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The monoclus of a coalitional game*

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Abstract

The analysis of single-valued solution concepts for coalitional games with transferable utilities has a long tradition. Opposed to most of this literature we will not deal with solution concepts that provide payoffs to the players for the grand coalition only, but we will analyze allocation scheme rules, which assign payoffs to all players in all coalitions. We introduce four closely related allocation scheme rules for coalitional games. Each of these rules results in a population monotonic allocation scheme (PMAS) whenever the underlying coalitional game allows for a PMAS. The driving force behind these rules are monotonicities, which measure the payoff difference for a player between two nested coalitions. From a functional point of view these monotonicities can best be compared with the excesses in the definition of the (pre-)nucleolus. Two different domains and two different collections of monotonicities result in four allocation scheme rules. For each of the rules we deal with nonemptiness, uniqueness, and continuity, followed by an analysis of conditions for (some of) the rules to coincide. We then focus on characterizing the rules in terms of subbalanced weights. Finally, we deal with computational issues by providing a sequence of linear programs.

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1 Introduction

The analysis of single-valued solution concepts for coalitional games with transferable utilities has a long tradition. Following the seminal introduction of the Shapley value (cf. Shapley (1953)), a lot of attention has been paid to this value as well as to other solution concepts. Well-known rules include the (pre-)nucleolus (cf. Schmeidler (1969)), τ-value (cf. Tijs (1981)) and the egalitarian solution (cf. Dutta and Ray (1989)).

Following comments in Dutta and Ray (1989), Sprumont (1990) deals with not only specifying payoffs for all players in the grand coalition, but also for any other coalition. His concern is to guarantee that once a group of players has decided on a coalition no player in this coalition is ever tempted to form a smaller coalition. This naturally results in the concept of *population monotonic allocation schemes* (PMAS). Two key issues in Sprumont (1990) are the issue of when a game allows for a PMAS and the issue how to construct such allocation schemes. The first issue is handled by providing sufficient conditions, e.g., convexity, and by characterizing the games with a PMAS as the games that can be written as a positive linear combination of monotonic simple games with at least one veto player. The second issue is partly tackled by Sprumont (1990) by considering an extended Shapley value for the class of games with increasing average marginal contributions.

The first issue also comes to the fore in Norde and Reijnierse (2002). They introduce vectors of subbalanced weights and prove that a game has a PMAS if and only if it satisfies all inequalities corresponding to these vectors of subbalanced weights. This parallels the result of Bondareva (1963) and Shapley (1967) who identified the class of games with a nonempty core as the class of games that satisfy all inequalities corresponding to vectors of balanced weights. Furthermore, relevant for constructing a PMAS, Norde and Reijnierse (2002) prove that up to 4 persons every integer-valued game with a PMAS has an integer-valued PMAS. Additionally, they present a 7-person integer-valued game with a PMAS that does not have an integer-valued PMAS. Slikker et al. (2003) show that the class of games with a PMAS corresponds to the class of games that result from information sharing situations.

As far as we know, the literature on rules that result in a payoff scheme, i.e. a scheme that does not only specify a payoff vector for the grand coalition but also for each subcoalition, for any coalitional game is scarce. Usually, these rules, which we will call allocation scheme rules, are based on allocation rules for coalitional games, simply applied to the game itself and all its subgames. Here, we depart from this approach and explicitly focus on allocation scheme rules themselves. Specifically, we are interested in a rule that results in a PMAS whenever the game under consideration has a PMAS. We will come up with four closely related allocation scheme rules that satisfy the required property. These four rules coincide on the class of games with a PMAS. We present a sequence of linear programs leading to one of those rules. This settles the second issue raised by Sprumont (1990).

The basic idea for the allocation scheme rules is similar to the basic idea underlying the (pre-)nucleolus. For any efficient payoff scheme we determine the change in payoff for a player if we enlarge the coalition he belongs to. These so-called monotonicities are then ordered increasingly and we select the payoff scheme that lexicographically maximizes this ordered vector of monotonicities. Four variants result by considering two possible domains for the payoff schemes, based on the imputation set and the preimputation set, and two collections of monotonicities.

We prove nonemptiness, uniqueness, and continuity of our allocation scheme rules. Furthermore, we show for each of these four rules that a resulting payoff scheme can be characterized in terms of the carriers of minimal monotonicities allowing for subbalanced weights. These results are in line with results for the (pre-)nucleolus of Schmeidler (1969) and Kohlberg (1971). For an extensive survey of results on the (pre-)nucleolus we refer to Maschler (1992).

The setup of this paper is as follows. Section 2 contains preliminaries, followed by the definition of four allocation scheme rules in Section 3. Section 4 deals with nonemptiness, uniqueness, continuity, and comparison between the rules. Moreover, this section contains characterizations in terms of subbalanced weights. The computation of the rules is addressed in Section 5. We conclude in section 6

2 Preliminaries

A *coalitional game* with transferable utility (TU-game) is a pair (N, v) where $N = \{1, \ldots, n\}$ denotes the set of players and v is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. The function v is called the *characteristic function* of the coalitional game (N, v).

A *payoff vector* $x = (x_i)_{i \in N} \in \mathbb{R}^N$ specifies for each player $i \in N$ the benefit (e.g., (extra) profit) x_i that this player can expect if he cooperates with the other players.

A payoff vector is called *efficient* if the payoffs to the various players add up to exactly $v(N)$. The set consisting of all efficient payoff vectors is the preimputation set $PI(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N)\}$. Note that not all these payoff vectors will be acceptable to the players, as each player will require that he gets at least as much as what he can obtain when staying alone. A payoff vector $x \in \mathbb{R}^N$ with the property that $x_i \geq v(\{i\})$ for all $i \in N$ is called *individually rational*. The set of all individually rational and efficient payoff vectors is the *imputation set* $I(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for each } i \in N\}$.

The *core* $\text{Core}(N, v)$ of a game (N, v) consists of the payoff vectors $x \in \mathbb{R}^N$ that satisfy the condition $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$ and $\sum_{i \in N} x_i = v(N)$. For a coalition $S \subseteq N$, $v_{|S}$ denotes the restriction of the characteristic function v to the player set S, i.e., $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$. The pair $(S, v_{|S})$ is a coalitional game with player set S, called a *subgame* of (N, v). A game is called *balanced* if it has a nonempty core and *totally balanced* if all its subgames are balanced. We will use the notions 'balancedness' and 'nonemptiness of the core' interchangeably. The terminology 'balanced' is due to Bondareva (1963) and Shapley (1967). They independently identified the class of games that have nonempty cores as the class of balanced games. To describe this last class, we define for all $S \subseteq N$ the vector $e^S \in \mathbb{R}^N$ by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \backslash S$. A map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) e^S = e^N$. Now, a game (N, v) is called *balanced* if for every balanced map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N \backslash \{\emptyset\}} \kappa(S) v(S) \leq v(N)$. We will refer to this last condition as a balancedness condition.

Sprumont (1990) considered payoff schemes rather than payoff vectors. A payoff scheme prescribes a payoff for all players, not only for the grand coalition, but for any coalition they belong to. So, a payoff scheme for game (N, v) can be represented by $y = (y_{S,i})_{S\subseteq N, i\in S} \in \prod_{S\subseteq N} \mathbb{R}^S$. The focus of Sprumont (1990) was on population monotonic allocation schemes (PMAS): a vector $(y_{S,i})_{S\subseteq N, i\in S}$ is a *population monotonic allocation scheme* for the coalitional game (N, v) if it satisfies the following conditions:

(a) $\sum_{i\in S} y_{S,i} = v(S)$ for all $S \subseteq N$;

(b) $y_{S,i} \leq y_{T,i}$ for all $S, T \subseteq N$ with $S \subseteq T$ and all $i \in S$.

We remark that condition (b) is equivalent to the condition where one considers pairs S, T with $|T| = |S| + 1$ only. Throughout this paper we will denote the set $\{(i, S, T) \mid i \in S \subset T \subseteq N, |T| = |S| + 1\}$ by $\mathcal{S}$.

Norde and Reijnierse (2002) gave a description of the class of games with a PMAS similar to the description of the class of games with a nonempty core as balanced games. They introduced the concept of *vector of subbalanced weights*, VSW for short: a vector $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$[1] is a VSW if there exist nonnegative weights $\mu_{(i,S,T)}$ for all $(i, S, T) \in \mathcal{S}$ such that

$$\sum_{R\in 2^N\setminus\{\emptyset\}:(i,R,S)\in\mathcal{S}} \mu_{(i,R,S)} - \sum_{T\in 2^N\setminus\{\emptyset\}:(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} = \gamma_S$$

for all $S \in 2^N\setminus\{\emptyset\}$ and $i \in S$. The collection $\{(i, S, T) \in \mathcal{S} \mid \mu_{(i,S,T)} > 0\}$ will be called a *carrier* of the VSW $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$[2].

The main result of Norde and Reijnierse (2002) states that a game (N, v) has a PMAS if and only if it obeys for any VSW $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$ the associated VSW inequality

$$\sum_{S\in 2^N\setminus\{\emptyset\}} \gamma_S v(S) \geq 0.$$

A (single-valued) *allocation rule* is a function γ that assigns a payoff vector $\gamma(N, v) \in \mathbb{R}^N$ to every coalitional game (or possibly to every game in some specific subclass of games). Similarly, a (single-valued) *allocation scheme rule* is a function η that assigns a payoff scheme $\eta(N, v) \in \prod_{S\subseteq N} \mathbb{R}^S$ to every coalitional game (or possibly to every game in some specific subclass of games). The two most-cited allocation rules are the *(pre-)nucleolus* (cf. Schmeidler (1969)) and the *Shapley value* (cf. Shapley (1953)). The (pre-)nucleolus always selects a core-element, whenever the core is nonempty. In order to define the nucleolus we first need the concept of 'ordering function'. If K is a finite set then the *ordering function on* $\mathbb{R}^K$ is the function $\eta^K : \mathbb{R}^K \to \mathbb{R}^{|K|}$, defined by the following subsequent steps: $\eta_1^K(x) = \min\{x_j | j \in K\}$. Choosing $j_1 \in K$ such that $\eta_1^K(x) = x_{j_1}$ we have $\eta_2^K(x) = \min\{x_j | j \in K\setminus\{j_1\}\}$, etcetera.[3]

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[1] The original definition of Norde and Reijnierse (2002) used disjoint subsets Λ and Δ of $2^N\setminus\{\emptyset\}$ in order to describe a VSW, where $\Lambda = \{S \in 2^N\setminus\{\emptyset\} \mid \gamma_S < 0\}$ and $\Delta = \{S \in 2^N\setminus\{\emptyset\} \mid \gamma_S > 0\}$.

[2] For a VSW $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$ the corresponding set of (positive) weights $\mu_{(i,S,T)}$ need not be uniquely determined.

[3] In a more formal way the function η^K is defined by $\eta_i^K(x) = \min\{\max\{x_j | j \in L\} | L \subseteq K, |L| = i\}$ for every $i \in \{1, \ldots, |K|\}$.

Now the (pre-)nucleolus is defined as follows. Let (N, v) be a coalitional game. For any payoff vector $x \in \mathbb{R}^N$ define the *satisfaction*[4] of coalition $S \subseteq N$ as

$$s(S, x) = \sum_{i \in S} x_i - v(S).$$

Let $\theta(x) \in \mathbb{R}^{2^n}$ have the satisfactions of payoff vector x ordered increasingly, i.e. $\theta(x) = \eta^{2^N}((s(S, x))_{S \subseteq N})$. Then the nucleolus $\nu(N, v)$ is defined, in case the imputation set is not empty, as the set of vectors in the imputation set whose θ's are lexicographically maximal. The prenucleolus is defined similarly, but considering the preimputation set, rather than the imputation set. Both the nucleolus and the prenucleolus always consist of a single payoff vector, which is referred to as the nucleolus or prenucleolus as well. In case the core of a game is nonempty, the nucleolus and the prenucleolus coincide.

3 The (pre-)monoclus: definitions

Both the nucleolus as well as the prenucleolus of a coalitional game result in an element of the core of this game, whenever this core is nonempty. The basic idea underlying this work is to look for a rule that leads to payoff schemes that are population monotonic, whenever a PMAS exists.

The vehicle, used by the (pre-)nucleolus, that measures whether a payoff vector is in the core is the set of satisfactions. Whenever all satisfactions are nonnegative for some efficient payoff vector, one obviously deals with a core element. Having the vector of satisfactions ordered increasingly and subsequently lexicographically maximized results in the unique (pre-) nucleolus.

Requiring a payoff scheme to be a PMAS can be considered in a similar way. Rather than making sure that satisfactions are nonnegative, one can look at so-called monotonicities, describing the change in payoff a player experiences if his coalition is supplemented with some other players. Formally, for a payoff scheme $x = (x_{S,i})_{S \subseteq N, i \in S} \in \prod_{S \subseteq N} \mathbb{R}^S$ and triple (i, S, T), with $i \in S \subset T \subseteq N$, the *monotonicity* of x with respect to (i, S, T) is defined as

$$mon(x, (i, S, T)) = x_{T,i} - x_{S,i}.$$

Obviously, for an efficient payoff scheme, being population monotonic corresponds to all monotonicities being nonnegative.

The main line of research on the nucleolus focuses on two variants. First the nucleolus itself, and second the prenucleolus. The difference between the two is the domain of payoff vectors that is taken into account. This domain is the preimputation set for the prenucleolus and the imputation set for the nucleolus. A similar distinction will be made in this paper, but for payoff schemes rather than payoff vectors. We distinguish between the product set of the preimputation sets and the product set of the imputation sets.

[4]The original definition of Schmeidler (1969) used the *excess* of coalition S, $e(S, x) = v(S) - \sum_{i \in S} x_i$, instead of the satisfaction.

An additional distinction will be made regarding the monotonicity conditions that should be taken into account. A PMAS requires a nondecreasing payoff for all $i \in N$ and all $S, T \subseteq N$ with $i \in S \subseteq T$. As remarked before, one could restrict to such S, T with $|T| = |S| + 1$ as well. This has no impact on the definition of PMAS, but it slightly changes the rules we develop if we restrict attention to a restricted set of monotonicities. Though defining four variants we will later on show that they oftentimes coincide, for example on the class of games that have a PMAS.

Let

$$\mathcal{T} = \{(i, S, T) \mid i \in S \subset T \subseteq N\}.$$

Note that

$$\mathcal{T} \supset \mathcal{S}(= \{(i, S, T) \mid i \in S \subset T \subseteq N; |T| = |S| + 1\}).$$

An arbitrary payoff scheme thus results in two sets of monotonicities, those associated with $\mathcal{T}$ and those associated with $\mathcal{S}$. We first focus on the second set. For all $x \in \prod_{S \subseteq N} PI(S, v_{|S})$

> $\theta^A(x) \in \mathbb{R}^{|\mathcal{S}|}$ has all monotonicities of x, corresponding to elements in $\mathcal{S}$, as its coordinates in a (weakly) increasing order, i.e. $\theta^A(x) = \eta^{\mathcal{S}}((mon(x, (i, S, T)))_{(i,S,T) \in \mathcal{S}})$.

So, the first coordinate of $\theta^A(x)$ is the lowest monotonicity that is encountered in payoff scheme x. Consequently, x is a PMAS if and only if this first coordinate of $\theta^A(x)$ is nonnegative.

We can now define the following two monocli.

Definition 3.1 Premonoclus A:

$$\mathcal{M}^{*,A}(N, v) = \{x \in \prod_{S \subseteq N} PI(S, v_{|S}) \mid \theta^A(x) \geq_L \theta^A(y) \text{ for every } y \in \prod_{S \subseteq N} PI(S, v_{|S})\}.$$

Definition 3.2 Monoclus A: If $\prod_{S \subseteq N} I(S, v_{|S}) \neq \emptyset$ then

$$\mathcal{M}^{A}(N, v) = \{x \in \prod_{S \subseteq N} I(S, v_{|S}) \mid \theta^A(x) \geq_L \theta^A(y) \text{ for every } y \in \prod_{S \subseteq N} I(S, v_{|S})\}.$$

Note that the difference between the two payoff scheme rules is the domain that is considered: the product of the preimputation sets for premonoclus A and the product of the imputation sets for monoclus A. Hence, monoclus A is well-defined only if all subgames have a nonempty imputation set.

Similarly, taking all monotonicities in $\mathcal{T}$ into account, for all $x \in \prod_{S \subseteq N} PI(S, v_{|S})$

> $\theta^B(x) \in \mathbb{R}^{|\mathcal{T}|}$ has all monotonicities of x, corresponding to elements in $\mathcal{T}$, as its coordinates in a (weakly) increasing order, i.e. $\theta^B(x) = \eta^{\mathcal{T}}((mon(x, (i, S, T)))_{(i,S,T) \in \mathcal{S}})$.

The first coordinate is again the lowest monotonicity that is encountered in payoff scheme x. Similar to our remarks above, x is a PMAS if and only if this first coordinate of $\theta^B(x)$ is nonnegative.

Using this, we can define two additional monocli.

Definition 3.3 Premonoclus B:

$$\mathcal{M}^{*,B}(N,v) = \{x \in \prod_{S\subseteq N} PI(S, v_{|S}) \mid \theta^B(x) \geq_L \theta^B(y) \text{ for every } y \in \prod_{S\subseteq N} PI(S, v_{|S})\}.$$

Definition 3.4 Monoclus B: If $\prod_{S\subseteq N} I(S, v_{|S}) \neq \emptyset$ then

$$\mathcal{M}^{B}(N,v) = \{x \in \prod_{S\subseteq N} I(S, v_{|S}) \mid \theta^B(x) \geq_L \theta^B(y) \text{ for every } y \in \prod_{S\subseteq N} I(S, v_{|S})\}.$$

The distinction between the two is similar to the difference between premonoclus A and monoclus A. As for monoclus A, we remark that monoclus B is well-defined only if all subgames have a nonempty imputation set.

Maschler et al. (1992) generalize the nucleolus to arbitrary pairs consisting of a topological space and a finite set of real continuous functions with this topological space as their domain. The four monocli described are all special cases of this general monoclus. The general setting allows them to characterize this general nucleolus using properties that make explicit use of the richness of their general approach. More specific, their characterization of general monocli does not correspond to a characterization of the nucleolus in the classical setting. Neither does it imply a characterization of the monocli in this classical setting.

It is well-known that the prenucleolus coincides with the nucleolus in case the prenucleolus is an element of the imputation set. The following theorem provides a similar result for (pre)monocli. The proof is obvious and therefore omitted.

Theorem 3.1 Let (N, v) be a coalitional game. If $\mathcal{M}^{*,A}(N,v)$ $(\mathcal{M}^{*,B}(N,v)) \subseteq \prod_{S\subseteq N} I(S, v_{|S})$ then $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{A}(N,v)$ $(\mathcal{M}^{*,B}(N,v) = \mathcal{M}^{B}(N,v))$.

4 Nonemptiness, uniqueness, continuity and comparison

In this section we focus on nonemptiness, uniqueness, and continuity of the monocli defined in the previous section. Our main focus of attention will be on premonoclus A. We will afterwards argue that similar results hold for the other monocli as well. We will end this section with a comparison between the different monocli.[5]

4.1 Premonoclus A

In this subsection, we first show that premonoclus A is nonempty. Subsequently, we prove that it always contains exactly one element and proceed then by presenting a characterizing property. We conclude with a result that states that premonoclus A is continuous.

The following theorem states that every game has a nonempty premonoclus A.

[5]We remark that Theorems 4.1 and 4.2 can be proven using results of Maschler et al. (1992). We have chosen for direct proofs for reasons of self-containedness.

Theorem 4.1 Every coalitional game (N,v) has a nonempty $\mathcal{M}^{*,A}(N,v)$. Moreover, $\mathcal{M}^{*,A}(N,v)$ is a compact subset of $\prod_{S\subseteq N} PI(S,v_{|S})$.

Proof: Let (N,v) be a coalitional game. First we prove the following claim.

Claim θ^A is a continuous function on $\prod_{S\subseteq N} PI(S,v_{|S})$.

Note that

1. $x\mapsto mon(x,(i,S,T))$ is a continuous function on $\prod_{S\subseteq N} PI(S,v_{|S})$ for any $(i,S,T)\in\mathcal{S}$;
2. $\theta_k^A(x)=\min\Big\{\max\{mon(x,(i,S,T))\mid(i,S,T)\in\mathcal{U}\}\mid\mathcal{U}\subseteq\mathcal{S};|\mathcal{U}|=k\Big\}$.

Since the maximum or minimum of a finite number of continuous functions is continuous, we derive that θ_k^A is continuous. This proves the claim.
Consider $\overline{x}\in\prod_{S\subseteq N} PI(S,v_{|S})$ defined by $\overline{x}_{S,i}=\frac{v(S)}{|S|}$ for all $S\subseteq N$ and all $i\in S$. Let $m=\min\{mon(\overline{x},(i,S,T))|(i,S,T)\in\mathcal{S}\}$. Let

$$M=\{x\in\prod_{S\subseteq N} PI(S,v_{|S})\mid x_{S,i}\geq v(\{i\})+(|S|-1)m \text{ for all } S\subseteq N \text{ and } i\in S\}.$$

Note that M is nonempty ($\overline{x}\in M$) and compact. Moreover, for any $x\in\prod_{S\subseteq N} PI(S,v_{|S})\backslash M$ it holds that $\theta_1^A(x)<m$ and hence, $\theta^A(x)<_L\theta^A(\overline{x})$. Hence, $\mathcal{M}^{*,A}(N,v)\subseteq M$.

Define

$$\begin{aligned}Y_1=&\{x\in M\mid\theta_1^A(x)\geq\theta_1^A(y)\ \text{ for every } y\in M\};\\ Y_k=&\big\{x\in Y_{k-1}\mid\theta_k^A(x)\geq\theta_k^A(y)\ \text{ for every } y\in Y_{k-1}\big\}\text{ for all } k\in\{2,\ldots,|\mathcal{S}|\big\}.\end{aligned}$$

Claim For any $k\in\{1,\ldots,|\mathcal{S}|\}$ it holds that Y_k is compact and nonempty.

We already argued that M is nonempty and compact. This and the continuity of θ_1^A implies that Y_1 is compact and nonempty. Using continuity of $\theta_2^A,\ldots,\theta_{|\mathcal{S}|}^A$ inductively implies that all $Y_2,\ldots,Y_{|\mathcal{S}|}$ are compact and nonempty. This concludes the proof of the claim.
We conclude that $\mathcal{M}^{*,A}(N,v)=Y_{|\mathcal{S}|}$ is nonempty and compact. □

In order to show that premonoclus A always contains one element we first need a lemma about ordering functions.

Lemma 4.1 Let K be a finite set and let η^K be the ordering function on $\mathbb{R}^K$. Then $\eta^K(a+b)\geq_L\eta^K(a)+\eta^K(b)$ for every $a,b\in\mathbb{R}^K$. Moreover, if $\eta^K(a+b)=\eta^K(a)+\eta^K(b)$, then there exist $j_1,\ldots,j_{|K|}\in K$ such that $K=\{j_1,\ldots,j_{|K|}\}$, $a_{j_1}\leq a_{j_2}\leq\cdots\leq a_{j_{|K|}}$ and $b_{j_1}\leq b_{j_2}\leq\cdots\leq b_{j_{|K|}}$.

Proof: Let $a,b\in\mathbb{R}^K$. Choose $j_1\in K$ such that $\eta_1^K(a+b)=\min\{a_j+b_j|j\in K\}=a_{j_1}+b_{j_1}$. Then clearly $\eta_1^K(a+b)=a_{j_1}+b_{j_1}\geq\min\{a_j|j\in K\}+\min\{b_j|j\in K\}=\eta_1^K(a)+\eta_1^K(b)$. If

$\eta_1^K(a+b) > \eta_1^K(a)+\eta_1^K(b)$ then $\eta^K(a+b) >_L \eta^K(a)+\eta^K(b)$ and we are done. Otherwise, we conclude that $a_{j_1} = \min\{a_j | j \in K\}$, $b_{j_1} = \min\{b_j | j \in K\}$ and we choose $j_2 \in K$ such that $\eta_2^K(a+b) = \min\{a_j+b_j | j \in K\backslash\{j_1\}\} = a_{j_2}+b_{j_2}$. Again note that $\eta_2^K(a+b) = a_{j_2}+b_{j_2} \geq \min\{a_j | j \in K\backslash\{j_1\}\} + \min\{b_j | j \in K\backslash\{j_1\}\} = \eta_2^K(a)+\eta_2^K(b)$. If $\eta_2^K(a+b) > \eta_2^K(a)+\eta_2^K(b)$ then $\eta^K(a+b) >_L \eta^K(a)+\eta^K(b)$ and we are done. Otherwise, we conclude that $a_{j_2} = \min\{a_j | j \in K\backslash\{j_1\}\}$, $b_{j_2} = \min\{b_j | j \in K\backslash\{j_1\}\}$. Proceeding in this way we find that either $\eta^K(a+b) >_L \eta^K(a)+\eta^K(b)$ or $\eta^K(a+b) = \eta^K(a)+\eta^K(b)$ and there exists a sequence of indices $j_1,\ldots,j_{|K|} \in K$ with the required properties. $\square$

Theorem 4.2 For every coalitional game (N,v) it holds that $|\mathcal{M}^{*,A}(N,v)| = 1$.

Proof: Let (N,v) be a coalitional game. By Theorem 4.1 we know that $\mathcal{M}^{*,A}(N,v)$ is nonempty and compact. Let $x,y \in \mathcal{M}^{*,A}(N,v)$ and assume that $x \neq y$. Let $a = (mon(x,(i,S,T)))_{(i,S,T)\in\mathcal{S}}$ and $b = (mon(y,(i,S,T)))_{(i,S,T)\in\mathcal{S}}$. Note that $\eta^{\mathcal{S}}(a) = \theta^A(x) = \theta^A(y) = \eta^{\mathcal{S}}(b)$. Let $z = \frac{1}{2}(x+y) \in \prod_{S\subseteq N} PI(S,v_{|S})$. Note that $(mon(z,(i,S,T)))_{(i,S,T)\in\mathcal{S}} = \frac{1}{2}(a+b)$. We have

$$2\theta^A(z) = 2\eta^{\mathcal{S}}(\frac{1}{2}(a+b)) = \eta^{\mathcal{S}}(a+b) \geq_L \eta^{\mathcal{S}}(a)+\eta^{\mathcal{S}}(b) = 2\theta^A(x),$$

so $\theta^A(z) \geq_L \theta^A(x)$. Since $x \in \mathcal{M}^{*,A}(N,v)$ we must have $\theta^A(z) = \theta^A(x)$ and hence $\eta^{\mathcal{S}}(a+b) = \eta^{\mathcal{S}}(a)+\eta^{\mathcal{S}}(b)$. According to Lemma 4.1 and the fact that $\eta^{\mathcal{S}}(a) = \theta^A(x) = \theta^A(y) = \eta^{\mathcal{S}}(b)$ we derive that $a = b$, i.e. for every $(i,S,T) \in \mathcal{S}$ we have

$$x_{T,i} - x_{S,i} = mon(x,(i,S,T))) = mon(y,(i,S,T))) = y_{T,i} - y_{S,i}.$$

Since for all $i \in N$

$$x_{\{i\},i} = v(\{i\}) = y_{\{i\},i}$$

we conclude that $x = y$, which contradicts our assumption.

This completes the proof. $\square$

Let (N,v) be a coalitional game and let $x \in \prod_{S\subseteq N} PI(S,v_{|S})$ be an associated payoff scheme. Define

$$b_1^A(N,v,x) = \underset{\mathcal{S}}{\operatorname{argmin}}\ \ mon(x,\cdot)$$

and, inductively, for any $k \geq 1$ while $\mathcal{S}\backslash(\cup_{r=1}^k b_r^A(N,v,x)) \neq \emptyset$

$$b_{k+1}^A(N,v,x) = \underset{\mathcal{S}\backslash\cup_{r=1}^k b_r^A(N,v,x)}{\operatorname{argmin}}\ \ mon(x,\cdot).$$

Note that this results in an ordered partition of $\mathcal{S}$. The index of the last element of this partition will usually be denoted by p_x.

Definition 4.1 An ordered partition $(b_1, \ldots, b_p)$ of $\mathcal{S}$ has **Property 1A**[6] if for all $k \in \{1, \ldots, p\}$ and any scheme $q \in \prod_{S \subseteq N} \mathbb{R}^S$

(1) $mon(q, (i, S, T)) \geq 0$ for all $(i, S, T) \in b_1 \cup \ldots \cup b_k$;

(2) $\sum_{i \in T} q_{T,i} = 0$ for all $T \subseteq N$

imply

(3) $mon(q, (i, S, T)) = 0$ for all $(i, S, T) \in b_1 \cup \ldots \cup b_k$.

The following theorem shows that a payoff scheme is premonoclus A if and only the associated ordered partition satisfies property 1A.

Theorem 4.3 Let (N, v) be a coalitional game and let $x \in \prod_{S \subseteq N} PI(S, v_{|S})$ be a payoff scheme. The following two statements are equivalent:

(i) $x = \mathcal{M}^{*,A}(N, v)$;[7]

(ii) $(b_1^A(N, v, x), \ldots, b_{p_x}^A(N, v, x))$ has property 1A.

Proof: Denote for all $k \in \{1, \ldots, p_x\}$ the number of elements of $b_k^A(N, v, x)$ by m_k.

(i)⇒(ii) Suppose (i) is satisfied and (ii) is not satisfied. Let k be minimal such that there exists a payoff scheme q that satisfies (1) and (2) of property 1A, but not (3). Consider $x + tq$ with $t > 0$. Note that $x + tq \in \prod_{S \subseteq N} PI(S, v_{|S})$. If t is small enough then the first m_1 elements of $\theta^A(x + tq)$ are the monotonicities associated with the elements of $b_1^A(N, v, x)$, followed by the monotonicities associated with the elements of $b_2^A(N, v, x)$, etc.

Using (1) of property 1A we have that for all $(i, S, T) \in \cup_{r=1}^k b_r(N, v, x)$ it holds that

$$mon(x + tq, (i, S, T)) = mon(x, (i, S, T)) + t\, mon(q, (i, S, T)) \geq mon(x, (i, S, T)).$$

Since (3) of property 1A does not hold there exists $(i, S, T) \in \cup_{r=1}^k b_r(N, v, x)$ with $mon(q, (i, S, T)) > 0$. Hence $\theta^A(x + tq) >_L \theta^A(x)$, which contradicts (i).

(ii)⇒(i) Suppose (ii) is satisfied. Suppose there exists a payoff scheme $y \in \prod_{S \subseteq N} PI(S, v_{|S})$ with $\theta^A(y) \geq_L \theta^A(x)$. Hence, $mon(y, (i, S, T)) \geq mon(x, (i, S, T))$ for all $(i, S, T) \in b_1^A(N, v, x)$. Consequently, $mon(y - x, (i, S, T)) \geq 0$ for all $(i, S, T) \in b_1^A(N, v, x)$. Moreover, for every $T \subseteq N$ we have $\sum_{i \in T}(y - x)_{T,i} = \sum_{i \in T} y_{T,i} - \sum_{i \in T} x_{T,i} = v(T) - v(T) = 0$. So, scheme $q = y - x$ satisfies (1) and (2) of property 1A for $k = 1$. Using this property 1A, we derive that $mon(y - x, (i, S, T)) = 0$ for all $(i, S, T) \in b_1^A(N, v, x)$. Applying a similar procedure inductively on $b_2^A(N, v, x), \ldots, b_p^A(N, v, x)$ eventually leads to $mon(y - x, (i, S, T)) = 0$ for all $(i, S, T) \in \cup_{k=1}^{p_x} b_k^A(N, v, x) = \mathcal{S}$. Using $x_{\{i\},i} = v(\{i\}) = y_{\{i\},i}$ for all $i \in N$, we conclude that $y = x$. □

The following result is a consequence of Theorem 4.3.

[6]Following the terminology in Kohlberg (1971).

[7]From now on we will, with a slight abuse of notation, denote the unique element of $\mathcal{M}^{*,A}(N, v)$ by $\mathcal{M}^{*,A}(N, v)$ as well.

Theorem 4.4 Let (N,v) be a coalitional game and let $x \in \prod_{S\subseteq N} PI(S, v_{|S})$ be a payoff scheme. If $\cup_{r=1}^{k} b_r(x,N,v)$ is the carrier of a VSW for every $k \in \{1,\dots,p_x\}$, then $x = \mathcal{M}^{*,A}(N,v)$.

Proof: It is sufficient to show that $(b_1^A(N,v,x),\dots,b_{p_x}^A(N,v,x))$ has property 1A. Let $k \in \{1,\dots,p_x\}$ and $q \in \prod_{S\subseteq N} \mathbb{R}^S$ be such that (1) and (2) of property 1A hold. Let $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$ be a VSW with carrier $\cup_{r=1}^{k} b_r(x,N,v)$ and let $\{\mu_{(i,S,T)}\}_{(i,S,T)\in\mathcal{S}}$ be the corresponding weights. Then

$$
\begin{aligned}
0 &= \sum_{S\in 2^N\setminus\{\emptyset\}} \gamma_S \sum_{i\in S} q_{S,i} \\
&= \sum_{S\in 2^N\setminus\{\emptyset\}} \sum_{i\in S} \gamma_S q_{S,i} \\
&= \sum_{S\in 2^N\setminus\{\emptyset\}} \sum_{i\in S} \Big(\sum_{R\in 2^N\setminus\{\emptyset\}:(i,R,S)\in\mathcal{S}} \mu_{(i,R,S)} - \sum_{T\in 2^N\setminus\{\emptyset\}:(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} \Big) q_{S,i} \\
&= \sum_{S\in 2^N\setminus\{\emptyset\}} \sum_{i\in S} \sum_{R\in 2^N\setminus\{\emptyset\}:(i,R,S)\in\mathcal{S}} \mu_{(i,R,S)} q_{S,i} - \sum_{S\in 2^N\setminus\{\emptyset\}} \sum_{i\in S} \sum_{T\in 2^N\setminus\{\emptyset\}:(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} q_{S,i} \\
&= \sum_{(i,R,S)\in\mathcal{S}} \mu_{(i,R,S)} q_{S,i} - \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} q_{S,i} \\
&= \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} q_{T,i} - \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} q_{S,i} \\
&= \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} (q_{T,i} - q_{S,i}) \\
&= \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} mon(q,(i,S,T)) \\
&= \sum_{(i,S,T)\in\cup_{r=1}^{k} b_r(x,N,v)} \mu_{(i,S,T)} mon(q,(i,S,T)).
\end{aligned}
$$

As $\mu_{(i,S,T)} > 0$ for all $(i,S,T) \in \cup_{r=1}^{k} b_r(x,N,v)$ we conclude that $mon(q,(i,S,T)) = 0$ for every $(i,S,T) \in \cup_{r=1}^{k} b_r(x,N,v)$. □

The following example illustrates the use of Theorem 4.3.

Example 4.1 Consider the game (N,v) with $N = \{1,2,3\}$ and characteristic function v described by

$$
v(S) = \begin{cases} 0 & \text{if } |S| \leq 1; \\ 1 & \text{if } S \in \{\{1,2\},\{1,2,3\}\}; \\ 2 & \text{if } S = \{1,3\}; \\ 6 & \text{if } S = \{2,3\}. \end{cases}
$$

The values of $x = \mathcal{M}^{*,A}(N,v)$ are given in Table 1. Note that $b_1^A(N,v,x) = \{(1,\{1\},\{1,2\}),(1,\{1\},\{1,3\}),(1,\{1,2\},N),(1,\{1,3\},N),(2,\{2,3\},N),(3,\{2,3\},N)\}$ (with associated monotonicity $-\frac{5}{4}$), $b_2^A(N,v,x) = \{(2,\{1,2\},N),(3,\{1,3\},N)\}$, $b_3^A(N,v,x) = \{(2,\{2\},\{1,2\})\}$, $b_4^A(N,v,x) = \{(2,\{2\},\{2,3\})\}$, $b_5^A(N,v,x) = \{(3,\{3\},\{1,3\})\}$ and

	1	2	3
{1}	0	*	*
{2}	*	0	*
{3}	*	*	0
{1,2}	$-\frac{5}{4}$	$\frac{9}{4}$	*
{1,3}	$-\frac{5}{4}$	*	$\frac{13}{4}$
{2,3}	*	$\frac{10}{4}$	$\frac{14}{4}$
N	$-\frac{10}{4}$	$\frac{5}{4}$	$\frac{9}{4}$

Table 1: $\mathcal{M}^{*,A}(N,v)$ in Example 4.1.

$b_6^A(N,v,x) = \{(3,\{3\},\{2,3\})\}$. In order to check that $(b_1^A(N,v,x),\ldots,b_6^A(N,v,x))$ satisfies property 1A we have to show that (1) and (2) imply (3) for every $k \in \{1,\ldots,6\}$ and $q \in \prod_{S\subseteq N}\mathbb{R}^S$. We will do this for $k=1$. Suppose $q \in \prod_{S\subseteq N}\mathbb{R}^S$ is such that $mon(q,(i,S,T)) \geq 0$ for all $(i,S,T) \in b_1^A(N,v,x)$ and $\sum_{i\in T} q_{T,i} = 0$ for all $T \subseteq N$. Then

$$\begin{aligned} 0 &= \sum_{i\in N} q_{N,i} - \sum_{i\in\{1\}} q_{\{1\},i} - \sum_{i\in\{2,3\}} q_{\{2,3\},i} \\ &= (q_{N,1} - q_{\{1,2\},1}) + (q_{\{1,2\},1} - q_{\{1\},1}) + (q_{N,2} - q_{\{2,3\},2}) + (q_{N,3} - q_{\{2,3\},3}) \\ &= (q_{N,1} - q_{\{1,3\},1}) + (q_{\{1,3\},1} - q_{\{1\},1}) + (q_{N,2} - q_{\{2,3\},2}) + (q_{N,3} - q_{\{2,3\},3}). \end{aligned}$$

Hence $mon(q,(i,S,T)) = 0$ for all $(i,S,T) \in b_1^A(N,v,x)$.
An alternative way to show that $x = \mathcal{M}^{*,A}(N,v)$ is by using Theorem 4.4 and showing that $\cup_{r=1}^k b_r(x,N,v)$ is the carrier of a VSW for every $k \in \{1,\ldots,p_x\}$. Again we will only show this for $k=1$: note that $b_1^A(N,v,x)$ is a carrier of VSW $(\gamma_S)_{S\in 2^N\setminus\{\emptyset\}}$, where $\gamma_N = 1$, $\gamma_{\{1\}} = \gamma_{\{2,3\}} = -1$ and $\gamma_S = 0$ otherwise. The corresponding weights can be chosen as $\mu_{(1,\{1\},\{1,2\})} = \mu_{(1,\{1\},\{1,3\})} = \mu_{(1,\{1,2\},N)} = \mu_{(1,\{1,3\},N)} = \frac{1}{2}$, $\mu_{(2,\{2,3\},N)} = \mu_{(3,\{2,3\},N)} = 1$ and $\mu_{(i,S,T)} = 0$ otherwise. The associated VSW inequality is $v(N) - v(\{1\}) - v(\{2,3\}) \geq 0$. $\diamond$

Theorem 4.5 $\mathcal{M}^{*,A} : TU^N \to \mathbb{R}^N$ is a continuous function.

Proof: Consider a sequence of games $(N,v^l)_{l=1}^\infty$ such that $v^l \to v$ if $l \to \infty$ for some game (N,v), i.e. $\lim_{l\to\infty} v^l(S) = v(S)$ for every $S \subseteq N$. Let $x^l = \mathcal{M}^{*,A}(N,v^l)$ for all $l \in \mathbb{N}$. We will show that $x^l \to \mathcal{M}^{*,A}(N,v)$ if $l \to \infty$.

Let $L > 0$ be such that $|v^l(S)| \leq L$ for every $l \in \mathbb{N}$ and $S \subseteq N$. Define the compact set $M \subseteq \prod_{S\subseteq N}\mathbb{R}^S$ by $M = \{x \in \prod_{S\subseteq N}\mathbb{R}^S | -(2n-1)L \leq x_{S,i} \leq (2n^2-3n+2)L$ for every $S \subseteq N$ and $i \in S\}$. We will show that $x^l \in M$ for every $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$. Define $y^l \in \prod_{S\subseteq N} PI(S, v^l_{|S})$ by $y^l_{S,i} = \frac{v^l(S)}{|S|}$ for every $S \subseteq N$ and $i \in S$. For every $(i,S,T) \in \mathcal{S}$ we have $mon(y^l,(i,S,T)) = y^l_{T,i} - y^l_{S,i} = \frac{v^l(T)}{|T|} - \frac{v^l(S)}{|S|} \geq -\frac{L}{|T|} - \frac{L}{|S|} \geq -2L$. Since $\theta^A(x^l) \geq_L \theta^A(y^l)$ we have $mon(x^l,(i,S,T)) \geq -2L$ for every $(i,S,T) \in \mathcal{S}$. So, for every $S \subseteq N$ and $i \in S$ we have $x^l_{S,i} = x^l_{\{i\},i} + (x^l_{S,i} - x^l_{\{i\},i}) \geq v^l(\{i\}) + (|S|-1)\cdot(-2L) \geq -L + (n-1)\cdot(-2L) = -(2n-1)L$. On the other hand, note that for every $S \subseteq N$ and $i \in S$ we have

$x^l_{S,i} = v^l(S) - \sum_{j\in S\setminus\{i\}} x^l_{S,j} \leq L + (|S|-1)(2n-1))L \leq L + (n-1)(2n-1))L = (2n^2-3n+2)L$. This finishes the proof that $x^l \in M$.

Hence, $(x^l)_{l=1}^\infty$ has a convergent subsequence. Consider an arbitrary convergent subsequence $(x^{l_k})_{k=1}^\infty$ of $(x^l)_{l=1}^\infty$, and denote its limit point by y. Since the number of partitions of $\mathcal{S}$ is finite there is at least one ordered partition that appears infinitely many times in the sequence $(b_1^A(N, v^{l_k}, x^{l_k}), \ldots, b^A_{p_{x^{l_k}}}(N, v^{l_k}, x^{l_k}))_{k=1}^\infty$. Select such an ordered partition and denote the corresponding subsequence by $(x^{m_k})_{k=1}^\infty$. Note that this sequence converges to y and that by Theorem 4.3 the associated ordered partition satisfies property 1A. Since all weak inequalities are preserved when taking the limit it follows immediately that the ordered partition $(b_1^A(N, v, y), \ldots, b^A_{p_y}(N, v, y))$ is a coarsening of the ordered partition $(b_1^A(N, v^{m_k}, x^{m_k}), \ldots, b^A_{p_{x^{m_k}}}(N, v^{m_k}, x^{m_k}))$, which does not depend on k. Clearly, a coarsening of an ordered partition, satisfying property 1A, satisfies property 1A as well. Theorem 4.3 implies that $y = \mathcal{M}^{*,A}(N, v)$. Using that the sequence $(x^{l_k})_{k=1}^\infty$ was chosen arbitrarily and the partition as well implies that $\mathcal{M}^{*,A}(N, v) = y = \lim_{l\to\infty} x^l = \lim_{l\to\infty} \mathcal{M}^{*,A}(N, v^l)$. □

4.2 Other monocli

This section deals with results for the other three monocli that are similar to the results for premonoclus A in the previous subsection.

Under the extra assumption that $\prod_{S\subseteq N} I(S, v_{|S}) \neq \emptyset$ Theorems 4.1 and 4.2 can be modified in a straightforward way to monoclus A. In order to modify Theorem 4.3 we need to adjust Definition 4.1.

Definition 4.2 Let $b_0 \subseteq \{(S, i)|i \in S \subseteq N\}$ and let $(b_1, \ldots, b_p)$ be an ordered partition of $\mathcal{S}$. The tuple $(b_0, b_1, \ldots, b_p)$ has **Property 2A** if for all $k \in \{1, \ldots, p\}$ and any scheme $q \in \prod_{S\subseteq N} \mathbb{R}^S$

(1) $q_{S,i} \geq 0$ for all $(S, i) \in b_0$;

(2) $mon(q, (i, S, T)) \geq 0$ for all $(i, S, T) \in b_1 \cup \ldots \cup b_k$;

(3) $\sum_{i\in T} q_{T,i} = 0$ for all $T \subseteq N$

imply

(4) $mon(q, (i, S, T)) = 0$ for all $(i, S, T) \in b_1 \cup \ldots \cup b_k$.

For a coalitional game (N, v) and an $x \in \prod_{S\subseteq N} I(S, v_{|S})$ we define

$$b_0^A(N, v, x) = \{(S, i)|i \in S \subseteq N, x_{S,i} = v(\{i\})\}.$$

The following theorem generalizes Theorem 4.3 to monoclus A. The proof is similar and therefore omitted.

Theorem 4.6 Let (N, v) be a coalitional game and let $x \in \prod_{S \subseteq N} I(S, v_{|S})$ be a payoff scheme. The following two statements are equivalent:

(i) $x = \mathcal{M}^A(N, v)$;

(ii) $(b_0^A(N, v, x), b_1^A(N, v, x), \ldots, b_{p_x}^A(N, v, x))$ has property 2A.

Continuity of $\mathcal{M}^A$ can be shown in a similar fashion as in Theorem 4.5.

Analogous results can be established for $\mathcal{M}^{*,B}$ and $\mathcal{M}^B$ by considering set $\mathcal{T}$ instead of set $\mathcal{S}$. Definitions 4.1 and 4.2 are generalized to Property 1B and 2B respectively and deal with ordered partitions of $\mathcal{T}$. The proofs of all relevant theorems can be adjusted in an obvious way.

The following example illustrates how to use Theorem 4.6.

Example 4.2 Consider again the game (N, v) in Example 4.1. The values of $y = \mathcal{M}^A(N, v)$ are given in Table 2. Note that $b_0^A(N, v, y) = \{(\{1\}, 1), (\{2\}, 2), (\{3\}, 3), (N, 1), (N, 2)\}$,

	1	2	3
{1}	0	*	*
{2}	*	0	*
{3}	*	*	0
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	*
{1,3}	$\frac{1}{2}$	*	$\frac{3}{2}$
{2,3}	*	$\frac{5}{2}$	$\frac{7}{2}$
N	0	0	1

Table 2: $\mathcal{M}^A(N, v)$ in Example 4.2.

$b_1^A(N, v, y) = \{(2, \{2, 3\}, N), (3, \{2, 3\}, N)\}$, $b_2^A(N, v, y) = \{(1, \{1, 2\}, N), (1, \{1, 3\}, N)\}$, $(2, \{1, 2\}, N), (3, \{1, 3\}, N)\}$, $b_3^A(N, v, y) = \{(1, \{1\}, \{1, 2\}), (1, \{1\}, \{1, 3\}), (2, \{2\}, \{1, 2\})\}$, $b_4^A(N, v, y) = \{(3, \{3\}, \{1, 3\})\}$, $b_5^A(N, v, y) = \{(2, \{2\}, \{2, 3\})\}$ and $b_6^A(N, v, y) = \{(3, \{3\}, \{2, 3\})\}$. In order to check that $(b_0^A(N, v, y), \ldots, b_6^A(N, v, y))$ satisfies property 2A we have to show that (1), (2) and (3) imply (4) for every $k \in \{1, \ldots, 6\}$ and $q \in \prod_{S \subseteq N} \mathbb{R}^S$. Again, we will only do this for $k = 1$. Suppose $q \in \prod_{S \subseteq N} \mathbb{R}^S$ is such that $q_{S,i} \geq 0$ for every $(S, i) \in b_0^A(N, v, y)$, $mon(q, (i, S, T)) \geq 0$ for all $(i, S, T) \in b_1^A(N, v, y)$ and $\sum_{i \in T} q_{T,i} = 0$ for all $T \subseteq N$. Then

$$\begin{aligned} 0 &= \sum_{i \in N} q_{N,i} - \sum_{i \in \{2,3\}} q_{\{2,3\},i} \\ &= q_{N,1} + (q_{N,2} - q_{\{2,3\},2}) + (q_{N,3} - q_{\{2,3\},3}) \\ &\geq (q_{N,2} - q_{\{2,3\},2}) + (q_{N,3} - q_{\{2,3\},3}). \end{aligned}$$

Hence $mon(q, (i, S, T)) = 0$ for all $(i, S, T) \in b_1^A(N, v, y)$.

In a similar fashion we find that $\mathcal{M}^B(N, v) = \mathcal{M}^A(N, v)$ and that $\mathcal{M}^{*,B}$ is given by the values in Table 3. $\diamond$

	1	2	3
{1}	0	*	*
{2}	*	0	*
{3}	*	*	0
{1,2}	$-\frac{9}{12}$	$\frac{21}{12}$	*
{1,3}	$-\frac{9}{12}$	*	$\frac{33}{12}$
{2,3}	*	$\frac{30}{12}$	$\frac{42}{12}$
N	$-\frac{20}{12}$	$\frac{10}{12}$	$\frac{22}{12}$

Table 3: $\mathcal{M}^{*,B}(N,v)$ in Example 4.2.

The observation that $\mathcal{M}^A(N,v) = \mathcal{M}^B(N,v)$ in the example above is not a coincidence. One can show that these two monocli coincide for any 3-person game. The following example illustrates that for 4-player games all monocli can be distinct.

Example 4.3 Consider the game (N,v) with $N = \{1,2,3,4\}$ and characteristic function v described by

$$v(S) = \begin{cases} 240 & \text{if } S = \{1,2,3,4\}; \\ 480 & \text{if } S \in \{\{1,2,3\},\{1,2,4\}\}; \\ 960 & \text{if } S \in \{\{1,2\},\{1,3,4\}\}; \\ 1200 & \text{if } S = \{2,3,4\}; \\ 0 & \text{for other } S. \end{cases}$$

The four monocli are given in Table 4. $\diamond$

	$\mathcal{M}^{*,A}(N,v)$				$\mathcal{M}^{A}(N,v)$				$\mathcal{M}^{*,B}(N,v)$				$\mathcal{M}^{B}(N,v)$			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
{1}	0	*	*	*	0	*	*	*	0	*	*	*	0	*	*	*
{2}	*	0	*	*	*	0	*	*	*	0	*	*	*	0	*	*
{3}	*	*	0	*	*	*	0	*	*	*	0	*	*	*	0	*
{4}	*	*	*	0	*	*	*	0	*	*	*	0	*	*	*	0
{1,2}	360	600	*	*	480	480	*	*	360	600	*	*	360	600	*	*
{1,3}	0	*	0	*	0	*	0	*	75	*	-75	*	0	*	0	*
{1,4}	0	*	*	0	0	*	*	0	75	*	*	-75	0	*	*	0
{2,3}	*	0	0	*	*	0	0	*	*	75	-75	*	*	0	0	*
{2,4}	*	0	*	0	*	0	*	0	*	75	*	-75	*	0	*	0
{3,4}	*	*	0	0	*	*	0	0	*	*	0	0	*	*	0	0
{1,2,3}	96	336	48	*	240	240	0	*	195	435	-150	*	120	360	0	*
{1,2,4}	96	336	*	48	240	240	*	0	195	435	*	-150	120	360	*	0
{1,3,4}	96	*	432	432	240	*	360	360	360	*	300	300	320	*	320	320
{2,3,4}	*	336	432	432	*	320	440	440	*	600	300	300	*	560	320	320
{1,2,3,4}	-168	72	168	168	0	0	120	120	30	270	-30	-30	0	240	0	0

Table 4: $\mathcal{M}^{*,A}(N,v), \mathcal{M}^{A}(N,v), \mathcal{M}^{*,B}(N,v)$ and $\mathcal{M}^{B}(N,v)$ in Example 4.3.

4.3 Comparison of monocli

If a game admits a PMAS all monocli coincide.

Theorem 4.7 Let (N,v) be a coalitional game with a PMAS. Then $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{A}(N,v) = \mathcal{M}^{B}(N,v) = \mathcal{M}^{*,B}(N,v)$.

Proof: Let $x = (x_{S,i})_{S\subseteq N, i\in S}$ be a PMAS of (N,v). Since $x_{S,i} \geq x_{\{i\},i} = v(\{i\})$ for every $S \subseteq N$ and $i \in S$ we have $x \in \prod_{S\subseteq N} I(S, v_{|S})$. Since both $\theta^A(x)$ and $\theta^B(x)$ are nonnegative we infer that $\mathcal{M}^{*,A}(N,v)$ and $\mathcal{M}^{*,B}(N,v)$ are PMAS too and hence both elements of $\prod_{S\subseteq N} I(S, v_{|S})$. Using Theorem 3.1 we have $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{A}(N,v)$ and $\mathcal{M}^{*,B}(N,v) = \mathcal{M}^{B}(N,v)$. We still have to show that $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{*,B}(N,v)$.

Let $y = \mathcal{M}^{*,B}(N,v)$. Then $(b_1^B(N,v,y), \ldots, b^B_{p_y^B}(N,v,y))$ satisfies property 1B, according to (the premonoclus B variant of) Theorem 4.3. It is sufficient to show that $(b_1^A(N,v,y), \ldots, b^A_{p_y^A}(N,v,y))$ satisfies property 1A, since then, according to Theorem 4.3, $y = \mathcal{M}^{*,A}(N,v)$.

Let $k \in \{1, \ldots, p_y^A\}$ and $q \in \prod_{S\subseteq N} \mathbb{R}^S$ be such that $mon(q,(i,S,T)) \geq 0$ for every $(i,S,T) \in \cup_{l=1}^k b_l^A(N,v,y)$ and $\sum_{i\in T} q_{T,i} = 0$ for every $T \subseteq N$. Let $u \in \mathbb{R}$ be such that $u = mon(y,(i,S,T))$ for every $(i,S,T) \in b_k^A(N,v,y)$. Let $k^* \in \{1, \ldots, p_y^B\}$ be such that $mon(y,(i,S,T)) \leq u$ for every $(i,S,T) \in b_l^B(N,v,y)$ with $l \leq k^*$ and $mon(y,(i,S,T)) > u$ for every $(i,S,T) \in b_l^B(N,v,y)$ with $l > k^*$. Clearly, since $\mathcal{S} \subseteq \mathcal{T}$, we have $\cup_{l=1}^k b_l^A(N,v,y) \subseteq \cup_{l=1}^{k^*} b_l^B(N,v,y)$.

Let $(i,S,T) \in \cup_{l=1}^{k^*} b_l^B(N,v,y)$. Then $y_{T,i} - y_{S,i} \leq u$. Since y is a PMAS we have $0 \leq y_{W,i} - y_{V,i} \leq u$ for all $V, W \subseteq N$ with $S \subseteq V \subseteq W \subseteq T$ and $|W| = |V|+1$. So, for all such V and W we get $(i,V,W) \in \cup_{l=1}^k b_l^A(N,v,y)$ and hence $q_{W,i} - q_{V,i} = mon(q,(i,V,W)) \geq 0$. Therefore $mon(q,(i,S,T)) = q_{T,i} - q_{S,i} \geq 0$. Since $(b_1^B(N,v,y), \ldots, b^B_{p_y^B}(N,v,y))$ satisfies property 1B we derive that $mon(q,(i,S,T)) = 0$ for all $(i,S,T) \in \cup_{l=1}^{k^*} b_l^B(N,v,y)$. Hence $mon(q,(i,S,T)) = 0$ for all $(i,S,T) \in \cup_{l=1}^k b_l^A(N,v,y)$, so $(b_1^A(N,v,y), \ldots, b^A_{p_y^A}(N,v,y))$ satisfies property 1A. $\square$

The following example shows that the converse of Theorem 4.7 is not true. This example is presented already in Sprumont (1990) as an example of a totally balanced game that lacks a PMAS. In fact it is the four player 'glove game' where two players possess a left-hand glove and the other two players a right-hand glove.

Example 4.4 Consider the game (N,v) with $N = \{1,2,3,4\}$ and characteristic function v described by

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \text{ or } S \in \{\{1,2\},\{3,4\}\}; \\ 1 & \text{if } |S| = 3 \text{ or } S \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}; \\ 2 & \text{if } |S| = \{1,2,3,4\}. \end{cases}$$

The values of $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{A}(N,v) = \mathcal{M}^{*,B}(N,v) = \mathcal{M}^{B}(N,v)$ are given in Table 5. $\diamond$

5 A sequence of minimization problems

In this section we follow Kohlberg (1971) and show that premonoclus A can be found by solving a sequence of LP problems.

	1	2	3	4
{1}	0	*	*	*
{2}	*	0	*	*
{3}	*	*	0	*
{4}	*	*	*	0
{1,2}	0	0	*	*
{1,3}	$\frac{1}{2}$	*	$\frac{1}{2}$	*
{1,4}	$\frac{1}{2}$	*	*	$\frac{1}{2}$
{2,3}	*	$\frac{1}{2}$	$\frac{1}{2}$	*
{2,4}	*	*	$\frac{1}{2}$	$\frac{1}{2}$
{3,4}	*	*	0	0
{1,2,3}	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	*
{1,2,4}	$\frac{1}{3}$	$\frac{1}{3}$	*	$\frac{1}{3}$
{1,3,4}	$\frac{1}{3}$	*	$\frac{1}{3}$	$\frac{1}{3}$
{2,3,4}	*	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
{1,2,3,4}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 5: $\mathcal{M}^{*,A}(N,v) = \mathcal{M}^{A}(N,v) = \mathcal{M}^{*,B}(N,v) = \mathcal{M}^{B}(N,v)$ in Example 4.4.

Let (N,v) be a coalitional game. Consider the following LP problem.

$$
\begin{array}{lll}
(P_1) & \min \sum_{S\subseteq N} \gamma(S)v(S) & \\
& \gamma(S) = \sum_{R:(i,R,S)\in\mathcal{S}} \mu_{(i,R,S)} - \sum_{T:(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} & \forall S \subseteq N\ \forall i \in S; \\
s.t. & \sum_{(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} = 1; & \\
& \mu_{(i,S,T)} \geq 0 & \forall (i,S,T) \in \mathcal{S}.
\end{array}
$$

We remark that the decision variables $\gamma(S)$, $S \subseteq N$, are superfluous from a modeling point of view, since they are fully dependent on the decision variables $\mu_{(i,S,T)}$, $(i,S,T) \in \mathcal{S}$. By construction, we have that $(\gamma(S))_{S\in 2^N\setminus\{\emptyset\}}$ is a VSW.

Note that the objective function is linear and that the feasible area, i.e. the collection of feasible $\mu_{(i,S,T)}$, $(i,S,T) \in \mathcal{S}$, is nonempty and compact. Hence, (P_1) is well-defined.[8] Consider an arbitrary optimal solution $((\overline{\gamma}(S))_{S\subseteq N}, (\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ of (P_1). Set $c_1 = \{(i,S,T) \in \mathcal{S} \mid \overline{\mu}_{(i,S,T)} > 0\}$ and set $a_{(i,S,T)}$ equal to the optimal value of (P_1) for all $(i,S,T) \in c_1$. Set a_1 equal to this value as well.[9]

Let $k \geq 2$ and suppose that LP problems (P_1) up to (P_{k-1}) have been well-defined and have resulted in sets $c_1, \ldots, c_{k-1}$, associated optimal values $a_1, \ldots, a_{k-1}$, and numbers $a_{(i,S,T)}$, $(i,S,T) \in \cup_{l=1}^{k-1} c_l$. Furthermore, suppose $\mathcal{S} \setminus \cup_{l=1}^{k-1} c_l \neq \emptyset$. Now define LP problem (P_k) by

[8] With well-defined we mean that (the feasible area is non-empty and that) the minimum exists.

[9] We remark that (N,v) has a PMAS if and only if $a_1 \geq 0$.

$$
\begin{array}{llll}
(P_k) & \min \sum_{S\subseteq N} \gamma(S)v(S) - \sum_{(i,S,T)\in \cup_{l=1}^{k-1} c_l} a_{(i,S,T)}\mu_{(i,S,T)} & & \\
& \gamma(S) = \sum_{U:(i,U,S)\in\mathcal{S}} \mu_{(i,U,S)} - \sum_{T:(i,S,T)\in\mathcal{S}} \mu_{(i,S,T)} & \forall S \subseteq N\ \forall i \in S; \\
s.t. & \sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{k-1} c_l} \mu_{(i,S,T)} = 1; & \\
& \mu_{(i,S,T)} \geq 0 & \forall (i,S,T) \in \mathcal{S}.
\end{array}
$$

We can identify two essential differences with respect to (P_1). First, the values $a_{(i,S,T)}$, determined in previous rounds, play an active role in the objective function. Secondly, the normalization requirement $\sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{k-1} c_l} \mu_{(i,S,T)} = 1$ takes only variables into account associated with elements of $\mathcal{S}$ that did not belong to a VSW that resulted in previous steps. All remarks regarding $\gamma(S)$ that we made directly after the definition of (P_1) are valid for (P_k) as well.

In Lemma 5.1 we show that (P_k) is well-defined. Consider an arbitrary optimal solution $((\overline{\gamma}(S))_{S\subseteq N}, (\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ of (P_k). Set $c_k = \{(i,S,T) \in \mathcal{S}\setminus \cup_{l=1}^{k-1} c_l \mid \overline{\mu}_{(i,S,T)} > 0\}$ and set $a_{(i,S,T)}$ equal to the optimal value of (P_k) for all $(i,S,T) \in c_k$. Set a_k equal to this value as well.

Lemma 5.1 The optimization problem (P_k) is well-defined.

Proof: First note that the objective function is affine. Secondly, note that the feasible set F_k is of the type $\{x \in \mathbb{R}^u \mid Ax = b; x \geq 0\}$ for some $m \times u$-matrix A and $b \in \mathbb{R}^m$.[10] Furthermore, since $\mathcal{S}\setminus\cup_{l=1}^{k-1} c_l \neq \emptyset$, it holds that F_k is nonempty. As a consequence of the fact that the system $Ax = b$ has solutions we can assume without loss of generality, by considering the row echelon form of A, that $\text{rank}(A) = m$. Representation Theorem 2.6.7 in Bazaraa et al. (1993) implies that, since F_k is nonempty, it has extreme points $x_1, \ldots, x_r$ and extreme directions $d_1, \ldots, d_t$ such that $x \in F_k$ if and only if x can be written as

$$
\begin{array}{ll}
x = \sum_{u=1}^{r} \alpha_u x_u + \sum_{v=1}^{t} \beta_v d_v & \text{with} \\
\quad \sum_{u=1}^{r} \alpha_u = 1; & \\
\quad \alpha_u \geq 0 & \text{for every } u \in \{1, \ldots, r\}; \\
\quad \beta_v \geq 0 & \text{for every } v \in \{1, \ldots, t\}.
\end{array}
$$

The structure of F_k implies that for all $v \in \{1, \ldots, t\}$ we have $Ad_v = 0$ and $d_v \geq 0$. Consider $x \in F_k$ such that $\beta_v > 0$ for some $v \in \{1, \ldots, t\}$. We will show that the objective function value does not increase if we go from x to $x - \beta_v d_v$. Since $d_v \geq 0$ and since condition $\sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{k-1} c_l} \mu_{(i,S,T)} = 1$ holds for both x and $x - \beta_v d_v$ we conclude that d_v can be written as $d_v = ((\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ with $\overline{\mu}_{(i,S,T)} = 0$ for all $(i,S,T) \in \mathcal{S}\setminus \cup_{l=1}^{k-1} c_l$. Let $(\overline{\gamma}(S))_{S\subseteq N}$ be the corresponding values of the variables $(\gamma(S))_{S\subseteq N}$. Since $d_v \neq 0$ we can let $l^* < k$ be the highest

[10] This requires a representation of the optimization problems without superfluous variables $\gamma(S)$. See the comments directly after the description of the optimization problems. We will use the representation without these superfluous variables in the proof of lemma 5.1 only.

index for which there exists $(i,S,T) \in c_{l^*}$ with $\overline{\mu}_{(i,S,T)} > 0$. Set

$$\delta = \frac{1}{\sum_{(i,S,T)\in c_{l^*}} \overline{\mu}_{(i,S,T)}},$$

implying that $\delta d_v \in F_{l^*}$, i.e., normalize d_v. Since a_{l^*} is the optimal value of (P_{l^*}) we have

$$\sum_{S\subseteq N} \delta\overline{\gamma}(S)v(S) - \sum_{(i,S,T)\in\cup_{l=1}^{l^*-1}c_l} a_{(i,S,T)}\delta\overline{\mu}_{(i,S,T)} \geq a_{l^*}.$$

Moreover, $a_{l^*} = a_{(i,S,T)}$ for all $(i,S,T) \in c_{l^*}$, but also $a_{l^*} = \sum_{(i,S,T)\in c_{l^*}} a_{(i,S,T)}\delta\overline{\mu}_{(i,S,T)}$. This implies

$$\sum_{S\subseteq N} \delta\overline{\gamma}(S)v(S) - \sum_{(i,S,T)\in\cup_{l=1}^{k-1}c_l} a_{(i,S,T)}\delta\overline{\mu}_{(i,S,T)} \geq 0.$$

The expression before the inequality represents the change in the objective function of (P_k) if we add δd_v to a candidate solution of (P_k). We conclude that the objective function of (P_k) does not increase it we go from x to $x - \beta_v d_v$. Hence, the optimal value for the objective function is obtained for some x that belongs to the set of convex combinations of the extreme points. This being a compact set completes the proof. $\square$

Let k^* be the index such that $\cup_{l=1}^{k^*} c_l = \mathcal{S}$. This k^* exists because of the finiteness of $\mathcal{S}$. The procedure described above results in a number $a_{(i,S,T)}$ for any element $(i,S,T) \in \mathcal{S}$ and in an ordered partition of $\mathcal{S}$. Any two elements in the same partition element have the same number, any partition element, together with the earlier partition elements, corresponds to a VSW, and the numbers associated with consecutive partition elements are weakly increasing. Consecutive partition elements may have the same associated number for all its elements. Merge those consecutive partition elements to end up with partition $(b_1,\ldots,b_p)$ in which any two elements in the same element of the partition have the same number, any partition element, together with the earlier partition elements, corresponds to a VSW (combining the VSWs of the sets of c_l this partition element is created from), and the numbers associated with consecutive partition elements are *strictly* increasing. If these numbers would correspond to a payoff vector scheme this payoff vector scheme would, in view of Theorem 4.4, be premonoclus A. The following lemma shows that the numbers indeed correspond to a payoff vector scheme.

Lemma 5.2 There exists a unique payoff vector scheme such that its monotonicities equal the numbers that form the outcome of the sequence of LP problems.

Proof: Let $(c_1,\ldots,c_{k^*})$ be the partition arising from the solution of LP problems $(P_1),\ldots,(P_{k^*})$, let $a_{(i,S,T)}$, $(i,S,T) \in \mathcal{S}$, be the corresponding numbers and $a_1,\ldots,a_{k^*}$ the corresponding optimal values. Finally let, for every $r \in \{1,\ldots,k^*\}$, x_r be a corresponding optimal solution.

Let $S \subset T$ and $i \in S$. An *i-path $\mathcal{J}$ from S to T* is a sequence of elements (i,S_0,S_1), (i,S_1,S_2), …, (i,S_{k-1},S_k) in $\mathcal{S}$ such that $S_0 = S$ and $S_k = T$. The sum $\sum_{j=1}^{k} a_{(i,S_{j-1},S_j)}$ is called the *value* of i-path $\mathcal{J}$. We will show that two different i-paths from S to T have the same value.

Suppose to the contrary that there exist two i-paths $\mathcal{J}$ and $\mathcal{K}$ from S to T with a different value. Without loss of generality assume that $\mathcal{J}$ and $\mathcal{K}$ are disjoint. Let $l^* \in \{1,\dots,k^*\}$ be the minimal element such that $\mathcal{J} \cup \mathcal{K} \subseteq \cup_{r=1}^{l^*} c_r$. Let $\mathcal{J}_r = \mathcal{J} \cap c_r$ and $\mathcal{K}_r = \mathcal{K} \cap c_r$ for all $r \in \{1,\dots,l^*\}$. Note that $\mathcal{J} = \cup_{r=1}^{l^*}\mathcal{J}_r$ and $\mathcal{K} = \cup_{r=1}^{l^*}\mathcal{K}_r$. Since i-paths $\mathcal{J}$ and $\mathcal{K}$ have a different value we have $\sum_{r=1}^{l^*}|\mathcal{J}_r|a_r \neq \sum_{r=1}^{l^*}|\mathcal{K}_r|a_r$. Without loss of generality assume that $0 < \sum_{r=1}^{l^*}|\mathcal{J}_r|a_r - \sum_{r=1}^{l^*}|\mathcal{K}_r|a_r =: \delta$. Now $x = \sum_{r=1}^{l^*} x_r = ((\overline{\gamma}(S))_{S\subseteq N}, (\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ is an optimal solution for (P_{l^*}) as well with the property that $\overline{\mu}_{(i,S,T)} > 0$ for all $(i,S,T) \in \cup_{r=1}^{l^*} c_r$.

Let $\epsilon = \frac{1}{2}\min_{(i,S,T)\in\cup_{r=1}^{l^*}c_r} \overline{\mu}_{(i,S,T)} > 0$.[11] For all $(i,S,T)\in\mathcal{S}$ define

$$\nu_{(i,S,T)} = \begin{cases} \overline{\mu}_{(i,S,T)} + \epsilon & \text{if } (i,S,T)\in\mathcal{J}; \\ \overline{\mu}_{(i,S,T)} - \epsilon & \text{if } (i,S,T)\in\mathcal{K}; \\ \overline{\mu}_{(i,S,T)} & \text{otherwise.} \end{cases}$$

The triple $((\overline{\gamma}(S))_{S\subseteq N}, (\nu_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ does not provide a feasible solution to (P_{l^*}) in general; the only requirement that might not be satisfied being $\sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{l^*-1}c_l}\mu_{(i,S,T)} = 1$. Going from $\overline{\mu}$ to ν has increased this sum by $(|\mathcal{J}_{l^*}| - |\mathcal{K}_{l^*}|)*\epsilon$ (note that this amount can be zero or negative as well). Moreover, the value of the objective function of (P_{l^*}) has increased by $(|\mathcal{J}_{l^*}| - |\mathcal{K}_{l^*}|)*\epsilon*a_{l^*} - \epsilon*\delta$. Dividing $((\overline{\gamma}(S))_{S\subseteq N}, (\nu_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ element-wise by $\sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{l^*-1}c_l}\nu_{(i,S,T)} = 1 + (|\mathcal{J}_{l^*}| - |\mathcal{K}_{l^*}|)*\epsilon$ provides a feasible solution with an associated objective function value that is strictly smaller than a_{l^*}. A contradiction.

From now on we will denote the value of an i-path from S to T by $w(i,S,T)$. Defining the payoff scheme $y = (y_{S,i})_{S\subseteq N, i\in S}$ by $y_{S,i} := v(i) + w(i,\{i\},S)$ for every $S\subseteq N$ and $i\in S$ it is clear that y is the only candidate for being an efficient payoff scheme whose monotonicities coincide with the number $a_{(i,S,T)}$ for every $(i,S,T)\in\mathcal{S}$. It remains to show that $\sum_{i\in S} y_{S,i} = v(S)$ for all $S\subseteq N$.

Suppose that $\sum_{i\in S^*} y_{S^*,i} \neq v(S^*)$ for some $S^*\subseteq N$. Let $\mathcal{J}_i$ be an i-path from i to S^* for every $i\in S^*$ and let $\mathcal{J} = \cup_{i\in S^*}\mathcal{J}_i$. Let l^* be the lowest index such that $\mathcal{J} \subseteq \cup_{l=1}^{l^*} c_l$. We distinguish two cases: $\sum_{i\in S^*} y_{S^*,i} < v(S^*)$ and $\sum_{i\in S^*} y_{S^*,i} > v(S^*)$.

First, assume $\sum_{i\in S^*} y_{S^*,i} < v(S^*)$. Let $x = ((\overline{\gamma}(S))_{S\subseteq N}, (\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ denote an optimal solution for (P_{l^*}). Let $\epsilon > 0$ and define

$$\nu_{(i,S,T)} = \begin{cases} \overline{\mu}_{(i,S,T)} + \epsilon & \text{if } (i,S,T)\in\mathcal{J}; \\ \overline{\mu}_{(i,S,T)} & \text{otherwise.} \end{cases}$$

Let $\mathcal{J}_{l^*} = \mathcal{J}\cap c_{l^*}$. Going from $\overline{\mu}$ to ν has increased $\sum_{(i,S,T)\in\mathcal{S}\setminus\cup_{l=1}^{l^*-1}c_l}\mu_{(i,S,T)}$ by $\epsilon|\mathcal{J}_{l^*}|$ and the

[11] Note that i is 'fixed'.

value of the objective function of (P_{l^*}) by

$$\begin{aligned}\epsilon\big(v(S^*) - \sum_{i\in S^*} v(\{i\}) - \sum_{(i,S,T)\in\mathcal{J}\setminus\mathcal{J}_{l^*}} a_{(i,S,T)}\big) &< \epsilon\big(\sum_{i\in S^*} y_{S^*,i} - \sum_{i\in S^*} y_{\{i\},i} - \sum_{(i,S,T)\in\mathcal{J}\setminus\mathcal{J}_{l^*}} a_{(i,S,T)}\big)\\ &= \epsilon\big(\sum_{(i,S,T)\in\mathcal{J}} a_{(i,S,T)} - \sum_{(i,S,T)\in\mathcal{J}\setminus\mathcal{J}_{l^*}} a_{(i,S,T)}\big)\\ &= \epsilon\big(\sum_{(i,S,T)\in\mathcal{J}_{l^*}} a_{(i,S,T)}\big)\\ &= \epsilon * a_{l^*} * |\mathcal{J}_{l^*}|\end{aligned}$$

Normalizing $(\nu_{(i,S,T)})_{(i,S,T)\in\mathcal{S}}$ to satisfy all constraints of (P_{l^*}) results with the appropriate $(\gamma(S))_{S\subseteq N}$ in a feasible solution for (P_{l^*}) with a lower value for the object function. A contradiction.

Second, assume $\sum_{i\in S^*} y_{S^*,i} > v(S^*)$. Since x_r is an optimal solution for (P_r), for every $r \in \{1,\dots,l^*\}$, we have that $x = \sum_{r=1}^{l^*} x_r = ((\overline{\gamma}(S))_{S\subseteq N}, (\overline{\mu}_{(i,S,T)})_{(i,S,T)\in\mathcal{S}})$ is an optimal solution for (P_{l^*}) with the property that $\overline{\mu}_{(i,S,T)} > 0$ for all $(i,S,T) \in \cup_{r=1}^{l^*} c_r$. Let $\epsilon = \frac{1}{2}\min_{(i,S,T)\in\cup_{r=1}^{l^*} c_r} \overline{\mu}_{(i,S,T)} > 0$. For all $(i,S,T) \in \mathcal{S}$ define

$$\nu_{(i,S,T)} = \begin{cases} \overline{\mu}_{(i,S,T)} - \epsilon & \text{if } (i,S,T) \in \mathcal{J};\\ \overline{\mu}_{(i,S,T)} & \text{otherwise.}\end{cases}$$

Similar as in the first case, normalizing $(\nu_{(i,S,T)})_{(i,S,T)\in\mathcal{S}}$ to satisfy all constraints of (P_{l^*}) results with the appropriate $(\gamma(S))_{S\subseteq N}$ in a feasible solution for (P_{l^*}) with a lower value for the objective function. A contradiction.

We conclude that y is efficient. This completes the proof. □

Using several results above the proof of the theorem below is easy.

Theorem 5.1 Let (N,v) be a coalitional game. An (efficient) payoff vector scheme x is pre-monoclus A if and only if $\cup_{r=1}^{k} b_r(x,N,v)$ allows for a VSW (i.e., there exists a VSW with $\mu_{(i,S,T)} > 0$ for $(i,S,T) \in \cup_{r=1}^{k} b_r(x,N,v)$ and $\mu_{(i,S,T)} = 0$ otherwise) for all $k = 1,\dots,p_x$.

Proof: The if-part follows by Theorem 4.4. It remains to prove the only-if-part. Let $x = \mathcal{M}^{*,A}(N,v)$. According to Lemma 5.2 there exists a unique payoff vector scheme y such that its monotonicities equal the numbers that form the outcome of the sequence of LP problems. By construction the associated ordered partition is such that every partition element, together with earlier partition elements, correspond to a VSW. By Theorem 4.4 we infer that $y = \mathcal{M}^{*,A}(N,v)$. Hence $x = y$ and therefore $\cup_{r=1}^{k} b_r(x,N,v)$ allows for a VSW for all $k = 1,\dots,p_x$. □

6 Concluding remarks

In this paper we have initiated the explicit analysis of allocation scheme rules. Opposed to allocation rules, an allocation scheme rule does not only provide a payoff vector for the grand coalition, but also for any subcoalition. Rather than constructing an allocation scheme by the usage of an allocation rule coalition-wise, we have constructed four allocation scheme rules that have no (known) direct connection with any existing allocation rule.

Our allocation scheme rules are inspired by optimization techniques underlying the nucleolus. Where this results for the nucleolus, as well as for the prenucleolus, in an allocation rule that provides a core-element whenever one exists, our allocation scheme rules provide a population monotonic allocation scheme whenever one exists. The four rules differ in the optimization domain and the monotonicities taken into account. The first difference is thereby similar to the difference between the nucleolus and the prenucleolus. The second difference is initiated by two (equivalent) formulations of population monotonicity.

This work is not the first to study allocations that are based on lexicographic optimization techniques in a similar spirit as the nucleolus. Two well-known variants are the per-capita nucleolus (cf. Young et al. (1982)) and the modiclus (cf. Sudhölter (1996)). The per-capita nucleolus satisfies an attractive monotonicity property that is not satisfied by the nucleolus, namely nondecreasing payoffs as a result of an increasing value of the grand coalition. Extending this requirement to other coalitions makes the per-capita nucleolus nonmonotonic as well. The modiclus is based on lexicographically optimizing with respect to differences between excesses (bi-excesses) and shares many properties with the per-capita nucleolus (see, e.g., Peleg and Sudhölter (2003)).

The research carried out in this work paves the way for different types of follow-up research. Our primary focus will be on the performance of monocli on specific types of coalitional games (e.g., airport games) and on attempts to find an axiomatic characterization. It is straightforward to show that the monocli satisfy properties dealing with efficiency, covariance, and anonymity (symmetry), which should of course be suitably defined for allocation scheme rules. As for the nucleolus, additivity (linearity) will not be satisfied. Attempts to characterize these rules along the lines of characterizations of the nucleolus would require a consistency type of property. Here, one encounters a crucial difference between nucleolus and monoclus: for every coalitional game a player has several (interrelated) associated payoffs. Taking this into account in a consistency property seems far from straightforward. Finally, we mention that this work might inspire others to initiate different types of allocation scheme rules, rather than restricting attention to coalition-wise extensions of allocation rules.

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