mixed lexicographic values. On the other hand, one can have weights on the players or a hierarchical weight system on the players leading to weighted lexicographic values. The outline of the paper is as follows.
Section 2 is devoted to preliminaries and notations. In section 3, μ-mixed lexicographic values are introduced. In section 4 and 5, we introduce p-weighted and (p, S)-weighted lexicographic values respectively and their relations with μ-mixed lexicographic values are studied. In section 6 the relations between (p, S)-weighted lexicographic values and weighted Shapley values of some classes of games are investigated. In section 7 monotonicitity of p-weighted lexicographic values with respect to the weights is studied.

2 Preliminaries and notations

An n-person cooperative game ([7]) $\langle N, v\rangle$ with player set $N = \{1, 2, ..., n\}$ is a map $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$, where 2^N is the collection of subsets of N. Let us denote by G^N the set of all n-person cooperative games. The core $C(v)$ of the game $\langle N, v\rangle$ is the bounded polyhedral set

$$C(v) = \{x \in \mathbb{R}^n \ \mid \ x(N) = v(N),\ x(S) \geq v(S) \text{ for each } S \subseteq N\},$$

where $x(S) = \Sigma_{i\in S} x_i$. Games with a non empty core are called balanced games. We denote by BA^N the set of all n-person balanced games.

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The imputation set of $\langle N, v\rangle$ is the set

$$I(v) = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{n} x_i = v(N), x_i \geq v(\{i\}) \forall i \in N \right. \right\},$$

Given $x \in \mathbb{R}^n$, we denote with x_{-j} the vector belonging to $\mathbb{R}^{n-1}$ obtained from x by deleting its j-th coordinate.

A game $\langle N, v\rangle$ is called:

- a **monotonic game** if $v(S) \leq v(T)$ for all $S \subseteq T$;
- a **convex game** if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$. We denote by CG^N the set of all n-person convex games;
- a **simplex game** if $I(v) = C(v)$;
- a **big boss game** (BBG for short) with big boss 1 if:
  1) $v(S) = 0$ if $1 \notin S$;
  2) v is monotonic;
  3) $v(N) - v(N \setminus S) \geq \Sigma_{i \in S}(v(N) - v(N \setminus \{i\}))$ if $1 \notin S$.
- a **clan game** with clan $T \subseteq N$ if:
  1) $v(S) = 0$ if $T \nsubseteq S$;
  2) v is monotonic;
  3) $v(N) - v(N \setminus S) \geq \Sigma_{i \in S}(v(N) - v(N \setminus \{i\}))$ if $S \subseteq N \setminus T$.
- an **exact game** if the core $C(v)$ of $\langle N, v\rangle$ is non empty and for every $S \subseteq N$ there exists $x \in C(v)$ such that $x(S) = v(S)$ (see [9]).

Given a balanced game $\langle N, v\rangle$, its exactification is the game $\langle N, v^E\rangle$ with $v^E(S) = \min_{x \in C(v)} x(S)$ for each $S \in 2^N$.

Given an ordering $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$ in N and a compact subset A of $\mathbb{R}^n$, the **Lexicographic maximum** of A with respect to σ is the vector $L^\sigma(A) \in A$ such that:
$(L^\sigma(A))_{\sigma(i)} = \max\{x_{\sigma(i)} \mid x \in A\ ,\ ((L^\sigma(A))_{\sigma(j)} = x_{\sigma(j)})\ \forall j \in N,\ j < i\}$.

The **Average Lexicographic maximum** $AL(A)$ of A is the average over all $L^\sigma(A)$ i.e. $AL(A) = \frac{1}{n!}\Sigma_{\sigma \in \Pi(N)} L^\sigma(A)$, where $\Pi(N)$ denotes the set of all possible orderings in N. Given a balanced game $\langle N, v\rangle$, we denote by $AL(v)$ the vector $AL(C(v))$ (see Tijs in [11]).

The **Lexicore**, $LEC(v)$ of $\langle N, v\rangle$ is defined (see [3]) as

$$LEC(v) = conv(\{L^\sigma(v) \mid \sigma \in \Pi(N)\}).$$

We denote by $\triangle^N = \{p = (p_1, p_2, ..., p_n) \in \mathbb{R}^n \mid 0 \leq p_i \leq 1, \Sigma_{i=1}^n p_i = 1\}$ and by $Int(\triangle^N) = \{p = (p_1, p_2, ..., p_n) \in \mathbb{R}^n \mid 0 < p_i < 1, \Sigma_{i=1}^n p_i = 1\}$

3 μ-mixed lexicographic values

The average lexicographic value was defined in [11] for balanced games and then extended in [2] to share sets, i.e. elements $C \in \mathcal{K}^n$, where

$$\mathcal{K}^n = \cup_{\alpha \in \mathbb{R}} \mathcal{K}^n_\alpha,$$

K^n_α being the family of all compact subsets of $H_\alpha = \{x \in \mathbb{R}^n \mid x(N) = \alpha\}$.

For $C \in \mathcal{K}^n$, here we denote the AL-value of C by $AL(C)$, $AL(C) = \frac{1}{n!}\Sigma_{\sigma \in \Pi(N)} L^\sigma(C)$, i.e. the average of all lexicographic maxima of C.

Inspired by this definition, here we give the definition of the μ-mixed lexicographic value of C. It is a weighted average of the lexicographic maxima of C.

Definition 3.1 *Given a system of non negative numbers $\mu = (\mu_\sigma)_{\sigma\in\Pi(N)}$, such that $0 \le \mu_\sigma \le 1$ and $\sum_{\sigma\in\Pi(N)} \mu_\sigma = 1$, we define the μ-mixed lexicographic value of C with system of weights μ, as the vector:*

$$M^\mu L(C) = \sum_{\sigma\in\Pi(N)} \mu_\sigma L^\sigma(C).$$

REMARK 3.2 If $\mu_\sigma = \frac{1}{n!}$ for every $\sigma \in \Pi(N)$, then the μ-mixed lexicographic value of C is $AL(C)$.

REMARK 3.3 The set of all μ-mixed lexicographic values of C is the convex hull of the set $\{L^\sigma(C) \mid \sigma \in \Pi(C)\}$. If C is the core of a balanced game $\langle N, v\rangle$ ($C = C(v)$), then the set of all μ-mixed lexicographic values of $C(v)$ is the Lexicore of $\langle N, v\rangle$.

4 p-weighted lexicographic values

For $i \in N$, and $C \in \mathcal{K}^n$,, let $M^i(C)$ be the set

$$M^i(C) = \arg\max\{x_i \mid x \in C\},$$

and let C^i be the set

$$C^i = \left\{a_{-i} \in \mathbb{R}^{n-1} \,|a \in \arg\max\{x_i \mid x \in C\}\right\} \subseteq \mathbb{R}^{n-1}.$$

Then $C^i \in \mathcal{K}^{n-1}_{\alpha^i}$ where $\alpha^i = \alpha - \max\{x_i \mid x \in C\}$. Let $\pi_i : M^i(C) \longrightarrow C^i$ the i-th projection defined as $\pi_i(x) = x_{-i}$ and let $\pi_i^{-1} : C^i \longrightarrow M^i(C)$ the inverse of π_i. Note that here π_i^{-1} is a function.

Observe that $AL(C)$ satisfies the following recursive formula:

$$AL\,(C) = \frac{1}{n}\sum_{i\in N} \pi_i^{-1}(AL(\pi_i(M^i(C)))). \tag{4.1}$$

In fact:
$AL\,(C) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} L^\sigma\,(C) = \frac{1}{n}\cdot\sum_{i\in N}\frac{1}{(n-1)!}\sum_{\sigma_{-i}\in\Pi(N\setminus\{i\})} \pi_i^{-1}(L^{\sigma_{-i}}(C^i)) =$
$\frac{1}{n}\sum_{i\in N}\pi_i^{-1}(AL(\pi_i(M^i(C))))$.

Inspired by the formula (4.1) for the average lexicographic value, we give, by induction on n, the following definition of the p-weighted lexicographic values.

Definition 4.1 *Let $C \in \mathcal{K}^n_\alpha$, $p \in Int(\triangle^n)$ if $n \ge 2$, $p = 1$ if $n = 1$. We define the p-weighted lexicographic value ($A^pL(C)$) by induction on n.*

If $n = 1$ then $p = 1$ and $A^1L\,(C) = \alpha$.

Suppose now we have defined the weighted lexicographic values for elements belonging to $\mathcal{K}^{n-1}_\alpha$. We will define the p-weighted lexicographic value for elements $C \in \mathcal{K}^n_\alpha$. Given $p \in Int(\triangle^n)$, we define the p-weighted lexicographic value of C with system of weights p as the vector

$$A^pL\,(C) = \sum_{i\in N} p_i\pi_i^{-1}(A^{p_{-i}}L(\pi_i(M^i(C)))),$$

where $\pi_i(M^i(C))$ are $(n-1)$-dimensional share sets and p_{-i} is the system of positive weights on $\pi_i(M^i(C))$ whose j-th component ($j \ne i$) is

$$(p_{-i})_j = \frac{p_j}{\sum_{k\ne i} p_k}.$$

The following result holds:

Theorem 4.2 *Given $p \in Int(\triangle^n)$, the following system of positive weights $\mu = (\mu_\sigma)_{\sigma \in \Pi(N)}$, where*

$$\mu_\sigma = \prod_{j=1}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N \setminus T_j} p_{\sigma(r)}}$$

with $T_j = \begin{cases} \emptyset & \text{if} \quad j = 1, \\ \{\sigma(1), \sigma(2), ..., \sigma(j-1)\} & \text{if} \quad j = 2, 3, ...n \end{cases}$
satisfies the equality

$$A^p(L(C)) = \sum_{\sigma \in \Pi(N)} \mu_\sigma L^\sigma(C) = M^\mu(L(C)).$$

Proof. Let us prove this result by induction on n. For $n = 2$ it is trivial. Let us suppose that it holds for $n-1$. Let us fix the ordering $\sigma = \{\sigma(1), \sigma(2), ..., \sigma(n)\} \in \Pi(N)$ and let us set $\sigma' = \{\sigma(2), ..., \sigma(n)\}$, $N' = N \setminus \{\sigma(1)\}$ and $T'_j = T_j \setminus \{\sigma(1)\}$. Then we have, by induction hypothesis in $(n-1)$-dimensions,

$$\mu_{\sigma'} = \prod_{j=2}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N' \setminus T'_j} p_{\sigma(r)}} = \prod_{j=2}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N \setminus T_j} p_{\sigma(r)}}$$

because, for $j \geq 2$, $N' \setminus T'_j = N \setminus T_j$.

The weight μ_σ is given by

$$\begin{aligned} \mu_\sigma &= p_{\sigma(1)} \mu_{\sigma'} = p_{\sigma(1)} \prod_{j=2}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N \setminus T_j} p_{\sigma(r)}} = \frac{p_{\sigma(1)}}{\sum\limits_{r \in N \setminus T_1} p_{\sigma(r)}} \prod_{j=2}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N \setminus T_j} p_{\sigma(r)}} = \\ &= \prod_{j=1}^{n} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N \setminus T_j} p_{\sigma(r)}}, \end{aligned}$$

being $T_1 = \emptyset$ and $\sum\limits_{r \in N \setminus T_1} p_{\sigma(r)} = 1$.

We prove now that μ satisfies the condition $0 < \mu_\sigma \leq 1$ for each $\sigma \in \Pi(N)$ and $\sum\limits_{\sigma \in \Pi(N)} \mu_\sigma = 1$. Since $0 < \mu_\sigma \leq 1$ is obvious, we prove by induction that $\sum\limits_{\sigma \in \Pi(N)} \mu_\sigma = 1$. For $n = 1$ it is obvious. Let us suppose it holds in $(n-1)$-dimensions, i.e. fixed $h \in N$, let σ_{-h} be the generic ordering of $N' = N \setminus \{h\}$ $(\sigma_{-h} \in \Pi(N'))$, $(T_j)_{-h} = T_j \setminus \{h\}$, we have

$$1 = \sum_{\sigma_{-h} \in \Pi(N')} \mu_{\sigma_{-h}} = \sum_{\sigma_{-h} \in \Pi(N')} \prod_{j \in N'} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N' \setminus (T_j)'} p_{\sigma(r)}}.$$

Let us consider the orderings in N defined by $\overline{\sigma}^h = (h, \sigma_{-h})$ and let us denote the set of orderings $\overline{\sigma}^h$ by Σ^h. Then, we have that

$$\sum_{\sigma \in \Pi(N)} \mu_\sigma = \sum_{h=1}^{n} \sum_{\overline{\sigma}^h \in \Sigma^h} \mu_{\overline{\sigma}^h} = \sum_{h=1}^{n} p_h \sum_{\sigma_{-h} \in \Pi(N')} \mu_{\sigma_{-h}} =$$
$$= \sum_{h=1}^{n} p_h \sum_{\sigma_{-h} \in \Pi(N')} \prod_{j \in N'} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N' \setminus (T_j)'} p_{\sigma(r)}} = \sum_{h=1}^{n} p_i = 1. \blacksquare$$

Remark 4.3 In the case where $C = C(v)$ is the core of a balanced game $\langle N, v \rangle$, Theorem 4.2 guarantees that every weighted lexicographic value of $C = C(v)$ belongs to the lexicore of $\langle N, v \rangle$.

5 (p, S)-weighted lexicographic values

Here we extend the definition of p-weighted lexicographic values to the case of nonnegative weights. The problem is that if we consider a system of weights $p = (p_1, ..., p_n)$ such that several of them are 0, we are not able to state how to divide the amount inside the coalitions containing only 0-weight players. To avoid this problem, we introduce (as in [4]) a partition $S = (S_1, ..., S_m)$ of N, $S_j \neq \emptyset$ for all j, and a hierarchy between the elements of the partition in the sense that all players belonging to S_j are "more important" then players belonging to S_i with $i < j$ (i.e. the weight of a player in S_i is 0 with respect to players in S_j).

Definition 5.1 *Consider the set $C \in \mathcal{K}^n_\alpha$. Let $p =\in Int(\triangle^n)$ if $n \geq 2$, $p = 1$ if $n = 1$ and let $S = (S_1, ..., S_m)$ be a partition of N that here and in the following has the property that $S_j \neq \emptyset$ for every $j \in \{1, ..., m\}$.*

If $n = 1$ then $p = 1$ and $A^{(p,S)}L(C) = \alpha$. Suppose now $n \geq 2$ and we have defined the (p, S)-weighted lexicographic values for elements belonging to $\mathcal{K}^{n-1}_\alpha$. We define the (p, S)-weighted lexicographic values for elements $C \in \mathcal{K}^n_\alpha$. Let

$$\lambda_i = \begin{cases} \frac{p_i}{\sum_{r \in S^n_k} p_r} & \text{if} \quad i \in S^n_k, \\ 0 & \text{if} \quad i \notin S^n_k, \end{cases}$$

where $k = \max \{j \,|S_j \neq \emptyset\}$. We define the (p, S)-weighted lexicographic value of C with $p \in Int(\triangle^n)$ and partition S as the vector

$$A^{(p,S)}L(C) = \sum_{i \in N} \lambda_i \pi_i^{-1}(A^{(p_{-i},S^i)}L(\pi_i(M^i(C)))),$$

where $\pi_i(M^i(C))$ are $(n-1)$-dimensional share set and p_{-i} is the system of positive weights on $\pi_i(M^i(C))$ whose j-th component ($j \neq i$) is:

$$(p_{-i})_j = \frac{p_j}{\sum_{k \neq i} p_k},$$

and the partition S^i of $N' = N \backslash \{i\}$ is given by the sets $S^i = (S^i_1, S^i_2, ...S^i_k)$ such that

$$S^i_j = \begin{cases} S_j \backslash \{i\} & \text{if} \quad S_r \neq \{i\} \quad \text{for all } r \leq j, \\ S_{j+1} & \text{if} \quad \text{there exists } r \leq j \text{ such that } S_r = \{i\}, \ j \leq m-1. \end{cases}$$

Given $p =\in Int(\triangle^n)$ if $n \geq 2$, $p = 1$ if $n = 1$ and a partition S of N, let

$$c_i = \begin{cases} |\ S_i\ | & \text{if} \quad i \in \{1, ..., m\}, \\ 0 & \text{if} \quad i = m+1 \end{cases}$$

and

$$k_j = \begin{cases} \sum_{i=j+1}^m c_i, & \text{if} \quad j \in \{0, ..., m-1\}, \\ 0 & \text{if} \quad j = m \end{cases}$$

(observe that $k_j < k_{j-1}$ and $k_0 = \sum_{i=1}^m c_i = n$). Let $\Gamma(N)$ be the set of $\sigma \in \Pi(N)$ such that $\sigma = (\sigma_{S_m}, \sigma_{S_{m-1}}, ..., \sigma_{S_1})$ with $\sigma_{S_j} \in \Pi(S_j)$, $j = 1, ...m$. The following result holds:

Remark 5.2 Observe that if $S = (N)$ then $A^{(p,S)}L(C) = A^p L(C)$

Theorem 5.3 *Given $p \in Int(\triangle^n)$ and a partition S of N, $\mu = (\mu_\sigma)_{\sigma \in \Pi(N)}$ where*

$$\mu_\sigma = \begin{cases} \prod_{j=1}^m \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j \backslash T^s_j} p_{\sigma(r)}} \right) & \text{if} \quad \sigma \in \Gamma(N) \\ 0 & \text{if} \quad \sigma \notin \Gamma(N) \end{cases}$$

with

$$T_j^s = \begin{cases} \emptyset & \text{if} \quad s = k_j + 1, \\ \{\sigma(k_j+1), \sigma(k_j+2), ..., \sigma(k_j+s)\} & \text{if} \quad k_{j+1} \le s \le k_j - 1 \end{cases}$$

satisfies the equality

$$A^{(p,S)}L(C) = \sum_{\sigma\in\Pi(N)} \mu_\sigma L^\sigma(C) = M^\mu(C).$$

Proof. Let us prove this result by induction on n. For $n = 2$ it is trivial. Let us suppose that the assertion holds for $n-1$. We want to prove that it holds for n. Let us fix the ordering $\sigma = \{\sigma(1), \sigma(2), ..., \sigma(n)\} \in \Gamma(N)$, and let $\sigma' = \{\sigma(2), ..., \sigma(n)\}$. Let us set $S'_m = S_m \setminus \{\sigma(1)\}$, $S'_j = S_j$, $j = 1, 2, ...m-1$ and in $N' = N \setminus \{\sigma(1)\}$ let us consider the partition $(S'_1, S'_2, ...S'_m)$ if $S_m \neq \{\sigma(1)\}$ or $(S'_1, S'_2, ...S'_{m-1})$ if $S_m = \{\sigma(1)\}$. Then, $\sigma' \in \Gamma(N')$ and, by the induction hypothesis, the coefficient of $L^{\sigma'}$ is

$$\mu_{\sigma'} = \begin{cases} \prod_{j=1}^{m-1} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S'_j \setminus (T_j^s)'} p_{\sigma(r)}} \right) & \text{if} \quad S'_m = \emptyset, \\ \\ \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S'_j \setminus (T_j^s)'} p_{\sigma(r)}} \right) & \text{if} \quad S'_m \neq \emptyset, \end{cases}$$

with $(T_j^s)' = T_j^s \setminus \{\sigma(1)\}$. Now, we want to calculate the coefficient given to L^σ. By definition of $A^{(p,S)}L(C)$, we have that $\mu_\sigma = \lambda_{\sigma(1)} \cdot \mu_{\sigma'}$, that is

$$\mu_\sigma = \lambda_{\sigma(1)} \cdot \mu_{\sigma'} = \begin{cases} \lambda_{\sigma(1)} \cdot \prod_{j=1}^{m-1} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S'_j \setminus (T_j^s)'} p_{\sigma(r)}} \right) & \text{if} \quad S'_m = \emptyset, \\ \\ \lambda_{\sigma(1)} \cdot \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S'_j \setminus (T_j^s)'} p_{\sigma(r)}} \right) & \text{if} \quad S'_m \neq \emptyset. \end{cases}$$

Observe that, for $j \le m-1$, $(T_j^s)' = T_j^s$ being $\sigma(1) \in S_m$. In the first case we have that

$$\begin{aligned} \lambda_{\sigma(1)} \cdot \mu_{\sigma'} &= \lambda_{\sigma(1)} \cdot \prod_{j=1}^{m-1} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S'_j \setminus (T_j^s)'} p_{\sigma(r)}} \right) = \frac{p_{\sigma(1)}}{p_{\sigma(1)}} \cdot \prod_{j=1}^{m-1} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S_j \setminus T_j^s} p_{\sigma(r)}} \right) = \\ &= \prod_{j=1}^{m-1} \left(\frac{p_{\sigma(1)}}{p_{\sigma(1)}} \cdot \prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S_j \setminus T_j^s} p_{\sigma(r)}} \right) = \\ &= \prod_{j=1}^{m-1} \left(\frac{p_{\sigma(1)}}{\sum\limits_{r\in S_m \setminus T_m^1} p_{\sigma(r)}} \cdot \prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S_j \setminus T_j^s} p_{\sigma(r)}} \right) = \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r\in S_j \setminus T_j^s} p_{\sigma(r)}} \right) \end{aligned}$$

being the third equality due to the fact that, in this first case, $S_m = \{\sigma(1)\}$, $T_m^1 = \emptyset$ and $\sum\limits_{r\in S_m \setminus T_m^1} p_{\sigma(r)} = p_{\sigma(1)}$. We can see that it coincides with μ_σ. Let us consider now the case $S'_m \neq \emptyset$.

Observe that here as well $T_m^1 = \emptyset$. In this second case we have

$$\lambda_{\sigma(1)} \cdot \mu_{\sigma'} = \lambda_{\sigma(1)} \cdot \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j' \setminus \left(T_j^s\right)'} p_{\sigma(r)}} \right) = \frac{p_{\sigma(1)}}{\sum\limits_{r \in S_m} p_{\sigma(r)}} \cdot \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j' \setminus \left(T_j^s\right)'} p_{\sigma(r)}} \right) =$$

$$= \frac{p_{\sigma(1)}}{\sum\limits_{r \in S_m \setminus T_m^1} p_{\sigma(r)}} \cdot \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j' \setminus \left(T_j^s\right)'} p_{\sigma(r)}} \right) =$$

$$= \prod_{j=1}^{m} \left(\frac{p_{\sigma(1)}}{\sum\limits_{r \in S_m \setminus T_m^1} p_{\sigma(r)}} \prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j' \setminus \left(T_j^s\right)'} p_{\sigma(r)}} \right) = \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j \setminus T_j^s} p_{\sigma(r)}} \right) = \mu_\sigma .$$

We prove now that μ satisfies the conditions $0 < \mu_\sigma \le 1$ and $\sum\limits_{\sigma \in \Pi(N)} \mu_\sigma = 1$. Since $0 < \mu_\sigma \le 1$ is obvious, then let us prove that $\sum\limits_{\sigma \in \Pi(N)} \mu_\sigma = 1$. If $n = 1$ it is trivial. Let us suppose it holds in $(n-1)$-dimensions, i.e. fixed $h \in N$, we define σ_{-h} the generic ordering of $N' = N \setminus \{h\}$ ($\sigma_{-h} \in \Pi(N')$), $S_j' = S_j$, $j = 1, 2, ...m$, and $\left(T_j^s\right)_{-h} = T_j^s \setminus \{h\}$. By the induction hypothesis

$$\sum_{\sigma_{-h} \in \Pi(N')} \mu_{\sigma_{-h}} = \sum_{\sigma_{-h} \in \Pi(N')} \mu_{\sigma_{-h}} = \sum_{\sigma_{-h} \in \Pi(N')} \prod_{j=1}^{m} \left(\prod_{s=k_j+1}^{k_{j-1}} \frac{p_{\sigma(s)}}{\sum\limits_{r \in S_j \setminus T_j^s} p_{\sigma(r)}} \right) = 1$$

holds. Let us consider the orderings in N defined by $\overline{\sigma}^h = (h, \sigma_{-h})$. Then we have that, being $\mu_\sigma = 0$ if $\sigma \notin \Gamma(N)$,

$$\sum_{\sigma \in \Pi(N)} \mu_\sigma = \sum_{h \in S_m} \mu_{\overline{\sigma}^h} = \sum_{h \in S_m} \lambda_h \sum_{\sigma_{-h} \in \Pi(N')} \mu_{\sigma_{-h}} =$$

$$= \sum_{h \in S_m} \frac{p_h}{\sum_{r \in S_m} p_r} \sum_{\sigma_{-h} \in \Pi(N')} \prod_{j \in N'} \frac{p_{\sigma(j)}}{\sum\limits_{r \in N' \setminus (T_j)'} p_{\sigma(r)}} = \sum_{h \in S_m} \frac{p_h}{\sum_{r \in S_m} p_r} = 1.$$

and the proof is complete. ■

Remark 5.4 As in the case with positive weights, if $C = C(v)$ is the core of a balanced game $\langle N, v \rangle$, Theorem 5.3 guarantees that every weighted lexicographic value of $C = C(v)$ belongs to the lexicore of $\langle N, v \rangle$.

The following results hold.

Theorem 5.5 *Let $\beta \in \mathbb{R}_+$ and $C_1, C_2 \in \mathcal{K}_\alpha^n$. Then for every $p \in Int(\triangle^n)$ and for every partition S of N:*

$$A^{(p,S)} L(\beta C_1) = \beta A^{(p,S)} L(C_1) \qquad ; \qquad A^{(p,S)} L(C_1 + C_2) = A^{(p,S)} L(C_1) + A^{(p,S)} L(C_2).$$

In [2] we studied continuity properties for the average lexicographic maximum which here we can easily extend to weighted average lexicographic maxima. First, we remind the definition of a perfect set ([2]).

Definition 5.6 *We say that $C \subseteq H_\alpha$ has a **perfect structure** if for each $S \in 2^N$ there exists $\beta_S \in \mathbb{R}$ such that*

$$C = \bigcap_{S \in 2^N} \{x \in \mathbb{R}^n \mid x(S) \ge \beta_S\}.$$

Let

$$\mathcal{P}^n_\alpha = \{D \in \mathcal{K}^n_\alpha \mid \; D \textit{ has a perfect structure}\}$$

and

$$\mathcal{P}^n = \cup_{\alpha \in \mathbb{R}} \mathcal{P}^n_\alpha.$$

The following theorem holds.

Theorem 5.7 *For every $p \in Int(\triangle(N))$ and for every partition S, $A^{(p,S)}L : \mathcal{P}^n \longrightarrow \mathbb{R}^n$ is continuous on $\mathcal{P}^n$.*

Proof. The proof can be easily obtained by induction on n using continuity of the multifunction argmax on $\mathcal{P}^n$ (see [2] Lemma1 and Lemma2). ∎

Definition 5.8 *Let $\langle N, v\rangle$ be a balanced game. We define*

$$A^{(p,S)}L\,(v) = A^{(p,S)}L\,(C\,(v))\,.$$

6 Relations between weighted weighted lexicographic values and weighted Shapley values

Let us remind the definition of unanimity games. Given the coalition $\emptyset \neq T \subseteq N$, the unanimity game $\langle N, u_T\rangle$ is the game s.t.

$$u_T\,(S) = \begin{cases} 1 & \text{if} \quad T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that every game $\langle N, v\rangle$ can be written as

$$v = \sum_{T \in 2^N \setminus \emptyset} \xi_T u_T,$$

with

$$\xi_T = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

Let us consider the set:

$$\overline{S} = T \cap S_m.$$

In [9] and [10] Shapley introduced the concept of Shapley value and weighted Shapley value with a system of positive weights. In [4] Kalai and Samet extended this definition to a system of nonnegative weights. Here, we remind this definition. Let u_T be a unanimity game. Then, the weighted Shapley value of u_T with system of weights $p = (p_1, p_2, ..., p_n) \in Int(\triangle^n)$ is defined as

$$\Phi^{(p,S)}\,(u_T)_i = \begin{cases} \frac{p_i}{\sum_{r \in \overline{S}} p_r} & \text{if} \quad i \in \overline{S} \\ 0 & \text{if} \quad i \notin \overline{S} \end{cases}$$

Given the game $\langle N, v\rangle$, the weighted Shapley value of $\langle N, v\rangle$ is defined by linearity:

$$\Phi^{(p,S)}\,(v) = \sum_{T \in 2^N} \xi_T \Phi^{(p,S)}\,(u_T)\,.$$

Lemma 6.1 $A^{(p,S)}L(u_T) = \Phi^{(p,S)}(u_T)$.

Proof. Let us prove this result by induction. Let us consider N, a partition S of N and $T \subseteq N$, and let us remind that in Definion 5.1, C is the core of the unanimity game u_T:

$$C(u_T) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in T} x_i = 1, \quad x_i \geq 0 \right\}.$$

First of all, we observe that for $n = 1$ the two definitions trivially coincide. Let us suppose then that the two definitions coincide for $n-1$ players. We must prove that they coincide for n players. According to Definition 5.1, the coefficients we must give to $\underset{x \in C(u_T)}{\arg\max} x_i$ are

$$\lambda_i = \begin{cases} \frac{p_i}{\sum_{k \in S_m} p_k} & \text{if} \quad i \in S_m, \\ 0 & \text{if} \quad i \notin S_m, \end{cases}$$

This means that the payoff for player i, when he chooses first, is

$$\lambda_i \cdot \max_{x \in C(u_T)} x_i = \begin{cases} \frac{p_i}{\sum_{k \in S_m} p_k} \cdot 1 & \text{if} \quad i \in S_m \cap T, \\ \frac{p_i}{\sum_{k \in S_m} p_k} \cdot 0 = 0 & \text{if} \quad i \in S_m \backslash T, \\ 0 \cdot 1 = 0 & \text{if} \quad i \in T \backslash S_m, \\ 0 \cdot 0 = 0 & \text{if} \quad i \notin T \cup S_m. \end{cases}$$

Let us fix $h \in S_m \cap T$, and let us try to calculate now the contribution to his payoff given by other players j (i.e. when j chooses first). Now, in cases $j \in S_m \cap T$, $j \in T \backslash S_m$, $j \notin T \cup S_m$, player h cannot receive anything else, being or the coefficient $\lambda_j = 0$ (cases 2 and 3) or the whole amount (1) already assigned (to player j itself). We must consider then the contribution to payoff of player h given by player $j \in S_m \backslash T$. In this case, let us consider the new $(n-1)$-dimensional unanimity game $u_{T'}$ with set of players $N' = N \backslash \{j\}$, $T' = T$, $S'_r = S_r$, $r = 1, 2, ..., m-1$, $S'_m = S_m \backslash \{j\}$ and the new partition of N', $(S'_1, S'_2, ... S'_m)$. The core of $u_{T'}$ is (if $j \notin T$, $\max_{x \in C(u_T)_j} x_j = 0$)

$$C(u_{T'}) = (\underset{x \in C(u_T)}{\arg\max} x_j)_{-j} = \left\{ x \in \mathbb{R}^{n-1} \mid \sum_{k \in T} x_k = 1, \quad x_k \geq 0 \right\}.$$

As in this case $T \cap S'_m = T \cap S_m \neq \emptyset$ ($\hat{\imath} \in T \cap S_m$), the contribution to the payoff of player i given by player $j \in S_m \backslash T$ is, by the induction hypothesis,

$$\frac{p_h}{\sum_{k \in S_m \cap T'} p_k} = \frac{p_h}{\sum_{k \in S_m \cap T} p_k}.$$

Let us calculate then the final payoff for player h:

$$\frac{p_h}{\sum_{k \in S_m} p_k} + \sum_{j \in S_m \backslash T} \frac{p_j}{\sum_{k \in S_m} p_k} \frac{p_h}{\sum_{k \in S_m \cap T} p_k} = \frac{p_h \left(\sum_{k \in S_m \cap T} p_k\right) + p_h \left(\sum_{j \in S_m \backslash T} \frac{p_j}{\sum_{k \in S_m} p_k}\right)}{\left(\sum_{k \in S_m} p_k\right)\left(\sum_{k \in S_m \cap T} p_k\right)} =$$

$$= \frac{p_h \left(\sum_{k \in S_m} p_k\right)}{\left(\sum_{k \in S_m} p_k\right)\left(\sum_{k \in S_m \cap T} p_k\right)} = \frac{p_h}{\left(\sum_{k \in S_m \cap T} p_k\right)}$$

this is the h-th component of $\Phi^{(p,S)}(u_T)$. Let us suppose now $h \notin S_m \cap T$. In this case his payoff is zero due to the fact that $\max_{x \in C(u_T)} x_h = 0$ if $h \notin T$ or if $h \in T \backslash S_m$, because his coefficient is zero with respect to coefficients of other players $j \in S_m \cap T$ who take all the amount and then $(\underset{x \in C(u_T)}{\arg\max} x_j) = 0$. ■

The following theorem is true:

Theorem 6.2 *If $\langle N,v\rangle$ is a convex game ($\langle N,v\rangle$ belongs to CG^N), $p\in Int(\triangle^n)$ and S a partition of N, then*

$$A^{(p,S)}L(v)=\Phi^{(p,S)}(v).$$

Proof. Let us suppose that $v\in CG^N$. Then

$$v=\sum_{T\in\Upsilon^-}\xi_T u_T+\sum_{T\in\Upsilon^+}\xi_T u_T$$

where Υ^+ and Υ^- are the sets of coalitions s.t. ξ_T are positive and negative respectively. Then, we have

$$v-\sum_{T\in\Upsilon^-}\xi_T u_T=\sum_{T\in\Upsilon^+}\xi_T u_T$$

and

$$A^{(p,S)}L(v+\sum_{T\in\Upsilon^-}-\xi_T u_T)=A^{(p,S)}L(\sum_{T\in\Upsilon^+}\xi_T u_T).$$

Being $v,u_T\in CG^N$, and being $A^{(p,S)}L$ additive and positively homogeneous on CG^N (5.5), we have

$$A^{(p,S)}L(v)+\sum_{T\in\Upsilon^-}-\xi_T A^{(p,S)}(u_T)=\sum_{T\in\Upsilon^+}\xi_T A^{p,S}L(u_T),$$

$$A^{(p,S)}L(v)+\sum_{T\in\Upsilon^-}-\xi_T\Phi^{(p,S)}(u_T)=\sum_{T\in\Upsilon^+}\xi_T\Phi^{(p,S)}(u_T)$$

and then

$$A^{(p,S)}L(v)=\sum_{T\in\Upsilon^+}\xi_T\Phi^{(p,S)}(u_T)-\sum_{T\in\Upsilon^-}-\xi_T\Phi^{(p,S)}(u_T)=\sum_{T\in\Upsilon}\xi_T\Phi^{(p,S)}(u_T)=\Phi^{(p,S)}(v).$$

■

Theorem 6.3 *If $\langle N,v\rangle$ is a convex game, then*

$$A^{(p,S)}L(v)=\Phi^{(p,S)}(v).$$

Proof. Let CG^N be the cone of convex games. Then, in CG^N the core is an additive correspondence (see[1]), and, using Theorem 6.2 the proof is completed. ■

Remark 6.4 Due to Theorems 6.2 and [5] (theorem A), for every convex game v and for every $x\in C(v)$ there exists a system of weights p such that $x=A^{(p,S)}(v)$.

Theorem 6.5 *If $\langle N,v\rangle$ is a simplex game or a dual simplex game, or a big boss game or a clan game, then*

$$A^{(p,S)}(v)=A^{(p,S)}\left(v^E\right)=\Phi^{(p,S)}\left(v^E\right).$$

Proof. Due to Theorems 3.1 and 4.2 of [11] the exactification of simplex, dual simplex bigboss and clan game is a convex game and, using Theorem 6.2, we obtain the result.

■

Remark 6.6 Let $\langle N,v\rangle$ be a monotonic game. Then, $[A^{(p,S)}L(v)]_i\geq 0$ for all $i\in N$. In fact, if $\langle N,v\rangle$ is monotonic, then $\min_{x\in C(v)}x_i\geq 0$, that implies $[L^\sigma(v)]_i\geq 0$ for all $\sigma\in\Pi(N)$. This means that $[A^{(p,S)}L(v)]_i\geq 0$ being a convex combination of $[L^\sigma(v)]_i$.

7 Monotonicity properties of weighted lexicographic values

In this section we want briefly analize what happens to the i-th component of the weighted average lexicographic value when the weight assiciated to the i-th player increases. At the beginning we are discussing the monotonicity properties of $A^pL(C)$ i.e. the case when $S = \{N\}$. From now on we consider $p = (p_1, p_2, ...p_n)$ and $p' = (p'_1, p'_2, ...p'_n)$, $p, p' \in Int\,(\Delta^n)$ and, without loss of generality, we us suppose that $p'_1 \geq p_1$. In general it is not true that $[A^{p'}L(C)]_1 \geq [A^pL(C)]_1$, also when C is a perfect set (i.e. the core of a game), as we can see in the following examples.

Example 7.1 *Let* $C = co\{(\frac{1}{2}, \frac{1}{2}, 0), (0, 0, 1)\}$, $p = (\frac{1}{100}, \frac{98}{100}, \frac{1}{100})$ *and* $p' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. *In this case* $A^pL(C) = (\frac{99}{200}, \frac{99}{200}, \frac{1}{100})$ *and* $A^{p'}L(C) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, *that is* $[A^{p'}L(C)]_1 = \frac{1}{3} < \frac{99}{200} = [A^pL(C)]_1$ *even if* $p_1 = \frac{1}{100} < \frac{1}{3} = p'_1$.

Example 7.2 *Let* $< N, v >$ *be the following game:* $N = \{1, 2, 3\}$ $v(\{i\}) = 0\ \forall i = 1, 2, 3$, $v\{2, 3\} = v\{1, 2\} = 0$, $v\{1, 3\} = \frac{1}{2}$ *let* C *the core of* $< N, v >$, *and* $p' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. *In this case* $A^{p'}L(C) = (\frac{5}{12}, \frac{2}{12}, \frac{5}{12})$. *Let* $p = (\frac{1}{3} + k, k, \frac{2}{3} - 2k)$. *Now* $p_{-2} = (\frac{1}{1-k}(\frac{1}{3} + k), \frac{1}{1-k}(\frac{2}{3} - k))$ *and the first coordinate of* $A^pL(C)$ *is:*

$$1 \cdot (\frac{1}{3} + k) + \frac{1}{2} \cdot \frac{k}{1-k}(\frac{1}{3} + k) = (\frac{1}{3} + k)(1 + \frac{1}{2} \cdot \frac{k}{1-k})$$

and the limit of this number when $k \longrightarrow 0$ *is* $\frac{1}{3} < \frac{5}{12}$.

We can conclude that in general we have no monotonicity of the component of the weighted average lexicographic value with respect to the associated weight, but we have monotonicity properties if we consider the case when one weight increases and all other weights decrease or do not change, as we can see in the following theorem.

Theorem 7.3 *Let us consider* $p, p' \in Int(\Delta^n)$ *such that* $p'_1 = p_1 + k_1$, $p'_j = p'_j - k_j$ *for* $j \neq 1$, $k_i \geq 0$ *for all* $i = 1, 2, ...n$, $k_1 = \Sigma_{j=2}^n k_j$. *Then*

$$[A^pL(C)]_1 \leq [A^{p'}L(C)]_1.$$

pf Let us prove this result by induction. For $n = 2$ it is obvious. Let us suppose now it is true for $n - 1$ and we prove it is true for n.

Let us observe first that $[\pi_1^{-1}(A^{p_{-1}}L\,(\pi_1(C)))]_1 = \max\{x_1 \mid x \in D\} = [\pi_1^{-1}(A^{p'_{-1}}L\,(\pi_1(C)))]_1$.

Let us consider now $[\pi_j^{-1}(A^{p_{-j}}L\,(\pi_j(C)))]_1$ and $[\pi_j^{-1}(A^{p'_{-j}}L\,(\pi_j(C)))]_1$ for $j \neq 1$.

First of all let us observe that $[\pi_j^{-1}(A^{p_{-j}}L(\pi_j(C)))]_1 = [A^{p_{-j}}L(\pi_j(C))]_1$ and $[\pi_j^{-1}\left(A^{p'_{-j}}L(\pi_j(C))\right)]_1 = [A^{p'_{-j}}L(\pi_j(C))]_1$. The weights we have used in calculating $[A^{p_{-j}}L(\pi_j(C))]_1$ are

$$\frac{p_i}{\Sigma_{h\neq j}p_h}, \qquad i \in \{1, 2, ...n\} \setminus \{j\},$$

while for $[A^{p'_{-j}}L(\pi_j(C))]_1$ are

$$\frac{p'_1}{\Sigma_{h\neq j}p'_h} = \frac{p_1 + k_1}{\Sigma_{h\neq j}p_h + k_j} \qquad \text{and} \qquad \frac{p'_i}{\Sigma_{h\neq j}p'_h} = \frac{p_i - k_i}{\Sigma_{h\neq j}p_h + k_j}$$

$i \in \{2, 3, ...n\} \setminus \{j\}$.

These weights satisfy the induction hypothesis as

$$\frac{p'_1}{\Sigma_{h\neq j}p'_h} \geq \frac{p_1}{\Sigma_{h\neq j}p_h} \qquad \text{and} \qquad \frac{p'_i}{\Sigma_{h\neq j}p'_h} \leq \frac{p_i}{\Sigma_{h\neq j}p_h}$$

for all $i \neq 1$, being

$$p'_1(\Sigma_{h\neq j}p_h) = (p_1 + k_1)\,(\Sigma_{h\neq j}p_h) \geq= p_1\,(\Sigma_{h\neq j}p_h - \Sigma_{h\neq j}k_h) = p_1(\Sigma_{h\neq j}p'_h).$$

As, by induction hypothesis, monotonicity holds in dimension $n-1$, that is, for all $j \neq 1$,

$$[\pi_j^{-1}(A^{p_{-j}} L(\pi_j(C))]_1 = [A^{p_{-j}} L(\pi_j(C))]_1 \leq [A^{p'_{-j}} L(\pi_j(C)]_1 = [\pi_j^{-1}\left(A^{p'_{-j}} L(\pi_j(C\right))]_1. \quad (7.2)$$

Let us set

$$w_j := [\pi_j^{-1}(A^{p_{-j}} L(\pi_j(C))]_1, \qquad w'_j := [\pi_j^{-1}\left(A^{p'_{-j}} L(\pi_j(C\right))]_1$$

and $\quad w_1 := [\pi_1^{-1}(A^{p_{-1}} L\pi_1(C))]_1 = \max\{x_1 \mid x \in C\} = [\pi_1^{-1}(A^{p'_{-1}} L\pi_1(C))]_1 =: w'_1$

and let us remark that by induction hypothesis (7.2)

$$w_j \leq w'_j \leq w_1 = w'_1 := \max\{x_1 \mid x \in C\}. \quad (7.3)$$

Then if we consider the first component of the n-dimensional average lexicographic value we have $[A^{p\prime} L(C)]_1 = \sum_{i=1}^n p'_i w'_i = p'_1[\pi_1^{-1}\left(A^{p'_{-1}} L\pi_1(C)\right)]_1 + p'_2[\pi_2^{-1}(A^{p'_{-2}} L(\pi_2(C))]_1 + ... p'_n[\pi_n^{-1}(A^{p'_{-n}} L(\pi_n(C))]_1 =$
$= (p_1+k_1)[\pi_1^{-1}\left(A^{p'_{-1}} L(\pi_1(C\right))]_1 + (p_2-k_2)[\pi_2^{-1}(A^{p'_{-2}} L(\pi_2(C))]_1 + ...(p_n-k_n)[\pi_n^{-1}\left(A^{p'_{-n}} L(\pi_n(C\right))]_1 =$
$= (p_1+k_1)w'_1 + \sum_{i=2}^n (p_i-k_i)w'_i = \sum_{i=1}^n p_i w'_i + k_1 w'_1 - \sum_{i=2}^n k_i w'_i =$
$= \sum_{i=1}^n p_i w'_i + \sum_{i=2}^n k_i w'_1 - \sum_{i=2}^n k_i w'_i = \sum_{i=1}^n p_i w'_i + \sum_{i=2}^n k_i (w'_1 - w'_i)$
(remember that $k_1 = \sum_{i=2}^n k_i$). Due to (7.3) and to non negativity of k_i we have that for all i $\sum_{i=2}^n k_i(w'_1 - w'_i) \geq 0$
and, if we consider positivity of p_i for all i and induction hypothesis (7.2), then $\sum_{i=1}^n p_i w'_i + \sum_{i=2}^n k_i (w'_1 - w'_i) \geq \sum_{i=1}^n p_i w_i = [A^p L(C)]_1$, that is our thesis.

If we consider the case of $A^{(p,S)}L(C)$ i.e. the case when the partition S of N is not trivial, also under the hypothesis of the previous theorem monotonicity does not hold, as we can see in the following example.

Example 7.4 $N = \{1,2,3\}$, $S_1 = \{1\}$, $S_2 = \{2,3\}$, $C = co\left\{\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}, 0\right)\right\}$, $p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. *In this case* $\lambda = (0, \frac{1}{2}, \frac{1}{2})$ *and* $A^{(p,S)}L(C) = (\frac{5}{8}, \frac{1}{8}, \frac{2}{8})$. *Let us consider now the new system of weights* $p' = (1-\varepsilon-\delta, \varepsilon, \delta)$ *with* ε, δ *small and positive. Now* $\lambda' = (0, \frac{\varepsilon}{\varepsilon+\delta}, \frac{\delta}{\varepsilon+\delta})$ *and* $A^{p',S)}L(C) = (\frac{3\varepsilon+2\delta}{4(\varepsilon+\delta)}, \frac{\varepsilon}{\varepsilon+\delta}\frac{1}{4}, \frac{\delta}{\varepsilon+\delta}\frac{1}{2})$. *We can observe that* $\frac{1}{3} < 1-\varepsilon-\delta$ *for* ε, δ *small enough, but* $\frac{3\varepsilon+2\delta}{4(\varepsilon+\delta)} > \frac{5}{8}$ *if and only if* $\varepsilon > 3\delta$. *If we choose, for example,* $\varepsilon = \frac{1}{12}$ *and* $\delta = \frac{4}{12}$, $p' = (\frac{7}{12}, \frac{1}{12}, \frac{4}{12})$, $\lambda' = (0, \frac{1}{5}, \frac{4}{5})$ *and* $[A^{(p',S)}L(C)]_1 = \frac{3\varepsilon+2\delta}{4(\varepsilon+\delta)} = \frac{11}{20} < \frac{5}{8}$, *that is* $A^{(p,S)}L(C)$ *is not monotonic.*

8 Concluding remarks

In Section 4 we have seen that every p-weighted lexicographic value of a balanced game $\langle N,v\rangle$ ($p \in Int(\Delta^n)$) belongs to the interior of the lexicore of $\langle N,v\rangle$ and in Section 5 we have seen that every (p,S)-weighted lexicographic value of $\langle N,v\rangle$ belongs to the lexicore of $\langle N,v\rangle$. Given an element x belonging to the lexicore of $\langle N,v\rangle$, the problem of the existence of $p \in Int(\Delta^n)$ and partition S such that the (p,S)-weighted lexicographic value of $\langle N,v\rangle$ coincides with x is still open.

For monotonicity prpperties we do not have complete results for $A^{(p,S)}L$ and the example is referred to share sets and not to games.

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