CentER

Discussion Paper

No. 2007–58

PUBLIC CONGESTION NETWORK SITUATIONS, AND RELATED GAMES

By John Kleppe, Hans Reijnierse

August 2007

ISSN 0924-7815

TILBURG UNIVERSITY

Public congestion network situations, and related games

John Kleppe[1,2] Hans Reijnierse[1]

August 16, 2007

Abstract

This paper analyzes congestion effects on network situations from a cooperative game theoretic perspective. In network situations players have to connect themselves to a source. Since we consider publicly available networks any group of players is allowed to use the entire network to establish their connection. We deal with the problem of finding an optimal network, the main focus of this paper is however to discuss the arising cost allocation problem. For this we introduce two different transferable utility cost games. For concave cost functions we use the *direct cost game*, where coalition costs are based on what a coalition can do in absence of other players.

This paper however mainly discusses network situations with convex cost functions, which are analyzed by the use of the *marginal cost game*. In this game the cost of a coalition is defined as the additional cost it induces when it joins the complementary group of players. We prove that this game is concave. Furthermore, we define a cost allocation by means of three egalitarian principles, and show that this allocation is an element of the core of the marginal cost game. These results are extended to a class of continuous network situations and associated games.

Keywords: Congestion, network situations, cooperative games, public.

JEL Classification Numbers: C61, C71.

1 Introduction

Economic congestion situations arise if a group of players uses facilities from a common pool and the cost of using a certain facility depends on the number of its users. As a result, a congestion situation creates interaction between players and involves the analysis of a cost allocation problem.

For many game theoretic problems in a cooperative setting congestion effects could be considered. In particular the branch of cooperative literature on Operations Research Games, surveyed by Borm et al. (2001), is well suited for this kind of approach. However, the stream of literature within this field of research on the topic of congestion is limited. An example is a recent paper by Matsubayashi et al. (2005) in which hub-spoke network systems with congestion effects are discussed using cooperative games.

[1]CentER and Department of Econometrics and Operations Research, Tilburg University.

[2]Corresponding author: PO Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: J.Kleppe@uvt.nl.

The inspiration for this paper however, comes from the papers by Quant et al. (2006) and Quant and Reijnierse (2004) on (convex) congestion network situations, which generalize the well known minimum cost spanning tree problems.

In congestion network situations a single source is considered to which all players have to be connected, and the cost of using an arc in order to achieve this depends on the number of its users.

Consider e.g. the symmetric congestion network situation of Figure 1.1 with two players. The numbers on the arcs represent the total usage costs for each number of players. Because in the work of Quant et al. (2006) and Quant and Reijnierse (2004) all arcs are

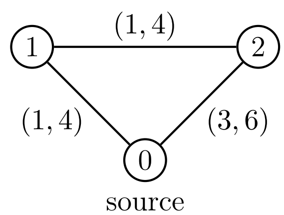

Figure 1.1: A congestion network situation.

considered to be private, a particular coalition of players is only allowed to use the arcs between the players of that coalition and the source in order to establish their connection. In this paper however we consider congestion network situations with public arcs, which means that each coalition of players is allowed to use any arc of the network.

For applications of public congestion network situations one could think e.g. of computer networks with one main server, or of a single distribution center with several suppliers on a publicly available road network.

In this paper we discuss congestion network situations with the underlying idea that players have to get from their initial nodes to the source (suppliers supply a single distribution center). Note however that we could equivalently think of a setup in which players start at the source and have to get to their final nodes (suppliers supply *from* a single distribution center).

In case the players are willing to cooperate, it is natural that (in principle) an optimal network is constructed. In order to find an appropriate allocation of the involved costs, two TU-games are introduced. In the so-called *direct cost game* a coalition that is formed must construct a network that connects all of its members, assuming that the other players do *not* make use of any arcs. This is in our opinion a convenient model for situations with concave congestion costs. Then, the presence of other players would decrease marginal costs of construction and it is reasonable to assume that a coalition formed does not benefit from the presence of other players (otherwise cooperation is not to be expected). In case of convex costs however, it is (by the same argument) reasonable to assume that a coalition must construct a network while the non-members have been connected already. This is modeled by the so-called *marginal cost game*, which turns out to be the dual of the direct cost game. We elaborate on this in Section 3. If costs are neither concave nor convex an appropriate model is still to be found.

Quant et al. (2006) show that if arcs are private and costs are concave, there exists an efficient network which is a tree. Since, for the grand coalition it does not matter whether arcs are private or public, this result still stands in our model. Furthermore, their example of a private congestion network situation with concave costs of which the corresponding TU-game is not balanced, gives the same result for the direct cost game in the case of public arcs. For linear costs, it is easy to verify that the direct cost game is linear (and hence coincides with the marginal cost game). Therefore we will focus on the case of convex costs. Our first main result is that the marginal cost game of a convex congestion network situation is concave. As a consequence, cooperation is likely to occur and stable allocations exist. On the other hand, concave cost games tend to have a relatively large core, making a refinement of the core desirable. Our second main result is the introduction of such a refinement. It is based on the principles that

- every player should pay for his own path to the source,
- two players whose paths share some arc should contribute an equal part of the cost of this arc,
- if there are several path decompositions possible for the optimal network, the average over all decompositions should be used to allocate the total cost.

The structure of the paper is as follows. Section 2 settles notation for public congestion network situations. In Section 3 we introduce the marginal cost game, show how to calculate coalition costs by the use of an algorithm, and prove that this game is concave. In the last part of that section we refine the core by the above properties. In Section 4 we let go of the restriction that players have to use a single path to the source, and in this continuous framework we extend the results of Section 3.

2 Public congestion network situations

A *public congestion network situation*, or congestion network situation as we call it from here on, is given by a triple $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$, where N is the set of players that has to be connected to the source 0. The set $N \cup \{0\}$ is denoted by N^0. By A_S we denote the set of all arcs between pairs in $S \subseteq N^0$, i.e. (S, A_S) is the complete digraph on S. For each arc $a \in A_{N^0}$ the function $\gamma_a : \{0, 1, \ldots, |N|\} \to \mathbb{R}_+$ is a nonnegative (weakly) increasing cost function. We assume that $\gamma_a(0) = 0$ for all $a \in A_{N^0}$. Elements of A_{N^0} are denoted by a or by (i, j), where $i, j \in N^0$. The arc (i, j) denotes the connection between i and j in the direction from i to j. If $a = (i, j)$, then a^{-1} denotes the arc in de opposite direction, hence $a^{-1} = (j, i)$. The cost function of an arc (i, j) is denoted by γ_{ij}. A congestion network situation is symmetric if $\gamma_{ij} = \gamma_{ji}$ for all $i, j \in N^0$.

In a congestion network situation each player has to connect himself to the source, which can be done by choosing a path from his initial node to the source. A path between any two nodes i and j is denoted by $P(i, j)$ and is a sequence of arcs $((i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p))$, such that $i_0 = i$, $i_p = j$ and $i_r \neq i_s$ for all $r, s \in \{0, \ldots, p\}$, $r \neq s$. Instead of $P(i, 0)$ we also write P_i. By $(k, \ell) \prec_P (r, s)$ we denote that arc (k, ℓ) is a predecessor of (r, s) on path

P.

Let $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$ be a congestion network situation. A network can be described by an integer valued function $f : A_{N^0} \to \{0, 1, \ldots, |N|\}$, such that f assigns to each arc a number of users. The indegree for a node $i \in N^0$ with respect to network f is defined by $\text{indegree}^f(i) = \sum_{j\in N^0\setminus\{i\}} f(j,i)$. Analogously, the outdegree is defined by $\text{outdegree}^f(i) = \sum_{j\in N^0\setminus\{i\}} f(i,j)$. For a coalition $S \subseteq N$ the collection of all feasible networks connecting the members of S to the source is given by

$$\begin{aligned} F_S = \{f : A_{N^0} \to \{0,\ldots,|N|\}| \quad & \text{outdegree}^f(i) - \text{indegree}^f(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}^f(i) - \text{indegree}^f(i) = 0 \text{ for all } i \in N\setminus S, \\ & f(a) \in \{0,\ldots,|S|\} \text{ for all } a \in A_{N^0}\}. \end{aligned}$$

Note that in a feasible network for S each player of S is connected to the source by a particular path. However, as all arcs are publicly available these paths may consist of arcs between any two nodes in N^0. Each network f induces a digraph (N^0, A_f), where A_f consists of all used arcs:

$$A_f = \{a \in A_{N^0} | f(a) > 0\}.$$

The cost of such a network f is defined by

$$\gamma(f) = \sum_{a\in A_{N^0}} \gamma_a(f(a)).$$

A frequently used method to analyze an interactive situation in a cooperative setting such as described above is to consider a corresponding TU-game. A TU-game consists of a pair (N, c) in which N is the set of players and c is a function assigning to each coalition $S \subseteq N$ a cost of $c(S)$, denoting the cost of which coalition S can guarantee itself without cooperating with players outside their coalition. By definition $c(\emptyset) = 0$. The core of a TU-game (N, c) consists of those cost allocation vectors for which no coalition would be better off if it would separate itself and pay its coalition costs. It is given by

$$\text{Core}(c) = \{x \in \mathbb{R}^N | x(N) = c(N), \ \sum_{i\in S} x_i \le c(S), \text{ for all } S \subseteq N\}.$$

With each congestion network situation $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$ one can associate a *direct cost game* (N, k^T), such that in case no players outside coalition S use the arcs of the network, $k^T(S)$ denotes the minimum cost of a network connecting all players of S to the source:

$$k^T(S) = \min_{f\in F_S} \gamma(f)$$

for all $S \subseteq N$. We omit the superscript T if no confusion can occur. As discussed in the previous section, this game only models the situation properly in case of concave cost functions on the arcs. Nevertheless, it turns out to be quite useful in the convex case as well.

3 Convex congestion network situations

In Section 1 we argued that for linear and concave cost functions the switch from private to public arcs has a minor influence, and all results derived by Quant et al. (2006) and Quant and Reijnierse (2004) for congestion network situations with private arcs are almost directly applicable for the setup with public arcs. This is however not the case for congestion network situations with convex cost functions.

A convex congestion network situation $T = (N, 0, (\gamma_a)_{a \in A_{N^0}})$ is a congestion network situation in which all γ_a are convex. A cost function γ_a, $a \in A_{N^0}$, is convex if for all $r \in \{1, \ldots, |N| - 1\}$

$$\gamma_a(r+1) - \gamma_a(r) \geq \gamma_a(r) - \gamma_a(r-1).$$

Then, if a group of players decides to form a coalition in this setup it will have to take into account the presence of the remaining players. As discussed before, we have the opinion that when determining the value of a coalition S it is natural to assume that the players in $N \backslash S$ are already on the network. However, this still leaves many options open. The most pessimistic view is that the remaining players will try to frustrate S as much as possible. Less rigorous is to assume that the players in $N \backslash S$ choose an optimal network for themselves. An optimistic approach is then to assume that the members of $N \backslash S$ are willing to change their paths to the source, as long as S compensates them for the additional costs. We have chosen for this approach, because given the fact that utility is transferable it seems the most natural setup. Furthermore, if an allocation is stable under this approach, it is stable under every less optimistic approach as well. Hence, we assume that if coalition S forms, it constructs a network feasible and optimal for the grand coalition. The complementary coalition $N \backslash S$ pays $k(N \backslash S)$ to make use of the network. This idea is formalized by the *marginal cost game* (N, c^T) (or (N, c) if no confusion can occur), which is given by

$$c^T(S) = k^T(N) - k^T(N \backslash S)$$

for all $S \subseteq N$. Let us illustrate this idea by means of the example of Figure 1.1. Suppose we would like to determine the value of coalition $\{1\}$. We assume that player 2 has been optimally connected to the source already, by the path $((2, 1), (1, 0))$ with cost 2. Then, player 1 could use the direct link, resulting in a cost of 3, or connect himself via node 2, which costs 4. In our setup coalition $\{1\}$ has a third option. He can ask player 2 to link himself directly to the source, making it possible for player 1 to form a less expensive direct connection. This results in a cost of 1 for himself and a cost of 1 to compensate player 2. It is straightforward to verify that this leads to a cost for coalition $\{1\}$ of $k(N) - k(\{2\}) = 4 - 2 = 2$.

3.1 Optimal networks and the marginal cost game

This section focuses on the marginal cost game introduced above. We start by presenting an algorithm to find the optimal network for each coalition of players, from which coalition costs follow immediately. After that we prove that (N, c) is a concave game.

Let $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$ be a convex congestion network situation and consider a feasible network f. We assume that this network satisfies $f(a^{-1}) = 0$ whenever $f(a) > 0$ for all $a \in A_{N^0}$, since if both $f(a)$ and $f(a^{-1})$ are positive, the network remains feasible if $f(a)$ and $f(a^{-1})$ are decreased by one. Because this new network is at least as cheap as f, it is reasonable to make the above assumption. Furthermore, we assume that A_f does not contain any circuits[1], since the network arising from f by decreasing the number of users of the arcs in a circuit by one yields a feasible network at least as cheap as f.

Then given f, we define a length function ℓ_f on the complete digraph (N^0, A_{N^0}) as follows:

$$\ell_f(a) = \begin{cases} \infty & \text{if} \quad f(a) = |N|, \\ \gamma_a(f(a)+1) - \gamma_a(f(a)) & \text{if} \quad f(a^{-1}) = 0 \text{ and } f(a) < |N|, \\ \gamma_{a^{-1}}(f(a^{-1})-1) - \gamma_{a^{-1}}(f(a^{-1})) & \text{if} \quad f(a^{-1}) > 0. \end{cases}$$

This function can be interpreted as the marginal cost of an extra user of an arc. Note that if the opposite a^{-1} of an arc is used, an extra user of arc a should be interpreted as the reduction of the number of users of a^{-1} by one.

An order on N is a bijective function $\sigma : \{1, \ldots, |N|\} \to N$. The player at position t in the order σ is denoted by $\sigma(t)$. The set of all orders on N is given by $\Pi(N)$. By $\Pi_S(N)$ we denote the set of all orders such that $\sigma(i)^{-1} < \sigma(j)^{-1}$ for all $i \in S, j \in N\backslash S$. Hence, in an order $\sigma \in \Pi_S(N)$ the players in coalition S are placed on the first $|S|$ positions.

Let f^1 and f^2 be two networks. The sum $f^1 \oplus f^2$ is defined by:

$$f^1 \oplus f^2(a) := \max\{f^1(a) + f^2(a) - f^1(a^{-1}) - f^2(a^{-1}), 0\}$$

for all $a \in A_{N^0}$. This operation takes into account that the usage of two oppositely directed arcs cannot be beneficial. If there is two way traffic between nodes, the numbers of users are subtracted instead of added.

Let us denote the network corresponding to a path P by f_P. The following algorithm provides an optimal network for a coalition $S \subseteq N$, which is denoted by f^*_S.

Algorithm 3.1

Input: a convex congestion network situation $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$, and an order $\sigma \in \Pi_S(N)$.

Output: an optimal network f^*_S for coalition $S \subseteq N$.

1. Initialize $V = \emptyset$, and $t = 1$.
2. Find a shortest path from $\sigma(t)$ to 0 in (N^0, A_{N^0}), given length function $\ell_{f^*_V}$, and call it $P^*_{V,\sigma(t)}$.
3. Set $f^*_{V\cup\{\sigma(t)\}} = f^*_V \oplus f_{P^*_{V,\sigma(t)}}$.
4. If $t \neq |S|$, set $t = t+1$, $V = V \cup \{\sigma(t)\}$ and return to step 2.

[1] A set of arcs A contains a circuit if there exists a sequence $((i_0, i_1), (i_1, i_2), \ldots, (i_{p-1}, i_p))$ such that $i_0 = i_p$ and $(i_r, i_{r+1}) \in A_{N^0}$ for all $r \in \{1, \ldots, p-1\}$.

Finding shortest paths can be done by the Floyd-Warshall algorithm (cf. Cormen et al. (1990)), which has a complexity of order $\mathcal{O}(|\mathrm{N}|^3)$. Since we have to do this procedure at most $|N|$ times the complete algorithm has a complexity of order $\mathcal{O}(|\mathrm{N}|^4)$.

The proof of the validity of the algorithm is a straightforward generalization of Theorem 3.2 of Quant and Reijnierse (2004), and is therefore omitted. Let us illustrate the algorithm by the use of the following example.

Example 3.1 Consider the symmetric convex congestion network situation of Figure 3.1 with three players. Let us calculate the direct cost of coalition $\{1,3\}$ by the use of order

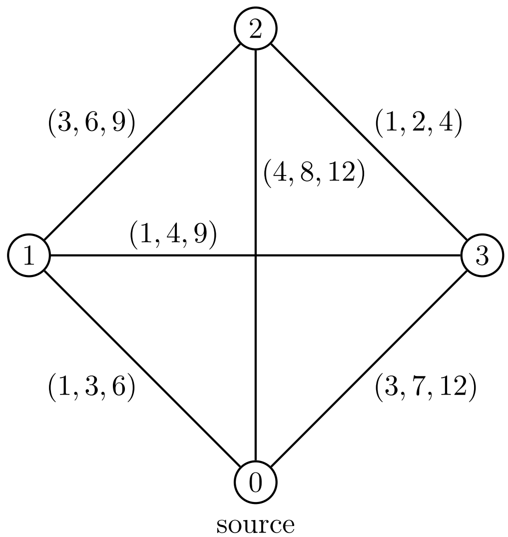

Figure 3.1: A convex congestion network situation.

$\sigma = \{3,1,2\}$. Then path P^*_3 is given by $((3,1),(1,0))$, which implies that $k(\{3\}) = 2$. The shortest path from player 1 to the source given $\ell_{f^*_{\{3\}}}$ is then $P^*_{\{3\},1} = ((1,3),(3,0))$. Adding this path to $f^*_{\{3\}}$ results in network $f^*_{\{1,3\}}$ with $f^*_{\{1,3\}}(1,0) = f^*_{\{1,3\}}(3,0) = 1$. Hence, $k(\{1,3\}) = 4$.

All coalition costs regarding the direct and marginal cost game for this situation are given in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$k(S)$	1	3	2	5	4	6	8
$c(S)$	2	4	3	6	5	7	8

$\triangleleft$

The following theorem links the convexity of the congestion network situation to the concavity of the corresponding cost game (N,c).

Theorem 3.1 Let $T = (N, 0, (\gamma_a)_{a\in A_{N0}})$ be a convex congestion network situation. Then (N,c) is a concave game.

Proof: We prove that (N, c) is a concave game by showing that the game (N, k) is convex. Algorithm 3.1 finds an optimal network with corresponding costs for a particular coalition by putting players one by one on the network. If we assume that the players of coalition S are already optimally on the network, the next player $i \in N\backslash S$ adds a cost which equals the length of path $P^*_{S,i}$ derived by the algorithm.

Let f^*_S be an optimal network for S. Let $i, j \in N\backslash S$ and let $T = S \cup \{j\}$. By $\ell_f(P) = \sum_{a \in P} \ell_f(a)$ we denote the length of a path P, given network f. Then, we show that $\ell_{f^*_T}(P^*_{T,i}) \geq \ell_{f^*_S}(P^*_{S,i})$. As $\ell_{f^*_T}(P^*_{T,i}) = k(T \cup \{i\}) - k(T)$ and $\ell_{f^*_S}(P^*_{S,i}) = k(S \cup \{i\}) - k(S)$, this result implies that $k(T \cup \{i\}) - k(T) \geq k(S \cup \{i\}) - k(S)$ for every $S \subset T \subseteq N\backslash\{i\}$, and hence that (N, k) is convex.

Let us first assume that there do not exist arcs $a \in P^*_{T,i}$ such that $a^{-1} \in P^*_{S,j}$. Then $f^*_T(a) \geq f^*_S(a)$ for all $a \in P^*_{T,i}$, which implies by the convexity of γ_a that

$$\begin{aligned} \ell_{f^*_T}(P^*_{T,i}) &\geq \ell_{f^*_S}(P^*_{T,i}) \\ &\geq \ell_{f^*_S}(P^*_{S,i}). \end{aligned}$$

Secondly, we assume that there do exist arcs $a \in P^*_{T,i}$ such that $a^{-1} \in P^*_{S,j}$. Let (k_1, ℓ_1) be the first arc on $P^*_{T,i}$ with this property, i.e., the inverse arc $(\ell_1, k_1) \in P^*_{S,j}$ and $a^{-1} \notin P^*_{S,j}$ for all $a \prec_{P^*_{T,i}} (k_1, \ell_1)$.

Let us here introduce another way to describe a path $P(i, j)$. A path from i to j can be given by a sequence of nodes, $\underline{P}(i, j) = (i_0, i_1, \ldots, i_p)$, with $i_0 = i$, $i_p = j$ and $i_r \neq i_s$ for all $r, s \in \{0, \ldots, p\}$, $r \neq s$. By $k \prec_{\underline{P}} \ell$ we denote that node k is a predecessor of node ℓ on path $\underline{P}$. We define $m_1 \in \underline{P}^*_{T,i} \cap \underline{P}^*_{S,j}$ as the first node on path $\underline{P}^*_{T,i}$ such that

$$\begin{aligned} k_1 &\prec_{P^*_{T,i}} m_1, \\ k_1 &\prec_{P^*_{S,j}} m_1. \end{aligned}$$

Take m_1 as the new starting node. If there exist arcs on $P^*_{T,i}$ beyond node m_1 used by $P^*_{S,j}$ in the opposite direction, the arc (k_2, ℓ_2) is defined similar to the way we defined arc (k_1, ℓ_1). Moreover, all nodes k_r and m_r, $r \in \{1, \ldots, R\}$, are sequentially defined analogously to the definitions of k_1 and m_1. Note that node m_r may coincide with k_{r+1} and that m_R may be equal to the source, 0. To get an idea how $P^*_{S,j}$ and $P^*_{T,i}$ may relate, see Figure 3.2. We can divide $P^*_{T,i}$ into $2R + 1$ pieces: $P^*_{T,i} = (P^*_T(i, k_1), P^*_T(k_1, m_1), P^*_T(m_1, k_2), \ldots, P^*_T(k_R, m_R), P^*_T(m_R, 0))$, where $P^*_V(x, y)$ denotes the shortest path from x to y, given length function $\ell_{f^*_V}$.

Then,

$$\begin{aligned} \ell_{f^*_T}(P^*_T(i, k_1)) &\geq \ell_{f^*_S}(P^*_T(i, k_1)) \\ &\geq \ell_{f^*_S}(P^*_S(i, k_1)), \end{aligned} \quad (1)$$

because $f^*_T(a) \geq f^*_S(a)$ for all $a \in P^*_T(i, k_1)$. By the same reasoning we obtain

$$\begin{aligned} \ell_{f^*_T}(P^*_T(m_R, 0)) &\geq \ell_{f^*_S}(P^*_T(m_R, 0)) \\ &\geq \ell_{f^*_S}(P^*_S(m_R, 0)). \end{aligned} \quad (2)$$

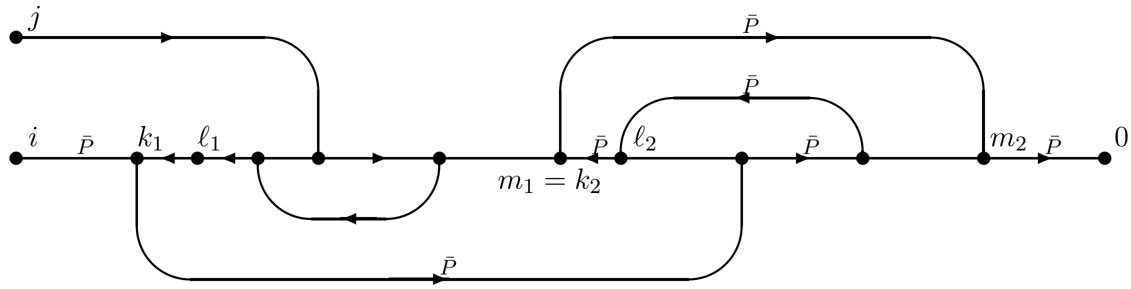

Figure 3.2: Paths $P^*_{S,j}$ (following the arrows) and $P^*_{T,i}$ (going in a straight line). Edges belonging to path $\bar{P}$ are labeled by a $_{\bar{P}}$.

Note further that from any node k_r, with $r \in \{1, \dots, R\}$, player j, when moving from k_r to m_r, could have chosen $P^*_T(k_r, m_r)$ with cost $\ell_{f^*_T}(P^*_T(k_r, m_r))$. Therefore,

$$\ell_{f^*_T}(P^*_T(k_r, m_r)) \geq \ell_{f^*_S}(P^*_S(k_r, m_r)) \tag{3}$$

for all $r \in \{1, \dots, R\}$. And as none of the arcs between any m_r and k_{r+1} are taken by player j in the reverse order, $f^*_T(a) \geq f^*_S(a)$ for all $a \in P^*_T(m_r, k_{r+1})$, for all $r \in \{1, \dots, R-1\}$ and hence,

$$\begin{aligned} \ell_{f^*_T}(P^*_T(m_r, k_{r+1})) &\geq \ell_{f^*_S}(P^*_T(m_r, k_{r+1})) \\ &\geq \ell_{f^*_S}(P^*_S(m_r, k_{r+1})) \end{aligned} \tag{4}$$

for all $r \in \{1, \dots, R-1\}$. Let us define path $\bar{P}$ as
$\bar{P} = (P^*_T(i, k_1), P^*_S(k_1, m_1), P^*_T(m_1, k_2), \dots, P^*_S(k_R, m_R), (P^*_T(m_R, 0))$. Combining (1) - (4) leads to

$$\begin{aligned} \ell_{f^*_T}(P^*_{T,i}) &\geq \ell_{f^*_S}(\bar{P}) \\ &\geq \ell_{f^*_S}(P^*_{S,i}). \end{aligned}$$

□

3.2 Cost allocation

Because the game (N, c) is concave it has a non-empty and relatively large core. In this subsection we consider a refinement of the core in which players pay for each arc proportionally to their average usage of the arc in the optimal network.

Let $T = (N, 0, (\gamma_a)_{a \in A_{N0}})$ be a convex congestion network situation and let $S \subseteq N$ with optimal network f^*_S. Let $D_S = \{D_{S(i)}\}_{i \in S}$ be a decomposition of f^*_S into $|S|$ paths. A decomposition of an optimal network f^*_N is denoted by D.

Consider Example 3.1, with optimal network f^*_N given by $f^*_N(2,3) = f^*_N(3,1) = f^*_N(3,0) = 1$ and $f^*_N(1,0) = 2$. Then $D(1) = ((1,0))$, $D(2) = ((2,3),(3,0))$, and $D(3) = ((3,1),(1,0))$ is a decomposition of the optimal network.

Recall the egalitarian principles given in Section 1, being:

- every player should pay for his own path to the source,
- two players whose paths share some arc should contribute an equal part of the cost of this arc,
- if there are several path decompositions possible for the optimal network, the average over all decompositions should be used to allocate the total cost.

The first two properties naturally lead to the idea that the contribution of each player should only depend on the arcs used by him in the optimal network, and furthermore that his contribution to the total cost of each arc is proportional to his usage of the arc. These ideas result in the following cost allocation, given decomposition D of optimal network f^*_N:

$$\phi_D(i) \quad = \quad \sum_{a \in A_{f^*_N}} \frac{f_{D(i)}(a)}{f_N(a)} \gamma_a(f^*_N(a)) \tag{5}$$

for all $i \in N$. For the decomposition given above this allocation equals $\phi_D = (1\frac{1}{2}, 3\frac{1}{2}, 3)$.

The content of the following theorem is that for an optimal network f^*_N of a convex congestion network situation with decomposition D the corresponding allocation ϕ_D is an element of the core of the game (N, c).

Theorem 3.2 Let $T = (N, 0, (\gamma_a)_{a \in A_{N0}})$ be a convex congestion network situation, let f^*_N be an optimal network with decomposition D, and let ϕ_D defined to be $\phi_D(i) = \sum_{a \in A_{f^*_N}} \frac{f_{D(i)}(a)}{f_N(a)} \gamma_a(f^*_N(a))$ for all $i \in N$. Then $\phi_D \in \text{Core}(c)$.

Proof: It follows from the definition that ϕ_D is efficient. Let us therefore focus on the stability constraints of the core. Let $S \subset N$ and let $\bar{f} = \sum_{i \in S} f_{D(i)}$. Since $\bar{f}$ is feasible for S we obtain $k(S) \leq \gamma(\bar{f})$, so

$$\begin{aligned} k(S) &\leq \sum_{a \in A_{\bar{f}}} \gamma_a(\sum_{i \in S} f_{D(i)}(a)) \\ &\leq \sum_{a \in A_{\bar{f}}} \frac{\sum_{i \in S} f_{D(i)}(a)}{f^*_N(a)} \gamma_a(f^*_N(a)) \\ &= \sum_{i \in S} \sum_{a \in A_{f^*_N}} \frac{f_{D(i)}(a)}{f^*_N(a)} \gamma(f^*_N(a)) \\ &= \sum_{i \in S} \phi_D(i), \end{aligned}$$

where the second inequality uses the convexity of the cost functions γ_a. The concluding argument is

$$\begin{aligned}\sum_{i\in S}\phi_D(i) &= \sum_{i\in N}\phi_D(i)-\sum_{i\in N\backslash S}\phi_D(i)\\ &\le k(N)-k(N\backslash S)\\ &= c(S).\end{aligned}$$

□

Hence, a decomposition of an optimal network gives rise to a core element of the marginal cost game. However, given an optimal network f^*_S the decomposition D_S need not be uniquely determined, in the sense that we cannot distinguish which arcs are used by which players. This is the case when the corresponding digraph contains a cycle[2].

Let us reconsider Example 3.1. The optimal network f^*_N given by $f^*_N(2,3) = f^*_N(3,1) = f^*_N(3,0) = 1$ and $f^*_N(1,0) = 2$ does not have a unique decomposition, because the optimal network can be decomposed into both D, which was already considered, and D', with $D'(1) = ((1,0))$, $D(2) = ((2,3),(3,1),(1,0))$, and $D(3) = ((3,0))$, resulting in allocation $\phi_{D'} = (1\frac{1}{2}, 4, 2\frac{1}{2})$. Note that D and D' are the only possible decompositions.

The fact that allocation ϕ_D depends on the decomposition chosen gives it a flavor of arbitrariness. E.g., players 2 and 3 have to move from node 3 to the source. There is however no reason to send player 2 via node 1 (resulting in decomposition D), or directly from node 3 to the source (resulting in decomposition D'). In order to overcome this drawback we use the third egalitarian principle and introduce, given an optimal network f^*_N, the allocation ϕ as the average over all allocations ϕ_D that follow from each of the possible decompositions:

$$\phi = \frac{1}{|\mathcal{D}|}\sum_{D\in\mathcal{D}}\phi_D, \tag{6}$$

with $\mathcal{D}$ the set of all path decompositions of an optimal network f^*_N. For the already discussed optimal network of Example 3.1, this cost allocation is given by $\phi = \frac{1}{2}(1\frac{1}{2}, 3\frac{1}{2}, 3)+\frac{1}{2}(1\frac{1}{2}, 4, 2\frac{1}{2}) = (1\frac{1}{2}, 3\frac{3}{4}, 2\frac{3}{4})$. As each allocation ϕ_D is an element of the core, the convexity of the core yields that ϕ is a core element as well.

A problem of ϕ at first sight is that it is not polynomially computable, as although each ϕ_D can be computed in polynomial time given a decomposition D, the number of possible decompositions of an optimal network may not be polynomial. Consider however the following algorithm.

Algorithm 3.2

Input:	a convex congestion network situation $T = (N, 0, (\gamma_a)_{a\in A_{N^0}})$,
	and an optimal network f^*_N.
Output:	cost allocation ϕ.

[2] We say that the digraph (N, A) contains a cycle if the non-directed graph $(N, E(A))$, with $E(A) = \{\{i,j\}|(i,j) \in A \text{ or } (j,i) \in A\}$ contains a cycle.

1. Initialize $V = N$, and for all $i \in N$, $\phi(i) = 0$ and

$$g_j(i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

   for all $j \in N$.

2. Find a node $j \in V$ such that $\sum_{i \in V \setminus \{j\}} f_N^*(i,j) = 0$.

3. Set $\phi(i) = \phi(i) + \frac{g_j(i)}{\sum_{\ell \in N} g_j(\ell)} \gamma_{(j,k)}(f_N^*(j,k))$ for all $i \in N$, and for all $k \in (V \cup 0) \setminus \{j\}$.

4. Set $g_k(i) = g_k(i) + \frac{g_j(i)}{\sum_{\ell \in N} g_j(\ell)} f_N^*(j,k)$ for all $i \in N$, and for all $k \in V \setminus \{j\}$.

5. If $|V| > 1$, set $V = V \setminus \{j\}$ and return to step 2.

The complexity of the third and fourth step of the algorithm is of order $\mathcal{O}(|N|^2)$, as it requires calculations for each player for at most $|N| - 1$ arcs. The complexity of the other steps is of order $\mathcal{O}(|N|)$. Since steps 2 - 5 are repeated $|N|$ times, the complexity of the complete algorithm is of order $\mathcal{O}(|N|^3)$.

Let us illustrate Algorithm 3.2 by the convex congestion network situation of Example 3.1 with optimal network $f_N^*(2,3) = f_N^*(3,1) = f_N^*(3,0) = 1$ and $f_N^*(1,0) = 2$.

We start by finding a node j with $\text{indegree}^f(j) = 0$. The only node satisfying this condition is node 2. Since $g_2(i) = 0$ for $i \in \{1,3\}$, player 2 is the only player contributing to the costs of the outgoing arcs of this node. The single outgoing arc is $(2,3)$ with cost 1, and hence, player 2 has to pay this amount. Then we update $g_3(2) = 1$ and delete node 2 from V. Among all nodes in V node 3 is the only node j such that $\sum_{i \in V \setminus \{j\}} f_N^*(i,j) = 0$. Since $g_3(1) = 0$ and $g_3(2) = g_3(3) = 1$, players 2 and 3 equally share the costs of all outgoing arcs, being arcs $(3,1)$ and $(3,0)$. As a consequence, they both contribute an amount of 2. Updating g leads to $g_1(2) = g_1(3) = \frac{1}{2}$. Then only the cost of arc $(1,0)$ has to be divided. Since $g_1(1)$ is twice as high as $g_1(i)$, with $i \in \{2,3\}$, player 1 contributes twice as much as both player 2 and 3 of the cost of this arc, resulting in the final allocation $\phi = (1\frac{1}{2}, 3\frac{3}{4}, 2\frac{3}{4})$.

Note that this cost allocation exactly coincides with the allocation ϕ derived from the definition of equation (6), which is a result formalized in the following proposition.

Proposition 3.3 *Let $T = (N, 0, (\gamma_a)_{a \in A_{N0}})$ be a convex congestion network situation with optimal network f_N^*. Then the output ϕ of Algorithm 3.2 equals $\phi = \frac{1}{|\mathcal{D}|} \sum_{D \in \mathcal{D}} \phi_D$.*

The proof is straightforward once it is noted that a decomposition can be seen as a description telling at each node which player takes which arc. Then it follows that the fraction of decompositions for which a player uses a particular arc exactly coincides with the fraction of the cost of this arc this player has to pay according to Algorithm 3.2.

4 Continuous congestion network situations

In the previous sections we considered network situations in which players had to be connected to the source by a single path. However, if we think in the context of continuous streams of traffic (e.g. data traffic from terminals to a mainframe, or road traffic from suppliers to a distribution center) it is quite plausible that a player uses several paths to the source. As a consequence, the capacity and usage of an arc need no longer be integer, and therefore we switch from discrete to continuous cost functions.

Congestion network situations arising from this relaxation are called *continuous congestion network situations* and are given by $\mathcal{T} = (N, 0, (\gamma_a)_{a \in A_{N^0}})$ in which $\gamma_a : [0, |N|] \to \mathbb{R}_+$ is a (weakly) increasing cost function for all $a \in A_{N^0}$, with $\gamma_a(0) = 0$. The set of all feasible networks for a coalition $S \subseteq N$ is given by

$$
\begin{aligned}
F_S = \{f : A_{N^0} \to [0, |N|] \mid \; & \text{outdegree}^f(i) - \text{indegree}^f(i) = 1 \text{ for all } i \in S, \\
& \text{outdegree}^f(i) - \text{indegree}^f(i) = 0 \text{ for all } i \in N \backslash S, \\
& f(a) \in [0, |S|] \text{ for all } a \in A_{N^0}\}.
\end{aligned}
$$

Note that although each player has the possibility to use several paths the difference between the outdegree and the indegree of his node is still one in case he is connected to the source. The corresponding direct continuous network cost game is denoted by $(N, k^{\mathcal{T}})$, the marginal cost game by $(N, c^{\mathcal{T}})$.

In the remainder of this section we call the congestion network situations with discrete cost functions, as discussed in the previous sections, discrete congestion network situations. Given a continuous congestion network situation $\mathcal{T} = (N, 0, (\gamma_a)_{a \in A_{N^0}})$, one can find the related discrete congestion network situation by restricting the function γ_a to the domain $\{0, 1, \ldots, |N|\}$. The congestion network situation derived in this way is denoted by $T(\mathcal{T})$. For each continuous concave congestion network problem we have $k^{\mathcal{T}}(S) = k^{T(\mathcal{T})}(S)$ for all $S \subseteq N$. This result follows from Theorem 4.1 of Quant et al. (2006), stating that for a discrete concave congestion network situation there exists an optimal network that is a tree. For convex congestion network situations these games do not coincide, as shown by the following example.

Example 4.1 Consider the continuous convex congestion network situation of Figure 4.1. The corresponding optimal network is given by $f_N^*(1,0) = f_N^*(1,2) = \frac{1}{2}$ and $f_N^*(2,0) = \frac{3}{2}$,

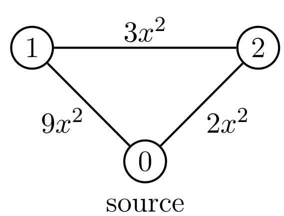

Figure 4.1: A continuous convex congestion network situation.

with a total cost of $7\frac{1}{2}$. However, an optimal network for $T(\mathcal{T})$ is given by $f_N^*(1,0) =$

$f^*_N(2,0) = 1$, with a total cost of 11. $\triangleleft$

Let us therefore discuss continuous convex congestion network situations in more detail and focus in particular on the two main results of Section 3.

First of all, the proof that $\phi_D \in C(c^T)$ is also valid to prove that $\phi_D \in C(c^{\mathcal{T}})$. Consequently, the allocation ϕ is also situated in the core of $(N, c^{\mathcal{T}})$ for continuous convex congestion network situations.

A drawback of the discrete setup is that a situation may have multiple optimal networks, e.g. Example 3.1 has two optimal networks. Besides the optimal network f^*_N given in Section 3, also the network $\bar{f}^*_N$, with $\bar{f}^*_N(1,0) = \bar{f}^*_N(2,0) = \bar{f}^*_N(3,0) = 1$ is efficient. As a consequence ϕ need not be unique. We calculated that for the first optimal network $\phi = (1\frac{1}{2}, 3\frac{3}{4}, 2\frac{3}{4})$, while for the latter this allocation is given by $\phi = (1, 4, 3)$. Hence, a discrete congestion network situation gives in general rise to a set of allocations ϕ. However, each continuous congestion network situation with strictly convex cost functions is a strictly convex optimization problem and therefore has a single optimal network. As a result ϕ is unique.

Analogous to the discrete setup, for the continuous framework ϕ_D can be defined for each decomposition by (5), with ϕ as the average over all decompositions. However, by means of Algorithm 3.2 we can calculate ϕ directly as well. For Example 4.1 this allocation is given by $\phi = (4\frac{1}{2}, 3)$.

The content of the following theorem is that also the other main result of Section 3 holds in the continuous framework.

Theorem 4.1 Let $\mathcal{T} = (N, 0, (\gamma_a)_{a \in A_{N^0}})$ be a continuous convex congestion network situation. Then $(N, c^{\mathcal{T}})$ is a concave game.

Proof: Let us first establish a general result. Consider two games (N, c) and $(\bar{N}, \bar{c})$, with $\bar{N} = (N \backslash V) \cup \{*\}$, for some $V \subset N$, and c and $\bar{c}$ related by:

$$\bar{c}(S) = \begin{cases} c(S) & \text{if} \quad * \notin S, \\ c((S \backslash \{*\}) \cup V) & \text{if} \quad * \in S. \end{cases}$$

Note that $(N, \bar{c})$ is derived from (N, c) by assuming that coalition V acts as a single player. The concavity of the game (N, c) implies the concavity of the game $(\bar{N}, \bar{c})$ as for the latter property only a subset of the constraints of the first has to be satisfied.

Let $\mathcal{T} = (N, 0, (\gamma_a)_{a \in A_{N^0}})$ be a continuous convex congestion network situation, with corresponding marginal cost game $(N, c^{\mathcal{T}})$. Based on this situation we define a special type of discrete congestion situation: $T_z = (N_z, 0, (\gamma^z_a)_{a \in A_{N^0_z}})$. In this situation there are $|N|$ *regular* nodes, forming the set N, and each of these regular nodes has its own group of $z-1$ *friends*, where Z_i, with $i \in N$, is a set of friends including himself. Furthermore, for fixed z, cost function γ^z_a is defined such that $\gamma^z_{ij}(r) = \gamma_{ij}(\frac{r}{z})$, with $i, j \in N^0$,

$r \in \{0, \ldots, |N_z|\}$. For each friend j of a regular node i we define $\gamma^z_{ij}(r) = \gamma^z_{ji}(r) = 0$, and $\gamma^z_{jk}(r) = \gamma^z_{kj}(r) = r \cdot M$ for all $r \in \{1, \ldots, |N_z|\}$, and for all $k \in N^0_z \backslash \{i, j\}$, with M sufficiently large. (We could define M as $M = \max_{a \in A_{N^0}} \gamma_a(|N|)$.)

Due to this construction friend j of a regular node i uses a path towards the source that starts with arc (j, i). From node i, each friend may use a different path, but note that such a path only visits regular nodes. Let (N_z, c^{T_z}) be the marginal cost game corresponding to T_z. By Theorem 3.1, for fixed z the game (N_z, c^{T_z}) is concave.

Due to the general result above, the game (N, c^z) defined by $c^z(S) = c^{T_z}(\bigcup_{i \in S} Z_i)$, $S \subseteq N$, is also concave. Finally, for every $S \subseteq N$, $c^{\mathcal{T}}(S) = \lim_{z \to \infty} c^z(S)$, which implies that $(N, c^{\mathcal{T}})$ is concave as well. □

References

Borm, P., H. Hamers, and R. Hendrickx (2001). Operations research games: a survey. *TOP*, **9**, 139–216.

Cormen, T., C. Leiserson, and L. Rivest (1990). *Introduction to algorithms.* London: MIT Press and McGraw-Hill.

Matsubayashi, N., M. Umezawa, Y. Masuda, and H. Nishino (2005). A cost allocation problem arising in Hub-Spoke network systems. *European Journal of Operational Research*, **160**, 821–838.

Quant, M., P. Borm, and H. Reijnierse (2006). Congestion network problems and related games. *European Journal of Operational Research*, **172**, 919–930.

Quant, M. and J. Reijnierse (2004). Convex congestion network problems. CentER Discussion Paper 2004–59, Tilburg University, Tilburg, The Netherlands.