Link monotonic allocation schemes[a]

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Abstract

A network is a graph where the nodes represent players and the links represent bilateral interaction between the players. A reward game assigns a value to every network on a fixed set of players. An allocation scheme specifies how to distribute the worth of every network among the players. This allocation scheme is link monotonic if extending the network does not decrease the payoff of any player. We characterize the class of reward games that have a link monotonic allocation scheme. Two allocation schemes for reward games are studied, the Myerson allocation scheme and the position allocation scheme, which are both based on allocation rules for communication situations. We introduce two notions of convexity in the setting of reward games and with these notions of convexity we characterize the classes of reward games where the Myerson allocation scheme and the position allocation scheme are link monotonic. As a by-product we find a characterization of the Myerson value and the position value on the class of reward games using potentials.

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1 Introduction

Network structures have been used in several contexts to describe the interaction between economic agents. The place of an agent in a network will affect not only his own productivity and bargaining position but also the productivities and bargaining positions of the other players in the network. *Jackson* and *Wolinsky* (1996) provide an overview of references on networks in a social science context. Subsequently, they study stability of networks. They describe the economic possibilities of the players depending on the network structure by a reward function.[1] This reward function is then used to study the relationship between the set of networks that are productively efficient and the set of networks that are stable, i.e. networks where self-interested agents do not form or break cooperation. *Jackson* and *Wolinsky* (1996) show that there exists a conflict between stability and efficiency of networks. This conflict is further studied by *Dutta* and *Mutuswami* (1997). They study a game in strategic form to describe the formation of a network. The possible structures in the strategic form game are evaluated by the agents using an exogenously given allocation rule assigning payoffs to all players. This allocation rule is a straightforward generalization of the Myerson value for communication situations (see *Myerson* (1977)).

In this paper we analyze networks from a cooperative point of view. The analysis of the stability of a network does not only require a specification of payoffs in this network, but also in all other possible networks between the agents. A scheme specifying the payoffs in all networks is called an allocation scheme. We will focus on allocation schemes where no agent is ever tempted to prevent the formation of additional cooperation between agents or to break down cooperation between players. These allocation schemes are in the same spirit as population monotonic allocation schemes for cooperative games, introduced by *Sprumont* (1990), and will be called link monotonic allocation schemes. As in a population monotonic allocation scheme no player is worse off with additional cooperation between the players. We will characterize the reward functions for which a link monotonic allocation scheme exists and additionally study when two specific allocation schemes are link monotonic. The first allocation scheme is based on the Myerson value. The second allocation scheme will be based on the position value, introduced by *Borm*, *Owen*, and *Tijs* (1992). As by-products we will characterize the extension of these allocation rules to the network setting described above using potentials. Additionally, we will introduce two notions of convexity in the setting of reward games.

The plan of this paper is as follows. Section 2 contains some preliminaries on coop-

[1] *Jackson* and *Wolinsky* (1996) refer to a reward function as a value function.

erative games and communication situations. In section 3 we describe reward games. In section 4 we introduce link monotonic allocation schemes, study the relations with population monotonic allocation schemes and provide a characterization of reward games with a link monotonic allocation scheme. In section 5 we introduce player convex and link convex reward games. We characterize two allocation rules and characterize the class of reward games for which the allocation schemes based on these rules are link monotonic. Finally, section 6 analyzes an example, the symmetric connections model, introduced by *Jackson* and *Wolinsky* (1996).

2 Preliminaries

A cooperative game with transferable utility (TU-game) is a pair (N, v) where $N = \{1, \ldots, n\}$ denotes the set of players and v is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. The function v is called the *characteristic function* of the cooperative game (N, v). A cooperative game (N, v) is *convex* if for all $S, T \subseteq N$ with $S \subseteq T$ and all $i \in S$

$$v(S) - v(S\backslash\{i\}) \leq v(T) - v(T\backslash\{i\}).$$

Shapley (1953) showed that every cooperative game (N, v) can be written as a unique linear combination of unanimity games[2] $(N, u_S)_{S \subseteq N}$, i.e. $v = \sum_{S \subseteq N} \alpha_S(v) u_S$, where $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. The coefficients $(\alpha_S(v))_{S \subseteq N}$ are called unanimity coefficients. If no confusion on the underlying game can arise we will simply write $(\alpha_S)_{S \subseteq N}$ instead of $(\alpha_S(v))_{S \subseteq N}$. Then the Shapley value Φ of the game (N, v) can be described by

$$\Phi_i(N, v) = \sum_{S \subseteq N:\ i \in S} \frac{\alpha_S}{|S|}, \text{ for all } i \in N.$$

A *communication situation* is a triple (N, v, L) where (N, v) is a TU-game as described above and (N, L) an undirected graph, i.e. $L \subseteq \bar{L} := \{\{i, j\} \mid \{i, j\} \subseteq L,\ i \neq j\}$ denotes a set of links. This undirected graph (N, L) partitions the player set into *communication components*, where two players are in the same communication component if and only if they are *connected*, i.e. there exists a path between the two players using only links in L. The resulting set of communication components will be denoted by N/L. The set of links in graph (N, L) within a coalition $S \subseteq N$ will be denoted by $L(S)$, i.e. $L(S) = \{\{i, j\} \in L \mid \{i, j\} \subseteq S,\ i \neq j\}$. The set of components in the graph $(S, L(S))$ will be denoted by S/L.

[2] $S \subseteq N$ denotes that S is a subset of N, $S \subset N$ denotes that S is a strict subset of N.

The *graph-restricted game* (N, v^L) associated with communication situation (N, v, L) is defined by

$$v^L(S) := \sum_{C \in S/L} v(C), \quad \text{for all } S \subseteq N.$$

The *Myerson value* μ (cf. *Myerson* (1977)) for a communication situation (N, v, L) coincides with the Shapley value of the graph-restricted game,

$$\mu(N, v, L) := \Phi(N, v^L).$$

Hence, if $(\beta_S)_{S \subseteq N}$ denote the unanimity coefficients of the game (N, v^L) then

$$\mu_i(N, v, L) = \sum_{S \subseteq N:\ i \in S} \frac{\beta_S}{|S|}, \text{ for all } i \in N.$$

With a slight abuse of notation we denote the set of links in which player i is involved by L_i.[3] Furthermore, denote the set of players involved in at least one link by $N(L) := \{i \in N \mid \exists j : \{i, j\} \in L\} = \{i \in N \mid L_i \neq \emptyset\}$. For notational convenience we denote the full cooperation structure on set S by $K_S = \{\{i, j\} \mid \{i, j\} \subseteq S,\ i \neq j\}\}$. Note that $\bar{L} = K_N$.

3 Reward games

In communication situations the profit that can be obtained by the players depends only on the (connected) components. In order to allow for influence of the internal structure within a component on the profit the players can obtain we consider reward games.

A pair (N, r), with N the player set and $r : 2^{\bar{L}} \to \mathbb{R}$ a *reward function*, will be called a *reward game*. For every cooperation structure (N, L) with $L \subseteq \bar{L}$ the value $r(L)$ represents the profit that can be obtained by all players together if they cooperate according to this cooperation structure. Throughout this paper we will assume that $r(\emptyset) = 0$, which states that no cooperation between the players implies that no profit can be made.

Note that the reward function of the reward game (N, r) can be seen as the characteristic function of the TU-game $(\bar{L}, r)$. We will refer to this TU-game as the *link game* associated with the reward game (N, r). Since every TU-game can be written as a unique linear combination of unanimity games, the reward function can be written as a unique linear combination of characteristic functions of unanimity (link) games, i.e.

$$r = \sum_{A \subseteq \bar{L}} \alpha_A u_A.$$

[3] In fact this set is a function that depends on the set of links L and the player index i.

Jackson and *Wolinsky* (1996) and *Dutta* and *Mutuswami* (1997) concentrate on *component additive* reward functions, i.e. $r(L) = \sum_{S \in N/L} r(L(S))$. Although we mainly concentrate on component additive reward games, we do not restrict ourselves to this class of reward games. A reward game is *monotonic* if $r(L_1) \leq r(L_2)$ for all $L_1 \subseteq L_2$.

A *reward communication situation* is a triple (N, r, L) where (N, r) is a reward game and (N, L) an undirected graph. The set of all reward communication situations with player set N will be denoted by RCS^N. The set of all communication situations will be denoted by RCS. *Dutta* and *Mutuswami* (1997) introduced the Myerson value for reward communication situations, a generalization of the Myerson value for communication situations:

$$\mu_i(N, r, L) := \sum_{A \subseteq L,\ A_i \neq \emptyset} \frac{\alpha_A}{|N(A)|}, \quad \text{for all } i \in N.$$

Furthermore, they provide a characterization of the Myerson value using component balancedness and fairness.

The Myerson value for reward communication situations is an example of an *allocation rule* for reward communication situations. Such an allocation rule, say γ, assigns a vector $\gamma(N, r, L) \in \mathbb{R}^N$ to every triple (N, r, L). An *allocation scheme* $(x_{i,L})_{i \in N, L \subseteq \bar{L}}$ for reward game (N, r) assigns payoffs to all players in all possible networks on the player set.

4 Link monotonic allocation schemes

In this section we will introduce link monotonic allocation schemes. We will relate link monotonic allocation schemes to population monotonic allocation schemes for cooperative games. Finally, we will characterize the class of reward games that have a link monotonic allocation scheme.

Dutta and *Mutuswami* (1997) analyze the formation of a cooperation structure. Given some allocation rule (e.g. the Myerson value, see section 3) they analyze a link formation game in strategic form. They conclude that in monotonic reward games the full cooperation structure will result or a structure that results in the same payoff division as the full cooperation structure $(N, \bar{L})$.

Aumann and *Myerson* (1988) describe a link formation game in extensive form, where links are formed sequentially. Furthermore, they study a superadditive TU-game where the full cooperation structure between the players does not result as a subgame perfect Nash equilibrium. The full cooperation structure need not form since in their example the formation of a link can decrease the payoff of several players. If the formation of a link between two players would not decrease the payoff of any player then the full cooperation structure would be a subgame perfect Nash equilibrium. In the following we will consider reward games and we will concentrate on reward games that have allocation schemes with the property that the formation of a link between two players would not decrease the payoff of any player.

Definition 1 A vector $(x_{i,L})_{i\in N, L\subseteq \overline{L}}$ is a *link monotonic allocation scheme* for the reward game (N, r) if it satisfies the following conditions:

(i) $x_{i,L} = 0$ for all $L \subseteq \overline{L}$, $i \notin N(L)$.

(ii) $\sum_{i\in N} x_{i,L} = r(L)$ for all $L \subseteq \overline{L}$.

(iii) $x_{i,L} \leq x_{i,L^*}$ for all $i \in N$ and $L \subseteq L^* \subseteq \bar{L}$.

The first condition makes sure that if a player does not cooperate with any other player then he receives zero payoff. The second condition states that the value of a network is divided among the players forming the network (efficiency). The third condition makes sure that no player ever has a reason to prevent the formation of any link (monotonicity).

The notion of link monotonic allocation scheme is inspired by the notion of population monotonic allocation scheme (cf. *Sprumont* (1990)) for cooperative games.

Definition 2 A vector $(y_{i,S})_{i\in S, S\subseteq N}$ is a *population monotonic allocation scheme* for the cooperative game (N, v) if it satisfies the following conditions:

(i) $\sum_{i\in S} y_{i,S} = v(S)$ for all $S \subseteq N$.

(ii) $y_{i,S} \leq y_{i,T}$ for all $i \in S$ and $S \subseteq T \subseteq N$.

The concepts of population monotonic allocation schemes (in short PMAS) and link monotonic allocation schemes (in short LMAS) appear to be related. This relation is made explicit in the following theorem which states that a reward game has a LMAS if the associated link game has a non-negative PMAS.

Theorem 4.1 Let (N, r) be a reward game. If the associated link game $(\overline{L}, r)$ has a non-negative PMAS then (N, r) has a LMAS.

Proof: Let $y = (y_{\{i,j\},L})_{\{i,j\}\in L,\ L\subseteq \bar{L}}$ be a non-negative PMAS for $(\overline{L}, r)$. For all $i \in N$ and all $L \subseteq \overline{L}$ define[4]

$$x_{i,L} = \sum_{j\in N:\{i,j\}\in L} \frac{1}{2} y_{\{i,j\},L}.$$

We show that $(x_{i,L})_{i\in N, L\subseteq \overline{L}}$ is a LMAS for (N, r) by checking the three conditions in definition 1.

[4]We define the empty sum to be zero.

(i) This property follows immediately by noting that

$$\{j \in N : \{i,j\} \in L\} = \emptyset \text{ if } i \notin N(L).$$

(ii) Let $L \subseteq \overline{L}$. Then

$$\sum_{i \in N} x_{i,L} = \sum_{i \in N(L)} x_{i,L} = \sum_{i \in N(L)} \sum_{j \in N : \{i,j\} \in L} \frac{1}{2} y_{\{i,j\},L} = \sum_{\{i,j\} \in L} y_{\{i,j\},L} = r(L),$$

where the first equality follows by (i) and the last equality by the fact that y is a PMAS for $(\bar{L}, r)$.

(iii) Let $L \subseteq L^* \subseteq \overline{L}$ and $i \in N$. Then

$$x_{i,L} = \sum_{j \in N : \{i,j\} \in L} \frac{1}{2} y_{\{i,j\},L} \leq \sum_{j \in N : \{i,j\} \in L} \frac{1}{2} y_{\{i,j\},L^*} \leq \sum_{j \in N : \{i,j\} \in L^*} \frac{1}{2} y_{\{i,j\},L^*} = x_{i,L^*},$$

where the first inequality follows since y is a PMAS for $(\bar{L}, r)$ and the last inequality follows by the non-negativity of y.

This completes the proof. □

In the following example we show that the non-negativity assumption is not superfluous.

Example 4.1 Consider the reward game $(\{1,2\}, r)$ with $r(\emptyset) = 0$ and $r(\{\{1,2\}\}) = -1$. Then $y_{\{1,2\},\{\{1,2\}\}} = -1$ is a PMAS for the (1-person) link game $(\{\{1,2\}\}, r)$. Suppose x is a LMAS for $(\{1,2\}, r)$. Then it should hold that

$$\begin{cases} x_{1,\{\{1,2\}\}} & \geq x_{1,\emptyset} & = & 0 \\ x_{2,\{\{1,2\}\}} & \geq x_{2,\emptyset} & = & 0 \\ x_{1,\{\{1,2\}\}} + x_{2,\{\{1,2\}\}} & = -1 & & \end{cases}.$$

Consequently, $0 \leq x_{1,\{\{1,2\}\}} + x_{2,\{\{1,2\}\}} = -1$, a contradiction. We conclude that $(\{1,2\}, r)$ does not have a LMAS.

Remark 4.1 Note that the non-negativity of the PMAS in theorem 4.1 can be replaced by the condition that r is non-negative.

In *Sprumont* (1990) it is shown that every convex game has a PMAS. Using theorem 4.1 and remark 4.1 above the following corollary follows directly.

Corollary 4.1 Let (N, r) be a reward game. If the associated link game $(\overline{L}, r)$ is non-negative and convex, then (N, r) has a LMAS.

Theorem 4.1 states that a reward game has a LMAS if the corresponding link game has a non-negative PMAS. The following example illustrates that the LMAS-concept is not just the equivalent for PMAS in link games associated with reward games. In this example we present a reward game with a LMAS, although the associated link game does not have a PMAS.

Example 4.2 Consider the 'glove game' with player 1 having a left glove and player 2 and 3 both having a right glove. The value of a left glove or a right glove alone is zero. The value of a pair of gloves, a left and a right glove, is one.

Computing the rewards that can be obtained for the various cooperation structures results in the reward game (N, r) with[5]

$$r(L) = \begin{cases} 1 & , \text{ if } 12 \in L \text{ or } 13 \in L \\ 0 & , \text{ otherwise} \end{cases} .$$

It is easily verified that the corresponding link game $(\bar{L}, r)$ does not have a PMAS since such a PMAS should satisfy $x_{12,\{12\}} = x_{13,\{13\}} = 1$ implying $x_{12,\{12,13\}} + x_{13,\{12,13\}} \geq 2$ which cannot hold since $r(\{12, 13\}) = 1$. Note that this link game is not even balanced.

Now consider the following allocation scheme for the reward game (N, r).

$$y_{i,L} = \begin{cases} 1 & , \text{ if } i = 1 \text{ and } r(L) = 1 \\ 0 & , \text{ otherwise} \end{cases} .$$

It is easily checked that $(y_{i,L})_{i \in N, L \subseteq \bar{L}}$ is a link monotonic allocation scheme.

The example above illustrates that the class of reward games with a link monotonic allocation scheme does not correspond to the class of reward games with a PMAS for the corresponding link game. *Sprumont* (1990) showed that a cooperative game with a PMAS has to be totally balanced. The example above shows that a reward game can have a LMAS while the corresponding link game is not even balanced.

After the results above one might expect the class of reward games with a LMAS to be a large class. However, we show in the following example that the class of reward games with a LMAS does not contain all reward games with a totally balanced associated link game.

[5]For notational convenience we will sometimes refer to a link $\{i, j\}$ as ij.

Example 4.3 Consider the reward game (N, r) with $N = \{1, 2, 3, 4\}$ and

$$r(L) = \begin{cases} 2 & , \text{ if } L = \bar{L} \\ 1 & , \text{ if } L \neq \bar{L} \text{ and } \exists i \in N \text{ such that } L \supset K_{N\setminus\{i\}} \\ 0 & , \text{ otherwise} \end{cases} .$$

For the associated link game, y with $y_{13} = y_{24} = 1$ and $y_{12} = y_{14} = y_{23} = y_{34} = 0$ is a core-element. For all subgames one can also find a core-element, e.g. for the subgame on $\{12, 13, 14, 23, 24\}$ we have that y with y with $y_{12} = 1$ and $y_{13} = y_{14} = y_{23} = y_{24} = 0$ is a core-element. Hence, $(\bar{L}, r)$ is totally balanced. We will show that (N, r) does not have a LMAS.

Suppose x is a LMAS for (N, r). From $x_{i,\emptyset} = 0$ for all $i \in N$ it follows by monotonicity of the allocation scheme that $x_{i,L} \geq 0$ for all $i \in N$ and all $L \subseteq \bar{L}$. From $x_{1,\{12,13,23\}} + x_{2,\{12,13,23\}} + x_{3,\{12,13,23\}} = 1$, and $\sum_{i \in \{1,2,3,4\}} x_{i,\{12,13,14,23,24\}} = 1$ it follows by monotonicity that $x_{4,\{12,13,14,23,24\}} = 0$. By using monotonicity of the allocation scheme we conclude that $x_{4,\{12,14,24\}} = 0$.

Interchanging the role of players 1 and 4 we get $x_{1,\{12,14,24\}} = 0$, while interchanging the role of players 2 and 4 would result in $x_{2,\{12,14,24\}} = 0$. So, $x_{1,\{12,14,24\}} + x_{2,\{12,14,24\}} + x_{4,\{12,14,24\}} = 0$ which contradicts. $x_{1,\{12,14,24\}} + x_{2,\{12,14,24\}} + x_{4,\{12,14,24\}} = 1$. We conclude that (N, r) does not have a LMAS.

In the following we will describe the class of reward games with a LMAS. First, we will introduce some definitions. Player i is called a *veto-player* in the reward game (N, r) if cooperation of player i is required to obtain profits, i.e. $r(L) = 0$, for all $L \subseteq K_{N\setminus\{i\}}$. A reward game is a *reward game with veto-control* if it is a reward game with at least one veto-player. The reward game (N, r) is called a *simple reward game* if $r(L) \in \{0, 1\}$ for all $L \subseteq \bar{L}$. Finally, recall that a reward game is *monotonic* if $r(L_1) \leq r(L_2)$ for all $L_1 \subseteq L_2$.

Sprumont (1990) showed that a TU-game has a PMAS if and only if it is a positive linear combination of monotonic simple games with veto-control. The following theorem provides a similar result with respect to reward games with a LMAS.

Theorem 4.2 A reward game (N, r) has a LMAS if and only if it is a positive linear combination of monotonic simple reward games with veto-control.

Proof: First assume (N, r) is a positive linear combination of monotonic simple reward games with veto-control. If x_1 is a LMAS for (N, r^1) and x_2 a LMAS for (N, r^2) then obviously $\alpha x_1 + \beta x_2$ is a LMAS for $(N, \alpha r^1 + \beta r^2)$ if $\alpha, \beta \geq 0$, where $(\alpha r^1 + \beta r^2)(L) =$

$\alpha r^1(L)+\beta r^2(L)$ for all $L \subseteq \bar{L}$. A monotonic simple reward game with veto-control has a LMAS, which is easily seen by attributing the reward of any structure completely to one specific veto-player. Since (N, r) is a positive linear combination of monotonic simple reward games with veto-control it follows that (N, r) has a LMAS.

Now, assume (N, r) has a LMAS $x = (x_{i,L})_{i \in N, L \subseteq \bar{L}}$. Then we can write (N, r) as a sum of monotonic reward games with veto-control $(N, r^i)_{i \in N}$, where $r^i(L) = x_{i,L}$ for all $i \in N$ and all $L \subseteq \bar{L}$.

It remains to show that every monotonic reward game with veto-control can be written as a positive linear combination of monotonic simple reward games with veto-control. Let (N, r^i) be a monotonic reward game with veto-player i. Define

$$K := |\{z \in \mathbb{R}_{++} \mid \exists L \in \bar{L} :\; r^i(L) = z\}|$$

and let $t_0 := r^i$. For $k = 1, \ldots, K$ define

$$\alpha_k := \min\{t_{k-1}(L) \mid t_{k-1}(L) > 0,\; L \subseteq \bar{L}\}$$

and

$$t_k := t_{k-1} - \alpha_k r^i_k,$$

where $r^i_k(L) = \begin{cases} 1 & , \text{ if } t_{k-1}(L) > 0 \\ 0 & , \text{ otherwise} \end{cases}$. By construction we have

$$r^i = \sum_{k=1}^{K} \alpha_k r^i_k, \quad \text{with } \alpha_k > 0 \text{ for all } k \in \{1, \ldots, K\}.$$

Since all (N, r^i_k) are monotonic simple reward games with veto-player i it follows that every monotonic reward game with veto-control can be written as a positive linear combination of monotonic simple reward games with veto-control. □

5 Convexity

In this section we will introduce two notions of convexity in the setting of reward games, *player convexity* and *link convexity*. We will show that there is a relation between player convexity and the allocation scheme based on the Myerson value being link monotonic. Similarly, we will show that there is a relation between link convexity and the allocation scheme based on the position value (cf. *Borm et al.* (1992)) being link monotonic.

In introducing a notion of convexity in the setting of reward games, we could simply focus on convexity of the corresponding link game. However, in reward games we focus on the players and convexity of the associated link games focuses on the links. Therefore, we will not consider convexity of the associated link game, but we will introduce two different notions of convexity for the class of reward games.

First, recall that a cooperative game (N, v) is convex if for all $S, T \subseteq N$ with $S \subseteq T$ and all $i \in S$

$$v(S) - v(S \backslash \{i\}) \leq v(T) - v(T \backslash \{i\}).$$

Convexity states that the marginal contribution of a player does not decrease if this player joins a larger coalition.

Translating the interpretation of convexity in cooperative games to reward games, it seems natural to look at the total contribution of all links a player is involved in. However, this can be interpreted in at least two different ways.

Definition 3 A reward game (N, r) is *player convex* if for all $L^1, L^2 \subseteq \bar{L}$ with $L^1 \subseteq L^2$ and all $i \in N$

$$r(L^1) - r(L^1 \backslash L^1_i) \leq r(L^2) - r(L^2 \backslash L^2_i).$$

Definition 4 A reward game (N, r) is *link convex* if for all $L^1, L^2 \subseteq \bar{L}$ with $L^1 \subseteq L^2$ and all $i \in N$

$$\sum_{l \in L^1_i} \left[r(L^1) - r(L^1 \backslash \{l\}) \right] \leq \sum_{l \in L^2_i} \left[r(L^2) - r(L^2 \backslash \{l\}) \right].$$

A reward game is player convex if the marginal contribution of the set of all the links a player is involved in does not decrease when the set of links is enlarged. A reward game is link convex if the sum of the marginal contributions of the links a player is involved in does not decrease when the set of links is enlarged.

These two notions of convexity will be used in analyzing two allocation rules defined on the class of reward games. We will use convexity in describing the set of reward games where a specific (extended) allocation rule is a link monotonic allocation scheme.

The first allocation rule is the Myerson value which was already described in section 3. The second allocation rule is the position value. Let (N, r) be a reward game with unanimity coordinates $(\alpha_A)_{A \subseteq \bar{L}}$. Then the position value $\pi(N, r, L)$ is defined by

$$\pi_i(N, r, L) = \sum_{A \subseteq L} \frac{\alpha_A |A_i|}{2|A|} = \sum_{A \subseteq L} \sum_{l \in A_i} \frac{\alpha_A}{2|A|}, \quad \text{for all} \quad i \in N.$$

This position value is a natural extension of the position value for cooperative games, introduced by *Borm*, *Owen*, and *Tijs* (1992). Recall that $\Phi_l(L, r_{|L}) = \sum_{A \subseteq L,\ l \in A} \frac{\alpha_A}{|A|}$, where Φ denotes the Shapley value. Hence,

$$\pi_i(N, v, L) = \sum_{l \in L_i} \frac{1}{2} \Phi_l(L, r_{|L}), \quad \text{for all} \quad i \in N.$$

We will characterize the two allocation rules above using potentials, similar to the characterization of the Shapley value by *Hart* and *Mas-Colell* (1989). The potential used in the characterization of the Myerson value focuses on the total marginal contribution of a player. The potential used in the characterization of the position value focuses on the marginal contributions of the links.

Consider a function P that assigns to every reward communication situation (N, r, L) a real number. The marginal contribution of a player can now be defined in two natural ways. First as the total marginal contribution of all his links, i.e.

$$D_i^1 P(N, r, L) := P(N, r, L) - P(N, r, L \backslash L_i), \quad \text{for all } (N, r, L) \in \text{RCS and all } i \in N. \quad (1)$$

Secondly, we can define the marginal contribution of a player as the sum of the marginal contributions of the links this player is involved in, i.e.

$$D_i^2 P(N, r, L) = \sum_{l \in L_i} [P(N, r, L) - P(N, r, L \backslash \{l\})] \quad \text{for all } (N, r, L) \in \text{RCS and all } i \in N. \quad (2)$$

A function P is called a *player potential function* if $P(N, r, \emptyset) = 0$ and

$$\sum_{i \in N} D_i^1 P(N, r, L) = r(L), \quad (3)$$

i.e the sum of the marginal contributions w.r.t D^1 equals the value of the cooperation structure.

A function P is called a *link potential function* if $P(N, r, \emptyset) = 0$ and

$$\sum_{i \in N} D_i^2 P(N, r, L) = r(L), \quad (4)$$

i.e the sum of the marginal contributions w.r.t D^2 equals the value of the cooperation structure.

The following theorem shows that there exists a unique player potential function and for all reward games it holds that the marginal contributions coincide with the Myerson value.

Theorem 5.1 There exists a unique player potential function P. For all reward communication situations (N, r, L) ∈RSC it holds that $D_i^1 P(N, r, L) = \mu_i(N, r, L)$ for all $i \in N$.

Proof: First we show that there exists a player potential function and that the marginal contributions of this player potential function coincide with the Myerson value.

Let (N, r) be a reward game. Since a reward function is the characteristic function of the associated link game, this function can be written as a unique linear combination of unanimity games, i.e. $r = \sum_{A \subseteq \bar{L}} \alpha_A u_A$. We define $(P(N, r, L))_{L \subseteq \bar{L}}$:

$$P(N, r, L) = \sum_{A \subseteq L} \frac{\alpha_A}{N(A)} \quad , \text{ for all } L \subseteq \bar{L}.$$

Obviously, $P(N, r, \emptyset) = 0$. Furthermore, for all $L \subseteq \bar{L}$

$$\begin{aligned} D_i^1 P(N, r, L) &= P(N, r, L) - P(N, r, L \backslash L_i) \\ &= \left[\sum_{A \subseteq L} \frac{\alpha_A}{N(A)} - \sum_{A \subseteq L \backslash L_i} \frac{\alpha_A}{N(A)} \right] \\ &= \sum_{A \subseteq L : A_i \neq \emptyset} \frac{\alpha_A}{N(A)} \\ &= \mu_i(N, r, L) \end{aligned} \tag{5}$$

Since the Myerson value is efficient it follows that the sum of the marginal contributions equals the value of the cooperation structure.

Since the arguments above hold for all reward games (N, r) it holds that $(P(N, r, L))_{(N,r,L) \in \text{RSC}}$ is a player potential function.

It remains to show that the player potential function is unique. If Q is a player potential function it follows by equations (1) and (3) that for all (N, r, L) with $L \neq \emptyset$ that

$$Q(N, r, L) = \frac{1}{|N|} \left[r(L) + \sum_{i \in N} Q(N, r, L \backslash L_i) \right]. \tag{6}$$

For all reward games (N, r) it holds that $Q(N, r, \emptyset) = 0$, so $Q(N, r, L)$ can be determined recursively using this equation.[6] This proves the uniqueness of the player potential function.

This completes the proof. □

Secondly, we consider D^2. Recall that P is a link potential function if $\sum_{i \in N} D_i^2 P(N, r, L) = r(L)$ for all (N, r, L) and $P(N, r, \emptyset) = 0$ for all (N, r). The following theorem shows that there exists a unique link potential function and the marginal contributions coincide with the position value.

[6]Note that for $i \in N$ with $L_i = \emptyset$ it holds that $L \backslash L_i = L$. So, equation (6) is not a recursive formula. However, since $L \neq \emptyset$ there exists $i \in N$ with $L_i \neq \emptyset$, so equation (6) can be rewritten to show that $Q(N, v, L)$ is uniquely determined by $\{Q(N, r, A) \mid A \subset L\}$.

Theorem 5.2 There exists a unique link potential function P. For all reward communication situations $(N, r, L) \in$RSC it holds that $D_i^2 P(N, r, L) = \pi_i(N, r, L)$ for all $i \in N$.

Proof: First we show that there exists a link potential function and that the marginal contributions of the link potential function coincide with the position value.

Let (N, r) be a reward game. Recall that a reward function is the characteristic function of the associated link game, which can be written as a unique linear combination of unanimity games, i.e. $r = \sum_{A \subseteq \bar{L}} \alpha_A u_A$. We define $(P(N, r, L))_{L \subseteq \bar{L}}$:

$$P(N, r, L) = \sum_{A \subseteq L} \frac{\alpha_A}{2|A|} \quad , \text{ for all } L \subseteq \bar{L}.$$

Obviously, $P(N, r, \emptyset) = 0$. Furthermore, for all $L \subseteq \bar{L}$

$$\begin{aligned} D_i^2 P(N, r, L) &= \sum_{l \in L_i} [P(N, r, L) - P(N, r, L \backslash \{l\})] \\ &= \sum_{l \in L_i} \left[\sum_{A \subseteq L} \frac{\alpha_A}{2|A|} - \sum_{A \subseteq L \backslash \{l\}} \frac{\alpha_A}{2|A|} \right] \\ &= \sum_{l \in L_i} \sum_{A \subseteq L : l \in A} \frac{\alpha_A}{2|A|} \\ &= \pi_i(N, r, L) \end{aligned} \tag{7}$$

Finally, note that the position value is efficient which implies that the sum of the marginal contributions equals the value of the cooperation structure. Since the arguments above hold for all reward games (N, r) it holds that $(P(N, r, L))_{(N,r,L) \in \text{RSC}}$ is a link potential function.

It remains to show that the link potential function is unique. If Q is a link potential function it follows by equations (2) and (4) that for all reward communication situations (N, r, L) with $L \neq \emptyset$

$$\begin{aligned} r(L) &= \sum_{i \in N} \sum_{l \in L_i} [Q(N, r, L) - Q(N, r, L \backslash \{l\})] \\ &= \sum_{l \in L} 2[Q(N, r, L) - Q(N, r, L \backslash \{l\})]. \end{aligned}$$

Hence,

$$Q(N, r, L) = \frac{1}{2|L|} \left[r(L) + 2 \sum_{l \in L} Q(N, r, L \backslash \{l\}) \right].$$

Since for all reward games $Q(N, r, \emptyset) = 0$ this determines $Q(N, r, L)$ recursively. This proves the uniqueness of the link potential function.

This completes the proof. □

In the remainder of this section we study allocation schemes resulting from the Myerson value and the position value. We concentrate on the conditions on the underlying reward game such that these allocation schemes are link monotonic. For a reward game (N, r) we refer to $(\mu_i(N, r, L))_{i \in N,\ L \subseteq \bar{L}}$ as the Myerson allocation scheme and to $(\pi_i(N, r, L))_{i \in N,\ L \subseteq \bar{L}}$ as the position allocation scheme. Furthermore, if P is the player potential function then we refer to $(N, P_{|(N,r)})$, where $P_{|(N,r)}$ is the restriction of P to $\{(N, r, L) | L \subseteq \bar{L}\}$, as the player potential reward game associated with (N, r). Similarly, if P is the link potential function then we refer to $(N, P_{|(N,r)})$ as the link potential reward game associated with (N, r).

Marín-Solano and *Rafels* (1996) show that the allocation scheme based on the Shapley value for a cooperative game is a population monotonic allocation scheme if and only if the associated potential game is convex. We will find a similar result regarding reward games, the Myerson value, and the position value.

The following theorem states that the Myerson allocation scheme is a link monotonic allocation scheme if and only if the associated player potential reward game is player convex.

Theorem 5.3 Let (N, r) be a reward game. The Myerson allocation scheme is a LMAS if and only if the associated player potential reward game $(N, P_{|(N,r)})$ is player convex.

Proof: For notational convenience we will write P instead of $P_{|(N,r)}$ and $P(L)$ instead of $P_{|(N,r)}(N, r, L)$.

Since the Myerson allocation scheme obviously satisfies conditions (i) and (ii) of definition 1 it suffices to show that the Myerson allocation scheme satisfies condition (iii) if and only if the player potential reward game associated with (N, r) is player convex. Denote the player potential reward game associated with (N, r) by (N, P).

(N, P) is player convex if and only if for all $i \in N$ and all $L^1 \subseteq L^2 \subseteq \bar{L}$

$$P(L^1) - P(L^1 \backslash L_i^1) \leq P(L^2) - P(L^2 \backslash L_i^2).$$

By (5) we find that this is equivalent to

$$\mu_i(N, r, L^1) \leq \mu_i(N, r, L^2).$$

for all $i \in N$ and all $L^1 \subseteq L^2 \subseteq \bar{L}$. This completes the proof. □

The following theorem contains a similar result for the position value and a link convex associated link potential reward game.

Theorem 5.4 Let (N, r) be a reward game. The position allocation scheme is a LMAS if and only if the associated link potential reward game $(N, P_{|(N,r)})$ is link convex.

Proof: For notational convenience we will write P in stead of $P_{|(N,r)}$ and $P(L)$ instead of $P_{|(N,r)}(N, r, L)$.

Since the position allocation scheme obviously satisfies conditions (i) and (ii) of definition 1 it suffices to show that the position allocation scheme satisfies condition (iii) if and only if the link potential reward game associated with (N, r) is link convex. Denote the link potential game associated with (N, r) by (N, P).

(N, P) is link convex if and only if for all $i \in N$ and all $L^1 \subseteq L^2 \subseteq \bar{L}$:

$$\sum_{l \in L_i^1} \left[P(L^1) - P(L^1 \backslash \{l\}) \right] \leq \sum_{l \in L_i^2} \left[P(L^2) - P(L^2 \backslash \{l\}) \right].$$

By (7) we find that this holds if and only if

$$\pi_i(N, r, L^1) \leq \pi_i(N, r, L^2),$$

for all $i \in N$ and all $L^1 \subseteq L^2 \subseteq \bar{L}$. This completes the proof. □

6 Symmetric connections model

In this section we analyze a specific example, the symmetric connections model described and analyzed by *Jackson* and *Wolinsky* (1996) and *Watts* (1997). We start with a description of this model. Subsequently, we analyze under what conditions a link monotonic allocation scheme exists and relate these results to the conclusions of *Jackson* and *Wolinsky* (1996). Finally, we show that in this model the Myerson allocation scheme and the position allocation scheme will in general not be link monotonic.

The connections model represents social communication between individuals. Players communicate with people they are connected with, however, the value of the communication between two players depends on the shortest path in the graph between these two players. If we denote by $t_{ij}(L)$ the length of the shortest path between i and j in the graph (N, L), where $t_{ij}(L) = \infty$ if i and j are not connected, then the utility of player i in graph (N, L) is

$$u_i(L) = \sum_{j \in N \backslash \{i\}} \delta^{t_{ij}(L)} - c|L_i|,$$

where δ $(0 < \delta < 1)$ represents the idea that the value of communication between two players decreases when the distance between the two players increases and c denotes the costs for a player to maintain a link. The value of communication structure (N, L) is

$$r(L) = \sum_{i \in N} u_i(L).$$

The following theorem shows that in the symmetric connections model with at least three players $(u_i(L))_{i \in N, L \subseteq \bar{L}}$ is a link monotonic allocation scheme if and only if $c \leq \delta - \delta^2$. Additionally, we show that (N, r) possesses no link monotonic allocation scheme if $c > \delta - \delta^2$.

Theorem 6.1 The symmetric connections model with $|N| \geq 3$ has a link monotonic allocation scheme if and only if $c \leq \delta - \delta^2$. Moreover, if $c \leq \delta - \delta^2$ then $(u_i(L))_{i \in N, L \subseteq \bar{L}}$ is a link monotonic allocation scheme.

Proof: First we will show that if $c > \delta - \delta^2$ there exists no link monotonic allocation scheme. Subsequently, we show that if $c \leq \delta - \delta^2$ then $(u_i(L))_{i \in N, L \subseteq \bar{L}}$ is a link monotonic allocation scheme.

Assume $c > \delta - \delta^2$. The value of the complete graph is given by $r(\bar{L}) = \sum_{\{i,j\} \in \bar{L}} [2\delta - 2c]$ since all pairs of players are connected directly. Deleting one link, between players $1, 2 \in N$ reduces the costs by $2c$ and the profits by $2(\delta - \delta^2)$ since the length of the shortest path between players 1 and 2 increases from $t_{12} = 1$ to $t_{12} = 2$. Hence, $r(\bar{L} \backslash \{\{1, 2\}\}) = \sum_{\{i,j\} \in N} [2\delta - 2c] + 2c - 2(\delta - \delta^2) > r(\bar{L})$ since $c > \delta - \delta^2$. So, cooperation structure $(N, \bar{L} \backslash \{\{1, 2\}\})$ has a larger value than the cooperation structure $(N, \bar{L})$ which contains one link more. This implies that (N, r) cannot have a link monotonic allocation scheme.

Now, assume $c \leq \delta - \delta^2$. It follows directly that $(u_i(L))_{i \in N, L \subseteq \bar{L}}$ satisfies conditions (i) and (ii) of definition 1 on page 6. It remains to show that it satisfies condition (iii). It suffices to show that $u_i(L) \leq u_i(L^*)$ for all $L, L^* \subseteq \bar{L}$ with $|L| = |L^*| - 1$. Let $i \in N$, $L \subset \bar{L}$, and $\{j, k\} \in \bar{L} \backslash L$. Denote $L^* = L \cup \{\{j, k\}\}$. Since adding a link can only reduce the length of the shortest path between two players it holds that

$$t_{rs}(L) \geq t_{rs}(L^*) \tag{8}$$

for all $r, s \in N$, which implies that for all $r, s \in N$

$$\delta^{t_{rs}(L)} \leq \delta^{t_{rs}(L^*)}. \tag{9}$$

We will distinguish two cases, (i) $i \in \{j, k\}$ and (ii) $i \notin \{j, k\}$.

(i) $i \in \{j,k\}$. Without loss of generality assume $i = j$. Then

$$\begin{aligned} u_i(L^*) &= \sum_{r\neq i} \delta^{t_{ir}(L^*)} - c|L_i^*| \\ &= \sum_{r\neq i,k} \delta^{t_{ir}(L^*)} - c|L_i| + \delta^{t_{ik}(L^*)} - c \\ &\geq \sum_{r\neq i,k} \delta^{t_{ir}(L)} - c|L_i| + \delta^{t_{ik}(L)} \\ &= \sum_{r\neq i} \delta^{t_{ir}(L)} - c|L_i| \\ &= u_i(L), \end{aligned}$$

where the second equality follows by rearranging the terms and $|L_i^*| = |L_i| + 1$. The inequality follows by equation (8) and $\delta^{t_{ik}(L^*)} - \delta^{t_{ik}(L)} = \delta - \delta^{t_{ik}(L)} \geq \delta - \delta^2 \geq c$.
(ii) $i \notin \{j,k\}$. Then

$$u_i(L^*) = \sum_{r\neq i} \delta^{t_{ir}(L^*)} - c|L_i^*| \geq \sum_{r\neq i} \delta^{t_{ir}(L)} - c|L_i| = u_i(L),$$

where the inequality follows by equation (8) and $|L_i^*| = |L_i|$. □

Jackson and *Wolinsky* (1996) find that if $c < \delta - \delta^2$ then the complete graph is the unique pairwise stable network, i.e. there is no player that can strictly improve his payoff by breaking a link he is involved in and there is no pair of players that can both improve their payoffs by forming an additional link between them, where at least one improvement should be strict. They do not consider the case $c = \delta - \delta^2$. In this case the corresponding LMAS improves the payoffs of the players that form a link, however, this improvement need not be strict.

It is easily seen that if $|N| = 2$ there exists a LMAS if and only if $\delta \geq c$. In that case both the Myerson allocation scheme and the position allocation scheme coincide with $(u_i(L))_{i\in N, L\subseteq \bar{L}}$, which is a LMAS. However, we will show that if $|N| \geq 3$ then the position allocation scheme and the Myerson allocation scheme are both not a link monotonic allocation scheme in this model.

Theorem 6.2 In the symmetric connections model with $|N| \geq 3$ both the extended Myerson value and the position allocation scheme are not a LMAS.

Proof: The Myerson allocation scheme is not a LMAS since $\mu_2(N, r, \{\{1,2\},\{2,3\}\}) = 2\delta - 2c + \frac{2}{3}\delta^2$ while $\mu_2(N, r, \{\{1,2\},\{2,3\},\{1,3\}\}) = 2\delta - 2c$. Similarly, the position allocation scheme is not a LMAS since $\pi_2(N, r, \{\{1,2\},\{2,3\}\}) = 2\delta - 2c + \delta^2$ while $\pi_2(N, r, \{\{1,2\},\{2,3\},\{1,3\}\}) = 2\delta - 2c$. □

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