CentER

Discussion Paper

No. 2008–84

A GAME THEORETICAL APPROACH TO SHARING PENALTIES AND REWARDS IN PROJECTS

By Arantza Estévez-Fernández

September 2008

ISSN 0924-7815

TILBURG UNIVERSITY

A game theoretical approach to sharing penalties and rewards in projects*

Arantza Estévez-Fernández[1]

Abstract:

This paper analyzes situations in which a project consisting of several activities is not realized according to plan. If the project is expedited, a reward arises. Analogously, a penalty arises if the project is delayed. This paper considers the case of arbitrary monotonic reward and penalty functions on the total expedition and delay, respectively. Attention is focused on how to divide the total reward (penalty) among the activities: the core of a corresponding cooperative project game determines a set of stable allocations of the total reward (penalty). In the definition of project games, surplus (cost) sharing mechanisms are used to take into account the specific characteristics of the reward (penalty) function at hand. It turns outs that project games are related to bankruptcy and taxation games. This relation allows us to establish the nonemptiness of the core of project games.

Keywords: Project planning, delay, expedition, cost sharing mechanism, surplus sharing mechanism, bankruptcy problems, taxation problems, cooperative game, core.

JEL classification: C71

1 Introduction

A project consists of a set of activities, which interconnections are known, being completed over a period of time and intended to achieve a particular aim. The project time is the minimum time needed to end all the activities in a project. A good planning of a project is important to reduce the project time. Two important methods to schedule and coordinate the activities in a project are the PERT (Program Evaluation Review

*The author thanks to René van der Brink, Harold Houba, Gerard van der Laan and Ines Lindner for their comments and suggestions on a first version of this paper. Special thanks go to Peter Borm and Herbert Hamers for their encouragement and their helpful discussion and comments.

[1]Department of Econometrics and Operations Research, VU University Amsterdam, De Boelelaan 1105, 1081HV Amsterdam, The Netherlands. *E-mail address*: mestevez-fernandez@feweb.vu.nl

Technique) and the CPM (Critical Path Method). Although PERT has represented a big advance in the field of project management, many issues have been addressed during the last decades as the question of assessing the project success in Pinto and Slevin (1988) and Tubig and Abeti (1990), the role of research in projects in Sherwin and Isenson (1967), behavioral functions in technology-based innovative projects in Roberts and Fusfeld (1981), the classification of projects in types depending on the management style in Shenhar and Dvi (1996), the search of adequate success criteria for project management in Atkinson (1999) and the study of project scheduling with resource constrains in Dorndorf, Pesch and Phan-Huy (2000) and Möhring, Schulz, Stork and Uetz (2003).

An important issue while planning a project is the estimation of the time of the different activities. In many real-life situations the estimated duration and the duration after realization (or real duration) of an activity may differ and as a result the real duration of the project may not coincide with its planned duration. Usually, a penalty (reward) arises if a project is delayed (expedited). Two motivating examples are given below.

A usual case of penalties associated to delayed projects can be found in the building of a house. When a company is hired to build a house, there is usually a clause in the contract in which a penalty for the company is agreed upon if the house is finished later than planned. If the company has contracted out some specialized jobs to other firms, and these firms have incurred some delay, it is important to know for which part of the penalty these firms can be hold responsible.

An example of rewards associated to expedited projects can be found in projects ordered by a government. During the term of office of a government, it has been decided to build a new hospital in a city. For political reasons (e.g. the approach of new elections), the government wants it to be finished before the planned time. As an incentive, a bonus is promised if the works are expedited. In such a case it is important to know which activities (and therefore the companies involved in such activity) are responsible of the final expedition in order to allocate the bonus.

In practice, activities are often entrusted to different companies. If the project is delayed or expedited, an allocation problem arises: how to share the penalty (in case of delay) or the reward (in case of expedition) among the various activities and its corresponding companies. Cooperative games are a mathematical tool to provide an answer to this type of allocation problems. The focus of our study will be on defining a game in an adequate way and analyzing the corresponding core. The core of a game (Gillies (1953)) provides allocations of the total penalty or the total reward that are stable, i.e. no group of activities can reasonably object to allocations in the core. The interaction between operations research and cooperative game theory already goes back to the seventies, for a survey on the topic see Borm, Hamers and Hendrickx (2001).

The main focus in the literature on project problems has been on delayed project problems. Branzêi, Ferrari, Fragnelli and Tijs (2002) study delayed project problems in the framework of taxation problems and propose a specific allocation rule. Bergantiños and Sánchez (2002), analyze two other allocation rules for delayed project problems. A common feature in these papers is however that game theoretical aspects are only indirectly present in analyzing the allocation problem at hand. Estévez-Fernández, Borm and Hamers (2007) is the first paper to approach the related allocation problem from a direct game theoretical point of view. Moreover, Estévez-Fernández et al. (2007) is the first article where both delayed and expedited project problems are analyzed. Still, the paper is restricted to project problems where the penalty (reward) function is proportional with respect to the total delay (expedition) of the project. An important aspect of Estévez-Fernández et al. (2007) is that it provides the tools to obtain "fair" allocations of the corresponding penalty or reward by explicitly considering the structure of the project, i.e. the interconnections among the different activities.

In this paper, we extend the work in Estévez-Fernández et al. (2007) by analyzing project problems with arbitrary but monotonic penalty and reward functions. As in Estévez-Fernández et al. (2007), this is done by defining project games associated to project problems and considering the corresponding core as the solution set to the underlying allocation problem. Here, two stages are needed to define project games for monotonic penalty (reward) functions. In the first stage cost (surplus) sharing problems are used. As a first approach to share the penalty (reward) of a project, we look at each path in the project separately and we share the penalty (reward) that the path can be held responsible of among its activities by making use of a cost (surplus) sharing mechanism which is chosen by taking into account the specific characteristics of the penalty (reward) function at hand. In this first stage, the total amount shared among all the activities may exceed the total penalty (reward) of the project, hence a second stage is needed to exactly obtain allocations of the total penalty (reward). In the second stage, a project game is defined in the spirit of Estévez-Fernández et al. (2007) using the allocations obtained in the first stage.

For clarity of exposition, we have divided our study on project problems in three parts. First, we study delayed project problems, which are project problems in which the project has been delayed and none of the activities has been expedited. Second, we analyze expedited project problems, where a project has been expedited and none of the activities has been delayed. To conclude, we turn our attention to general project problems, in which activities in a project may be either delayed or expedited, possibly bringing a difference on the overall planned time for the project.

Our main result is that project games have a nonempty core. Moreover, it turns out that the games associated to delayed project problems can be described as the maximum of as many taxation games as paths in the project. On the other hand, the games associated to expedited project problems are convex and can be described as the maximum of a number of bankruptcy games (see O'Neill (1982)).

The structure of this paper is as follows. Section 2 recalls some basic notions from project problems, cost and surplus sharing problems, cooperative games, and bankruptcy and taxation problems. Sections 3 and 4 are dedicated to delayed project problems and expedited project problems, respectively. Section 5 analyzes general project problems. Section 6 concludes.

2 Preliminaries

2.1 Projects

A project consists of a set of activities for which the inter-connections are known. These activities are completed over a period of time and intended to achieve a particular aim. Let N denote the set of activities of a project. Given an activity $i \in N$, let P_i denote the set of *predecessors* of i, i.e. the set of activities that have to be processed before i can start. Analogously, let F_i be the set of *followers* of i, i.e. the set of activities that need i to be completed before starting. A *project* is defined as a collection of *ordered* subsets of N or paths, $\{N_1, \ldots, N_m\}$, where a bijection $\sigma_a : \{1, \ldots |N_a|\} \to N_a$ describes the order in N_a, $a \in \{1, \ldots, m\}$, satisfying the following conditions:

(i) $N = \bigcup_{a=1}^{m} N_a$;

(ii) $F_{\sigma_a(|N_a|)} = \emptyset$, $P_{\sigma_a(1)} = \emptyset$, and $P_{\sigma_a(r)} = \{\sigma_a(1), \ldots, \sigma_a(r-1)\}$ for every $a \in \{1, \ldots, m\}$ and every $r \in \{2, \ldots, |N_a|\}$;

(iii) for $a, b \in \{1, \ldots, m\}$, if $i, j \in N_a \cap N_b$ with $\sigma_a^{-1}(i) < \sigma_a^{-1}(j)$, then $\sigma_b^{-1}(i) < \sigma_b^{-1}(j)$.

Throughout, there is no specific need to explicitly keep track of the ordering. Therefore, $\sigma_1, \ldots, \sigma_m$ are suppressed from the notations.

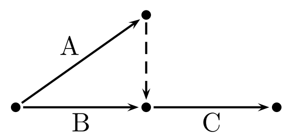
Note that a project can be represented by a directed graph where the set of arcs corresponds to the set of activities. In order to avoid multiple arcs, dummy activities are introduced in the graph (a *dummy activity* is an activity that does not consume neither time nor resources). Dummy activities are represented by a dashed arc.

Example 2.1. Table 1 gives the set of activities of a project with their corresponding predecessors.

Activity	Predecessors
A	
B	
C	A,B

Table 1: Predecessors of activities in Example 2.1.

Here, the set of activities is $N = \{\mathrm{A, B, C}\}$ and the collection of paths is $\{N_1, N_2\}$, with $N_1 = \{\mathrm{A, C}\}$, and $N_2 = \{\mathrm{B, C}\}$. The graphical representation of this project is given in Figure 1.



◇

Figure 1: Representation of the project given in Table 1.

Associated to a project $\{N_1, \ldots, N_m\}$, there is a duration function $l : N \to \mathbb{R}_+$ with $l(i)$ denoting the length or *duration* of activity $i \in N$. Given a project $\{N_1, \ldots, N_m\}$ and a duration function l, we define the *duration of a path* N_a according to l, $D(N_a, l)$, as the sum of the duration of its activities, i.e. $D(N_a, l) = \sum_{i \in N_a} l(i)$. The *duration of the project* according to l, $D(l)$, is the maximum duration of its paths, i.e. $D(l) = \max_{1 \leq a \leq m} \{D(N_a, l)\}$. The *slack of* N_a according to l, $\mathrm{slack}(N_a, l)$, is the maximum time that the activities of N_a can be delayed without altering the duration of the project, i.e. $\mathrm{slack}(N_a, l) = D(l) - D(N_a, l)$. We say that a path is *critical* if it has slack zero.

Example 2.2. Consider the project given in Example 2.1 and let $l : N \to \mathbb{R}_+$ be given by $l(\mathrm{A}) = 15$, $l(\mathrm{B}) = 10$, and $l(\mathrm{C}) = 8$. Table 2 summarizes the duration and slack of the paths. Note that $D(l) = 23$.

N_a	$D(N_a, l)$	$\mathrm{slack}(N_a, l)$
AC	23	0
BC	18	5

Table 2: Duration and slack of the paths in Example 2.2. ◇

Throughout, we use a fixed notation for two specific duration functions. We denote by $p : N \to \mathbb{R}_+$ the function representing the *planned* or estimated *time* of the activities and by $r : N \to \mathbb{R}_+$ the function giving the *real time* of the activities after the realization of the project. We define the *delay function* $d : N \to \mathbb{R}_+$ as $d(i) = (r(i) - p(i))_+ (:= \max\{r(i) - p(i), 0\})$, i.e. $d(i)$ represents the delay of activity i. Analogously, we define the *expedition function* $e : N \to \mathbb{R}_+$ as $e(i) = (p(i) - r(i))_+$, i.e. $e(i)$ represents the expedition of activity i.

Example 2.3. Consider the project given in Example 2.1; let the planned time $p : N \to \mathbb{R}_+$ be given by $p(\mathrm{A}) = 15$, $p(\mathrm{B}) = 10$, and $p(\mathrm{C}) = 8$, and let the real time $r : N \to \mathbb{R}_+$ be given by $r(\mathrm{A}) = 16$, $r(\mathrm{B}) = 3$, and $r(\mathrm{C}) = 10$. The delay and expedition functions are given in Table 3. ◇

	A	B	C
$d(i)$	$(16-15)_+ = 1$	$(3-10)_+ = 0$	$(10-8)_+ = 2$
$e(i)$	$(15-16)_+ = 0$	$(10-3)_+ = 7$	$(8-10)_+ = 0$

Table 3: Delay and expedition functions in Example 2.3.

2.2 Cost sharing and surplus sharing problems

A *cost sharing problem* is defined by a tuple (N, q, c) where $N = \{1, 2, \ldots, n\}$ is the set of agents (or players), $q \in \mathbb{R}^N_+$ is a vector of nonnegative numbers, with q_i representing the demand of agent $i \in N$, and $c : \mathbb{R} \to \mathbb{R}_+$ is a nondecreasing cost function satisfying $c(t) = 0$ for $t \leq 0$. A *cost sharing mechanism* on a class of cost sharing problems $\mathcal{C}$ is a mapping y that assigns to each $(N, q, c) \in \mathcal{C}$ a vector of cost shares $y(N, q, c) \in \mathbb{R}^N_+$, i.e. $\sum_{i \in N} y_i(N, q, c) = c(\sum_{i \in N} q_i)$, satisfying that if $q_i = 0$, then $y_i(N, q, c) = 0$.

Several mechanisms can be found in the literature (see e.g. Koster (1999)); we recall here one of the most studied cost sharing mechanisms in the literature, the serial cost sharing mechanism, y^s. Let (N, q, c) be a cost sharing problem and assume without loss of generality that $q_1 \leq q_2 \leq \ldots \leq q_n$, then the *serial cost sharing mechanism* is defined by

$$y^s_1(N, q, c) = \frac{c(nq_1)}{n},$$
$$y^s_i(N, q, c) = \frac{c(nq_1)}{n} + \sum_{k=2}^{i} \frac{c(\sum_{j=1}^{k-1} q_j + (n-k+1)q_k) - c(\sum_{j=1}^{k-2} q_j + (n-k+2)q_{k-1})}{n-k+1} \text{ for every } i \in N \setminus \{1\}.$$

Example 2.4. Let (N, q, c) be a cost sharing problem with $N = \{1, 2, 3\}$, $q = (2, 5, 6)$, and c defined by

$$c(t) = \begin{cases} 0 & \text{if } t \leq 0; \\ t^2 + 100, & \text{if } t > 0. \end{cases}$$

In this case, $c(2 + 5 + 6) = 269$ and $y^s(N, q, c) = (45\frac{1}{3}, 99\frac{1}{3}, 124\frac{1}{3})$. Table 4 summarizes the computation of the serial cost sharing rule.

t	$c(t)$
$3q_1 = 6$	136
$q_1 + 2q_2 = 12$	244
$q_1 + q_2 + q_3 = 13$	269

i	$\frac{c(6)}{3}$	$\frac{c(12)-c(6)}{2}$	$\frac{c(13)-c(12)}{1}$	y^s
1	$\frac{136}{3} = 45\frac{1}{3}$			$45\frac{1}{3}$
2	$\frac{136}{3} = 45\frac{1}{3}$	$\frac{108}{2} = 54$		$99\frac{1}{3}$
3	$\frac{136}{3} = 45\frac{1}{3}$	$\frac{108}{2} = 54$	$\frac{25}{1} = 25$	$124\frac{1}{3}$

Table 4: Computation of the cost sharing rule in Example 2.4.

◇

Analogously to cost sharing problems, one can think of surplus sharing problems. All definitions given above for cost sharing problems can easily be translated to surplus sharing problems.

2.3 TU games

A *cooperative (TU) game* in characteristic function form is an ordered pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. In general, $v(S)$ represents the value of coalition S, i.e. the joint payoff that can be obtained by this coalition when its members decide to cooperate.

A cooperative game can reflect costs or rewards. A game reflecting costs is denoted by a map c, while a game reflecting rewards is denoted by a map v.

The *core* of a game (N, v) is defined by

$$\text{Core}(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\},$$

i.e. the core is the set of efficient allocations of $v(N)$ to which no coalition can reasonably object. An important subclass of games with nonempty core is the class of convex games (see Shapley (1971)). A game (N, v) is said to be *convex* if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $i \in N$ and every $S \subset T \subset N \setminus \{i\}$.

Let (N, c) be a cost game. The corresponding *cost savings game* (N, v) is defined by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S)$$

for all $S \subset N$.

The properties and solutions concepts for cooperative cost games can consequently be derived from the above definitions. The equivalent of a convex game for a cost game is a *concave* game.

2.4 Bankruptcy and taxation problems

A *bankruptcy problem* is defined by a tuple (N, E, c) where $N = \{1, \ldots, n\}$ is the set of agents (or players), E is the estate that must be shared among the agents, and $c \in \mathbb{R}^N$ is the vector of claims of the agents, satisfying $\sum_{i \in N} c_i > E$. Bankruptcy problems have being firstly studied from a game theoretical viewpoint in O'Neill (1982). Given a bankruptcy problem (N, E, c), the corresponding *bankruptcy game*, $(N, v_{(N,E,c)})$ is defined by

$$v_{(N,E,c)}(S) = \left(E - \sum_{i \in N \setminus S} c_i\right)_+$$

for every $S \subset N$. In Curiel et al. (1987) it is shown that bankruptcy games are convex.

Taxation problems can be seen as dual of bankruptcy problems. A *taxation problem* is defined by a tuple (N, E, c) where $N = \{1, \ldots, n\}$ is the set of agents (or players), E is the tax that must be collected among the agents, and $c \in \mathbb{R}^N$ is the vector of the abilities to pay of the players, satisfying $\sum_{i \in N} c_i > E$. Given a taxation problem (N, E, c), the corresponding *taxation game*, $(N, c_{(N,E,c)})$, is defined by

$$c_{(N,E,c)}(S) = \min\left\{E, \sum_{i \in S} c_i\right\}$$

for every $S \subset N$. In Branzêi et al. (2002) it is shown that taxation games are concave.

3 Delayed project games

A *delayed project problem* arises when the planned time of the activities in a project was underestimated, incurring delay of the project. Associated to the delay of the project, there is a non-decreasing penalty function $K : \mathbb{R} \to \mathbb{R}_+$ satisfying $K(t) = 0$ for every $t \leq 0$. A delayed project problem can be described by a 4-tuple $(\{N_1, \ldots, N_m\}, p, r, K)$, where p and r satisfy $p \leq r$.

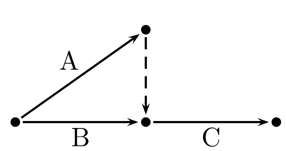
When a penalty forms due to the delay of the project, one can think of sharing the delay of the project among the activities in a first (linear) stage and allocating the (possibly nonlinear) penalty among the activities according to the delay they have been held responsible of in a second stage. This approach has already been suggested in Branzêi et al. (2002). The problem with this procedure is that the specific characteristics of the penalty function may be neglected. We show the inadequacy of this procedure in the following example in which the allocation of the total delay is obtained by considering the core of the game defined in Estévez-Fernández et al. (2007), which we subsequently recall.

Let $(\{N_1, \ldots, N_m\}, p, r, \tilde{K})$ be a delayed project problem with $\tilde{K}(t) = t$ if $t > 0$ and $\tilde{K}(t) = 0$ if $t \leq 0$. The associated *(linear) delayed project game*, $(N, \tilde{c})$, is defined by

$$\tilde{c}(S) = \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_a \cap S} d(i), \left(\sum_{i \in N_a} d(i) - \text{slack}(N_a, p) \right)_+ \right\} \right\}$$

for $S \subset N$, where $\mathcal{P}(S)$ is the set (of indices) of paths in which activities in S are involved, i.e. $\mathcal{P}(S) = \{a \in \{1, \ldots, m\} |\ N_a \cap S \neq \emptyset\}$.

Example 3.1. Consider the delayed project problem $(\{N_1, N_2\}, p, r, K)$ with $N_1 = \{\text{A}, \text{C}\}$, $N_2 = \{\text{B}, \text{C}\}$, $p(\text{A}) = 3$, $p(\text{B}) = 5$, $p(\text{C}) = 2$, $r(\text{A}) = 8$, $r(\text{B}) = 9$, $r(\text{C}) = 3$, and $K(t) = 0$ if $t \leq 0$, $K(t) = t^2$ if $0 < t \leq 4$, and $K(t) = 2t^2$ if $t > 4$. The project is represented in Figure 2 together with the duration of the paths according to the planned and real times.



N_a	$D(N_a, p)$	slack(N_a, p)	$D(N_a, r)$
AC	5	2	11
BC	7	0	12

Figure 2: Representation of the project in Example 3.1 and durations of the paths.

In this delay problem, $D(p) = 7$ and $D(r) = 12$, therefore there is a total delay of $D(r) - D(p) = 5$, with an associated cost of $K(5) = 50$.

Note that path $N_1 = \{\mathrm{A}, \mathrm{C}\}$ gives a delay of $11 - 7 = 4$ to the project with an associated penalty of $K(4) = 16$, while path $N_2 = \{\mathrm{B}, \mathrm{C}\}$ is responsible of a delay of $12 - 7 = 5$ with an associated penalty of $K(5) = 50$.

Following the proposed two-stage approach, we first share the total delay of 5 among the activities by making use of the associated linear delayed project game, which coalitional values are: $\tilde{c}(\{\mathrm{A}\}) = 4$, $\tilde{c}(\{\mathrm{B}\}) = 4$, $\tilde{c}(\{\mathrm{C}\}) = 1$, $\tilde{c}(\{\mathrm{A}, \mathrm{B}\}) = 4$, $\tilde{c}(\{\mathrm{A}, \mathrm{C}\}) = 4$, $\tilde{c}(\{\mathrm{B}, \mathrm{C}\}) = 5$, $\tilde{c}(N) = 5$. It can be checked that

$$\mathrm{Core}(\tilde{c}) = conv\{(0, 4, 1), (3, 1, 1)\}.$$

Let us consider $(3, 1, 1)$ as the allocation of the total delay among the activities. In a second stage, we divide the total penalty proportionally to this allocation of responsibilities, as suggested in Branzêi et al. (2002), giving an allocation of $(30, 10, 10)$. Note that this allocation assigns a total penalty of $30 + 10 = 40$ to A and C together, which is more than 16, the penalty associated to the delay induced by $N_1 = \{\mathrm{A}, \mathrm{C}\}$. $\diamond$

The inadequacy in first sharing delay "responsibilities" among the activities and then the penalty in a second stage is that the characteristics of the penalty function are not reflected on the final share of the total penalty. Our approach will follow the opposite reasoning. We first allocate the penalty associated to the gross delay of each path (i.e. the total delay created by its activities, without taking into account the slack of the path) among its activities. In a second step, we use cooperative games to share the penalty of the project among all its activities, using the initial allocations as reference points.

If activities in a path are delayed and they cause a delay of the project, we can consider this as a cost sharing problem where the demands of the activities are their delays and the cost function is the penalty function. Given a delayed project problem $(\{N_1, \ldots, N_m\}, p, r, K)$, for each $a \in \{1, \ldots, m\}$ we consider the cost sharing problem

$$(N_a, q^a, K)$$

with $q_i^a = d(i)$ for every $i \in N_a$.

Selecting a cost sharing mechanism y by taking into account the type of penalty function at hand, we denote by y^a the allocation proposed by our mechanism to the cost sharing problem (N_a, q^a, K), i.e. $y^a = y(N_a, q^a, K)$. Here, we pessimistically assume that the activities in N_a are not allowed to make use of the planned slack in the path and have to pay the cost associated to their total delay. In this way, y_i^a represents the cost that i is held responsible of for the cost associated to the total delay of the activities in N_a. By using these allocations as starting points, we associate a delayed project game to a delayed project problem where the set of players is the set of activities and the cost of a coalition is the maximal amount the coalition can be held responsible of with respect to the different paths involved in the coalition. Formally, given a delayed project problem $(\{N_1, \ldots, N_m\}, p, r, K)$ and a cost sharing mechanism y, we define the associated cost game (N, c_y) by

$$c_y(S) = \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_a \cap S} y_i^a, K\left(\sum_{i \in N_a} d(i) - \text{slack}(N_a, p) \right) \right\} \right\}$$

for every $S \subset N$. Recall that $\mathcal{P}(S) = \{a \in \{1, \ldots, m\} | \ N_a \cap S \neq \emptyset\}$ represents the set of paths in which activities of S are involved.

The game pessimistically assigns to a coalition the maximum penalty that the coalition can be held responsible of, taking into account that the activities in a path should never pay more than neither their initial allocation in the path nor the penalty associated to the net delay of the path (i.e. the total delay of the activities in the path minus the planned slack of the path).

Example 3.2. Consider the delayed project problem given in Example 3.1. Recall that $N_1 = \{\text{A}, \text{C}\}$, $N_2 = \{\text{B}, \text{C}\}$, $p(\text{A}) = 3$, $p(\text{B}) = 5$, $p(\text{C}) = 2$, $r(\text{A}) = 8$, $r(\text{B}) = 9$, $r(\text{C}) = 3$, and $K(t) = 0$ if $t \leq 0$, $K(t) = t^2$ if $0 < t \leq 4$, and $K(t) = 2t^2$ if $t > 4$. Note that $d(\text{A}) = 5$, $d(\text{B}) = 4$, $d(\text{C}) = 1$. As cost sharing mechanism, we take the serial cost sharing mechanism, then $y^1 = y(\{\text{A}, \text{C}\}, (5, 1), K) = (70, 2)$ and $y^2 = y(\{\text{B}, \text{C}\}, (4, 1), K) = (48, 2)$. Table 5 gives the values of the associated delayed project game.

S	$\{A\}$	$\{\text{B}\}$	$\{\text{C}\}$	$\{\text{A}, \text{B}\}$	$\{\text{A}, \text{C}\}$	$\{\text{B}, \text{C}\}$	$\{\text{A}, \text{B}, \text{C}\}$
$c_y(S)$	16	48	2	48	16	50	50

Table 5: Values of the delayed project game in Example 3.2.

Next, we show how to compute the value of coalition {A,C}. Note that $\mathcal{P}(S) = \{1, 2\}$ since A,C $\in N_1$ and C $\in N_2$. Then,

$$\begin{aligned} c_y(\{A, C\}) &= \max\{\min\{y_A^1 + y_C^1, K(5 + 1 - 2)\}, \min\{y_C^2, K(4 + 1 - 0)\}\} \\ &= \max\{\min\{70 + 2, 16\}, \min\{2, 50\}\} \\ &= \max\{16, 2\} = 16. \end{aligned}$$

It can be checked that the core of the game is $\text{Core}(c_y) = \text{conv}\{(0, 48, 2), (14, 34, 2)\}$. Note that this game is not concave by taking $i = \text{B}$, $S = \{\text{A}\}$, and $T = \{\text{A}, \text{C}\}$. $\diamond$

Given a delayed project problem $(\{N_1, \ldots, N_m\}, p, r, K)$ and a cost sharing mechanism y, the associated delayed project game, (N, c_y), can be described as the maximum of as many taxation games as paths in the project, where the taxation problem associated to path $a \in \{1, \ldots, m\}$ is

$$(N, E^a, c^{y,a}) \tag{3.1}$$

with $E^a = K(\sum_{i \in N_a} d(i) - \text{slack}(N_a, p))$, $c_i^{y,a} = y_i^a$ if $i \in N_a$ and $c_i^{y,a} = 0$ otherwise.

Theorem 3.1. *Let $(\{N_1, \ldots, N_m\}, p, r, K)$ be a delayed project problem, let y be a cost sharing mechanism, and let (N, c_y) be the associated delayed game. Then,*

$$c_y(S) = \max_{a \in \{1,\ldots,m\}} \{c_{(N,E^a,c^{y,a})}(S)\}$$

for every $S \subset N$.

Proof: Let $S \subset N$, then

$$\begin{aligned}
c_y(S) &= \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_a \cap S} y_i^a, K \left(\sum_{i \in N_a} d(i) - \text{slack}(N_a, p) \right) \right\} \right\} \\
&= \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_a \cap S} c_i^{y,a}, E^a \right\} \right\} \\
&= \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in S} c_i^{y,a}, E^a \right\} \right\} \\
&= \max_{a \in \mathcal{P}(S)} \left\{ c_{(N,E^a,c^{y,a})}(S) \right\} \\
&= \max_{a \in \{1,\ldots,m\}} \{ c_{(N,E^a,c^{y,a})}(S) \}
\end{aligned}$$

where the second equality is an immediate consequence of the definitions of $c^{y,a}$ and E^a, the third one follows because $c_i^{y,a} = 0$ for every $i \in N \setminus N_a$, and the last equality is a direct consequence of $c_{(N,E^a,c^{y,a})}(S) = 0$ for every $a \in \{1, \ldots, m\} \setminus \mathcal{P}(S)$ since $c_i^{y,a} = 0$ for every $i \in N_a$ with $a \in \{1, \ldots, m\} \setminus \mathcal{P}(S)$. $\square$

Example 3.3. Consider the delayed project game in Example 3.2. Associated to each path N_a we have a taxation problem $(N, E^a, c^{y,a})$:

$$\begin{aligned}
N_1 &= \{\text{A}, \text{C}\}, \quad E^1 = K(5 + 1 - 2) = 16, \quad c^{y,1} = (y_\text{A}^1, 0, y_\text{C}^1) = (70, 0, 2), \\
N_2 &= \{\text{B}, \text{C}\}, \quad E^2 = K(4 + 1 - 0) = 50, \quad c^{y,2} = (0, y_\text{B}^2, y_\text{C}^2) = (0, 48, 2).
\end{aligned}$$

Table 6 gives the values of the corresponding taxation games and delayed project game. $\diamond$

S	$\{A\}$	{B}	{C}	{A, B}	{A, C}	{B, C}	{A, B, C}
$c_{(N,E^1,c^1)}(S)$	16	0	2	16	16	2	16
$c_{(N,E^2,c^2)}(S)$	0	48	2	48	2	50	50
$c_y(S)$	16	48	2	48	16	50	50

Table 6: Values of the taxation games and delayed project game in Example 3.3.

Theorem 3.2. *Delayed project games have a nonempty core.*

Proof: Let $(\{N_1, \ldots, N_m\}, p, r, K)$ be a delayed project problem and let y be a cost sharing mechanism. Let $\bar{a} \in \{1, \ldots, m\}$ be such that $c_y(N) = c_{(N,E^{\bar{a}},c^{y,\bar{a}})}(N)$. The core of $(N, c_{(N,E^{\bar{a}},c^{y,\bar{a}})})$ is nonempty since taxation games are concave, then we can take $x \in \text{Core}(c_{(N,E^{\bar{a}},c^{y,\bar{a}})})$ and

$$\sum_{i \in N} x_i = c_{(N,E^{\bar{a}},c^{y,\bar{a}})}(N) = c_y(N) \tag{3.2}$$

where the first equality follows by definition of core element and the second one by assumption. Moreover, for every $S \subset N$ we have

$$\sum_{i \in S} x_i \leq c_{(N,E^{\bar{a}},c^{y,\bar{a}})}(S) \leq \max_{a \in \{1,\ldots,m\}} \{c_{(N,E^{a},c^{y,a})}(S)\} = c_y(S) \tag{3.3}$$

where the first inequality follows by definition of core element and the equality is a direct consequence of Lemma 3.1. By equations (3.2) and (3.3) we have that $x \in \text{Core}(c_y)$. □

4 Expedited project games

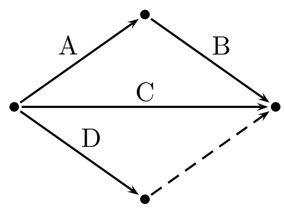
An *expedited project problem* forms when the planned time of the activities was overestimated, i.e. the real time of an activity is at most its planned time, bringing expedition to the project. Associated to the expedition of the project, we have a non-decreasing reward function $R : \mathbb{R} \to \mathbb{R}_+$ satisfying $R(t) = 0$ for every $t \leq 0$. An expedited project problem can be then described by a 4-tuple $(\{N_1, \ldots, N_m\}, p, r, R)$ where p and r satisfy $p \geq r$. At this stage, one may think about using the duality between delayed and expedited project problems: an expedited project problem can be viewed as a delayed project problem by interchanging the planned and real time vectors. This dual approach however is inadequate to solve expedited project games. For an example see Example 4.1 in Estévez-Fernández et al. (2007).

As in the case of delayed project problems, when a reward arises due to an expedition of the project, one can think of sharing the expedition of the project among the activities in a first (linear) stage, and allocating the (possibly nonlinear) reward among the activities according to the expedition they have been held responsible of in a second stage. Similarly to delayed project problems, this approach may be inadequate since the specific characteristics of the reward function may be disregarded in the final allocation of the total reward.

In defining expedited project games, we follow the ideas in Estévez-Fernández et al. (2007). Before starting with the description of expedited project games, we give the following example that illustrates the ideas behind the model.

Example 4.1. Consider the expedited project problem $(\{N_1, N_2, N_3\}, p, r, R)$ with $N_1 = \{\text{A}, \text{B}\}$, $N_2 = \{\text{C}\}$, $N_3 = \{\text{D}\}$, $p(\text{A}) = 9$, $p(\text{B}) = 12$, $p(\text{C}) = 17$, $p(\text{D}) = 16$, $r(\text{A}) = 6$, $r(\text{B}) = 8$, $r(\text{C}) = 15$, $r(\text{D}) = 13$, and

$R(t) = 0$ if $t \leq 0$, $R(t) = t^4$ if $0 < t \leq 4$, and $R(t) = t^2 + 240$ if $t > 4$. The project is represented in Figure 3 together with the duration of the paths according to the planned and real times.



N_a	$D(N_a, p)$	slack(N_a, p)	$D(N_a, r)$
AB	21	0	14
C	17	4	15
D	16	5	13

Figure 3: Representation of the project in Example 4.1 and durations of the paths.

In this expedited project problem, $D(p) = 21$ and $D(r) = 15$, therefore there is a total expedition of $D(p) - D(r) = 6$ and a reward of $R(6) = 276$.

Note that the critical path $N_1 = \{\mathrm{A}, \mathrm{B}\}$ is indispensable to expedite the project. We subsequently explain how this expedition is achieved. First, suppose that only the activities in N_1 act according to realization while activities in N_2 and N_3 act according to plan, then the project is expedited in 4 with an associated reward of $R(4) = 256$, N_2 becomes critical, and N_3 has a slack of 1. Note that N_1 is responsible by itself of a reward of 256 and that N_2 becomes indispensable to continue expediting. Second, suppose that only the activities in N_1 and N_2 act according to realization while activities in N_3 act according to plan, then the project is expedited in one extra unit of time with an associated marginal reward of $R(5) - R(4) = 265 - 256 = 9$, and N_3 becomes critical. Note that N_1 and N_2 are exclusively responsible of an extra reward of 9 and that N_3 becomes indispensable to continue expediting. Finally, suppose that all activities act according to realization, then there is an additional expedition of 1 with a marginal reward of $R(6) - R(5) = 276 - 265 = 11$ of which N_1, N_2, and N_3 are responsible. The contribution of the paths to the reward obtained by the expedition of the project is summarized in Table 7.

	Phase 1	Phase 2	Phase 3
N_1	256	9	11
N_2	0	9	11
N_3	0	0	11

Table 7: Durations of the paths in Example 4.1.

Note that the sum of the first row gives the total reward. This type of decomposition into levels of expedition plays an important role in the definition of expedited project games. $\diamond$

In order to solve expedited project problems, we first optimistically allocate the part of the expedition that each path could contribute to among its activities. In a second step, we use cooperative games to share the total reward among all the activities in the project using for this the initial allocations as reference points.

Before defining expedited project games, we need to introduce some notation. Let $(\{N_1,\ldots,N_m\},p,r,R)$ be an expedited project problem. We denote by I_1 the set (of indices) of critical paths according to the planned time. Formally,

$$I_1=\{a\in\{1,\ldots,m\}|\ \text{slack}(N_a,p)=0\}.$$

Recursively, we define for $k\geq 2$,

$$I_k=\left\{a\in\{1,\ldots,m\}\setminus\bigcup_{l=1}^{k-1}I_l|\ \text{slack}(N_a,p)\leq\text{slack}(N_b,p)\text{ for all }b\in\{1,\ldots,m\}\setminus\bigcup_{l=1}^{k-1}I_l\right\},$$

i.e. I_k corresponds to all paths that would be critical in the (sub)project if all the paths in $I_1,\ldots,I_{k-1}$ were not present. By slack(I_k) we denote the slack of the paths in I_k according to the planned time, i.e. $\text{slack}(I_k)=\text{slack}(N_a,p)$ for each $a\in I_k$. Let $h\geq 1$ be such that $\text{slack}(I_h)<D(p)-D(r)\leq\text{slack}(I_{h+1})$. For $k=1,\ldots,h$, we define F^k as the marginal contribution of the paths in $I_1,\ldots,I_k$ to the total reward associated to the expedition. Formally,

$$F^k=\begin{cases}R(\text{slack}(I_{k+1}))-R(\text{slack}(I_k)) & \text{if } 1\leq k<h;\\ R(D(p)-D(r))-R(\text{slack}(I_h)) & \text{if } k=h.\end{cases}$$

Note that $\sum_{k=1}^{h}F^k=R(D(p)-D(r))$ since $R(\text{slack}(I_1))=0$.

Next, we define the maximal amount of reward that an activity can claim for itself. For this, we consider for each $a\in\{1,\ldots,m\}$ the surplus sharing problem

$$(N_a,p^a,R^a)$$

with $p_i^a=e(i)$ for every $i\in N_a$ and

$$R^a(t):=\begin{cases}R(t+\text{slack}(N_a,p))-R(\text{slack}(N_a,p)) & \text{if } t\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Selecting a surplus sharing mechanism z (taking into account the type of reward function at hand), we denote by z^a the allocation proposed by our mechanism to the surplus sharing problem (N_a,p^a,R^a), i.e. $z^a=z(N_a,p^a,R^a)$. Here, z_i^a is the maximum amount that i can claim according to the surplus sharing mechanism if its path is awarded with the total expedition that it can bring to the project. Then, we optimistically define the *vector of maximal rewards*, f^z, by

$$f_i^z=\max_{a:N_a\ni i}\{z_i^a\},$$

i.e. f_i^z is the maximum reward that activity i can claim from the expedition of the project when the surplus sharing mechanism z is considered.

Following the underlying ideas in Estévez-Fernández et al. (2007), to an expedited project problem we associate an expedited project game where the set of players is the set of activities and the value of a coalition is the sum over all $k \in \{1, \ldots, h\}$ of those specific parts of the contribution to the total reward F^k for which the activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} I_l$ cannot be held responsible for anymore at that phase. Formally, given an expedited project problem $(\{N_1, \ldots, N_m\}, p, r, R)$ we define the associated game (N, v_z), where v_z is defined by

$$v_z(S) = \sum_{k=1}^{h} \left(F^k - w_z^k(S) \right)$$

for every $S \subset N$, where for all $k \in \{1, \ldots, h\}$, $w_z^k(S)$ is recursively defined by

$$w_z^k(S) = \min \left\{ \sum_{i \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} f_i^z - \sum_{l=1}^{k-1} w_z^l(S),\ F^k \right\},$$

where $N_{I_l} := \cup_{a \in I_l} N_a$. Here, $w_z^k(S)$ represents the part of the contribution to the total reward F^k that players in S maximally would have to concede to players in the paths corresponding to $\bigcup_{l=1}^{k} I_l$ outside S, taking into account earlier concessions from the previous phases. Note that w_z^k is non-negative. Moreover, $v_z(N)$ equals the total expedition of the project because $w_z^k(N) = 0$ for any $k \in \{1, \ldots, h\}$.

Example 4.2. Consider the expedited project problem given in Example 4.1. Recall that the problem was given by $(\{N_1, N_2, N_3\}, p, r, K)$ with $N_1 = \{\text{A}, \text{B}\}$, $N_2 = \{\text{C}\}$, and $N_3 = \{\text{D}\}$, $p(\text{A}) = 9$, $p(\text{B}) = 12$, $p(\text{C}) = 17$, $p(\text{D}) = 16$, $r(\text{A}) = 6$, $r(\text{B}) = 8$, $r(\text{C}) = 15$, $r(\text{D}) = 13$, and $R(t) = 0$ if $t \leq 0$, $R(t) = t^4$ if $0 < t \leq 4$, and $R(t) = t^2 + 240$ if $t > 4$.

Here, $D(p) = 21$ and $D(r) = 15$, and then the total expedition is $D(p) - D(r) = 6$ with an associated reward of $R(6) = 276$. Besides, $e(\text{A}) = 3$, $e(\text{B}) = 4$, $e(\text{C}) = 2$ and $e(\text{D}) = 3$; $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$; $h = 3$; $F^1 = R(4) - R(0) = 256$, $F^2 = R(5) - R(4) = 9$, and $F^3 = R(6) - R(5) = 11$.

For the computation of f^z, we consider the serial surplus sharing mechanism. Associated to each path N_a we have the surplus problem:

(N_1, p^1, R^1): $N_1 = \{\text{A}, \text{B}\}$, $p^1 = (3, 4)$, $R^1 = R$, and then $z^1 = (138, 151)$,

(N_2, p^2, R^2): $N_2 = \{\text{C}\}$, $p^2 = (2)$, $R^2(t) = \begin{cases} R(t+4) - R(4) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ and then $z^2 = (20)$,

(N_3, p^3, R^3): $N_3 = \{\text{D}\}$, $p^3 = (3)$, $R^3(t) = \begin{cases} R(t+5) - R(5) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ and then $z^3 = (39)$, which gives $f^z = (138, 151, 20, 39)$.

Let (N, v_z) be the associated expedited project game. For coalition $\{A, C\}$ we have:

$$\begin{aligned} w_z^1(\{\mathrm{A},\mathrm{C}\}) &= \min\left\{f_{\mathrm{B}}^z, F^1\right\} = \min\{151, 256\} = 151,\\ w_z^2(\{\mathrm{A},\mathrm{C}\}) &= \min\left\{f_{\mathrm{B}}^z - w_z^1(\{\mathrm{A},\mathrm{C}\}), F^2\right\} = \min\{151-151, 9\} = 0,\\ w_z^3(\{\mathrm{A},\mathrm{C}\}) &= \min\left\{f_{\mathrm{B}}^z + f_{\mathrm{D}}^z - w_z^1(\{\mathrm{A},\mathrm{C}\}) - w_z^2(\{\mathrm{A},\mathrm{C}\}), F^3\right\}\\ &= \min\{151+39-151-0, 11\} = 11, \end{aligned}$$

and therefore

$$\begin{aligned} v_z(\{\mathrm{A},\mathrm{C}\}) &= (F^1 - w_z^1(\{\mathrm{A},\mathrm{C}\})) + (F^2 - w_z^2(\{\mathrm{A},\mathrm{C}\})) + (F^3 - w_z^3(\{\mathrm{A},\mathrm{C}\}))\\ &= (256-151) + (9-0) + (11-11) = 114. \end{aligned}$$

All coalitional values are given in Table 8.

S	{A}	{B}	{C}	{D}	{A, B}	{A, C}	{A, D}	{B, C}	{B, D}	{C, D}
$v_z(S)$	105	118	0	0	256	114	105	127	118	0

S	{A, B, C}	{A, B, D}	{A, C, D}	{B, C, D}	{A, B, C, D}
$v_z(S)$	265	256	125	138	276

Table 8: Values of the expedited project game in Example 4.2.

◇

Given an expedited project problem $(\{N_1, \ldots, N_m\}, p, r, R)$ and a surplus sharing mechanism z, the associated project game (N, v_z) can be expressed as the maximum of as many bankruptcy games as levels of expedition in the project, where the bankruptcy problem associated to level of expedition $k \in \{1, \ldots, h\}$ is

$$(N, E^k, c^{z,k}) \tag{4.1}$$

with $E^k = R(\mathrm{slack}(I_{k+1}))$ if $k < h$ and $E^h = R(D(p) - D(r))$, and $c_i^{z,k} = f_i^z$ if $i \in \cup_{l=1}^k N_{I_l}$ and $c_i^{z,k} = 0$ otherwise.

Theorem 4.1. *Let $(\{N_1, \ldots, N_m\}, p, r, R)$ be an expedited project problem, let z be a surplus sharing mechanism, and let (N, v_z) be the associated expedited project game. Then,*

$$v_z(S) = \max_{k \in \{1,\ldots,h\}} \{v_{(N,E^k,c^{z,k})}(S)\}$$

for every $S \subset N$.

Proof: We proceed by induction on h. Let $h = 1$ and $S \subset N$, then

$$v_z(S) = F^1 - w_z^1(S) = F^1 - \min\left\{\sum_{i \in N\setminus S} f_i^z,\ F^1\right\} = \max\left\{F^1 - \sum_{i \in N\setminus S} f_i^z,\ 0\right\} = v_{(N,E^1,c^{z,1})}(S).$$

Next, assume that the result is satisfied for $1, \ldots, h-1$. Let $S \subset N$, then

$$\begin{aligned}
v_z(S) &= \sum_{k=1}^{h} \left(F^k - w_z^k(S)\right) \\
&= \sum_{k=1}^{h} F^k - \sum_{k=1}^{h-1} w_z^k(S) - \min\left\{ \sum_{i \in (\bigcup_{k=1}^{h} N_{I_k}) \setminus S} f_i^z - \sum_{k=1}^{h-1} w_z^k(S),\ F^k \right\} \\
&= \max\left\{ \sum_{k=1}^{h} F^k - \sum_{i \in (\bigcup_{k=1}^{h} N_{I_k}) \setminus S} f_i^z,\ \sum_{k=1}^{h-1} F^k - \sum_{k=1}^{h-1} w_z^k(S) \right\} \\
&= \max\left\{ \sum_{k=1}^{h} F^k - \sum_{i \in (\bigcup_{k=1}^{h} N_{I_k}) \setminus S} f_i^z,\ \max_{k \in \{1,\ldots,h-1\}} \{v_{(N,E^k,c^{z,k})}(S)\} \right\} \\
&= \max\left\{ \left(\sum_{k=1}^{h} F^k - \sum_{i \in (\bigcup_{k=1}^{h} N_{I_k}) \setminus S} f_i^z \right)_+,\ \max_{k \in \{1,\ldots,h-1\}} \{v_{(N,E^k,c^{z,k})}(S)\} \right\} \\
&= \max\left\{ \left(E^h - \sum_{i \in N \setminus S} c_i^{z,h} \right)_+,\ \max_{k \in \{1,\ldots,h-1\}} \{v_{(N,E^k,c^{z,k})}(S)\} \right\} \\
&= \max\left\{ v_{(N,E^h,c^{z,h})}(S),\ \max_{k \in \{1,\ldots,h-1\}} \{v_{(N,E^k,c^{z,k})}(S)\} \right\} \\
&= \max_{k \in \{1,\ldots,h\}} \{v_{(N,E^k,c^{z,k})}(S)\}
\end{aligned}$$

where the fourth equality follows by induction, the fifth one is a consequence of $\max_{k \in \{1,\ldots,h-1\}} \{v_{(N,E^k,c^{z,k})}(S)\} \geq 0$, and the sixth one follows because $\sum_{k=1}^{h} F^k = R(D(p) - D(r)) = E^h$ and because, by definition of $c^{z,h}$, we have $c_i^{z,h} = f_i^z$ if $i \in \bigcup_{k=1}^{h} N_{I_k}$ and $c_i^{z,h} = 0$ otherwise. □

Example 4.3. Consider the expedited project game in Example 4.2. Associated to each level of expedition I_k we have a bankruptcy problem $(N, E^k, c^{z,k})$:

$$\begin{aligned}
&E^1 = R(\text{slack}(I_2)) = R(4) = 256, && c^{z,1} = (f_{\mathrm{A}}^z, f_{\mathrm{B}}^z, 0, 0) = (138, 151, 0, 0), \\
&E^2 = R(\text{slack}(I_3)) = R(5) = 265, && c^{z,2} = (f_{\mathrm{A}}^z, f_{\mathrm{B}}^z, f_{\mathrm{C}}^z, 0) = (138, 151, 20, 0), \\
&E^3 = R(D(p) - D(r)) = R(6) = 276, && c^{z,3} = (f_{\mathrm{A}}^z, f_{\mathrm{B}}^z, f_{\mathrm{C}}^z, f_{\mathrm{D}}^z) = (138, 151, 20, 39).
\end{aligned}$$

Table 9 gives the values of the corresponding bankruptcy games and expedited project game. ◇

Note that, by definition of $c^{z,k}$, we have that $c_i^{z,k'} \leq c_i^{z,k}$ for every $i \in N$ and every $k' \leq k$. For $U \subset N$, let $\hat{k}(U, v_z)$ denote the smallest index satisfying $v_z(U) = v_{(N,E^{\hat{k}(U,v_z)},c^{z,\hat{k}(U,v_z)})}(U)$, i.e.

$$\hat{k}(U, v_z) = \min\{k \in \{1, \ldots, h\} \mid v_z(U) = v_{(N,E^k,c^{z,k})}(U)\}.$$

S	{A}	{B}	{C}	{D}	{A, B}	{A, C}	{A, D}	{B, C}	{B, D}	{C, D}
$v_{(N,E^1,c^{z,1})}(S)$	105	118	0	0	256	105	105	118	118	0
$v_{(N,E^2,c^{z,2})}(S)$	94	107	0	0	245	114	94	127	107	0
$v_{(N,E^3,c^{z,3})}(S)$	66	79	0	0	217	86	105	99	118	0
$v_z(S)$	105	118	0	0	256	114	105	127	118	0

S	{A, B, C}	{A, B, D}	{A, C, D}	{B, C, D}	{A, B, C, D}
$v_{(N,E^1,c^{z,1})}(S)$	256	256	105	118	256
$v_{(N,E^2,c^{z,2})}(S)$	265	245	114	127	265
$v_{(N,E^3,c^{z,3})}(S)$	237	256	125	138	276
$v_z(S)$	265	256	125	138	276

Table 9: Values of the bankruptcy games and expedited project game in Example 4.3.

Lemma 4.2. *Let $(\{N_1,\ldots,N_m\},p,r,R)$ be an expedited project problem and let z be a cost sharing mechanism. Then, $\hat{k}(S,v_z) \leq \hat{k}(T,v_z)$ for every $S \subset T \subset N$.*

Proof: Since z is fixed, we denote $c^{z,k}$ by c^k. By Lemma 4.3 it follows

$$v_z(T) = \max_{k\in\{1,\ldots,h\}}\{v_{(N,E^k,c^k)}(T)\} = \max_{k\in\{1,\ldots,h\}}\left\{\left(E^k - \sum_{j\in N\setminus T} c_j^k\right)_+\right\} = \max_{k\in\{1,\ldots,h\}}\left\{\left(E^k - \sum_{j\in N\setminus S} c_j^k + \sum_{j\in T\setminus S} c_j^k\right)_+\right\}$$

We proceed by contradiction. Assume that $\hat{k}(S,v_z) > \hat{k}(T,v_z) \geq 1$, then $v_z(S) > 0$ and

$$\begin{aligned}
v_z(T) &= v_{(N,E^{\hat{k}(T,v_z)},c^{\hat{k}(T,v_z)})}(T)\\
&= \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(T,v_z)}\right)_+\\
&= \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(T,v_z)} + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)}\right)_+\\
&\leq \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(T,v_z)}\right)_+ + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)}\\
&< E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)} + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)}\\
&\leq E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)} + \sum_{j\in T\setminus S} c_j^{\hat{k}(S,v_z)}\\
&= \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(S,v_z)}\right)_+\\
&= v_{(N,E^{\hat{k}(S,v_z)},c^{\hat{k}(S,v_z)})}(T)
\end{aligned}$$

which contradicts the definition of $\hat{k}(T,v_z)$. Here, the first and second equalities follow by definition of $\hat{k}(T,v_z)$ and $v_{(N,E^{\hat{k}(T,v_z)},c^{\hat{k}(T,v_z)})}(T)$, respectively. The second inequality follows because $\hat{k}(T,v_z) < \hat{k}(S,v_z)$, and then $v_z(S) = v_{(N,E^{\hat{k}(S,v_z)},c^{\hat{k}(S,v_z)})}(S) > v_{(N,E^{\hat{k}(T,v_z)},c^{\hat{k}(T,v_z)})}(S)$ by definition of $\hat{k}(S,v_z)$, together with $v_z(S) > 0$. The last inequality follows because $\hat{k}(T,v_z) < \hat{k}(S,v_z)$, and then $c^{\hat{k}(T,v_z)} \leq c^{\hat{k}(S,v_z)}$ by definition of c^k. The fourth equality is a consequence of $E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)} \geq 0$ and $c^{\hat{k}(S,v_z)} \geq 0$. □

Theorem 4.3. *Expedited project games are convex.*

Proof: Let $(\{N_1,\ldots,N_m\},p,r,R)$ be an expedited project problem, let z be a cost sharing mechanism, and let (N,v_z) be the associated expedited project game. Since z is fixed, we denote $c^{z,k}$ by c^k. Let $i \in N$ and $S \subset T \subset N \setminus \{i\}$, we have to show that $v_z(S\cup\{i\}) - v_z(S) \leq v_z(T\cup\{i\}) - v_z(T)$.

If $v_z(S\cup\{i\}) - v_z(S) = 0$ or $v_z(S) = v_z(T)$, then the condition is satisfied by monotonicity of (N,v_z). We can then assume without loss of generality that $v_z(S\cup\{i\}) > v_z(S)$ and $v_z(T) > v_z(S)$. Note that then $v_z(S\cup\{i\}) > 0$ and $v_z(T) > 0$ since v_z is nonnegative. We distinguish between two cases.

Case 1: $\hat{k}(S\cup\{i\},v_z) \leq \hat{k}(T,v_z)$. Then,

$$
\begin{aligned}
v_z(S\cup\{i\}) - v_z(S) &= E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S\cup\{i\},v_z)} + c_i^{\hat{k}(S\cup\{i\},v_z)} - \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)}\right)_+ \\
&\leq E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S\cup\{i\},v_z)} + c_i^{\hat{k}(S\cup\{i\},v_z)} - \left(E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S\cup\{i\},v_z)}\right)_+ \\
&\leq c_i^{\hat{k}(S\cup\{i\},v_z)} \\
&\leq c_i^{\hat{k}(T,v_z)} \\
&= E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(T,v_z)} + c_i^{\hat{k}(T,v_z)} - E^{\hat{k}(T,v_z)} + \sum_{j\in N\setminus T} c_j^{\hat{k}(T,v_z)} \\
&\leq E^{\hat{k}(T\cup\{i\},v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(T\cup\{i\},v_z)} + c_i^{\hat{k}(T\cup\{i\},v_z)} - \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(T,v_z)}\right) \\
&= v_z(T\cup\{i\}) - v_z(T)
\end{aligned}
$$

where the first equality is a direct consequence of the definition of $\hat{k}$ applied to $S\cup\{i\}$ and S and because $v(S\cup\{i\}) > v(S)(\geq 0)$ by assumption; the first inequality follows by definition of $\hat{k}$ applied to coalition S; the second inequality is a direct consequence of $(x)_+ = \max\{0,x\}$; the third inequality follows because $\hat{k}(S\cup\{i\},v_z) \leq \hat{k}(T,v_z)$ and by definition of c^k we have $c^{\hat{k}(S\cup\{i\},v_z)} \leq c^{\hat{k}(T,v_z)}$; the last inequality follows by definition of $\hat{k}$ applied to coalition $T\cup\{i\}$ and $v_z(T\cup\{i\}) \geq v_z(T) > 0$; and the last equality is a direct consequence of the definition of $\hat{k}$ and $v_z(T\cup\{i\}) \geq v_z(T) > 0$ by assumption.

Case 2: $\hat{k}(S\cup\{i\},v_z) > \hat{k}(T,v_z)$. In this case we show that the equivalent condition $v_z(S\cup\{i\}) + v_z(T) \leq$

$v_z(T \cup \{i\}) + v_z(S)$ is satisfied.

$$
\begin{aligned}
v_z(S \cup \{i\}) + v_z(T) &= E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(S\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus T} c_j^{\hat{k}(T,v_z)}\right)_+ \\
&= E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(S\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(T,v_z)} + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)}\right)_+ \\
&\leq E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(S\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(T,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(T,v_z)}\right)_+ + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)} \\
&\leq E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(S\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)}\right)_+ + \sum_{j\in T\setminus S} c_j^{\hat{k}(T,v_z)} \\
&\leq E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(S\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)}\right)_+ + \sum_{j\in T\setminus S} c_j^{\hat{k}(S\cup\{i\},v_z)} \\
&= E^{\hat{k}(S\cup\{i\},v_z)} - \sum_{j\in N\setminus(T\cup\{i\})} c_j^{\hat{k}(S\cup\{i\},v_z)} + \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)}\right)_+ \\
&\leq E^{\hat{k}(T\cup\{i\},v_z)} - \sum_{j\in N\setminus(T\cup\{i\})} c_j^{\hat{k}(T\cup\{i\},v_z)} + \left(E^{\hat{k}(S,v_z)} - \sum_{j\in N\setminus S} c_j^{\hat{k}(S,v_z)}\right)_+ \\
&= v_z(T \cup \{i\}) + v_z(S),
\end{aligned}
$$

where the first equality is a direct consequence of the definition of $\hat{k}$ applied to $S \cup \{i\}$ and T together with the assumption $v_z(S \cup \{i\}) > v_z(S)(\geq 0)$; the second inequality follows by definition of $\hat{k}$ applied to S; the third inequality is a direct consequence of $\hat{k}(T, v_z) < \hat{k}(S \cup \{i\}, v_z)$ and by definition of c^k we have $c^{\hat{k}(T,v_z)} \leq c^{\hat{k}(S\cup\{i\},v_z)}$; the fourth inequality follows by definition of $\hat{k}$ applied to coalition $T \cup \{i\}$ and $v_z(T \cup \{i\}) \geq v_z(T) > 0$; finally, the last equality follows by definition of $\hat{k}$ together with $v_z(T \cup \{i\}) \geq v_z(S \cup \{i\})(> 0)$ by monotonicity of v_z. □

5 Project games

A *project problem* arises when the planned time of the activities has been incorrectly estimated, possibly bringing delay or expedition to the project. A non-decreasing reward function $R : \mathbb{R} \to \mathbb{R}$ is associated to the difference between the planned and real times of the project, satisfying $R(t) \leq 0$ for $t < 0$, $R(t) = 0$ for $t = 0$, and $R(t) \geq 0$ for $t > 0$. A project problem can be described by a 4-tuple $(\{N_1, \ldots, N_m\}, p, r, R)$ where p and r satisfy $p \neq r$.

Associated to a project problem $(\{N_1, \ldots, N_m\}, p, r, R)$ we define a project game where the set of players is the set of activities and the value of a coalition combines the underlying ideas from Sections 3 and 4. In determining the value of a coalition we pessimistically assume that all delayed activities have indeed acted according to realization and that all expedited activities outside the coalition have acted according to plan. Then, if the expedition given by the expedited activities in the coalition itself is not enough to expedite the duration of the project, the value of the coalition is negative and is determined along the lines of delayed project games. Otherwise, the value of the coalition is positive and is determined along the lines of expedited project games. Formally, given a project problem $(\{N_1, \ldots, N_m\}, p, r, R)$, a cost sharing mechanism y, and a surplus sharing mechanism z, we denote by (N, u_{yz}) the associated *project game*, still to be defined formally. Let $\mathcal{E}$ denote the set of expedited activities, i.e. $\mathcal{E} = \{i \in N |\, e(i) > 0\}$.

If $D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) \geq D(p)$, then the expedition carried by the expedited activities in S is not enough to expedite the project and $\bar{c}_y(S)$ reflects the maximum delay the coalition can be held responsible of. For every $a \in \{1, \ldots, m\}$, consider the cost sharing problem

$$(N_a, q^a, K)$$

with $q_i^a = d(i)$ for every $i \in N_a$ and $K(t) = -R(-t)$ if $t > 0$ and $K(t) = 0$ otherwise. Let $y^a = y(N_a, q^a, K)$, then coalition S cannot be held responsible neither for more than the total cost assigned to it by the cost sharing mechanism, nor for more than the net delay of the path as a consequence of the delay of activities in the path and the expedition of the activities within the coalition. Formally,

$$\bar{c}_y(S) = \max_{a \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_a \cap S} y_i^a,\ K \left(\sum_{i \in N_a} d(i) - \sum_{i \in N_a \cap S} e(i) - \mathrm{slack}(N_a, p) \right) \right\} \right\}. \tag{5.1}$$

If $D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) < D(p)$, then the expedition carried by the expedited activities in S is enough to expedite the project and $\bar{v}_z(S)$ reflects the amount of reward from the expedition that the coalition may claim. In order to define $\bar{v}_z(S)$ we need to introduce some notation.

We denote by $\mathrm{rslack}(N_a, p, r)$ the amount of *remaining slack* of a path with respect to the planned duration if only its delayed activities act according to realization, i.e. $\mathrm{rslack}(N_a, p, r) = \mathrm{slack}(N_a, p) - \sum_{i \in N_a} d(i)$. Note that $\mathrm{rslack}(N_a, p, r)$ can be negative, meaning that the delayed activities have consumed all the initial slack and would produce a delay on the project, as a whole, of $-\mathrm{rslack}(N_a, p, r)$ if the expedited activities had acted according to plan. We denote by J_1 the set (of indexes) of paths with remaining slack less than or equal to zero:

$$J_1 = \{a \in \{1, \ldots, m\} |\ \mathrm{rslack}(N_a, p, r) \leq 0\}.$$

Recursively, we define for $k \geq 2$

$$J_k = \left\{ a \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l |\ \mathrm{rslack}(N_a, p, r) \leq \mathrm{rslack}(N_b, p, r) \text{ for all } b \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l \right\},$$

i.e. J_k contains all paths that would have smallest remaining slack if the paths in $J_1, \ldots, J_{k-1}$ where not present. Set $\mathrm{rslack}(J_1) := 0$ and let $\mathrm{rslack}(J_k)$ denote the remaining slack of the paths in J_k for $k \geq 2$, i.e.

$\text{rslack}(J_k) = \text{rslack}(N_a, p, r)$ for each $a \in J_k$, $k \geq 2$. Let g be such that $\text{rslack}(J_g) < D(p) - D(r) \leq \text{rslack}(J_{g+1})$ if $D(p) - D(r) > 0$ and $g = 0$ otherwise. For $k = 1, \ldots, g$, we define F^k as the marginal contribution of the paths in $J_1, \ldots, J_k$ to the total reward associated to the expedition. Formally,

$$F^k = \begin{cases} R(\text{rslack}(J_{k+1})) - R(\text{rslack}(J_k)) & \text{if } 1 \leq k < g; \\ R(D(p) - D(r)) - R(\text{rslack}(J_g)) & \text{if } k = g. \end{cases}$$

Note that $\sum_{k=1}^{g} F^k = R(D(p) - D(r))$ since $R(\text{rslack}(J_1)) = 0$.

For every $a \in \{1, \ldots, m\}$, consider the surplus sharing problem

$$(N_a, p^a, R^a)$$

with $p_i^a = e(i)$ for every $i \in N_a$ and $R^a(t) = R(t + (\text{rslack}(N_a, p))_+) - R((\text{rslack}(N_a, p))_+)$ if $t \geq 0$ and $R^a(t) = 0$ otherwise.

Let $z^a = z(N_a, p^a, R^a)$. Similarly as in Section 4, z_i^a is the maximum amount that i can claim according to the surplus sharing mechanism if its path is awarded with the total expedition that it can bring to the project. Then, we optimistically define f^z by

$$f_i^z = \max_{a : N_a \ni i} \{z_i^a\}, \tag{5.2}$$

i.e. f_i^z is the maximum reward that activity i can claim from the expedition of the project when the surplus sharing mechanism z is considered.

Next, we define $\bar{v}_z(S)$ representing the sum over all $k = 1, \ldots, g$ of those specific parts of the corresponding level of expedition F^k for which expedited activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} J_l$ cannot be held responsible of. Formally,

$$\bar{v}_z(S) = \sum_{k=1}^{g} \left(F^k - \bar{w}_z^k(S)\right) \tag{5.3}$$

where $\bar{w}_z^k(S)$ represents the part of the level of expedition F^k that players in S maximally would have to concede to players in $\bigcup_{l=1}^{k} J_l$ outside S, taking into account concessions from the previous phases. Formally,

$$\bar{w}_z^k(S) = \min \left\{ \sum_{i \in (\bigcup_{l=1}^{k} N_{J_l}) \setminus S} f_i^z - \sum_{l=1}^{k-1} \bar{w}_z^l(S),\ F^k \right\}, \tag{5.4}$$

for all $k \in \{1, \ldots, g\}$, where $N_{J_l} = \bigcup_{a \in J_l} N_a$.

Finally, we define the associated project game (N, u_{yz}) by

$$u_{yz}(S) = \begin{cases} -\bar{c}_y(S), & \text{if } D(p_{|\mathcal{E} \setminus S}, r_{|N \setminus (\mathcal{E} \setminus S)}) \geq D(p); \\ \bar{v}_z(S), & \text{if } D(p_{|\mathcal{E} \setminus S}, r_{|N \setminus (\mathcal{E} \setminus S)}) < D(p). \end{cases} \tag{5.5}$$

for every $S \subset N$.

Note that if the project problem is a delayed project problem, then $\bar{c}_y = c_y$ and $u_{yz} = -c_y$ since $D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) = D(r) \geq D(p)$. Besides, if the project problem is an expedited project problem, then $\bar{v}_z = v_z$ and $u_{yz} = v_z$ since, for all $S \subset N$, $D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) \geq D(p)$ implies $v_z(S) = 0 (= u_{yz}(S))$.

Given a project problem $(\{N_1, \ldots, N_m\}, p, r, R)$, and a surplus sharing mechanism z, the corresponding game $(N, \bar{v}_z)$ can be described as the maximum of as many bankruptcy games as levels of expedition in the project, where the bankruptcy problem associated to level of expedition $k \in \{1, \ldots, g\}$ is

$$(N, \bar{E}^k, \bar{c}^k)$$

with $\bar{E}^k = R(\text{rslack}(J_{k+1}))$ if $k < g$ and $\bar{E}^g = R(D(p) - D(r))$, and $\bar{c}^k_i = f^z_i$ if $i \in \cup_{l=1}^k N_{J_l}$ and $\bar{c}^k_i = 0$ otherwise. Besides, it turns out that $(N, \bar{v}_z)$ is convex. The proofs of both results follow the same lines of those in Theorems 4.1 and 4.3 and are therefore omitted.

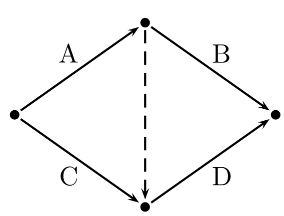
Lemma 5.1. *Let $(\{N_1, \ldots, N_m\}, p, r, R)$ be a project problem and let z be a surplus sharing mechanism. Then, $(N, \bar{v}_z)$ is convex and*

$$\bar{v}_z(S) = \max_{k\in\{1,\ldots,g\}} \{v_{(N,\bar{E}^k,\bar{c}^k)}(S)\}$$

for every $S \subset N$.

The following example illustrates the computation of a project game.

Example 5.1. Consider the expedited project problem $(\{N_1, N_2, N_3\}, p, r, R)$ with $N_1 = \{\text{A}, \text{B}\}$, $N_2 = \{\text{A}, \text{D}\}$, $N_3 = \{\text{C}, \text{D}\}$, $p(\text{A}) = 10$, $p(\text{B}) = 20$, $p(\text{C}) = 15$, $p(\text{D}) = 15$, $r(\text{A}) = 12$, $r(\text{B}) = 7$, $r(\text{C}) = 16$, $r(\text{D}) = 8$, and $R(t) = -t^4 - 100$ if $t < 0$, $R(0) = 0$, and $R(t) = t^2 + 200$ if $t > 0$. The project is represented in Figure 4.



N_a	$D(N_a, p)$	slack(N_a, p)	$D(N_a, r)$
AB	30	0	19
AD	25	5	20
CD	30	0	24

Figure 4: Representation of the project given in Example 4.1 and duration of the paths.

Here, $D(p) = 30$ and $D(r) = 24$, and then the total expedition is $D(p) - D(r) = 6$ with an associated reward of $R(6) = 236$. Besides, $d(\text{A}) = 2$, $d(\text{B}) = 0$, $d(\text{C}) = 1$ and $d(\text{D}) = 0$; $e(\text{A}) = 0$, $e(\text{B}) = 13$, $e(\text{C}) = 0$ and $e(\text{D}) = 7$; rslack$(\text{AB}, p, r) = -2$, rslack$(\text{AD}, p, r) = 3$, rslack$(\text{CD}, p, r) = -1$; $J_1 = \{1, 3\}$ and $J_2 = \{2\}$; $F^1 = R(3) - R(0) = 209$ and $F^2 = R(6) - R(3) = 27$.

For the computation of (N, u_{yz}), we first compute $(N, \bar{c}_y)$ and $(N, \bar{v}_z)$. For the computation of $(N, \bar{c}_y)$ we use the serial cost sharing mechanism. Associated to each path N_a, we have the cost sharing problem (N_a, q^a, K) with $q_i^a = d(i)$ and $K(t) = 0$ if $t \leq 0$ and $K(t) = t^4 + 100$ if $t > 0$. Then, $y^1 = y(\{\text{A, B}\}, (2,0), K) = (116, 0)$, $y^2 = y(\{\text{A,D}\}, (2,0), K) = (116, 0)$, and $y^3 = y(\{\text{C, D}\}, (1,0), K) = (101, 0)$.

For the computation of $(N, \bar{v}_z)$ we use the serial surplus sharing mechanism. Associated to each path N_a we have the surplus problem:

(N_1, p^1, R^1): $N_1 = \{\text{A, B}\}$, $p^1 = (0, 13)$, $R^1(t) = R(t)$ if $t \geq 0$ and $R^1(t) = 0$ otherwise, and $z^1 = (0, 369)$,

(N_2, p^2, R^2): $N_2 = \{\text{A, D}\}$, $p^2 = (0, 7)$, $R^2(t) = \begin{cases} R(t+3) - R(3) & \text{if } t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ $z^2 = (0, 91)$,

(N_3, p^3, R^3): $N_3 = \{\text{C, D}\}$, $p^3 = (0, 7)$, $R^3(t) = R(t)$ if $t \geq 0$ and $R^3(t) = 0$ otherwise, and $z^3 = (0, 249)$,

which gives $f^z = (0, 369, 0, 249)$. All coalitional values are given in Table 10.

S	$\{\text{A}\}$	$\{\text{B}\}$	$\{\text{C}\}$	$\{\text{D}\}$	$\{\text{A, B}\}$	$\{\text{A, C}\}$	$\{\text{A, D}\}$	$\{\text{B, C}\}$	$\{\text{B, D}\}$	$\{\text{C, D}\}$
$D(p_{\vert\mathcal{E}\setminus S}, r_{\vert N\setminus(\mathcal{E}\setminus S)})$	32	31	32	32	31	32	32	31	24	32
$\bar{c}_y(S)$	116	0	101	0	0	116	116	101	0	0
$\bar{v}_z(S)$	0	0	0	0	0	0	0	0	236	0
$u_{yz}(S)$	−116	0	−101	0	0	−116	−116	−101	236	0

S	$\{\text{A, B, C}\}$	$\{\text{A, B, D}\}$	$\{\text{A, C, D}\}$	$\{\text{B, C, D}\}$	$\{\text{A, B, C, D}\}$
$D(p_{\vert\mathcal{E}\setminus S}, r_{\vert N\setminus(\mathcal{E}\setminus S)})$	31	24	32	24	24
$\bar{c}_y(S)$	101	0	116	0	0
$\bar{v}_z(S)$	0	236	0	236	236
$u_{yz}(S)$	−101	236	−116	236	236

Table 10: Computation of the cost sharing rule in Example 4.2.

It can be checked that the core of the game is

$$\begin{aligned}\text{Core}(u_{yz}) = \text{conv}\{ &(-15, 135, -101, 217), (-116, 236, 0, 116), (0, 0, 0, 236), \\ &(-116, 352, 0, 0), (-15, 251, -101, 101), (0, 236, 0, 0)\}\end{aligned}$$

Note that this game is not convex by taking $i = \text{D}$, $S = \{\text{C}\}$, and $T = \{\text{A,C}\}$. ◇

Theorem 5.2. *Project games have a nonempty core.*

Proof: Let $(\{N_1, \ldots, N_m\}, p, r, R)$ be a project problem, let y and z be a cost and surplus sharing mechanism, respectively, and let (N, u_{yz}) be the associated project game. We distinguish between two cases.

Case 1: $D(p) \leq D(r)$.

In this case, $D(p) \leq D(r) \leq D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)})$ for every $S \subset N$, and therefore $u_{yz}(S) = -\bar{c}_y(S)$ for every $S \subset N$. Consequently, (N, u_{yz}) has a nonempty core if and only if $(N, \bar{c}_y)$ has a nonempty core.

Let $\hat{a} \in \{1, \ldots, m\}$ be such that $D(r) = D(N_{\hat{a}}, r)$, i.e. $N_{\hat{a}}$ is responsible of the total delay of the project. Consider the taxation problem $(N, E^{\hat{a}}, c^{\hat{a}})$ given by $E^{\hat{a}} = K(\sum_{i\in N_{\hat{a}}} d(i) - \sum_{i\in N_{\hat{a}}} e(i) - \text{slack}(N_{\hat{a}}, p))$, $c_i^{\hat{a}} = y_i^{\hat{a}}$ if $i \in N_{\hat{a}}$ and $c_i^{\hat{a}} = 0$ if $i \in N \setminus N_{\hat{a}}$. Then,

$$\begin{aligned}
\bar{c}_y(N) &= \max_{a\in\mathcal{P}(N)} \left\{ \min \left\{ \sum_{i\in N_a} y_i^a,\ K\left(\sum_{i\in N_a} d(i) - \sum_{i\in N_a} e(i) - \text{slack}(N_a, p) \right) \right\} \right\} \\
&= \min \left\{ \sum_{i\in N_{\hat{a}}} y_i^{\hat{a}},\ K\left(\sum_{i\in N_{\hat{a}}} d(i) - \sum_{i\in N_{\hat{a}}} e(i) - \text{slack}(N_{\hat{a}}, p) \right) \right\} \\
&= \min \left\{ \sum_{i\in N} c_i^{\hat{a}},\ E^{\hat{a}} \right\} \\
&= c_{(N,E^{\hat{a}},c^{\hat{a}})}(N)
\end{aligned}$$

where the second equality follows because $D(r) = D(N_{\hat{a}}, r)$ and the third one is a direct consequence of the definition of $E^{\hat{a}}$ and $c^{\hat{a}}$. Moreover, for any $S \subset N$ we have

$$\begin{aligned}
\bar{c}_y(S) &= \max_{a\in\mathcal{P}(S)} \left\{ \min \left\{ \sum_{i\in N_a\cap S} y_i^a,\ K\left(\sum_{i\in N_a} d(i) - \sum_{i\in N_a\cap S} e(i) - \text{slack}(N_a, p) \right) \right\} \right\} \\
&\geq \min \left\{ \sum_{i\in N_{\hat{a}}\cap S} y_i^{\hat{a}},\ K\left(\sum_{i\in N_{\hat{a}}} d(i) - \sum_{i\in N_{\hat{a}}\cap S} e(i) - \text{slack}(N_{\hat{a}}, p) \right) \right\} \\
&\geq \min \left\{ \sum_{i\in N_{\hat{a}}\cap S} y_i^{\hat{a}},\ K\left(\sum_{i\in N_{\hat{a}}} d(i) - \sum_{i\in N_{\hat{a}}} e(i) - \text{slack}(N_{\hat{a}}, p) \right) \right\} \\
&= \min \left\{ \sum_{i\in S} c_i^{\hat{a}},\ E^{\hat{a}} \right\} \\
&= c_{(N,E^{\hat{a}},c^{\hat{a}})}(S)
\end{aligned}$$

where the second inequality follows because R is nondecreasing and therefore K is also nondecreasing, and the second equality is a direct consequence of the definitions of $E^{\hat{a}}$ and $c^{\hat{a}}$.

Since $(N_{\hat{a}}, c_{(N,E^{\hat{a}},c^{\hat{a}})})$ is concave, we know that there is an $x \in \text{Core}(c_{(N,E^{\hat{a}},c^{\hat{a}})})$. Then, $\sum_{i\in N} x_i = c_{(N,E^{\hat{a}},c^{\hat{a}})}(N) = \bar{c}_y(N)$ and $\sum_{i\in S} x_i \leq c_{(N,E^{\hat{a}},c^{\hat{a}})}(S) \leq \bar{c}_y(S)$ for every $S \subset N$, and therefore $x \in \text{Core}(\bar{c}_y)$.

Case 2: $D(p) < D(r)$.

In this case, $u_{yz}(N) = \bar{v}_z(N)$ and $u_{yz}(S) \leq \bar{v}_z(S)$ for every $S \subset N$. By Lemma 5.1 we know that $(N, \bar{v}_z)$ is convex and therefore $\text{Core}(\bar{v}_z) \neq \emptyset$. Let $x \in \text{Core}(\bar{v}_z)$, then $\sum_{i\in N} x_i = \bar{v}_z(N) = u_{yz}(N)$ and $\sum_{i\in S} x_i \geq \bar{v}_z(S) \geq u_{yz}(S)$ for every $S \subset N$, and therefore $x \in \text{Core}(u_{yz})$. □

We now show that the core of project games satisfies some basic and desirable properties for solutions of project problems.

In many project problems the general manager of the project does not have legal authority to oblige delayed activities to compensate expedited activities for their contribution to decrease the total delay of the project. In this situation, a set-valued solution should satisfy: if the project is neither delayed, nor expedited, then there should be a solution in which nobody is neither punished, nor rewarded, i.e. if $D(r) = D(p)$, then the zero vector should be a possible solution; if the project is delayed, then there should be a solution in which the delayed activities pay exactly the total cost associated to the total delay, i.e. expedited activities are not compensated; if the project is expedited, then there should be a solution in which the delayed activities don't have to compensate expedited activities, i.e. expedited activities get exactly the total reward associated to the total expedition.

Let $\mathcal{D}$ be the set of delayed activities, i.e. $\mathcal{D} = \{i \in N | p(i) < r(i)\}$, and recall that $\mathcal{E}$ is the set of expedited activities, i.e. $\mathcal{E} = \{i \in N | p(i) > r(i)\}$.

Theorem 5.3. *Let $(\{N_1, \ldots, N_m\}, p, r, R)$ be a project problem, let y and z be a cost and surplus sharing mechanism, respectively, and let (N, u_{yz}) be the associated project game.*

(i) If $D(p) < D(r)$, then there exist $x \in \mathrm{Core}(u_{yz})$ such that $x_i = 0$ for every $i \in N \setminus \mathcal{D}$.

(ii) If $D(p) = D(r)$, then $0 \in \mathrm{Core}(u_{yz})$.

(iii) If $D(p) > D(r)$, then there exist $x \in \mathrm{Core}(u_{yz})$ such that $x_i = 0$ for every $i \in N \setminus \mathcal{E}$.

Proof: (i) If $D(p) < D(r)$, then $D(p) \leq D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) \leq D(r)$ for every $S \subset N$, and therefore $u_{yz}(S) = -\bar{c}_y(S)$ for every $S \subset N$. Let $\hat{a} \in \{1, \ldots, m\}$ be such that $D(r) = D(N_{\hat{a}}, r)$, i.e. $N_{\hat{a}}$ is responsible of the total delay of the project. Consider the taxation problem $(N, E^{\hat{a}}, c^{\hat{a}})$ given by $E^{\hat{a}} = K(\sum_{i \in N_{\hat{a}}} d(i) - \sum_{i \in N_{\hat{a}}} e(i) - \mathrm{slack}(N_{\hat{a}}, p))$, $c_i^{\hat{a}} = y_i^{\hat{a}}$ if $i \in N_{\hat{a}}$ and $c_i^{\hat{a}} = 0$ if $i \in N \setminus N_{\hat{a}}$. Note that $c_i^{\hat{a}} = 0$ for every $i \in N_{\hat{a}} \setminus \mathcal{D}$. By the proof of Theorem 5.2 we know that $\mathrm{Core}(c_{(N,E^{\hat{a}},c^{\hat{a}})}) \subset \mathrm{Core}(\bar{c}_y)$. Moreover, it is well known that any $x \in \mathrm{Core}(c_{(N,E^{\hat{a}},c^{\hat{a}})})$ satisfies $0 \leq x \leq c^{\hat{a}}$, and therefore $x_i = 0$ for every $i \in N \setminus (N_{\hat{a}} \cap \mathcal{D})$.
(ii) If $D(p) = D(r)$, then $D(p) \leq D(p_{|\mathcal{E}\setminus S}, r_{|N\setminus(\mathcal{E}\setminus S)}) \leq D(r)$ for every $S \subset N$, and therefore $u_{yz}(S) = -\bar{c}_y(S) \leq 0$ for every $S \subset N$. Moreover, $u_{yz}(N) = 0$, and then $0 \in \mathrm{Core}(u_{yz})$.
(iii) If $D(p) > D(r)$, then $u_{yz}(N) = \bar{v}_z(N)$ and $u_{yz}(S) \leq \bar{v}_z(S)$ for every $S \subset N$. Then, $\mathrm{Core}(\bar{v}_z) \subset \mathrm{Core}(u_{yz})$. Since $f_i^z = 0$ for every $i \in N \setminus \mathcal{E}$, we have $\bar{v}_z(S \cup \{i\}) = \bar{v}_z(S)$ for every $i \in N \setminus \mathcal{E}$ and therefore $x_i = 0$ for every $x \in \mathrm{Core}(\bar{v}_z)$ and $i \in N \setminus \mathcal{E}$. $\square$

6 Final remarks

We have used cooperative games to find solutions to project problems. The associated project game, in our opinion, provides an adequate thought experiment to evaluate coalitional influence and the core of this game provides a suitable answer to the allocation problem at hand.

Contrary to our focus on finding suitable allocations satisfying some basic properties, Castro, Gómez and Tejada (2007) concentrate on finding a game related to project problems satisfying some “desirable” properties. They put forward the properties of separability, non-manipulability by splitting, and independent slack and propose a cooperative game to share the total delay or expedition of a project satisfying these three properties. One can question the desirability of these three properties when we concentrate on the allocation of the rewards (or penalties) created by a project that has not performed as planned. For instance, the property of separability says that if a project can be decomposed in two different (sub)projects (i.e. if there is a node used by all paths in the project), then the associated game can be decomposed as the sum of the two games associated to the two corresponding (sub)projects. Note that the total reward of the project does not need to equal the sum of the rewards of the (sub)projects since we allow for non additive reward functions too. Hence, in our setting, separability does not need to be satisfied by project problems, let alone by the associated games.

In our opinion, it is not the properties of the game as a whole that are relevant, but rather the properties of the derived solutions (except of course from adequately modeling the coalitional possibilities).

As mentioned above, Castro, Gómez and Tejada (2007) define a cooperative game to share the total delay or expedition of a project. If one uses the core of this game to share the total reward in a project problem where the reward function is proportional to the total expedition or delay of the project, one encounters unwanted features in the allocations proposed by the core. It turns out that for projects in which the corresponding graph is a line, the game in Castro et al. (2007) is additive (i.e. the value of a coalition equals the sum of the individual values of its members). Therefore, the core of this game does not need to satisfy any of the properties proposed in Section 5.

Example 6.1. Consider the project problem $(\{N_1\}, p, r, R)$ with $N_1 = \{\mathrm{A}, \mathrm{B}\}$, $p(\mathrm{A}) = 10$, $p(\mathrm{B}) = 15$, $r(\mathrm{A}) = 13$, $r(\mathrm{B}) = 12$, and $R(t) = t$. The project is represented in Figure 5.

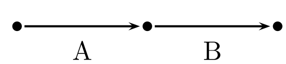


Figure 5: Representation of the project in Example 6.1.

In this problem, $D(p) - D(r) = 0$, $d(\mathrm{A}) = 2$, $d(\mathrm{B}) = 0$, $e(\mathrm{A}) = 0$, $e(\mathrm{B}) = 2$, the values of their corresponding game are $v(\{\mathrm{A}\}) = -2$, $v(\{\mathrm{B}\}) = 2$, $v(\{\mathrm{A}, \mathrm{B}\}) = 0$, and $\mathrm{Core}(v) = \{(-2, 2)\}$. $\diamond$

References

Atkinson, R. (1999), 'Project management: cost, time and quality, two best guesses and a phenomenon, its time to accept other success criteria', *International Journal of Project Management* **17**, 337–342.

Bergantiños, G. and Sánchez, E. (2002), 'How to distribute costs associated with a delayed project', *Annals of Operations Research* **109**, 159–174.

Borm, P., Hamers, H. and Hendrickx, R. (2001), 'Operations research games: A survey', *Sociedad de Estadística e Investigación Operativa Top* **9**(2), 139–216.

Branzêi, R., Ferrari, G., Fragnelli, V. and Tijs, S. (2002), 'Two approaches to the problem of sharing delay costs in joint projects', *Annals of Operations Research* **109**, 359–374.

Castro, J., Gómez, D. and Tejada, J. (2007), 'A project game for pert networks', *Operations Research Letters* **35**, 791–798.

Curiel, I. J., Maschler, M. and Tijs, S. (1987), 'Bankruptcy games', *Zeitschrift für Operations Research* **31**, A 143–A 159.

Dorndorf, U., Pesch, E. and Phan-Huy, T. (2000), 'Time-orientebd ranch-and-bound algorithm for resource-conprojsectt rained scheduling with generalised precedence constraints', *Management Science* **46**, 1365–1384.

Estévez-Fernández, A., Borm, P. and Hamers, H. (2007), 'Project games', *International Journal of Game Theory* **36**, 149–176.

Gillies, D. (1953), Some theorems on n-person games, PhD thesis, Princeton, University Press Princeton, New Jersey.

Koster, M. (1999), Cost Sharing in Production Situations and Network Exploitation, PhD thesis, Tilburg University. I.S.B.N. 90-5668-057-9.

Möhring, R. H., Schulz, A. S., Stork, F. and Uetz, M. (2003), 'Solving project scheduling problems by minimum cut computations', *Management Science* **49**, 330–350.

O'Neill, B. (1982), 'A problem of rights arbitration from the talmud', *Mathematical Social Sciences* **2**, 345–371.

Pinto, J. and Slevin, D. (1988), 'Project success: Definitions and measurements techniques', *Project Management Journal* **19**, 67–72.

Roberts, E. and Fusfeld, A. (1981), 'Staffing the innovative technology based organization', *Sloan Management Review* **22**, 19–34.

Shapley, L. S. (1971), 'Cores of convex games', *International Journal of Game Theory* **1**, 11–26.

Shenhar, A. and Dvi, D. (1996), 'Toward a typological theory of project management', *Sloan Management Review* **25**, 607–632.

Sherwin, C. and Isenson, R. (1967), 'Project hindsight: A defense department study of the utility of research', *Science* **156**, 1571–1577.

Tubig, S. and Abeti, P. (1990), 'Variables influencing the performance of defense r&d contractors', *IEEE Transactions on Engineering Management* **37**, 22–30.