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BELIEFS IN NETWORK GAMES

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Beliefs in Network Games

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Abstract

Networks can have an important effect on economic outcomes. Given the complexity of many of these networks, agents will generally not know their structure. We study the sensitivity of game-theoretical predictions to the specification of players' (common) prior on the network in a setting where players play a fixed game with their neighbors and only have local information on the network structure. We show that two priors are close in a strategic sense if and only if (1) the priors assign similar probabilities to all events that involve a player and his neighbors, and (2) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are similar, for any number of iterations. Also, we show that the common but unrealistic assumptions that the size of the network is common knowledge or that the types of players are independent are far from innocuous: if these assumptions are violated, small probability events can have a large effect on outcomes through players' conditional beliefs.

JEL classification: C72, D82, L14, Z13

Keywords: Network games, incomplete information, higher order beliefs, continuity, random networks, population uncertainty.

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1 Introduction

In many contexts, an agent's well being primarily depends on his own action and on the actions of those with whom he shares a direct relationship, rather than on the actions of all agents in the population. Also, an agent's connections provide access to various resources such as information, knowledge and capital.[1] Hence, in a variety of settings, the networks formed by agents' relations are important in determining economic outcomes. These networks are generally large and complex, and evolve rapidly over time.[2] This suggests that agents often will not know the exact structure of the network they belong to.[3] At the same time, it is unclear what beliefs agents have about their networks.[4] We consider a setting where agents interact strategically with their neighbors in the network and are uncertain about the network structure. We study the sensitivity of game-theoretical predictions in such games to the specification of players' belief on the network.

More specifically, suppose that players are located on a network and play a fixed game with their neighbors. Payoffs only depend on a player's own action and characteristics and on the actions and characteristics of his neighbors. Players have a common prior over the network, and, in addition, they have some local information: they know the number of neighbors they have in the network, i.e., a player's type is *connectivity*. We say that two priors are close in a strategic sense if for any game with bounded payoff functions in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the associated game with the other prior such that ex ante expected payoffs are close under both equilibria. If that is the case, players can obtain approximately the same payoffs under both priors. We thus study lower hemicontinuity of the correspondence of (interim) approximate equilibria in network games.

[1] For instance, key success factors for a firm in a high tech sector such as the biotechnology industry are its position in a network of R&D partnerships (Powell et al., 1996) and the collaboration with its R&D partners (Littler et al., 1995). Other empirical studies that highlight the role of networks of relations include Coleman et al. (1966) and Conley and Udry (2005) on the diffusion of new technologies in medicine and agriculture, respectively, Granovetter (1974) on job search, Tucker (2005) on adoption decisions, and Fafchamps and Lund (2003) on informal insurance networks in developing countries.

[2] For instance, several empirical emphasize the flexibility of R&D collaborations, with firms having many short term projects with many different partners (e.g. Hagedoorn, 2002; Powell et al., 2005).

[3] Indeed, Krackhardt and Hanson (1993) report that informal networks are mostly unobservable to senior executives. Also, Powell et al. (1996, p.120) observe that in R&D collaborations in biotechnology, "beneath most formal ties [...] lies a sea of informal relations".

[4] Evidence suggests that agents use simple heuristics (Janicik and Larrick, 2005), and that their perception of the network is biased (e.g. Kumbasar et al., 1994), even in an environment with strong incentives (Johnson and Orbach, 2002).

Our main result (Theorem 5.2) shows that two priors are close in a strategic sense if and only if (1) the priors assign similar probabilities to all local events, i.e., events that involve a player and his neighbors, and (2) with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are similar, and so on, for any number of iterations. This latter condition can also be stated in terms of the correlation among types: an equivalent formulation is that the set of types for which conditional beliefs are similar has to have high probability, and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on.

This result can be interpreted as follows. On the one hand, we can analyze a network game as a set of overlapping "local games" as far as ex ante beliefs are concerned: priors only need to assign similar probabilities to local events. On the other hand, these local games are interlaced through players' *conditional* beliefs: players need to form beliefs on the beliefs of his neighbors about the beliefs of their neighbors, and so on. This means that events that have small probability ex ante can have a large effect on outcomes through players' conditional beliefs: even if with high probability, each player has a type such that his conditional beliefs are similar under the two priors, it may be the case that with high probability, a player thinks it is likely, given his type, that his neighbors think it is likely, given their type,... the conditional beliefs of their neighbors are very different. Players' higher order beliefs can thus have a large impact on outcomes if condition (2) is not satisfied.

To establish our results, we introduce a new class of games, the class of *network games of incomplete information*. This class of games allows for uncertainty over the network size and for arbitrary correlations among player types. So far, strategic interactions on a network have been modeled as a Bayesian game.[5] In particular, it is assumed that the size of the network is commonly known. Moreover, it is often assumed in this literature that players' types are (asymptotically) independent. We refer to the games studied in this literature as *Bayesian network games*, to distinguish this class from the class of network games of incomplete information that we introduce.

Network games of incomplete information are not Bayesian games, as they allow for

[5] Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) provide a general framework for analyzing strategic interactions on networks under complete and incomplete information about the network structure. Papers that study specific games include Jackson and Yariv (2007) and Sundararajan (2005). For a survey, see Ioannides (2004) or Jackson (2006).

uncertainty over the player set, and nor does the class of network games of incomplete information contain the class of Bayesian network games. The reason is that uncertainty about the player set forces us to treat players symmetrically in network games of incomplete information (cf. Myerson, 1998). When the player set is not commonly known, players' perceptions about each others' strategic behavior cannot be formulated in terms of a strategy profile which assigns a distinct strategy to each individual in the game. Players are aware, though, of the possible types in the game, so that they can form perceptions about how the strategic behavior of players depend on their types. Strategies can thus only depend on players' types, not on their identity.[6]

Allowing for uncertainty over the network size is a natural step. The observation of Myerson (1998) that in some contexts, it is natural to assume that players are uncertain about the number of other players in the game holds *a fortiori* for network games, as in these games, players only interact with a small subset of players. Games with "population uncertainty" in which players interact globally have been studied by a number of authors (e.g. Kalai, 2004; McAfee and McMillan, 1987; Milchtaich, 2004; Myerson, 1998), but population uncertainty has not been studied in settings where players interact locally, as in the current paper. Population uncertainty plays a distinctly different role here than in games with global interactions: unlike in games with population uncertainty with global interactions, a player knows precisely the number of players he interacts with. However, a player does not know the number of players his *neighbors* interact with.

Introducing population uncertainty in network games and allowing for arbitrary correlations among types may seem to be innocuous extensions. However, we show that these assumptions can have large ramifications. When the number of players is fixed (or, more generally, when it is bounded), there is a precise bound on the number of players each player can interact with: when the number of players is n, the maximum connectivity is $n-1$. By contrast, when the number of players is potentially unbounded (i.e., exceeding each integer n with positive probability), the type set is infinite. In that case, a prior can be *sensitive to small probability events*: an event that has small prior probability can have a large effect on outcomes through players' conditional beliefs: a player may think it is likely, given his type, that his neighbors think it is likely, given their types. . . that the small probability event is true. When the type set is finite, this is ruled out. In that case, closeness of the two priors in terms of prior probabilities (condition (1)) implies that there is a sufficiently large set of

[6]However, in Appendix A we show that the set of equilibria is *invariant to payoff-irrelevant type-splitting*: when we allow for payoff-irrelevant subdivisions of types, the set of equilibria remains essentially unchanged. In that sense, there is no loss of generality in assuming that players only base their behavior on their type.

players whose conditional beliefs are close. A similar argument holds for the assumption of independent types. Hence, to explore the full range of strategic outcomes, one needs to go beyond network games with a fixed number of players and independent types. The class of network games of incomplete information provides a flexible framework to analyze the effects of different assumptions on players' priors.

The current paper builds on a literature relating higher order beliefs to the equilibria of incomplete information games, in particular Monderer and Samet (1989) and Kajii and Morris (1998), and we use extensively concepts and techniques from this literature. Kajii and Morris (1998) study lower hemicontinuity of the approximate equilibrium correspondence in Bayesian games with a (fixed) finite player set and a countably infinite state space.[7] They show that two priors over this state space are strategically close if and only if the prior probabilities of events are similar under the two priors and with high probability, it is approximate common knowledge that all players attach similar conditional probabilities to all events, i.e., with high probability, each player believes with high conditional probability that the conditional beliefs of all players are similar under the two priors and that all players believe with high conditional probability that the conditional beliefs of all players are similar, and that all players believe with high conditional probability that all players believe with high conditional probability... that the conditional beliefs of all players are similar under the two priors (for any number of iterations). Our result can thus be seen as a "spatial" analogue of this result: rather than requiring that all players believe that all players believe... that the conditional beliefs of all players are similar, we require that a player believes that his neighbors believe that their neighbors believe... that the conditional beliefs of their neighbors are similar.

Although we study the same issues as Kajii and Morris (1998), and follow their line of argument in our proofs,[8] conceptually, there are marked differences. To establish our results, we introduce the local p-belief operator. The local p-belief operator associates with each set of types a set of types that with conditional probability at least p interact exclusively with types in that set. It thus provides a measure of the "cohesiveness" of a set of types. We show that this operator also quantifies players' higher order beliefs regarding local events

[7] Monderer and Samet (1996) study the related question under what conditions two information partitions are close in a strategic sense. That is, they fix the probability distribution over the states and vary players' information partitions. Milgrom and Weber (1985) study upper hemicontinuity of the Bayesian equilibrium correspondence. Kets (2007b) applies the ideas of Kajii and Morris (1998) to the context of Bayesian network games.

[8] Also see Rothschild (2005).

in network games, i.e., a player's beliefs about his neighbors' beliefs about their neighbors' beliefs, and so on.

The local p-belief operator is thus closely related to the p-belief operator of Monderer and Samet (1989), which quantifies players' higher order beliefs in Bayesian games. The p-belief operator of Monderer and Samet associates with each event E the event in which all players believe E with conditional probability at least p. In the current context, an event would be a set of networks. The p-belief operator is thus defined at a global level, and characterizes players' beliefs about global events. Indeed it can be shown that the local p-belief operator and the p-belief operator of Monderer and Samet (extended to the context of network games of incomplete information) are complementary in this respect (see Kets, 2007a). While the p-belief operator of Monderer and Samet can be used to characterize players' higher order beliefs over the global structure of the network, the local p-belief operator is well suited to characterize players' higher order beliefs over local events.

The local p-belief operator is also related to the neighborhood operator of Morris (1997, 2000). Morris introduces the neighborhood operator in the context of games on a *fixed* network. For a given network, the neighborhood operator assigns to each subset of players the set of players in that subset for whom at least proportion p of their interactions is only with players in that subset. That is, the neighborhood operator relates to the cohesiveness of a group of players, just like the local p-belief operator relates to the cohesiveness of a set of types.

Hence, the local p-belief operator shares features of both the p-belief operator of Monderer and Samet (1989) and the neighborhood operator of Morris (1997, 2000). Like the p-belief operator, the local p-belief operator pertains to players' (higher order) beliefs in incomplete information games. Like the neighborhood operator, the local p-belief operator refers to the local interactions of players.

The current paper is thus at the interface of the literature on higher order beliefs and on network games. A notable paper that is also at this interface is Morris (2000). Morris considers a coordination game on a fixed network with infinitely many players. He studies the conditions under which an action that is initially played by a finite set of players will eventually spread to the whole population through myopic best reply dynamics. If that is the case, an action spreads contagiously. A necessary and sufficient condition for there to be no contagion starting from some finite group of players X is that the network of players outside that group contains a large group of players Y that is sufficiently cohesive, in the sense that players from Y interact mostly with other players from Y, who in turn interact mostly with other players from Y, and so on. This result is directly related to our main result. The second condition we identify for priors to be close in a strategic sense states that there should be a set of types of considerable measure that is sufficiently cohesive. Hence, our result can be seen as a stochastic analogue of this result. This underlines the formal relation between network games and incomplete information games first highlighted by Mailath et al. (1997) and Morris (1997) and exploited in Morris (2000).

The outline of this paper is as follows. Preliminaries are discussed in Section 2. In Section 3, we introduce the class of network games of incomplete information. The local p-belief operator and players' higher order beliefs in network games are discussed in Section 4. Section 5 contains our main result and a discussion of its implications. Section 6 concludes.

2 Preliminaries

In our framework, players are located on a network. A *network* $g = (V, E)$ consists of a finite, nonempty set of *vertices* V and a finite, nonempty set of *edges* E, with an edge an unordered pair of two distinct vertices. Players are identified with vertices, with edges representing the relations between players. If $\{i, j\} \in E$, where $i, j \in V, i \neq j$, then i and j are *neighbors*.

We consider a setting where the network is drawn from a class of networks according to some probability distribution. Let $n \in \mathbb{N}$, and let $V^{(n)} := \{1, \ldots, n\}$. Let $\mathcal{G}^{(n)}$ be the set of all networks with vertex set $V^{(n)}$ and let

$$\mathcal{G} := \bigcup_{n \in \mathbb{N}} \mathcal{G}^{(n)}$$

be the countable set of all networks with a finite vertex set. Let $\mathscr{F}$ be the σ-algebra generated by the set of singletons of $\mathcal{G}$. Let $\mathcal{M}$ denote the set of all probability measures on $(\mathcal{G}, \mathscr{F})$, and let $\mu \in \mathcal{M}$. We refer to the probability space $(\mathcal{G}, \mathscr{F}, \mu)$ as a *network belief system*.

A network belief system can be induced by a *random network*.[9] A random network is a network formed in a random construction procedure and is thus a measurable mapping from some measurable (sample) space to the measurable (outcome) space $(\mathcal{G}, \mathscr{F})$. For instance, in the Erdős-Rényi or binomial random network (Erdős and Rényi, 1959) an edge is created between two distinct vertices from a fixed vertex set with probability $p \in [0, 1]$, independently of the other edges. The probability measure on the sample space induces a probability distribution μ on the outcome space $(\mathcal{G}, \mathscr{F})$. In the current paper, we only

[9]See Jackson (2006) or Vega-Redondo (2007) for a discussion of random networks and their applications in economics.

consider the outcome space $(\mathcal{G}, \mathscr{F})$ and the probability distribution μ, and we do not specify the "experiment" that induces the probability space $(\mathcal{G}, \mathscr{F}, \mu)$. Example 2.1 gives a simple example of a random network with a random number of vertices; for a particularly elegant model of a random network with a random number of vertices, see Bollobás et al. (2007).

Example 2.1. Suppose that a population evolves in (discrete) generations, indexed by $m \in \{0, 1, \ldots\}$. Each member of the mth-generation gives birth to a family (possibly empty) of members of the $(m+1)$th generation. The number of offspring that each individual produces is a random variable. We assume that the number of offspring of each individual in the population is independent of the number of offspring of all other individuals, and that the probability distribution $(p_k)_{k \in \mathbb{N} \cup \{0\}}$ of the number of offspring is the same for each individual. This is a simple branching process (e.g. Grimmett and Stirzaker, 1992). If we associate a vertex with each individual and if we interpret ancestry relations as (undirected) edges, this random process gives rise to a network with a random number of vertices. $\triangleleft$

Given a network belief system $(\mathcal{G}, \mathscr{F}, \mu)$, the vertex set and the edge set are random variables. Let $\mathcal{Q}$ be the countable set of all finite subsets of $\mathbb{N}$, and let $V : \mathcal{G} \to \mathcal{Q}$ be the function that assigns to each network $g \in \mathcal{G}$ its vertex set $V(g)$. That is, if $g \in \mathcal{G}^{(n)}$ for some $n \in \mathbb{N}$, then $V(g) = V^{(n)}$. Let $\mathcal{Q}^{(2)} \subset \mathcal{Q}$ be the set of all subsets of $\mathbb{N}$ with two elements, and let $\mathcal{P}^{(2)}$ be the countable set of all finite subsets of $\mathcal{Q}^{(2)}$. Define the function $E : \mathcal{G} \to \mathcal{P}^{(2)}$, which assigns to each network $g \in \mathcal{G}$ its edge set $E(g)$.

We are interested in the local environment of vertices. Let $i \in \mathbb{N}$, and define the function $N_i : \mathcal{G} \to \mathcal{Q}$ as follows. For $g \in \mathcal{G}$,

$$N_i(g) := \begin{cases} \{j \in V(g) \mid \{i, j\} \in E(g)\}, & \text{if } i \in V(g); \\ \emptyset, & \text{otherwise}, \end{cases}$$

is the set of neighbors of vertex i in network g. We refer to the random variable N_i as the *neighborhood of* i, and to $N_i(g), g \in \mathcal{G}$, as the *neighborhood of* i *in* g. Also, define the function $D_i : \mathcal{G} \to \mathbb{N} \cup \{0\}$ by

$$D_i(g) := |N_i(g)|$$

for $g \in \mathcal{G}$. That is, $D_i(g)$ is the number of neighbors of vertex i in network g. We refer to $D_i(g)$ as the *connectivity of* i *in* g, and to the random variable D_i as the *connectivity of* i. Note that the connectivity of i in g can be 0 for two distinct reasons. It could be that i is a vertex in the network, but does not have any neighbors, or i is not a vertex of the network.

We will also consider the number of neighbors the neighbors of a given vertex have. Loosely speaking, the neighbor connectivity profile of a vertex in a given network is a list of

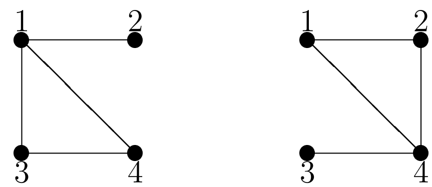


Figure 2.1: (a) The network g of Example 2.2; (b) A network isomorphic to g. To see that this network is isomorphic to g, note that there are two permutations of the vertex set $V^{(4)} = \{1, 2, 3, 4\}$ that renders g into this network: $\pi(i) = 5 - i$ for each $i \in V^{(4)}$, or $\pi'(1) = 4, \pi'(2) = 3, \pi'(3) = 1, \pi'(4) = 2$.

the connectivities of the neighbors of the vertex, in a nonincreasing order. For $t \in \mathbb{N}$, let

$$\Omega_K^t := \{(k_1, \ldots, k_t) \in \mathbb{N}^t \mid k_1 \geq k_2 \geq \ldots \geq k_{t-1} \geq k_t\}.$$

For $t = 0$, let $\Omega_K^t = \{0\}$, and define $\Omega_K := \bigcup_{t \in \mathbb{N} \cup \{0\}} \Omega_K^t$. Let $\mathscr{F}_K$ be the σ-field generated by the set of all singletons of Ω_K. For $g \in \mathcal{G}$ and $i \in V(g)$ such that $D_i(g) = 0$, we set $K_i(g) := 0$. Otherwise, define

$$\begin{aligned} N_1 &:= N_i(g), \\ j(1) &:= \max\{j \in N_1 \mid D_j(g) \geq D_k(g) \text{ for all } k \in N_1\}, \\ K_{i,1}(g) &:= D_{j(1)}(g), \end{aligned}$$

and for $\ell = 2, \ldots, D_i(g)$:

$$\begin{aligned} N_\ell &:= N_{\ell-1} \setminus \{j(\ell - 1)\}, \\ j(\ell) &:= \max\{j \in N_\ell \mid D_j(g) \geq D_k(g) \text{ for all } k \in N_\ell\}, \\ K_{i,\ell}(g) &:= D_{j(\ell)}(g). \end{aligned}$$

Then, $K_i(g) := (K_{i,1}(g), \ldots, K_{i,D_i(g)}(g))$ is the neighbor connectivity profile of i in g.

Example 2.2. Suppose we draw network g in Figure 2.1(a) from the set $\mathcal{G}$. Its vertex set is $V(g) = \{1, 2, 3, 4\}$, and its edge set is $E(g) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}$. The connectivity of vertex 1 in g is $D_1(g) = 3$, the neighborhood of vertex 1 is $N_1(g) = \{2, 3, 4\}$ and the neighbor connectivity profile of vertex 1 in g is $K_1(g) = (D_4(g), D_3(g), D_2(g)) = (2, 2, 1)$. $\triangleleft$

The following definition will be useful when specifying players' beliefs in the next section. Let $n \in \mathbb{N}$. Two networks $g, g' \in \mathcal{G}^{(n)}$ are *isomorphic* if there is a permutation π of $V^{(n)}$ such that $\{i, j\} \in E(g)$ if and only if $\{\pi(i), \pi(j)\} \in E(g')$. This defines an equivalence relation; hence, the set of all networks with n vertices $\mathcal{G}^{(n)}$ can be partitioned into a finite number of *isomorphism classes*, i.e., sets of isomorphic networks. Let $\mathscr{C}^{(n)}$ be the collection of isomorphism classes of $\mathcal{G}^{(n)}$, and let $\mathscr{C} := \bigcup_{n\in\mathbb{N}} \mathscr{C}^{(n)}$ be the collection of isomorphism classes of $\mathcal{G}$. Figure 2.1(a) and (b) depict two networks that are isomorphic.

Throughout this paper, we make the following two assumptions on network belief systems:

Assumption 1. (Finite expected number of vertices)
The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that the expected number of vertices is finite, i.e.,

$$\sum_{n\in\mathbb{N}} n\mu\big(\mathcal{G}^{(n)}\big) < \infty.$$

$\triangleleft$

Assumption 2. (No isolated vertices)
The network belief system $(\mathcal{G}, \mathscr{F}, \mu)$ is such that with probability 1, each vertex has at least one neighbor. That is,

$$\mu(\{g \in \mathcal{G} \mid D_i(g) > 0 \text{ for all } i \in V(g)\}) = 1.$$

$\triangleleft$

Assumption 2 is for notational convenience only and can easily be relaxed.

3 Network games of incomplete information

A network game of incomplete information is a game on a network, in which players are associated with a vertex in the network, and each player's payoff depends on the types and actions of himself and his neighbors. Players have incomplete information on the network: they have a common prior over the class $\mathcal{G}$ of all finite networks, and they know the number of neighbors they have, i.e., their connectivity. In particular, they may not know the number of players in the network. Here we introduce the class of network games of incomplete information.

3.1 Game

Let $(\mathcal{G}, \mathscr{F}, \mu)$ be a network belief system satisfying Assumptions 1 and 2. A network $g \in \mathcal{G}$ is drawn according to $(\mathcal{G}, \mathscr{F}, \mu)$. Each vertex in the set $V(g)$ represents a player, and we refer to a player by his vertex label. Players do not know their vertex label, however.[10] Each player $i \in V(g)$ knows the number of neighbors he has in the network: his *type* is his connectivity. Hence, the *type set* is $T = \mathbb{N} \cup \{0\}$. Henceforth, we will speak of type and *neighbor type profile*, rather than of connectivity and neighbor connectivity profile. Each player is endowed with a finite, nonempty set A of pure strategies or *actions*. For each $t \in T$, the *payoffs* of a player of type t are given by a function v_t. For $t > 0$, the payoffs of a player of type t are given by a function $v_t : A \times A^t \times T^t \to \mathbb{R}$ that is symmetric in A^t and T^t, i.e., for all permutations π_1, π_2 on $\{1, \ldots, t\}$, for all $a \in A$, $(a_1, \ldots, a_t) \in A^t$, $(\theta_1, \ldots, \theta_t) \in T^t$,

$$v_t\big(a, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t)\big) = v_t\big(a, (a_{\pi_1(1)}, \ldots, a_{\pi_1(t)}), (\theta_{\pi_2(1)}, \ldots, \theta_{\pi_2(t)}\big),$$

with $v_t\big(a, (a_1, \ldots, a_t), \big(\theta_1, \ldots, \theta_t)\big)$ the payoffs to a player of type t with neighbor type profile $(\theta_1, \ldots, \theta_t)$ when he chooses action $a \in A$, and his neighbors play according to the action profile $(a_1, \ldots, a_t)$. The payoffs to a player of type $t = 0$ are given by a function $v_0 : A \to \mathbb{R}$, i.e., the payoffs to an isolated player only depend on his own action.

Definition 3.1. A *network game of incomplete information* is a tuple

$$\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T} \rangle$$

with its elements defined as above. $\triangleleft$

We fix the action set A. A network game of incomplete information is then fully characterized by its probability measure on $(\mathcal{G}, \mathscr{F})$ and its profile of payoff functions. We henceforth denote a network game of incomplete information $\langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T} \rangle$ by the pair (μ, v), where $v := (v_t)_{t \in T}$.

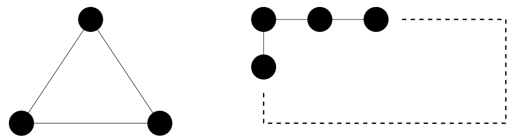
Let $B \in \mathbb{R}$. A profile v of payoff functions is *bounded by B* if for all $t \in T, t \neq 0$, $\theta \in \Omega_K^t$ and for all $a, a' \in A^{t+1}$,

$$\max\big\{|v_t(a, \theta) - v_t(a', \theta)|, |v_t(a, \theta)|\big\} \leq B.$$

If there exists $B \in \mathbb{R}$ such that the profile v is bounded by B, we say that it is *bounded*.

As in games with population uncertainty and random-player games, the player set is not commonly known, so that players are not aware of the particular identities of the other players in the game. Hence, we cannot assign a separate strategy to each individual player.

[10]The vertex labeling is introduced merely to be able to define random variables such as the connectivity of vertices. However, this labeling carries no meaning.



(a) (b)

Figure 3.1: The networks of Example 3.1. (a) The network $g^{(3)}$; (b) The network $g^{(300)}$.

Rather, a strategy can only depend on a player's type (cf. Myerson, 1998; Milchtaich, 2004). Hence, for each type $t \in T$, let σ_t be a real function defined on A which satisfies

$$\sigma_t(a) \geq 0$$

for all $a \in A$, and

$$\sum_{a \in A} \sigma_t(a) = 1,$$

with $\sigma_t(a)$ the probability that a player of type t chooses action a. The set of all probability distributions on A is denoted by Σ. An element $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots) \in \Sigma^T$ is referred to as a *strategy function*.

3.2 Beliefs

To calculate expected payoffs, we need to specify players' beliefs. There are two issues that should be noted. Firstly, as in games with population uncertainty and random-player games, players condition on their type, as well as on the fact that they are selected to play. That is, from a player's perspective, even if all networks in the support of μ have the same probability ex ante, he is more likely to belong to a network with many players: there are simply more vertices to be associated with in large networks. This is illustrated by Example 3.1.

Example 3.1. Suppose that the network belief system assigns probability $\frac{1}{2}$ to the network $g^{(3)}$ consisting of a triangle of three players, and probability $\frac{1}{2}$ to the network $g^{(300)}$ consisting of 300 players, connected in a cycle (see Figure 3.1. Though the prior probability of the two networks is $\frac{1}{2}$, from the perspective of a player, it is much more likely that network $g^{(300)}$ is realized, as to each "player position" in $g^{(3)}$, there are 100 player positions in $g^{(300)}$. Using Bayes' rule, a player's belief that $g^{(300)}$ is realized is

$$\frac{300 \cdot \frac{1}{2}}{3 \cdot \frac{1}{2} + 300 \cdot \frac{1}{2}} = \frac{300}{303}.$$

$\triangleleft$

Secondly, a player cannot distinguish between networks in a given isomorphism class, as he does not know his vertex label or the vertex labels of his opponents. Hence, to calculate players' beliefs that they have a given neighbor type profile, we need to consider the probability distribution over isomorphism classes induced by μ, and for each isomorphism class, we need to take into account the number of vertices with that neighbor type profile in the isomorphism class.

Formally, recall that $\mathscr{C}$ is the collection of isomorphism classes of $\mathcal{G}$, and that $\mathscr{F}_K$ is the σ-field associated with the set of all neighbor type profiles Ω_K. For each $C \in \mathscr{C}$, and each $F \in \mathscr{F}_K$, let $n_C(F)$ be the number of vertices in a network in C with their neighbor type profile in F. Note that $n_C(F)$ is well defined: for any two networks $g, g' \in C$, the number of vertices with their neighbor type profile in F is identical. Let

$$\bar{n} := \sum_{n \in \mathbb{N}} n\mu\big(\mathcal{G}^{(n)}\big)$$

be the expected number of players in the network belief system. By Assumption 1, $\bar{n}$ is finite. Consider a player who is called upon to play, but who does not know his type yet. The probability that the neighbor type profile of such a player lies in the set F is

$$q_\mu(F) = \frac{1}{\bar{n}} \sum_{C \in \mathscr{C}} \mu(C) n_C(F),$$

where we recall that $\mu(C)$ is the prior probability that a network from the isomorphism class C is realized. In words, $q_\mu(F)$ is equal to the expected fraction of players with a neighbor type profile in F. We refer to $q_\mu(F)$ as the *prior probability* that a player's neighbor type profile is in F. In particular, for each $t \in T$,

$$q_\mu(t) := q_\mu(\Omega_K^t)$$

denotes the prior probability that a player's type is t.

Remark 3.1. Tacitly we have assumed that there is some pool of candidate players from which (actual) players are drawn. We have not specified this pool, nor have we specified the method by which players are selected. There is no need to specify this, however, as we are solely interested in players' beliefs *given* that they have been selected to play. Hence, the probability measure q_μ gives the probability that an arbitrary player has a certain neighbor type profile. Also see Myerson (1998, p.382-384) on this point. $\triangleleft$

It can be readily checked from the definitions that q_μ is indeed a probability measure on the measurable space $(\Omega_K, \mathscr{F}_K)$ of neighbor type profiles:

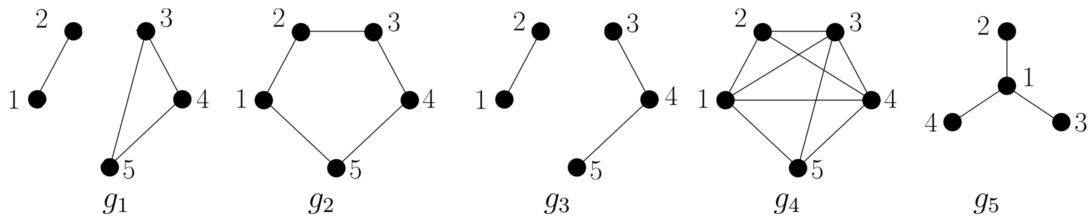


Figure 3.2: The networks representing the isomorphism classes of Example 3.2 that have positive probability.

(a) $q_\mu(\emptyset) = 0$, and $q_\mu(\Omega_K) = 1$;

(b) q_μ satisfies σ-additivity: for $A_1, A_2, \ldots$ a collection of disjoint members of $\mathscr{F}_K$,

$$q_\mu\Big(\bigcup_{k\in\mathbb{N}} A_k\Big) = \sum_{k\in\mathbb{N}} q_\mu(A_k).$$

Example 3.2. Suppose that a network belief system assigns positive probability only to the networks $g_1, g_2, \ldots, g_5$ in Figure 3.2 or to networks isomorphic to them. Suppose that all isomorphism classes associated with the networks in Figure 3.2 have equal probability, i.e., for each isomorphism class $C \in \mathscr{C}$ of $\mathcal{G}$, $\mu(C) = \frac{1}{5}$ if there is a network $g \in \{g_1, g_2, \ldots, g_5\}$ such that $g \in C$, and $\mu(C) = 0$ otherwise. To calculate a player's prior belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$, we now simply need to count the number of vertices in $g_1, \ldots, g_5$ with their neighbor type profile in F, and compare this to the total number of vertices in $g_1, \ldots, g_5$. For instance, a player's prior belief that his neighbor type profile is $\theta = (2, 2)$ is given by

$$q_\mu(\theta) = \frac{\frac{1}{5}\cdot 5 + \frac{1}{5}\cdot 3}{4\cdot\frac{1}{5}\cdot 5 + \frac{1}{5}\cdot 4} = \frac{8}{24},$$

and a player's prior belief that his type is $t = 2$ is $q_\mu(t) = \frac{9}{24}$. This is intuitive: from a player's perspective, he is equally likely to be associated with each of the vertices in $g_1, \ldots, g_5$. $\triangleleft$

Conditional probabilities can be calculated in the usual way. Let $t \in T$ be such that $q_\mu(t) > 0$. A player's belief that his neighbor type profile is in the set $F \in \mathscr{F}_K$ given that his type is t is given by

$$\begin{aligned} q_\mu(F|t) &:= \frac{q_\mu(F \cap \Omega_K^t)}{q_\mu(\Omega_K^t)}, \\ &= \frac{\sum_{C\in\mathscr{C}} \mu(C) n_C(F \cap \Omega_K^t)}{\sum_{C\in\mathscr{C}} \mu(C) n_C(\Omega_K^t)}. \end{aligned}$$

With minor abuse of notation, we write $q_\mu(\theta|t)$ to denote $q_\mu(\{\theta\}|t)$ for $\theta \in \Omega_K$. We refer to $q_\mu(F|t)$ as the *conditional belief* of (a player of) type t that his neighbor type profile is in F.

Example 3.2 (continued) To calculate a player's conditional belief that his neighbor type profile is in some set $F \in \mathscr{F}_K$ given that his type is $t \in T$, we need to count the number of vertices in $g_1, \dots, g_5$ with type t and neighbor type profile in F, and compare this to the total number of vertices in $g_1, \dots, g_5$ with type t. For instance, a player's conditional belief that his neighbor type profile is $\theta = (2, 2)$ given that his type is $t = 2$ is

$$q_\mu(\theta|t) = \frac{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3}{\frac{1}{5} \cdot 5 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 1} = \frac{8}{9}.$$

Indeed, eight out of the nine vertices in $g_1, \dots, g_5$ with type $t = 2$ have neighbor type profile $\theta = (2, 2)$. ◁

3.3 Payoffs and equilibrium

We now define expected payoffs. Let $t \in T, t \neq 0$, $\theta = (\theta_1, \dots, \theta_t) \in \Omega_K^t$, and define $\sigma_{(\theta)} := (\sigma_{\theta_1}, \dots, \sigma_{\theta_t}) \in \Sigma^t$. Let

$$v_t(a, \sigma_{(\theta)}) := \sum_{a^{(t)} \in A^t} \left(\prod_{\ell=1}^{t} \sigma_{\theta_\ell}\big(a_\ell^{(t)}\big) \right) v_t\big(a, a^{(t)}, \theta\big).$$

For each type $t \in T$ such that $q_\mu(t) > 0$, the *expected payoffs* to a player of type t of an action $a \in A$ when the other players play according to the strategy function $\sigma \in \Sigma^T$ are

$$\varphi_t(a, \sigma; \mu) := \sum_{\theta \in \Omega_K^t} q_\mu(\theta|t) v_t(a, \sigma_{(\theta)}, \theta). \tag{3.1}$$

For $t \in T$ such that $q_\mu(t) = 0$, set $\varphi_t(a, \sigma; \mu) := 0$ for all $a \in A$ and $\sigma \in \Sigma^T$. Also, for each $t \in T$ and $\sigma \in \Sigma^T$, let

$$\varphi_t(\sigma; \mu) := \sum_{a \in A} \sigma_t(a) \varphi_t(a, \sigma; \mu). \tag{3.2}$$

The *type-averaged (expected) payoffs* of strategy function $\sigma \in \Sigma^T$ are

$$\Phi(\sigma; \mu) := \sum_{t \in T} q_\mu(t) \varphi_t(\sigma; \mu). \tag{3.3}$$

The type-averaged payoff of a strategy function $\sigma \in \Sigma^T$ is the weighted average of the expected payoffs of the different types under the strategy function σ, and gives the expected payoff of a player who is called upon to play the game, but does not know his type yet. Hence, the expected payoffs of a type correspond to the interim expected payoffs of a player in standard Bayesian games, while the type-averaged payoffs correspond to the ex ante expected payoffs in Bayesian games.

Definition 3.2. Let $\varepsilon \geq 0$. An *ε-equilibrium* is a strategy function $\sigma \in \Sigma^T$ such that for each $t \in T$ such that $q_\mu(t) > 0$, for each action $a \in A$ such that $\sigma_t(a) > 0$,

$$\varphi_t(a, \sigma; \mu) \geq \varphi_t(b, \sigma; \mu) - \varepsilon$$

for all $b \in A$. We refer to a 0-equilibrium as an *equilibrium*. ◁

Proposition 3.1. *Let (μ, v) be a network game of incomplete information. If the profile of payoff functions v is bounded, the game has an equilibrium.*

Proof. See Appendix B. □

Let (μ, v) be a network game of incomplete information. Then, $\mathcal{N}^\varepsilon(\mu, v)$ denotes the set of ε-equilibria of (μ, v). In particular, $\mathcal{N}^0(\mu, v)$ denotes the set of equilibria of (μ, v).

In network games of incomplete information and other games with some form of population uncertainty, all players of the same type must be predicted to behave similarly in equilibrium, as players have no behaviorally relevant characteristics other than their type that are recognized by other players in the game. This raises the question how equilibria might change if we allow players to base their behavior on characteristics other than their type. As shown in Appendix A, the set of equilibria remains essentially unchanged when we allow for such payoff-irrelevant subdivisions of types, as long as the payoff-irrelevant characteristics of players do not provide them with additional information about their neighbors, given their type.

4 The local p-belief operator and higher order beliefs

Let $\mu \in \mathcal{M}$, and let $p \in [0, 1]$. The *local p-belief operator* B^p_μ associates with each set of types the subset of types that with conditional probability at least p interact exclusively with types in that set (whenever they have positive probability). Formally, let $S \subseteq T$. Then,

$$B^p_\mu(S) := \{t \in S \mid q_\mu(t) > 0 \Rightarrow q_\mu(S^t|t) \geq p\}. \tag{4.1}$$

Note that $B^p_\mu(S)$ includes the types in S that have zero probability. By definition, $B^p_\mu(S) \subseteq S$. If also

$$B^p_\mu(S) \supseteq S, \tag{4.2}$$

we say that the set of types S is *p-closed* (under μ).[11] If a set of types is p-closed, then each type in the set interacts with high conditional probability only with types in that set, who in turn interact with high conditional probability only with types in that set, and so on.

The local p-belief operator can be iterated any finite number of times. For instance, $B^p_\mu\big(B^p_\mu(S)\big)$ is the set of types $t \in B^p_\mu(S)$ such that with conditional probability at least p, they interact exclusively with types in $B^p_\mu(S)$, that is, with types in S that with conditional probability at least p interact exclusively with types in S. Define $\left[B^p_\mu\right]^1(S) := B^p_\mu(S)$ and, for each $\ell \in \mathbb{N}$, let $\left[B^p_\mu\right]^{\ell+1} = B^p_\mu \circ \left[B^p_\mu\right]^\ell$. Let

$$C^p_\mu(S) := \bigcap_{\ell \in \mathbb{N}} \left[B^p_\mu\right]^\ell(S)$$

be the set of types that with conditional probability at least p interact exclusively with types that with conditional probability at least $p\ldots$ interact exclusively with types in S, for any number of iterations.

Example 3.2 (continued) Let $S := \{1,2,3\}$. It is easy to check that the conditional belief of a player with type $t = 1$ or $t = 2$ that he interacts exclusively with players with types in S is $q_\mu(S^t|t) = 1$, while the conditional belief of a player with type $t = 3$ that he interacts exclusively with players with types in S is $q_\mu\big(S^3|3\big) = \frac{1}{3}$. Hence, for $p \in [0, \frac{1}{3}]$, we have $B^p_\mu(S) = S$, while for $p \in (\frac{1}{3}, 1]$, it holds that $B^p_\mu(S) = \{1,2\}$. Now consider the conditional beliefs of players with types in the set $B^p_\mu(S)$ that they only interact with players with a type in $B^p_\mu(S)$. For instance, for $p \in (\frac{1}{3}, 1]$, it is easy to check that $q_\mu\big(B^p_\mu(S)|1\big) = \frac{2}{3}$, while $q_\mu\big((B^p_\mu(S))^2|2\big) = 1$. Hence, for $p \in (\frac{1}{3}, \frac{2}{3})$, it holds that $B^p_\mu(B^p_\mu(S)) = \{1,2\}$, while for $p \in (\frac{2}{3}, 1]$, we have $B^p_\mu(B^p_\mu(S)) = \{2\}$. $\triangleleft$

The local p-belief operator satisfies the following desirable properties:[12]

Monotonicity: For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then $B^p_\mu(T') \subseteq B^p_\mu(T'')$.

Continuity: Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k\in\mathbb{N}}$ is a (weakly) decreasing sequence and $\bigcap_{k\in\mathbb{N}} T_k = S$, then $B^p_\mu(T_k) \downarrow B^p_\mu(S)$.

Continuity in p: If $p_k \uparrow p$, then, for any $S \subseteq T$, $B^{p_k}_\mu(S) \downarrow B^p_\mu(S)$.

For proofs, see Appendix B.

[11] We follow the convention in the literature on higher order beliefs of making the one-sided implications explicit, as it is the one-sided implication in (4.2) that captures the nature of a set being p-closed.

[12] See Monderer and Samet (1989, 1996) for a discussion. Note that the axiom of monotonicity implies the axiom of subpotency in the current context: for all $S \subseteq T$, $B^p_\mu\big(B^p_\mu(S)\big) \subseteq B^p_\mu(S)$.

The following two results, which we will use later on, have well known counterparts in the literature on higher order beliefs (Monderer and Samet, 1989, Prop. 3).

Lemma 4.1. *Let $S \subseteq T$, and let $p \in [0,1]$. The set of types $C^p_\mu(S)$ is p-closed, i.e.,*

$$B^p_\mu\big(C^p_\mu(S)\big) = C^p_\mu(S).$$

Lemma 4.2. *Let $p \in [0,1]$. Let $t \in T$, and let $R \subseteq T$. We have that $t \in C^p_\mu(R)$ if and only if there exists a subset of types $S \subseteq T$ that is p-closed such that $t \in S$ and $S \subseteq B^p_\mu(R)$.*

The proofs of Lemmas 4.1 and 4.2 can be found in Appendix B.

Though at first sight the local p-belief operator seems to refer primarily to the "cohesiveness" of a set of types, we can use the local p-belief operator to characterize players' higher order beliefs, i.e., the beliefs players have over the beliefs of other players over the beliefs of other players, and so on. For instance, consider the set $B^p_\mu(S)$ of types for some $S \subseteq T$. We have said that with conditional probability at least p, a player with type $t \in B^p_\mu(S)$ interacts exclusively with players whose types lie in S. An alternative formulation is that a player with a type $t \in B^p_\mu(S)$ believes, given his type, that with probability at least p, all his neighbors have their types in the set S. That is, the local p-belief operator is a belief operator restricted to events of the form "the types of all neighbors of an arbitrary player are in a given set".

When the local p-belief operator is iterated, we obtain statements about players' higher order beliefs. When $t \in B^p_\mu(B^p_\mu(S))$, a player with type t believes (with high conditional probability) that his neighbors believe that their neighbors' types are in S, i.e., the set $B^p_\mu(B^p_\mu(S))$ characterizes a player's beliefs about his neighbors' beliefs about their neighbors (see Figure 4.1(a)). Similarly, when $t \in B^p_\mu(B^p_\mu(B^p_\mu(S)))$, a player believes that his neighbors believe that their neighbors believe that their neighbors' types are in S. That is, the set $B^p_\mu(B^p_\mu(B^p_\mu(S)))$ characterizes a player's beliefs about his neighbors' beliefs about their neighbors' beliefs about their neighbors (see Figure 4.1(b)).

The local p-belief operator also allows us to characterize a player's beliefs over others' beliefs about himself and his beliefs. Indeed, a player is a neighbor of his neighbors, so that when a player believes (with high conditional probability) that his neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_\mu(B^p_\mu(S))$), then he believes that the players he interacts with believe that his type is in S. Similarly, if a player believes that his neighbors believe that their neighbors believe that their neighbors' types are in S (i.e., a player's type is in $B^p_\mu(B^p_\mu(B^p_\mu(S))))$, then he believes that his neighbors believe that he believes that their types are in S.

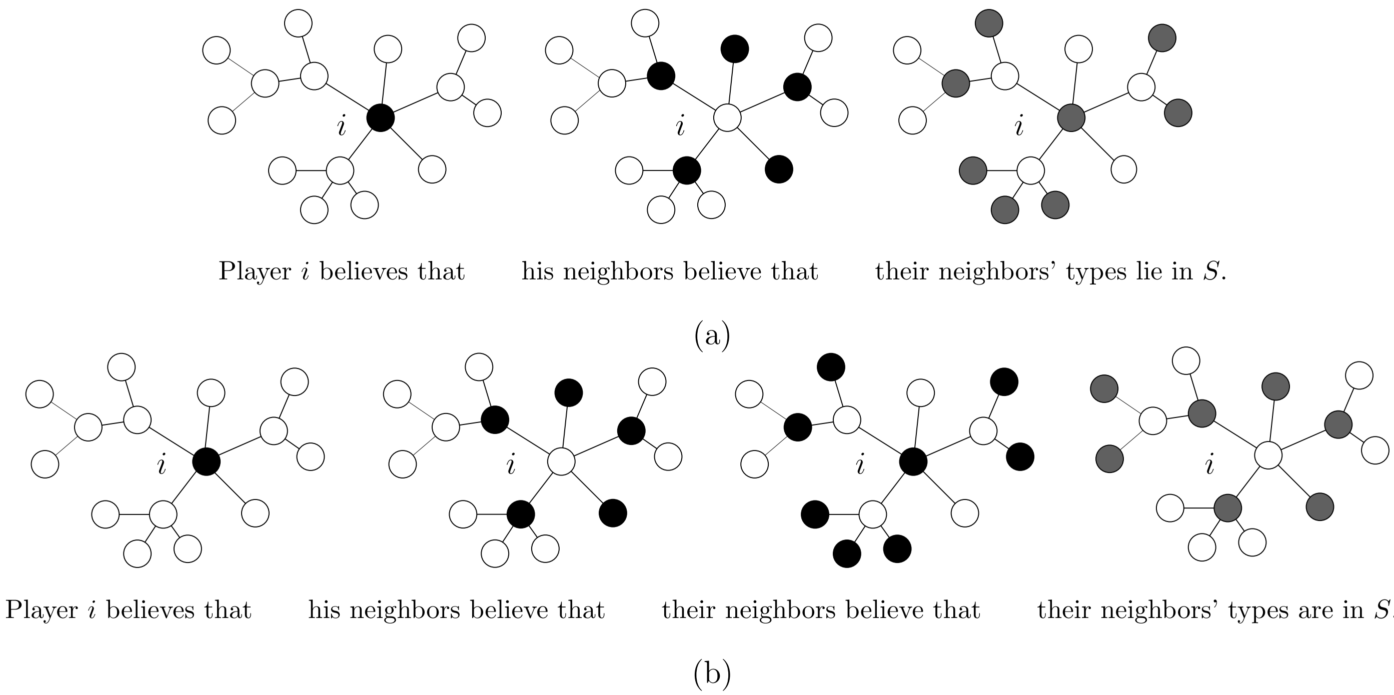


Figure 4.1: Higher order beliefs in a network. (a) Suppose player i has a type in $B^p_\mu\big(B^p_\mu(S)\big)$. Then, with conditional probability at least p, he believes that his neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors' types lie in S. (b) Suppose player i has a type in $B^p_\mu\big(B^p_\mu\big(B^p_\mu(S)\big)\big)$. Then, with conditional probability at least p, he believes that his neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors have a type in S, and that with conditional probability at least p, they believe that their neighbors' types lie in S.

We will use the local p-belief operator extensively in the next section to analyze players' beliefs in network games of incomplete information.

5 Strategic convergence

5.1 Main result

Our objective is to define a "measure" of similarity of network belief systems such that if two network belief systems are similar according to this measure, then, for each network game of incomplete information, for each equilibrium of the game in which beliefs are given by one of these network belief systems, there exists an approximate equilibrium of the game with beliefs given by the other network belief system, such that type-averaged payoffs are close in both equilibria. If that is the case, then, for each possible payoff function, each player who is called upon to play can obtain approximately the same payoffs (in an ex ante sense) under both network belief systems: from a players' (ex ante) perspective, the two network belief systems are similar. At the same time, we do not want to make the conditions on network belief systems to be similar too strict – when we say that two network belief systems are similar if and only if they are identical, the above holds trivially. Hence, we want to define a measure that guarantees that the above holds, but that is no stricter than necessary.

Formally, let $\mu, \mu' \in \mathcal{M}$, and let $v := (v_t)_{t \in T}$ be a profile of payoff functions. For each $\varepsilon \geq 0$, define

$$\chi(\mu, \mu'; v, \varepsilon) := \sup_{\sigma \in \mathcal{N}^0(\mu, v)} \inf_{\sigma' \in \mathcal{N}^\varepsilon(\mu', v)} |\Phi(\sigma; \mu) - \Phi(\sigma'; \mu')|,$$

where Φ is the type-averaged payoff given profile v of payoff functions. Hence, $\chi(\mu, \mu'; v, \varepsilon)$ is a measure of the difference in outcomes under μ and μ' in terms of type-averaged payoffs. That is, for a given $\varepsilon \geq 0$, for each equilibrium under μ, we first find an ε-equilibrium under μ' which minimizes the (absolute) difference in type-averaged payoffs under both equilibria, and we then look for the equilibrium under μ which maximizes this difference. This formalizes the idea that for *each* equilibrium of the network game of incomplete information with one network belief system, there exists *some* approximate equilibrium of the network game of incomplete information with the other network belief system, such that type-averaged payoffs are similar under both equilibria. However, the function $\chi(\mu, \mu'; v, \varepsilon)$ is not symmetric in μ and μ', as we would want. To obtain a symmetric function of μ and μ', let

$$\chi^*(\mu, \mu'; v, \varepsilon) := \max \left\{ \chi(\mu, \mu'; v, \varepsilon), \chi(\mu', \mu; v, \varepsilon) \right\}.$$

Note that when ε increases, the set of approximate equilibria weakly increases, as more and more strategies will satisfy the equilibrium criterion, and the (absolute) difference in type-averaged expected payoffs will decrease weakly. Hence, the interesting case is when ε comes arbitrarily close to 0. This leads us to the following definition (cf. Kajii and Morris, 1998):

Definition 5.1. Take any $\mu \in \mathcal{M}$, and consider a sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$. The sequence $(\mu^k)_{k\in\mathbb{N}}$ *converges strategically* to μ if for each profile v of payoff functions that is bounded, for each $\varepsilon > 0$, we have that

$$\lim_{k\to\infty} \chi^*(\mu, \mu^k; v, \varepsilon) = 0.$$

$\triangleleft$

A natural requirement for strategic convergence is that priors attach similar probabilities to the event that a player has a neighbor type profile in a certain set, i.e., that priors converge in the weak topology on Ω_K. Hence, define

$$d_0(\mu, \mu') := \sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu'}(F)|. \tag{5.1}$$

We also need to consider players' conditional beliefs, i.e., the beliefs they have over their neighbors' types and beliefs, given their own type. For $\delta \in [0, 1]$, let

$$T^\delta_{\mu,\mu'} := \left\{ t \in T \mid \begin{array}{c} q_\mu(t) > 0 \\ q_{\mu'}(t) > 0 \end{array} \right\} \Rightarrow \sup_{F\in\mathscr{F}_K} |q_\mu(F|t) - q_{\mu'}(F|t)| \leq \delta \right\} \tag{5.2}$$

be the set of types such that players' conditional beliefs on their neighbors' types are within δ, whenever the type has positive probability under μ and μ'. If δ is small, the conditional beliefs of a player with a type $t \in T^\delta_{\mu,\mu'}$ over the types of his neighbors are close under μ and μ'. If a player has a type $t \notin T^\delta_{\mu,\mu'}$, then his optimal actions under μ and μ' may differ substantially, as he believes that his local environment is very different under μ and μ'.

However, even if with high (prior) probability, a player has a type such that his conditional beliefs on his neighbors' types are similar under μ and μ' (i.e., $t \in T^\delta_{\mu,\mu'}$), outcomes can be very different under the two priors. The reason is that a player may believe with high conditional probability that the conditional beliefs of some of his neighbors on their neighbors' types are very different under μ and μ' (i.e., $t \notin B^p_\mu(T^\delta_{\mu,\mu'})$ for some $p \in [0, 1]$), or that some of his neighbors believe with high conditional probability that the conditional beliefs of some of their neighbors are very different under μ and μ' (i.e., $t \notin B^p_\mu\big(B^p_\mu(T^\delta_{\mu,\mu'})\big)$), and so on. Hence, we need to require that with high probability, a player has a type in the set $C^p_\mu(T^\delta_{\mu,\mu'})$, for some large $p \in [0, 1]$. In that case, a player's conditional beliefs are similar under μ and μ', and, he believes with high conditional probability that the conditional beliefs of his neighbors are similar under the two priors and that his neighbors believe with high conditional probability that the conditional beliefs of their neighbors are similar under the two priors, and so on. This makes that the actions that are optimal for a player of type $t \in C^p_\mu(T^\delta_{\mu,\mu'})$ under μ will be (almost) optimal under μ', as he expects his neighbors to behave similarly under μ and μ' (as his neighbors expect their neighbors to behave similarly, as the neighbors of his neighbors expect their neighbors...).

However, requiring that with high prior probability, a player has a type in $C^p_\mu(T^\delta_{\mu,\mu'})$ may still not be sufficient. Even if a player believes, given his type, that with high probability his neighbors will choose the same actions under μ and μ' (allowing for ε-best responses), they may not do so if in fact their type is not in $C^p_\mu(T^\delta_{\mu,\mu'})$! That is, if with high probability, some of the neighbors of a player have a type $t \notin C^p_\mu(T^\delta_{\mu,\mu'})$, the payoff to a player with type $t \in C^p_\mu(T^\delta_{\mu,\mu'})$ can be very different under μ and μ'. However, Lemma 5.1 shows that, if the probability is high that a player has a type in the set $C^p_\mu(T^\delta_{\mu,\mu'})$, then in fact also the probability that his neighbors have a type in $C^p_\mu(T^\delta_{\mu,\mu'})$ will be high:

Lemma 5.1. *Let $\mu \in \mathcal{M}$, and fix $\alpha, p \in [0,1]$. For each $S \subseteq T$, if the probability that a player has a type in the set $C^p_\mu(S)$ is at least α, i.e., if*

$$q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \Omega^t_K\Big) \geq \alpha,$$

then the probability that this player and his neighbors have their types in $C^p_\mu(S)$ is at least αp:

$$q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \big(C^p_\mu(S)\big)^t\Big) \geq \alpha p.$$

Proof. See Appendix B. □

Hence, by Lemma 5.1, it is sufficient to require that with high probability, a player has a type in $C^p_\mu(T^\delta_{\mu,\mu'})$. Formally, for each $S \subseteq T$, let

$$\Theta(S) := \bigcup_{t \in S} \Omega^t_K$$

be the set of neighbor type profiles in which the type of the "central" player belongs to the set S. Then, define

$$d_1(\mu,\mu') := \inf\big\{\delta \in [0,1] \mid q_\mu\big(\Theta(C^{1-\delta}_\mu(T^\delta_{\mu,\mu'}))\big) \geq 1-\delta\big\}. \tag{5.3}$$

If $d_1(\mu, \mu')$ is small, then, with high prior probability (under μ), a player has a type such that his conditional beliefs are similar under μ and μ', and with high conditional probability, he interacts exclusively with players whose conditional beliefs are close, and who, with high conditional probability, interact exclusively with players whose conditional beliefs are close, and so on.

We can combine (5.1) and (5.3) to obtain

$$d^*(\mu, \mu') := \max \{d_0(\mu, \mu'), d_1(\mu, \mu'), d_1(\mu', \mu)\}. \tag{5.4}$$

It is immediate that d^* is nonnegative and symmetric. Moreover, $d^*(\mu, \mu') = 0$ if and only if $\mu = \mu'$. However, d^* need not satisfy the triangle inequality, so that it is not a metric. However, d^* generates a topology on the set $\mathcal{M}$ of probability measures on $(\mathcal{G}, \mathscr{F})$: a sequence $(\mu^k)_{k\in\mathbb{N}}$ converges to μ if and only if for any $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that $d^*(\mu^k, \mu) \leq \varepsilon$ for all $k > K_\varepsilon$.

We are now ready to state our main result.

Theorem 5.2. *Let $\mu \in \mathcal{M}$ and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then, $(\mu^k)_{k\in\mathbb{N}}$ converges strategically to μ if and only if*

$$\lim_{k\to\infty} d^*(\mu, \mu^k) = 0.$$

Theorem 5.2 follows from Proposition 5.4 - 5.6. Proposition 5.4 uses Lemma 5.3.

Lemma 5.3. *Let $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Let v be a profile of payoff functions. If $\sigma \in \Sigma^T$ is an equilibrium of the game (μ, v) and if v is bounded by B, then there exists a $5\delta B$-equilibrium σ' of the game (μ', v), with $\sigma'_t = \sigma_t$ for all $t \in C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$.*

Proof. For ease of notation, define $Q := C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$. For each $t \in Q$, set $\sigma'_t = \sigma_t$. For $t \notin Q$ such that $q_{\mu'}(t) > 0$, let σ'_t be such that $(\sigma'_t)_{t\in T}$ is an equilibrium of the reduced game where each player with a type $t \in Q$ is required to play $\sigma'_t = \sigma_t$. Such an equilibrium exists by Proposition 3.1. By construction, σ'_t is a best response to σ' for $t \notin Q$. Hence, it remains to show that σ'_t is a $5\delta B$-best response for a type $t \in Q$. Hence, let $t \in Q$ such that $q_\mu(t) > 0$ and $q_{\mu'}(t) > 0$. By Lemma 4.1,

$$q_{\mu'}\left(Q^t \middle| t\right) \geq 1 - \delta. \tag{5.5}$$

Furthermore, by the definition of $Q = C^{1-\delta}_{\mu'}(T^\delta_{\mu,\mu'})$, for each $F \in \mathscr{F}_K$,

$$|q_\mu(F|t) - q_{\mu'}(F|t)| \leq \delta. \tag{5.6}$$

Let $a \in A$ such that $\sigma_t(a) > 0$, and let $b \in A$. Then,

$$|\varphi_t(a, \sigma'; \mu') - \varphi_t(b, \sigma'; \mu')| \leq \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta|t) \big| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \big| + \sum_{\theta \in Q^t} q_{\mu'}(\theta|t) \big| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \big|. \quad (5.7)$$

The first sum in (5.7) can be evaluated directly. Using (5.5) and that v is bounded by B,

$$\sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu'}(\theta|t) \big| v_t(a, \sigma'_{(\theta)}) - v_t(b, \sigma'_{(\theta)}) \big| < \delta B. \quad (5.8)$$

To evaluate the second sum in (5.7), first note that for $\theta \in Q^t$, all neighbors play according to σ. As σ is an equilibrium of (μ, v),

$$\sum_{\theta \in Q^t} q_{\mu}(\theta|t) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \leq \sum_{\theta \in \Omega_K^t \setminus Q^t} q_{\mu}(\theta|t) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \quad (5.9)$$

Also, by (5.5) and (5.6), we have that

$$q_{\mu}\big(\Omega_K^t \setminus Q^t \big| t\big) \leq 2\delta. \quad (5.10)$$

Combining (5.9) and (5.10), we obtain

$$\sum_{\theta \in Q^t} q_{\mu}(\theta|t) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \leq 2\delta B. \quad (5.11)$$

Let $P^t := \{\theta \in Q^t \mid q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \geq 0\}$ be the set of neighbor type profiles θ in Q^t such that the conditional probability of θ under μ' is at least as high as under μ. Then, by (5.6),

$$\sum_{\theta \in Q^t} \Big| \big(q_{\mu'}(\theta|t) - q_{\mu}(\theta|t)\big) \big(v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)})\big) \Big| = \sum_{\theta \in P^t} \big(q_{\mu'}(\theta|t) - q_{\mu}(\theta|t)\big) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| + \sum_{\theta \in Q^t \setminus P^t} \big(q_{\mu}(\theta|t) - q_{\mu'}(\theta|t)\big) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \leq 2\delta B. \quad (5.12)$$

Combining (5.11) and (5.12), we obtain

$$\sum_{\theta \in Q^t} q_{\mu'}(\theta|t) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \leq \sum_{\theta \in Q^t} q_{\mu}(\theta|t) \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| + \sum_{\theta \in Q^t} \big| q_{\mu'}(\theta|t) - q_{\mu}(\theta|t) \big| \big| v_t(a, \sigma_{(\theta)}) - v_t(b, \sigma_{(\theta)}) \big| \leq 4\delta B. \quad (5.13)$$

Combining (5.7), (5.8) and (5.13) gives

$$|\varphi_t(a,\sigma';\mu') - \varphi_t(b,\sigma';\mu')| \leq 5\delta B.$$

$\square$

Proposition 5.4 establishes the sufficiency of the condition in Theorem 5.2.

Proposition 5.4. *Let $\mu,\mu' \in \mathcal{M}$, and let $\delta \in [0,1]$. Let v be a profile of payoff functions. Suppose that $d^*(\mu,\mu') \leq \delta$. Then, if σ is an equilibrium of the game (μ, v) and v is bounded by B, then there exists a $5\delta B$-equilibrium σ' of the game (μ', v) such that*

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| \leq \big(4-\delta\big)\delta B.$$

Proof. For ease of notation, define $Q := C_{\mu'}^{1-\delta}(T^{\delta}_{\mu,\mu'})$. As $d^*(\mu,\mu') \leq \delta$,

$$|q_\mu(F) - q_{\mu'}(F)| \leq \delta \tag{5.14}$$

for all $F \in \mathscr{F}_K$, and

$$q_{\mu'}\big(\Theta(Q)\big) \geq 1-\delta. \tag{5.15}$$

Let $\sigma \in \Sigma^T$ be an equilibrium of (μ, v). By Lemma 5.3, there exists a $5\delta B$-equilibrium $\sigma' \in \Sigma^T$ of (μ', v) such that $\sigma'_t = \sigma_t$ for all $t \in Q$. Hence, using (5.15) and Lemma 5.1 (with $\alpha = p = 1-\delta$),

$$\begin{aligned}
\big|\Phi(\sigma';\mu') - \Phi(\sigma;\mu')\big| \leq{} & \sum_{\substack{t\in Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta\in Q^t} q_{\mu'}(\theta|t) \sum_{a\in A} |\sigma'_t(a) v_t(a,\sigma'_{(\theta)}) - \sigma_t(a) v_t(a,\sigma'_{(\theta)})| + \\
& \sum_{\substack{t\in Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta\in \Omega_K^t\setminus Q^t} q_{\mu'}(\theta|t) \sum_{a\in A} |\sigma'_t(a) v_t(a,\sigma'_{(\theta)}) - \sigma_t(a) v_t(a,\sigma'_{(\theta)})| + \\
& \sum_{\substack{t\in T\setminus Q:\\ q_{\mu'}(T)>0}} q_{\mu'}(t) \sum_{\theta\in \Omega_K^t} q_{\mu'}(\theta|t) \sum_{a\in A} |\sigma'_t(a) v_t(a,\sigma'_{(\theta)}) - \sigma_t(a) v_t(a,\sigma'_{(\theta)})|, \\
<{} & 0 + \big(1-(1-\delta)^2\big)B, \\
={} & \big(2-\delta\big)\delta B.
\end{aligned}$$

Let $P := \{\theta \in \Omega_K \mid q_{\mu'}(t) - q_\mu(t) \geq 0\}$, and define the function $\zeta : \Omega_K \to T$ by $\zeta(\theta) = t$ whenever $\theta \in \Omega_K^t$. That is, the function ζ gives the type of a player for each possible neighbor type profile he may have. Then,

$$\begin{aligned}\left|\Phi(\sigma;\mu') - \Phi(\sigma;\mu)\right| &\le \sum_{\theta\in P}\big(q_{\mu'}(t) - q_\mu(t)\big)\sum_{a\in A}\sigma_{\zeta(\theta)}|v_{\zeta(\theta)}(a,\sigma_{(\theta)})|+\\ &\qquad\sum_{\theta\in\Omega_K\setminus P}\big(q_\mu(t) - q_{\mu'}(t)\big)\sum_{a\in A}\sigma_{\zeta(\theta)}|v_{\zeta(\theta)}(a,\sigma_{(\theta)})|,\\ &\le 2\delta B.\end{aligned}$$

Combining (5.16) and (5.16) gives the desired result. ☐

We now establish necessity. Proposition 5.5 establishes that $d_0(\mu,\mu')$ should be small for strategic outcomes to be similar (in the sense defined above).

Proposition 5.5. *Let $\delta\in[0,1]$, and let $\mu,\mu'\in\mathcal{M}$. If*

$$d_0(\mu,\mu') > \delta,$$

then there exists a profile v of payoff functions with bound $B=1$ and an equilibrium σ of the game (μ,v) such that for any δ-equilibrium σ' of (μ',v), it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta.$$

Proof. If $d_0(\mu,\mu') > \delta$, there exists a set of neighbor type profiles $F\in\mathscr{F}_K$ such that $|q_\mu(F) - q_{\mu'}(F)| > \delta$. For each $t\in T$, $a\in A, a^{(t)}\in A^t$ and $\theta\in\Omega_K^t$, let

$$v_t(a,a^{(t)},\theta) = \begin{cases} 1, & \text{if } \theta\in F;\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to see that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta$$

for *any* two strategy functions $\sigma,\sigma'\in\Sigma^T$. ☐

Proposition 5.6 establishes that strategic outcomes can be very different if $d_1(\mu,\mu')$ is large.

Proposition 5.6. *Let $\delta\in[0,1]$, and let $\mu,\mu'\in\mathcal{M}$. If*

$$d_1(\mu,\mu') > \delta,$$

then there exists a profile v of payoff functions with bound $B=3$ and an equilibrium σ of the game (μ,v) such that for any δ-equilibrium σ' of the game $(\mu' v)$, it holds that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2.$$

Proof. As $d_1(\mu, \mu') > \delta$, we have

$$q_\mu\big(\Theta(C_\mu^{1-\delta}(T^\delta_{\mu,\mu'}))\big) > 1 - \delta$$

or

$$q_{\mu'}\big(\Theta(C_{\mu'}^{1-\delta}(T^\delta_{\mu,\mu'}))\big) > 1 - \delta. \tag{5.16}$$

Without loss of generality, assume that (5.16) holds. Recall that for each $t \notin T^\delta_{\mu,\mu'}$, there exists a set of neighbor type profiles $F_t \in \mathscr{F}_K$ such that

$$q_{\mu'}(F_t|t) - q_\mu(F_t|t) > \delta.$$

Write $A = \{b^1, b^2, \ldots, b^m\}$, where $m \in \mathbb{N}$, and let payoffs be defined as follows.[13] For each $t \in T$, $a^{(t)} \in A^t$ and $\theta \in \Omega^t_K$, let

$$v_t(b^1, a^{(t)}, \theta) := 0,$$

$$v_t(b^2, a^{(t)}, \theta) := \begin{cases} 2, & \text{if } t \in T^\delta_{\mu,\mu'} \text{ and } a^{(t)}_j = b^2 \text{ for some } j \in \{1, \ldots, t\}; \\ -\delta, & \text{if } t \in T^\delta_{\mu,\mu'} \text{ and } a^{(t)}_j = b^1 \text{ for all } j \in \{1, \ldots, t\}; \\ 1 - q_\mu(F_t|t), & \text{if } t \notin T^\delta_{\mu,\mu'} \text{ and } \theta \in F_t; \\ -q_\mu(F_t|t), & \text{if } t \notin T^\delta_{\mu,\mu'} \text{ and } \theta \notin F_t; \end{cases}$$

and for $\ell \in \{3, \ldots, m\}$, let

$$v_t(b^\ell, a^{(t)}, \theta) := -2.$$

Hence, action b^1 always gives a payoff of 0, regardless of the actions and types of a player and his neighbors. For players with type $t \in T^\delta_{\mu,\mu'}$, action b^2 is only profitable if there is at least one neighbor who also takes action b^2. By contrast, the payoffs of b^2 to players with type $t \notin T^\delta_{\mu,\mu'}$ only depends on their neighbor type profile θ: action b^2 is profitable only if θ belongs to F_t. All other actions than b^1 and b^2 are strictly dominated.

Consider the network game of incomplete information (μ, v). In this game, there is an equilibrium $\sigma \in \Sigma^T$ in which all types $t \in T$ choose action b^1 with probability 1. For each type t, expected payoffs are 0, so that type-averaged payoffs are 0. Now consider the game (μ', v). By definition, for each type $t \notin T^\delta_{\mu,\mu'}$, $q_{\mu'}(F_t|t) - q_\mu(F_t|t) > \delta$. The interim expected payoffs of playing b^2 are then

$$\varphi_t(b^2, \sigma; \mu') = q_{\mu'}(F_t|t)\big(1 - q_\mu(F_t|t)\big) - \big(1 - q_{\mu'}(F_t|t)\big)q_\mu(F_t|t) > \delta$$

for any strategy function $\sigma \in \Sigma^T$. Hence, in any δ-equilibrium, players with type $t \notin T^\delta_{\mu,\mu'}$ will play action b^2. Let $\hat{T}^\delta_{\mu,\mu'} := \{t \in T^\delta_{\mu,\mu'} \mid q_\mu(t) > 0\}$ be the set of types in $T^\delta_{\mu,\mu'}$ that have

[13]This game is based on the "infection game" of Kajii and Morris (1998).

positive probability. Let $t \in \hat{T}^{\delta}_{\mu,\mu'}$. If $q_{\mu}\big((T^{\delta}_{\mu,\mu'})^t|t\big) < 1-\delta$, then, with conditional probability at least δ, a player with type t has at least one neighbor who plays b^2. Hence, the interim expected payoffs of b^2 to such a type are at least

$$\delta \cdot 2 - (1-\delta) \cdot \delta > \delta,$$

so that in any δ-equilibrium, players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\big((T^{\delta}_{\mu,\mu'})^t|t\big) < 1-\delta$ will play b^2. By a similar argument, players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\Big(\big(B^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\big)^t|t\Big) < 1-\delta$ will play b^2 in any δ-equilibrium. It is easy to see that this argument can be iterated any finite number of times. Hence, all players with type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\Big(\big(C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\big)^t|t\Big) < 1-\delta$ will play b^2 in any δ-equilibrium.

By (5.16), the probability that a player has a type $t \notin C_{\mu'}(T^{\delta}_{\mu,\mu'})$ is greater than δ. As by Lemma 4.1 the set $C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})$ is $(1-\delta)$-closed, the probability that a player has a type $t \in \hat{T}^{\delta}_{\mu,\mu'}$ such that $q_{\mu}\Big(\big(C^{1-\delta}_{\mu'}(T^{\delta}_{\mu,\mu'})\big)^t|t\Big) < 1-\delta$ is greater than δ. Hence, in any δ-equilibrium $\sigma' \in \Sigma^T$ of (μ', v), type-averaged expected payoffs are greater than δ^2, so that

$$|\Phi(\sigma;\mu) - \Phi(\sigma';\mu)| > \delta^2.$$

□

We can now prove Theorem 5.2.

Proof. **(If)** Let v be a profile of payoff functions. By Proposition 5.4, for v bounded by B, and for $k \in \mathbb{N}$ such that $5Bd^*(\mu,\mu^k) \leq \varepsilon$,

$$\chi^*(\mu,\mu^k;v,\varepsilon) \leq \big(4 - d^*(\mu,\mu^k)\big)d^*(\mu,\mu^k)B.$$

Hence, for all profiles of payoff functions v that are bounded and for all $\varepsilon > 0$, if $d^*(\mu,\mu^k) \to 0$, then $\chi^*(\mu,\mu^k;v,\varepsilon) \to 0$.

(Only if) Let $\mu, \mu' \in \mathcal{M}$. For $\delta \in [0,1)$, if $d_0(\mu,\mu') > \delta$ or $d_1(\mu,\mu') > \delta$, then, by Propositions 5.5 and 5.6, there exists a profile of payoff functions v bounded by $B = 3$ and an equilibrium $\sigma \in \Sigma^T$ of (μ, v) such that for any δ-equilibrium $\sigma' \in \Sigma^T$ of (μ', v), $|\Phi(\sigma;\mu) - \Phi(\sigma';\mu')| > \delta^2$. □

Before we discuss the implications of this result in more detail, some discussion of our assumptions is in order. Firstly, we assume that all players with the same payoff function independently implement the same strategies, i.e., strategies do not depend on a player's identity in our framework. Our results do not depend on this assumption. Firstly, as shown in Appendix A, the set of equilibria remains essentially unchanged when we allow for payoff-irrelevant subdivisions of types. That is, allowing for payoff-irrelevant characteristics that may affect a player's behavior does not substantively change the set of equilibria, as long as these characteristics do not provide a player with additional information about his neighbors, given his connectivity. This means that we could derive a result similar to Theorem 5.2 for games in which we allow for such payoff-irrelevant subdivisions of types. Secondly, Kets (2007b) shows that a counterpart of Theorem 5.2 holds for Bayesian network games (where the player set is fixed and strategies may depend on a player's identity) when one defines strategic convergence in terms of symmetric Bayesian ε-equilibria.[14] Noting that priors in Bayesian network games are insensitive to small probability events (since the number of players is fixed), a necessary and sufficient condition for a sequence of priors to converge strategically to a prior in Bayesian network games is that the sequence converges uniformly to the prior over events in $\mathscr{F}_K$.

Secondly, the assumption that a player's payoffs only depend on the actions and types of his direct neighbors is not crucial. Under some suitable modifications and some additional technical assumptions, one could obtain similar results for games in which players' payoffs depend on the actions and types of players that are less than k steps away from them in the network, for arbitrary $k \in \mathbb{N}$.

Thirdly, our definition of strategic convergence requires that players choose approximate best responses given their type. If, alternatively, we would only have required that they choose approximate best responses before learning their type, i.e., if we would have considered some ex ante or type-averaged notion of approximate equilibrium, then convergence in the weak topology on Ω_K (i.e., $d_0(\mu, \mu^k) \to 0$) is sufficient for strategic convergence. This directly follows from a translation of the results of Engl (1995) to the current context.

We now proceed to discuss the implications of Theorem 5.2 in more detail.

5.2 Conditional beliefs and strategic convergence

Theorem 5.2 shows that it is not sufficient if two priors assign similar (prior) probabilities to all events in the space of neighbor type profiles for them to be strategically close. In addition, it needs to hold that with high probability, a player has a type such that his conditional beliefs are similar under the two priors, and that he thinks it is likely, given his type, the conditional beliefs of his neighbors are close, and that they think it is likely, given their type, ... that the conditional beliefs of their neighbors are similar, for any number of

[14]Of course, also the result of Kajii and Morris (1998) for general Bayesian games applies. However, the conditions of Kajii and Morris are stricter than necessary for Bayesian network games, as it does not exploit the symmetry and local nature of network games. Furthermore, Kajii and Morris (1998) do not discuss symmetric Bayesian ε-equilibria, while the literature on network games focuses on these type of equilibria.

iterations. When will this latter condition be binding?

To shed some light on this, we first investigate when this condition plays no role. We adopt the following definition from Kajii and Morris (1998):

Definition 5.2. A prior $\mu \in \mathcal{M}$ is *insensitive to small probability events* if for each sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$,

$$\lim_{k\to\infty} d_0(\mu, \mu^k) = 0 \Rightarrow \lim_{k\to\infty} d^*(\mu, \mu^k) = 0.$$

$\triangleleft$

In words, a prior $\mu \in \mathcal{M}$ is insensitive to small probability events if a necessary and sufficient condition for strategic convergence of any sequence $(\mu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$ to μ is that $d_0(\mu, \mu^k)$ converges to zero when k goes to ∞. The next Proposition establishes that a necessary and sufficient condition for a prior to be insensitive to small probability events is that it can be approximated on a finite subset of T:

Proposition 5.7. *A prior $\mu \in \mathcal{M}$ is insensitive to small probability events if and only if for each $\varepsilon > 0$, there exists a finite set of types $S_\varepsilon \subseteq T$ that is $(1-\varepsilon)$-closed under μ such that the probability that a player has a type in S_ε is at least $1-\varepsilon$, i.e.,*

$$q_\mu\big(\Theta(S_\varepsilon)\big) \geq 1 - \varepsilon.$$

The proof can be found in Appendix B.

It is easy to see that the following conditions are sufficient for a prior μ to be insensitive to small probability events:

Finite support: The set of types that have positive probability under μ is finite, i.e., $|\{t \in T \mid q_\mu(t) > 0\}| < \infty$;

Independent types: Players' types are independent, i.e., for all $t \in T$, all $\theta = (\theta_1, \ldots, \theta_t) \in \Omega_K^t$, $q_\mu(\theta|t) = q_\mu(\{\theta\})$.

Perfect correlation over types: Players only interact with players of their own type, i.e., for all $t \in T$ such that $q_\mu(t) > 0$, $q_\mu((t, \ldots, t)|t) = 1$, where $(t, \ldots, t)$ is a vector in T of length t.

One case of interest in which a prior has finite support is when the number of players is fixed, as in Bayesian network games. An example of a network belief system with an unbounded number of players and independent types is given in Example 2.1. Finally, network belief systems in which types are perfectly correlated are studied by e.g. Ellison (1993).

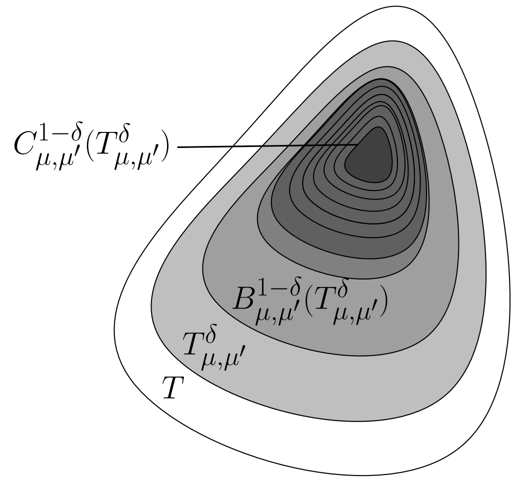


Figure 5.1: Even if with high probability, a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ may be small.

Proposition 5.7 also gives some insight into the question under which conditions a prior is *most* sensitive to small probability events. Consider two priors $\mu, \mu' \in \mathcal{M}$, and let $\delta \in [0, 1]$. Suppose that with probability at least $1 - \delta$, a player has a type $t \in T^{\delta}_{\mu,\mu'}$, i.e., a type such that his conditional beliefs under μ and μ' are within δ. Let $\Theta_0 \subseteq T^{\delta}_{\mu,\mu'}$ be the (possibly empty) set of types in $T^{\delta}_{\mu,\mu'}$ that with high conditional probability interact with types that do *not* belong to $T^{\delta}_{\mu,\mu'}$, and, for $\ell = 1, 2, \ldots$, let $\Theta_\ell \subseteq (T^{\delta}_{\mu,\mu'} \setminus \Theta_{\ell-1})$ be the set of types in $T^{\delta}_{\mu,\mu'} \setminus \Theta_{\ell-1}$ that interact with high conditional probability with types that do not belong to $T^{\delta}_{\mu,\mu'} \setminus \Theta_{\ell-1}$. If a player has a type in one of the sets Θ_ℓ, his own conditional beliefs are close under the two priors, but, with high conditional probability, he interacts with types whose conditional beliefs are very different under μ and μ', or who, with high conditionally probability, interact with types whose conditional beliefs are very different under μ and μ', and so on. If the probability is high that a player has such a type, then even if it is a high probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, the probability that he has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$ will be small, as illustrated in Figure 5.1.

This situation is ruled out under the following two conditions. The first condition is that the set $T^{\delta}_{\mu,\mu'}$ is sufficiently cohesive, in the sense that all types in $T^{\delta}_{\mu,\mu'}$ interact (with high conditional probability) only with types in $T^{\delta}_{\mu,\mu'}$, who in turn interact only with types in $T^{\delta}_{\mu,\mu'}$, and so on. In that case, if it is a high probability event that a player has a type in $T^{\delta}_{\mu,\mu'}$, it will be a high probability event that a player has a type in $C^{1-\delta}_{\mu}(T^{\delta}_{\mu,\mu'})$. The second condition is that players' types are independent. In that case, players' conditional beliefs play no role: if priors assign similar prior probabilities to all events, then players' conditional beliefs will also be similar. Hence, when there is some correlation among types, but the set $T^{\delta}_{\mu,\mu'}$ is not sufficiently cohesive, players' conditional beliefs play an important role so that small probability events can have a large effect on outcomes.

This means that one should be careful in defining the game. In particular, it is often assumed in the literature on network games that the size of the network is fixed and that types are independent. The current analysis shows that these assumptions are not innocuous. If players believe that there is some correlation among types, or if there is uncertainty about the size of the network, then priors may be very sensitive to small probability events, which is not the case when the number of players is fixed or when types are independent. This implies that small differences in the specification of players' prior can have a large effect on outcomes.

6 Conclusions

Given the complexity of many networks, it is important to study whether game-theoretical predictions are robust to assumptions on players' beliefs and information. We have studied the robustness of game-theoretical predictions to assumptions on players' (common) prior in network games of incomplete information. We have asked under what conditions on two priors it holds that for any bounded network game of incomplete information in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the game with the other prior such that ex ante expected payoffs are close. Our main result (Theorem 5.2) shows that two priors are close in a strategic sense if and only if they assign similar prior probabilities to all events involving a player and his neighbors, and, in addition, the set of types for which conditional beliefs are similar has high probability, and is sufficiently cohesive in the sense that with high conditional probability, a type in that set interacts only with types in that set that, with high conditional probability, only interact with types in that set, and so on. This latter condition can also be formulated in terms of players' higher order beliefs: with high probability, a player believes, given his type, that his neighbors' conditional beliefs are similar under the two priors, and that his neighbors believe, given their type, that... the conditional beliefs of their neighbors are similar, for any number of iterations.

To establish our results, we have used ideas and concepts from the literature on higher order beliefs. There are other important questions in the setting of network games of incomplete information that can be answered using ideas from this literature. One important question is how sensitive game-theoretical predictions are to the assumptions on players' information about the network structure. As in much of the literature on network games, we have assumed that players only know their connectivity. Indeed, Friedkin (1983) finds that the "observational horizon" of individuals is limited in communication networks in organizations: individuals only know their local environment in the network. However, there is a large variability among individuals. In addition, players can also represent entities like firms or countries, whose horizon is likely to be larger. For these reasons, it is important to investigate the sensitivity of predictions to informational assumptions. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) study the effect of gradually varying players' information about the network in a specific setting. Their results indicate that informational assumptions can have an important effect on results. However, to date, there is no systematic investigation how assumptions players' information affects results. The link with the literature on higher order beliefs may also be helpful here, as this literature contains numerous results on the effect of perturbing information structures. The current results suggest that such robustness questions are important to study in network games of incomplete information, and they illustrate how one can utilize ideas from the literature on higher order beliefs to answer such questions.

Appendix A Invariance under type-splitting

In this appendix, we show that the set of approximate equilibria is invariant under type-splitting, i.e., payoff-irrelevant subdivisions of types do not substantively change the set of ε-equilibria, for any $\varepsilon \geq 0$ (cf. Myerson, 1998, Th. 4).

First we need some more notation. Let Λ be a nonempty, finite set with complete order $\succeq_\Lambda$. Let $g \in \mathcal{G}$ be a network with vertex set $V(g)$. An *extended network associated with* g (given Λ) can be constructed by assigning to each vertex $i \in V(g)$ a *vertex state* $\lambda \in \Lambda$. Let $\Xi(g)$ be the set of extended networks associated with g, and define

$$\widetilde{\mathcal{G}}^{(n)} := \bigcup_{g \in \mathcal{G}^{(n)}} \Xi(g),$$

and

$$\widetilde{\mathcal{G}} := \bigcup_{n \in \mathbb{N}} \widetilde{\mathcal{G}}^{(n)}.$$

We refer to the members of $\widetilde{\mathcal{G}}$ as *extended networks*. Note that while there may be multiple extended networks associated with a given network $g \in \mathcal{G}$, there is a unique network $g \in \mathcal{G}$ for each extended network $\tilde{g} \in \widetilde{\mathcal{G}}$ such that $\tilde{g}$ is an extended network associated with g.

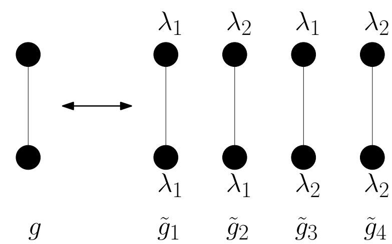


Figure A.1: When $\Lambda = \{\lambda_1, \lambda_2\}$, the network $g \in \mathcal{G}$ corresponds to the extended networks $\tilde{g}_1, \ldots, \tilde{g}_4$, i.e., $\Xi(g) = \{\tilde{g}_1, \ldots, \tilde{g}_4\}$, and $\eta(\tilde{g}_\ell) = g$ for $\ell = 1, \ldots, 4$.

Denote this network by $\eta(\tilde{g})$. For each $\tilde{g} \in \widetilde{\mathcal{G}}$, for each $i \in V(\eta(\tilde{g}))$, let $\kappa(i, \tilde{g}) \in \Lambda$ be the vertex state of i in the extended network $\tilde{g}$. An illustration is provided in Figure A.1.

Let $\tilde{\mathscr{F}}$ be the σ-field generated by the set of singletons of $\widetilde{\mathcal{G}}$, and let $\tilde{\mathcal{M}}$ denote the set of all probability measures on $(\widetilde{\mathcal{G}}, \tilde{\mathscr{F}})$. For $\tilde{\mu} \in \tilde{\mathcal{M}}$, the probability space $(\widetilde{\mathcal{G}}, \tilde{\mathscr{F}}, \tilde{\mu})$ is an *extended network belief system*.

Each vertex in an extended network is thus characterized by his connectivity and his vertex state. Define

$$\tilde{P} := \{(t, \lambda) \mid t \in \mathbb{N} \cup \{0\}, \lambda \in \Lambda\}$$

to be the set of all ordered pairs (t, λ). Since Λ is endowed with the complete order $\succeq_\Lambda$, there exists a complete order on $\tilde{P}$. We define a complete order $\succeq$ on $\tilde{P}$ by

$$\forall (t, \lambda), (t', \lambda') \in \tilde{P} : \quad (t, \lambda) \succeq (t', \lambda') \iff \big(t > t'\big) \text{ or } \big(t = t' \text{ and } \lambda \succeq_\Lambda \lambda\big).$$

For $t > 0$ and $\lambda \in \Lambda$, let

$$\widetilde{\Omega}_K^{(t,\lambda)} := \big\{(\lambda, (\theta_1, \lambda_1), \ldots, (\theta_t, \lambda_t) \in \Lambda \times \tilde{P}^t \mid (\theta_1, \lambda_1) \succeq \ldots \succeq (\theta_t, \lambda_t)\big\}.$$

For $t = 0$, define $\widetilde{\Omega}_K^{(t,\lambda)} = \{(0, \lambda)\}$ for all $\lambda \in \Lambda$. Let

$$\widetilde{\Omega}_K^t := \bigcup_{\lambda \in \Lambda} \widetilde{\Omega}_K^{(t,\lambda)},$$

and

$$\widetilde{\Omega}_K := \bigcup_{t \in \mathbb{N} \cup \{0\}} \widetilde{\Omega}_K^t.$$

Let $\tilde{\mathscr{F}}_K$ be the σ-field generated by the set of singletons of $\widetilde{\Omega}_K$.

We can now define the extended local profile of a vertex in an extended network, a concept akin to the neighbor connectivity profile of a vertex in a network (see Section 2).

For $\tilde{g} \in \widetilde{\mathcal{G}}$ and $i \in V(\eta(\tilde{g}))$ such that $D_i(\eta(\tilde{g})) = 0$, define $\tilde{K}_i(\tilde{g}) := (0, \kappa(i, \tilde{g}))$. Otherwise, define

$$\begin{aligned}
\tilde{N}_1 &:= N_i(\eta(\tilde{g})), \\
j(1) &:= \max\{j \in \tilde{N}_1 \mid (D_j(\eta(\tilde{g})), \kappa(j, \tilde{g})) \succeq (D_k(\eta(\tilde{g})), \kappa(k, \tilde{g})) \text{ for all } k \in \tilde{N}_1\}, \\
\tilde{K}_{i,1}(\tilde{g}) &:= (D_{j(1)}(\eta(\tilde{g})), \kappa(j(1), \tilde{g})),
\end{aligned}$$

and for $\ell = 2, \ldots, D_i(\eta(tg))$:

$$\begin{aligned}
\tilde{N}_\ell &:= \tilde{N}_{\ell-1} \setminus \{j(\ell - 1)\}, \\
j(\ell) &:= \max\{j \in N_\ell \mid (D_j(\eta(\tilde{g})), \kappa(j, \tilde{g})) \succeq (D_k(\eta(\tilde{g})), \kappa(k, \tilde{g})) \text{ for all } k \in \tilde{N}_\ell\}, \\
\tilde{K}_{i,\ell}(\tilde{g}) &:= (D_{j(\ell)}(\eta(\tilde{g})), \kappa(j(\ell), \tilde{g})).
\end{aligned}$$

Then, $\tilde{K}_i(\tilde{g}) := (\tilde{K}_{i,1}(\tilde{g}), \ldots, \tilde{K}_{i,D_i(\eta(tg))}(\tilde{g}))$ is the *extended local profile of i in $\tilde{g}$.*

Let $\Gamma = \langle T, A, (\mathcal{G}, \mathscr{F}, \mu), (v_t)_{t \in T}\rangle$ be a network game of incomplete information. An *extension of* Γ (given Λ) is a tuple

$$\tilde{\Gamma} = \langle \tilde{T}, A, (\widetilde{\mathcal{G}}, \tilde{\mathscr{F}}, \tilde{\mu}), (\tilde{v}_t)_{t \in \tilde{T}}\rangle$$

defined as follows. First, a network $g \in \mathcal{G}$ is drawn according to $(\mathcal{G}, \mathscr{F}, \mu)$. An extended network $\tilde{g} \in \Xi(g)$ is then created by assigning to each vertex $i \in V(g)$ a vertex state $\lambda \in \Lambda$ with probability $p_{\tilde{\mu}}(\lambda)$, independently of the other vertices, where $p_{\tilde{\mu}}(\lambda) \geq 0$ for all $\lambda \in \Lambda$ and $\sum_{\lambda \in \Lambda} p_{\tilde{\mu}}(\lambda) = 1$. A player is associated with each vertex of the extended network. Each player is endowed with the action set A and is informed of the number of neighbors he has in the extended network and the vertex state of the vertex he is associated with. That is, his *extended type* is a pair $(t, \lambda) \in \tilde{P}$. Hence, the *extended type set* is $\tilde{T} = \tilde{P}$. For each $(t, \lambda) \in \tilde{T}$, the payoffs to a player with extended type (t, λ) are given by a function $\tilde{v}$. For each $(t, \lambda) \in \tilde{T}, t > 0$, the function $\tilde{v}_{(t,\lambda)}$ maps $A \times A^t \times \tilde{T}^t$ to $\mathbb{R}$ in the following way. For each $a \in A, a^{(t)} \in A^t$, $((\theta_1, \lambda_1), \ldots, (\theta_t, \lambda_t)) \in \tilde{T}^t$,

$$\tilde{v}_{(t,\lambda)}\big(a, a^{(t)}, (\theta_1, \lambda_1), \ldots, (\theta_t, \lambda_t))\big) := v_t\big(a, a^{(t)}, (\theta_1, \ldots, \theta_t))\big)$$

are the payoffs to a player with extended type (t, λ) of action a when his neighbors have extended types $(\theta_1, \lambda_1), \ldots, (\theta_t, \lambda_t)$ and play according to the action profile $(a_1, \ldots, a_t)$. The payoffs to a player with an extended type $(t, \lambda) \in \tilde{T}$ with $t = 0$ are given by $\tilde{v}_{(t,\lambda)}(a) = v_0(a)$ for each $a \in A$. Hence, for each $(t, \lambda) \in \tilde{T}$, each player with extended type (t, λ) in the extension $\tilde{\Gamma}$ of Γ has the same payoff function as a player of type t in the original game Γ.

For each $(t, \lambda) \in \tilde{T}$, let $\tilde{\sigma}_{(t,\lambda)}$ be a real function defined on A which satisfies

$$\tilde{\sigma}_{(t,\lambda)}(a) \geq 0$$

for all $a \in A$, and

$$\sum_{a \in A} \tilde{\sigma}_{(t,\lambda)}(a) = 1,$$

with $\tilde{\sigma}_{(t,\lambda)}(a)$ the probability that a player with extended type (t, λ) chooses action a. Recall that the set of all probability distributions on A is denoted by Σ. Then, a profile of functions $\tilde{\sigma} \in \Sigma^{\tilde{T}}$ is referred to as an *extended strategy function.*

The difference between the games Γ and $\tilde{\Gamma}$ is thus that the types in Γ have been "splitted", with the subdivisions of types being payoff-irrelevant. The interest in these "extended" games lies in the fact that now players with the same connectivity may choose different probability distributions over actions if they differ in their vertex state.[15] We want to know whether equilibria change substantively when we allow for such payoff-irrelevant subdivisions.

To analyze the equilibria of these extended games, we first need to specify players' beliefs. Players need to form beliefs about the extended type of their neighbors, given their own extended type. Recall that $\mathscr{C}$ is the class of all isomorphism classes of $\mathcal{G}$. Let $t \in \mathbb{N}$, and let $\tilde{\theta} = (\lambda, (\theta_1, \lambda_1), \ldots, (\theta_t, \lambda_t)) \in \widetilde{\Omega}^t_K$ be an extended local profile of a player with connectivity t. Let $C \in \mathscr{C}$. The expected number of players with extended local profile $\tilde{\theta}$ in an extended network $\tilde{g} \in \bigcup_{g \in C} \Xi(g)$ derived from a network in C is given by

$$p_{\tilde{\mu}}(\lambda) p_{\tilde{\mu}}(\lambda_1) \cdots p_{\tilde{\mu}}(\lambda_t) n_C\big(\{(\theta_1, \ldots, \theta_t)\}\big),$$

where we recall that for $F \in \mathscr{F}_K$, $n_C(F)$ is the number of players in a network $g \in \mathcal{G}$ with neighbor type profile F. Then, the expected number of players with extended type $(t, \lambda) \in \tilde{T}$ in an extended network derived from a network in C is

$$\sum_{\lambda_1 \in \lambda} p_{\tilde{\mu}}(\lambda_1) \ldots \sum_{\lambda_t \in \lambda} p_{\tilde{\mu}}(\lambda_t) p_{\tilde{\mu}}(\lambda) n_C\big(\{(\theta_1, \ldots, \theta_t)\}\big) = p_{\tilde{\mu}}(\lambda) n_C\big(\{(\theta_1, \ldots, \theta_t)\}\big).$$

For each $(t, \lambda) \in \tilde{T}$, the conditional beliefs of a player with extended type (t, λ) such that $q_\mu(t) > 0$ and $p_{\tilde{\mu}}(\lambda) > 0$ that his extended local profile is $\tilde{\theta} \in \widetilde{\Omega}^t_K$ are then

$$\begin{aligned}
\tilde{q}_{\tilde{\mu}}(\tilde{\theta}|(t,\lambda)) &:= \frac{\sum_{C \in \mathscr{C}} p_{\tilde{\mu}}(\lambda) \cdot p_{\tilde{\mu}}(\lambda_1) \cdot \ldots p_{\tilde{\mu}}(\lambda_t) \cdot n_C\big(\{(\theta_1, \ldots, \theta_t)\}\big)}{\sum_{C \in \mathscr{C}} p_{\tilde{\mu}}(\lambda) \cdot n_C\big(\Omega^t_K\big)}, \\
&= p_{\tilde{\mu}}(\lambda_1) \cdot \cdots \cdot p_{\tilde{\mu}}(\lambda_t) \cdot q_\mu(\theta|t).
\end{aligned}$$

We can now define expected payoffs. The *extended expected payoff* to a player with extended type $(t, \lambda) \in \tilde{T}$ such that $q_\mu(t) > 0$ and $p_{\tilde{\mu}}(\lambda) > 0$ of an action $a \in A$ when other players

[15]Note that for each network game of incomplete information Γ, there is an extension of Γ that is essentially equivalent to Γ: when there is a $\lambda \in \Lambda$ such that $p_{\tilde{\mu}}(\lambda) = 1$, the extension of Γ is strategically equivalent to Γ.

play according to the extended strategy function $\tilde{\sigma} \in \Sigma^{\tilde{T}}$ is given by

$$\begin{aligned}
\tilde{\varphi}_{(t,\lambda)}(a, \tilde{\sigma}; \tilde{\mu}) &:= \sum_{\tilde{\theta} \in \tilde{\Omega}_K^t} \tilde{q}_{\tilde{\mu}}(\tilde{\theta} | (t, \lambda)) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \tilde{\sigma}_{(t_\ell, \lambda_\ell)}\big(a_\ell^{(t)}\big) \Big) \tilde{v}_{(t,\lambda)}\big(a, a^{(t)}, \tilde{\theta}\big), \\
&= \sum_{\lambda^{(t)} \in \Lambda^t} p_{\tilde{\mu}}\big(\lambda_1^{(t)}\big) \cdots p_{\tilde{\mu}}\big(\lambda_t^{(t)}\big) \sum_{\theta \in \Omega_K^t} q_\mu(\theta | t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \tilde{\sigma}_{(t_\ell, \lambda_\ell)}\big(a_\ell^{(t)}\big) \Big) v_t(a, a^{(t)}, \theta).
\end{aligned}$$

Definition A.1. Let $\tilde{\Gamma}$ be an extension of some network game of incomplete information, and let $\varepsilon \geq 0$. An *extended ε-equilibrium* of $\tilde{\Gamma}$ is an extended strategy profile $\tilde{\sigma} \in \Sigma^{\tilde{T}}$ such that for each $(t, \lambda) \in \tilde{T}$ such that $q_\mu(t) > 0$ and $p_{\tilde{\mu}}(\lambda) > 0$, for each action $a \in A$ such that $\tilde{\sigma}_{(t,\lambda)}(a) > 0$,

$$\tilde{\varphi}_{(t,\lambda)}(a, \tilde{\sigma}; \tilde{\mu}) \geq \tilde{\varphi}_{(t,\lambda)}(b, \tilde{\sigma}; \tilde{\mu}) - \varepsilon$$

for all $b \in A$. An extended 0-equilibrium is an *extended equilibrium.* $\triangleleft$

By an argument similar to that used in the proof of Proposition 3.1, an extended equilibrium exists in an extension of a network game of incomplete information with bounded payoffs.

The next two propositions establish that the set of approximate equilibria remains essentially unchanged when we allow for payoff-irrelevant type-splitting.

Proposition A.1. *Let $\varepsilon \geq 0$. Let Γ be a network game of incomplete information, and let $\tilde{\Gamma}$ be an extension of Γ. If the strategy function $\sigma \in \Sigma^T$ is an ε-equilibrium of Γ, then the extended strategy function $\tilde{\sigma} \in \Sigma^{\tilde{T}}$ defined by*

$$\tilde{\sigma}_{(t,\lambda)}(a) = \sigma_t(a) \qquad \text{for all } (t, \lambda) \in \tilde{T}, a \in A$$

is an extended ε-equilibrium of $\tilde{\Gamma}$.

Proof. The proof follows directly from the definitions. For each $(t, \lambda) \in \tilde{T}$ and each $a \in A$,

$$\begin{aligned}
\tilde{\varphi}_{(t,\lambda)}(a, \tilde{\sigma}; \tilde{\mu}) &= \sum_{\lambda^{(t)} \in \Lambda^t} p_{\tilde{\mu}}\big(\lambda_1^{(t)}\big) \cdots p_{\tilde{\mu}}\big(\lambda_t^{(t)}\big) \sum_{\theta \in \Omega_K^t} q_\mu(\theta | t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \tilde{\sigma}_{(t_\ell, \lambda_\ell)}\big(a_\ell^{(t)}\big) \Big) v_t(a, a^{(t)}, \theta), \\
&= \sum_{\theta \in \Omega_K^t} q_\mu(\theta | t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \sigma_{\theta_\ell}\big(a_\ell^{(t)}\big) \Big) v_t\big(a, a^{(t)}, \theta\big), \\
&= \varphi_t(a, \sigma; \mu).
\end{aligned}$$

Hence, if σ is an ε-equilibrium of Γ, then $\tilde{\sigma}$ is an extended ε-equilibrium of $\tilde{\Gamma}$. $\square$

Proposition A.2. *Let $\varepsilon \geq 0$. Let Γ be a network game of incomplete information, and let $\tilde{\Gamma}$ be an extension of Γ. If the extended strategy function $\tilde{\sigma} \in \Sigma^{\tilde{T}}$ is an extended ε-equilibrium of $\tilde{\Gamma}$, then the strategy function $\sigma \in \Sigma^T$ defined by*

$$\sigma_t(a) = \sum_{\lambda \in \Lambda} p_{\tilde{\mu}}(\lambda) \tilde{\sigma}_{(t,\lambda)}(a) \qquad \text{for all } t \in T, a \in A$$

is an ε-equilibrium of Γ.

Proof. Again, the proof is a direct consequence of the definitions. For each $t \in T$ and each $a \in A$,

$$\begin{aligned}
\varphi_t(a, \sigma; \mu) &= \sum_{\theta \in \Omega_K^t} q_\mu(\theta|t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \sigma_{\theta_\ell}\big(a_\ell^{(t)}\big) \Big) v_t\big(a, a^{(t)}, \theta\big), \\
&= \sum_{\theta \in \Omega_K^t} q_\mu(\theta|t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \Big[\sum_{\lambda \in \Lambda} p_{\tilde{\mu}}(\lambda) \tilde{\sigma}_{(\theta_\ell, \lambda)}\big(a_\ell^{(t)}\big) \Big] \Big) v_t\big(a, a^{(t)}, \theta\big).
\end{aligned}$$

Since

$$\begin{aligned}
\prod_{\ell=1}^{t} \Big[\sum_{\lambda \in \Lambda} p_{\tilde{\mu}}(\lambda) \tilde{\sigma}_{(\theta_\ell, \lambda)}\big(a_\ell^{(t)}\big) \Big] &= \sum_{\lambda^{(t)} \in \Lambda^t} \prod_{\ell=1}^{t} p_{\tilde{\mu}}\big(\lambda_\ell^{(t)}\big) \tilde{\sigma}_{(\theta_\ell, \lambda_\ell^{(t)})}\big(a_\ell^{(t)}\big), \\
&= \sum_{\lambda^{(t)} \in \Lambda^t} p_{\tilde{\mu}}\big(\lambda_1^{(t)}\big) \cdots p_{\tilde{\mu}}\big(\lambda_t^{(t)}\big) \prod_{\ell=1}^{t} \tilde{\sigma}_{(\theta_\ell, \lambda_\ell^{(t)})}\big(a_\ell^{(t)}\big),
\end{aligned}$$

we have

$$\begin{aligned}
\varphi_t(a, \sigma; \mu) &= \sum_{\lambda^{(t)} \in \Lambda^t} p_{\tilde{\mu}}\big(\lambda_1^{(t)}\big) \cdots p_{\tilde{\mu}}\big(\lambda_t^{(t)}\big) \sum_{\theta \in \Omega_K^t} q_\mu(\theta|t) \sum_{a^{(t)} \in A^t} \Big(\prod_{\ell=1}^{t} \tilde{\sigma}_{(\theta_\ell, \lambda_\ell^{(t)})}\big(a_\ell^{(t)}\big) \Big) v_t\big(a, a^{(t)}, \theta\big), \\
&= \tilde{\varphi}_{(t,\lambda)}(a, \tilde{\sigma}; \tilde{\mu})
\end{aligned}$$

for any $\lambda \in \Lambda$. Hence, if $\tilde{\sigma}$ is an extended ε-equilibrium of $\tilde{\Gamma}$, then σ is an ε-equilibrium of Γ. $\square$

Since the equilibrium definitions in network games of incomplete information and their extensions differ, the set of equilibria in a network game of incomplete information does not coincide with the set of equilibria in its extension. However, Proposition A.1 and A.2 establish the important result that allowing for payoff-irrelevant characteristics that might affect players' behavior does not substantively change the set of equilibria. The intuition is that for each equilibrium in a network game of incomplete information, we can find an extended strategy function such that an extended type in the extension faces the same distribution over opponents' actions as the corresponding type in the original game under the equilibrium. Conversely, for each extended equilibrium in an extension of a network game of incomplete information, we can find a strategy function in the original game such that a type in that game faces the same distribution over opponents' actions as a corresponding extended type under the extended equilibrium. In that sense, there is essentially no loss of generality in assuming that all players with the same payoff function independently implement the same strategy, as long as these payoff-irrelevant characteristics do not provide a player with information about his opponents given his connectivity.

Appendix B Proofs

B.1 Proof of Proposition 3.1

Proposition 3.1 uses Lemma B.1.

Lemma B.1. *Let (μ, v) be a network game of incomplete information such that the profile v of payoff functions is bounded. For each $t \in T$, let the function $\varphi_t(\cdot;\mu)$ on Σ^T be defined as in* (3.2)*. Then, $\varphi_t(\cdot;\mu)$ is continuous on the (topological) product space Σ^T.*

Proof. For each $t \in T$ and $n \in \mathbb{N}$, let

$$\Omega_K^{t,n} := \{(k_1, \ldots, k_t) \in \{1, \ldots, n\}^t \mid k_1 \geq k_2 \geq \ldots \geq k_{t-1} \geq k_t\}$$

be the set of neighbor type profiles of a player of type t such that the type of each neighbor is at most n. Clearly, $\Omega_K^{t,n}$ is a finite subset of the countable set Ω_K^t. For each $t \in T$ and $\sigma \in \Sigma^T$, define

$$\varphi_t^{(n)}(\sigma;\mu) := \begin{cases} \sum_{a\in A} \sigma_t(a) \sum_{\theta\in\Omega_K^{t,n}} q_\mu(\theta|t) v_t\big(a, \sigma_{(\theta)}, \theta\big), & \text{if } q_\mu(t) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

For $t \in T$ such that $q_\mu(t) = 0$, it holds that $\varphi_t^{(n)}(\sigma;\mu) = \varphi_t(\sigma;\mu) = 0$ for all $\sigma \in \Sigma^T$. Let $t \in T$ such that $q_\mu(t) > 0$. By the triangle inequality, for each $\sigma \in \Sigma^T$,

$$\big|\varphi_t\big(\sigma;\mu\big) - \varphi_t^{(n)}\big(\sigma;\mu\big)\big| \leq \sum_{\theta\in\Omega_K^t\setminus\Omega_K^{t,n}} q_\mu(\theta|t)\big|v_t\big(a, \sigma_{(\theta)}, \theta\big)\big|.$$

As v is bounded, there exists $B \geq 0$ such that

$$\sum_{\theta\in\Omega_K^t\setminus\Omega_K^{t,n}} q_\mu(\theta|t)\big|v_t\big(a, \sigma_{(\theta)}, \theta\big)\big| \leq B \sum_{\theta\in\Omega_K^t\setminus\Omega_K^{t,n}} q_\mu(\theta|t)$$

for all $\sigma \in \Sigma^T$. Moreover,

$$\lim_{n\to\infty} \sum_{\theta \in \Omega_K^t \setminus \Omega_K^{t,n}} q_\mu(\theta|t) = 0.$$

Hence, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $\sigma \in \Sigma^T$,

$$\left|\varphi_t\big(\sigma;\mu\big) - \varphi_t^{(n)}\big(\sigma;\mu\big)\right| \leq \varepsilon \tag{B.1}$$

for all $n > N_\varepsilon$. That is, for each $t \in T$, the sequence $\big(\varphi_t^{(n)}(\cdot;\mu)\big)_{n\in\mathbb{N}}$ converges uniformly on Σ^T to $\varphi_t(\cdot;\mu)$. As for each $n \in \mathbb{N}$, the function $\varphi_t^{(n)}(\cdot;\mu)$ is continuous on Σ^T, the function $\varphi_t(\cdot;\mu)$ is continuous on Σ^T. $\square$

We are now ready to prove Proposition 3.1.

Proof.

Consider a network game of incomplete information (μ, v) such that v is bounded, and fix some strategy function $\tau \in \Sigma^T$. Let $n \in \mathbb{N}$, and let $T^{(n)} := \{1, \dots, n\}$. Recall the definition of the function $\varphi_t(\cdot;\mu)$ on Σ^T in (3.2).

Consider the strategic game $G^{(n)} = \langle T^{(n)}, \Sigma, \big(\tilde{\varphi}_t^{(n)}(\cdot;\mu)\big)_{t\in T^{(n)}}\rangle$, where for each $t \in T^{(n)}$, $\tilde{\varphi}_t^{(n)}(\cdot;\mu)$ is a real-valued payoff function on Σ^n defined by

$$\tilde{\varphi}_t^{(n)}\big(\sigma^{(n)};\mu\big) = \varphi_t\big(\sigma_1^{(n)}, \dots, \sigma_n^{(n)}, \tau_{n+1}, \tau_{n+2}, \dots;\mu\big)$$

for all $\sigma^{(n)} \in \Sigma^n$. That is, the payoff of a player $t \in T^{(n)}$ in the game $G^{(n)}$ is the expected payoff of a player of type t in the original game (μ, v), given that players with type $t \in T \setminus T^{(n)}$ play according to τ. The set Σ is a nonempty, convex, compact subset of a finite-dimensional Euclidean space, and for each $t \in T^{(n)}$, $\tilde{\varphi}_t^{(n)}(\cdot;\mu)$ is a continuous real-valued function on Σ^n that is quasi-concave in σ_t on Σ. Hence, the best-response correspondence $b_t : \Sigma^n \rightrightarrows \Sigma^n$ of each player $t \in T^{(n)}$ is nonempty, convex-valued, and upper-hemicontinuous, so that by Kakutani's fixed point theorem, a Nash equilibrium $\big(\bar{\sigma}_1^{(n)}, \dots, \bar{\sigma}_n^{(n)}\big) \in \Sigma^n$ exists for $G^{(n)}$.

For each $n \in \mathbb{N}$, define

$$\bar{\sigma}^{(n)} := \big(\bar{\sigma}_1^{(n)}, \dots, \bar{\sigma}_n^{(n)}, \tau_{n+1}, \tau_{n+2}, \dots\big).$$

The set Σ is compact; hence, by the Cantor diagonal method (e.g. Ok, 2007, p. 197-198), there exists a subsequence $(\bar{\sigma}^{(n_j)})_{j\in\mathbb{N}}$ of the sequence $(\bar{\sigma}^{(n)})_{n\in\mathbb{N}}$ that converges to some $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, \dots) \in \Sigma^T$. We claim that $\bar{\sigma}$ is an equilibrium of the original game (μ, v). Suppose not. Then there exists $t \in T$ and $\sigma_t \in \Sigma$ such that

$$\varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_{t-1}, \bar{\sigma}_t, \bar{\sigma}_{t+1}, \dots;\mu) < \varphi_t(\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_{t-1}, \sigma_t, \bar{\sigma}_{t+1}, \dots;\mu).$$

By Lemma B.1, φ_t is continuous on the topological product space Σ^T. Hence, there exists $j \in \mathbb{N}$ such that $n_j \geq t$ and

$$\varphi_t\big(\bar{\sigma}_1^{(n_j)}, \ldots, \bar{\sigma}_t^{(n_j)}, \ldots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \ldots; \mu\big) < \varphi_t\big(\bar{\sigma}_1^{(n_j)}, \ldots, \sigma_t, \ldots, \bar{\sigma}_{n_j}^{(n_j)}, \tau_{n_j+1}, \tau_{n_j+2}, \ldots; \mu\big).$$

But this contradicts that $\big(\bar{\sigma}_1^{(n_j)}, \ldots, \bar{\sigma}_{n_j}^{(n_j)}\big)$ is a Nash equilibrium of the game $G^{(n_j)}$. □

B.2 Properties of the local p-belief operator

In this section, we prove the properties of the local p-belief operator as listed in Section 4, and we prove Lemma 4.1 and 4.2.

Lemma B.2. (Continuity)
Let $S \subseteq T$, and for $k \in \mathbb{N}$, let $T_k \subseteq T$. If $T_k \downarrow S$, i.e., if $(T_k)_{k\in\mathbb{N}}$ is a decreasing sequence and $\bigcap_{k\in\mathbb{N}} T_k = S$, then $B_\mu^p(T_k) \downarrow B_\mu^p(S)$.

Proof. First note that $B_\mu^p(T_{k+1}) \subseteq B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, i.e., $(B_\mu^p(T_k))_{k\in\mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k\in\mathbb{N}} B_\mu^p(T_k) = B_\mu^p\big(\bigcap_{k\in\mathbb{N}} T_k\big).$$

First suppose $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. We need to distinguish two cases. First suppose that $q_\mu(t) = 0$. Then, by definition, $t \in B_\mu^p(\bigcap_{k\in\mathbb{N}} T_k)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(T_k^t|t) \geq p$ for all $k \in \mathbb{N}$. Furthermore, $(T_k^t)_{k\in\mathbb{N}}$ is a decreasing sequence, and $\bigcap_{k\in\mathbb{N}} T_k^t = S^t$. Hence,

$$\lim_{k\to\infty} q_\mu(T_k^t|t) = q_\mu\Big(\bigcap_{k\in\mathbb{N}} T_k^t\big|t\Big).$$

Combining these results gives

$$q_\mu\Big(\bigcap_{k\in\mathbb{N}} T_k^t\big|t\Big) \geq p,$$

and hence $t \in B_\mu^p\big(\bigcap_{k\in\mathbb{N}} T_k^t\big)$.

Secondly, suppose that $t \in B_\mu^p\big(\bigcap_{k\in\mathbb{N}} T_k\big)$. Then, obviously, $t \in T_k$ for all $k \in \mathbb{N}$. Again, we need to consider two cases. If $q_\mu(t) = 0$, then it follows directly from the definition of B_μ^p that $t \in B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, and therefore $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t|t) \geq p$ implies that $q_\mu(T_k|t) \geq p$ for all $k \in \mathbb{N}$. Hence, $t \in B_\mu^p(T_k)$ for all $k \in \mathbb{N}$, and $t \in \bigcap_{k\in\mathbb{N}} B_\mu^p(T_k)$. □

Lemma B.3. (Monotonicity)
For any $T', T'' \subseteq T$, if $T' \subseteq T''$, then $B^p_\mu(T') \subseteq B^p_\mu(T'')$.

Proof. If $T' \subseteq T''$, then $T' \cap T'' = T'$. Hence,

$$B^p_\mu(T') = B^p_\mu(T' \cap T'') = B^p_\mu(T') \cap B^p_\mu(T'') \subseteq B^p_\mu(T'').$$

$\square$

Lemma B.4. (Continuity in p) *If $p_k \uparrow p$, then, for any $S \subseteq T$, $B^{p_k}_\mu(S) \downarrow B^p_\mu(S)$.*

Proof. Let $S \subseteq T$. It follows directly from the definition of the local p-belief operator that $\big(B^{p_k}_\mu(S)\big)_{k\in\mathbb{N}}$ is a decreasing sequence. It remains to show that

$$\bigcap_{k\in\mathbb{N}} B^{p_k}_\mu(S) = B^p_\mu(S).$$

Suppose $t \in \bigcap_{k\in\mathbb{N}} B^{p_k}_\mu(S)$. If $q_\mu(t) = 0$, then it follows directly from the definition that $t \in B^p_\mu(S)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t|t) \geq p_k$ for all $k \in \mathbb{N}$, and therefore $q_\mu(S^t|t) \geq p$. Hence, $t \in B^p_\mu(S)$.

Conversely, suppose $t \in B^p_\mu(S)$. If $q_\mu(t) = 0$, then it follows directly that $t \in \bigcap_{k\in\mathbb{N}} B^{p_k}_\mu(S)$. So suppose $q_\mu(t) > 0$. Then, $q_\mu(S^t) \geq p$, and hence $q_\mu(S^t) \geq p_k$ for all $k \in \mathbb{N}$. We conclude that $t \in B^{p_k}_\mu(S)$ for all k, and hence $t \in \bigcap_{k\in\mathbb{N}} B^{p_k}_\mu(S)$. $\square$

Finally, we present the proofs of Lemmas 4.1 and 4.2.

Proof of Lemma 4.1:
By definition, $B^p_\mu\big(C^p_\mu(S)\big) \subseteq C^p_\mu(S)$. It remains to show that $B^p_\mu\big(C^p_\mu(S)\big) \supseteq C^p_\mu(S)$. Obviously, $\big([B^p_\mu]^\ell(S)\big)_{k\in\mathbb{N}}$ is a weakly decreasing sequence, and, by definition, $\bigcap_{\ell\in\mathbb{N}} \big[B^p_\mu\big]^\ell(S) = C^p_\mu(S)$. Hence, using that the local p-belief operator is continuous,

$$C^p_\mu(S) = \bigcap_{\ell\in\mathbb{N}} \big[B^p_\mu\big]^\ell(S) \subseteq \bigcap_{\ell\in\mathbb{N}:k\geq 2} \big[B^p_\mu\big]^\ell(S) = B^p_\mu\big(\bigcap_{\ell\in\mathbb{N}} \big[B^p_\mu\big]^\ell(S)\big) = B^p_\mu\big(C^p_\mu(S)\big).$$

$\square$

Proof of Lemma 4.2:
Suppose $t \in C^p_\mu(R)$. By Lemma 4.1, the set $C^p_\mu(R)$ is p-closed. Also, by definition, $C^p_\mu(R) \subseteq B^p_\mu(R)$. Hence, we can set $S = C^p_\mu(R)$, and the statement follows.

Conversely, let $S \subseteq T$ be such that $t \in S$, and

$$S \subseteq B^p_\mu(S), \tag{B.2}$$
$$S \subseteq B^p_\mu(R). \tag{B.3}$$

We show by induction on ℓ that $S \subseteq \left[B^p_\mu\right]^\ell(R)$ for all $\ell \in \mathbb{N}$, from which it follows that $t \in C^p_\mu(R)$. By (B.3), $S \subseteq \left[B^p_\mu\right]^1(R)$. For each $\ell \in \mathbb{N}$, if $S \subseteq \left[B^p_\mu\right]^\ell(R)$, then by (B.2) and by monotonicity of the local p-belief operator,

$$S \subseteq B^p_\mu(S) \subseteq B^p_\mu\Big(\left[B^p_\mu\right]^\ell(R)\Big) = \left[B^p_\mu\right]^{\ell+1}(R).$$

$\square$

B.3 Proof of Lemma 5.1

By Lemma 4.1, $C^p_\mu(S)$ is p-closed. Hence, for all $t \in C^p_\mu(S)$ such that $q_\mu(t) > 0$, $q_\mu\big((C^p_\mu(S))^t \big| t\big) \geq p$. This yields

$$\begin{aligned} q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \big(C^p_\mu(S)\big)^t\Big) &= \sum_{\substack{t' \in C^p_\mu(S):\\ q_\mu(t')>0}} q_\mu\Big(\bigcup_{t \in C^p_\mu(S)} \big(C^p_\mu(S)\big)^t \big| t'\Big) q_\mu(t'), \\ &= \sum_{\substack{t' \in C^p_\mu(S):\\ q_\mu(t')>0}} q_\mu\big((C^p_\mu(S))^{t'} \big| t'\big) q_\mu(t'), \\ &\geq p \sum_{t' \in C^p_\mu(S)} q_\mu(t'), \\ &\geq \alpha p. \end{aligned}$$

$\square$

Remark B.1. Note that Lemma 5.1 can be generalized: we can replace $C^p_\mu(S)$ in the lemma by any subset of T that is p-closed. We have presented it in its current form for expositional reasons. $\triangleleft$

B.4 Proof of Proposition 5.7

Proposition 5.7 uses Lemma B.5.

Lemma B.5. *Let $\mu \in \mathcal{M}$, and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. If*

$$\lim_{k\to\infty} d_0(\mu,\mu^k) = 0 \qquad \textit{and} \qquad \lim_{k\to\infty} d_1(\mu,\mu^k) = 0,$$

then

$$\lim_{k\to\infty} d_1(\mu^k,\mu) = 0.$$

Proof. Let $\varepsilon > 0$. By assumption, there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\mu^k}(F)| \le \frac{\varepsilon}{2}, \tag{B.4}$$

and

$$\inf\Big\{\delta \in [0,1] \mid q_\mu\big(\Theta(C^{1-\delta}_\mu(T^\delta_{\mu,\mu^k}))\big)\Big\} \le \frac{\varepsilon}{2}. \tag{B.5}$$

Let $k > K$. Recall that for $t \in T^{\varepsilon/2}_{\mu,\mu^k}$ such that $q_\mu(t) > 0$ and $q_{\mu^k}(t) > 0$,

$$\sup_{F\in\mathscr{F}_K} |q_\mu(F|t) - q_{\mu^k}(F|t)| \le \frac{\varepsilon}{2}, \tag{B.6}$$

and define

$$\hat{T}^{\varepsilon/2}_{\mu,\mu^k} := \{t \in T^{\varepsilon/2}_{\mu,\mu^k} \mid q_\mu(t) > 0\}.$$

Note that, unlike $T^{\varepsilon/2}_{\mu,\mu^k}$, the set $\hat{T}^{\varepsilon/2}_{\mu,\mu^k}$ is not symmetric in μ and μ^k, i.e., $\hat{T}^{\varepsilon/2}_{\mu^k,\mu} \neq \hat{T}^{\varepsilon/2}_{\mu,\mu^k}$. Using (B.6) and the fact that the local p-belief operator is monotonic, we obtain

$$B^{1-\varepsilon/2}_\mu\big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\big) \subseteq B^{1-\varepsilon}_{\mu^k}\big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\big) \subseteq B^{1-\varepsilon}_{\mu^k}\big(\hat{T}^{\varepsilon}_{\mu,\mu^k}\big).$$

Hence,

$$C^{1-\varepsilon/2}_\mu\big(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}\big) \subseteq C^{1-\varepsilon}_{\mu^k}\big(\hat{T}^{\varepsilon}_{\mu,\mu^k}\big).$$

Using this and (B.5), we obtain

$$q_\mu\big(\Theta(C^{1-\varepsilon}_{\mu^k}(T^\varepsilon_{\mu,\mu^k}))\big) \ge q_\mu\big(\Theta(C^{1-\varepsilon}_{\mu^k}(\hat{T}^\varepsilon_{\mu,\mu^k}))\big) \ge \\ q_\mu\big(\Theta(C^{1-\varepsilon/2}_\mu(\hat{T}^{\varepsilon/2}_{\mu,\mu^k}))\big) = q_\mu\big(\Theta(C^{1-\varepsilon/2}_\mu(T^{\varepsilon/2}_{\mu,\mu^k}))\big) \ge 1 - \frac{\varepsilon}{2},$$

so that by (B.4),

$$q_{\mu^k}\big(\Theta(C^{1-\varepsilon}_{\mu^k}(T^\varepsilon_{\mu,\mu^k}))\big) \ge 1 - \varepsilon.$$

Combining these results gives

$$\inf\Big\{\delta \in [0,1] \mid q_{\mu^k}\big(\Theta(C^{1-\delta}_{\mu^k}(T^\delta_{\mu,\mu^k}))\big) \ge 1 - \delta\Big\} \le \varepsilon.$$

$\square$

Proof of Proposition 5.7

(If) Let $\varepsilon > 0$, and let $(\mu^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Suppose that $S_\varepsilon \subseteq T$ is such that

$$|S_\varepsilon| < \infty, \tag{B.7}$$

$$S_\varepsilon \subseteq B_\mu^{1-\varepsilon}(S_\varepsilon), \tag{B.8}$$

$$q_\mu\big(\Theta(S_\varepsilon)\big) \geq 1-\varepsilon. \tag{B.9}$$

By Lemma B.5, if $d_0(\mu,\mu^k) \to 0$ and $d_1(\mu,\mu^k) \to 0$, then also $d_1(\mu^k,\mu) \to 0$. Hence, it is sufficient to show that $d_1(\mu,\mu^k) \to 0$ whenever $d_0(\mu,\mu^k) \to 0$.

Let $\hat{S}_\varepsilon := \{t \in S_\varepsilon \mid q_\mu(t) > 0\}$ be the set of types in S_ε that have positive probability under μ. By (B.7), there exists $c > 0$ such that $q_\mu(t) = q_\mu(\Omega_K^t) \geq c$ for all $t \in \hat{S}_\varepsilon$. Then, for all $k \in \mathbb{N}$, for all $t \in \hat{S}_\varepsilon$,

$$\begin{aligned}
\sup_{F\in\mathscr{F}_K} |q_\mu(F|t) - q_{\mu^k}(F|t)| &= \sup_{F\in\mathscr{F}_K}\left|\frac{q_\mu(F\cup\Omega_K^t)}{q_\mu(\Omega_K^t)} - \frac{q_{\mu^k}(F\cup\Omega_K^t)}{q_\mu(\Omega_K^t)} + \right.\\
&\qquad \left.\frac{q_{\mu^k}(F\cup\Omega_K^t)}{q_\mu(\Omega_K^t)} - \frac{q_{\mu^k}(F\cup\Omega_K^t)}{q_{\mu^k}(\Omega_K^t)}\right|,\\
&\leq \sup_{F\in\mathscr{F}_K} \frac{1}{q_\mu(t)}\big|q_\mu(F\cup\Omega_K^t) - q_{\mu^k}(F\cup\Omega_K^t)\big| + \\
&\qquad \sup_{F\in\mathscr{F}_K} \frac{q_{\mu^k}(F|t)}{q_\mu(t)}\big|q_\mu(\Omega_K^t) - q_{\mu^k}(\Omega_K^t)\big|,\\
&\leq \left(\frac{2}{c}\right)\cdot \sup_{F\in\mathscr{F}_K}|q_\mu(F) - q_{\mu^k}(F)|.
\end{aligned}\tag{B.10}$$

Suppose that $\lim_{k\to\infty} d_0(\mu,\mu^k) = 0$. Then there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F\in\mathscr{F}_K}|q_\mu(F) - q_{\mu^k}(F)| \leq \left(\frac{c}{2}\right)\cdot\varepsilon.$$

Let $k > K$. Then, by (B.10), for all $t \in \hat{S}_\varepsilon$ such that $q_{\mu^k}(t) > 0$, it holds that

$$\sup_{F\in\mathscr{F}_K}|q_\mu(F|t) - q_{\mu^k}(F|t)| \leq \varepsilon,$$

so that $S_\varepsilon \subseteq T^\varepsilon_{\mu,\mu^k}$. By monotonicity of the local p-belief operator and (B.8),

$$S_\varepsilon = B_\mu^{1-\varepsilon}(S_\varepsilon) \subseteq B_\mu^{1-\varepsilon}\big(T^\varepsilon_{\mu,\mu^k}\big).$$

Using Lemma 4.2 and (B.8), we obtain

$$t \in S_\varepsilon \Rightarrow t \in C_\mu^{1-\varepsilon}\big(T^\varepsilon_{\mu,\mu^k}\big),$$

so that (using (B.9))

$$q_\mu\big(\Theta(C_\mu^{1-\varepsilon}(T^\varepsilon_{\mu,\mu^k}))\big) \geq q_\mu\big(\Theta(S_\varepsilon))\big) \geq 1-\varepsilon.$$

Hence, $d_1(\mu,\mu^k) \leq \varepsilon$ whenever $d_0(\mu,\mu^k) \leq \big(\frac{c}{2}\big)\varepsilon$.

(Only if) Suppose that

$$\lim_{k\to\infty} d_0(\mu,\mu^k) = 0 \Rightarrow \lim_{k\to\infty} d_1(\mu,\mu^k).$$

First we show that there exists a sequence $(\nu^k)_{k\in\mathbb{N}}$ in $\mathcal{M}$ such that

(a) for each $k \in \mathbb{N}$, the set of types $\{t \in T \mid q_{\nu^k}(t) > 0\}$ that have positive probability under ν^k is finite;

(b) $(\nu^k)_{k\in\mathbb{N}}$ converges to μ in the weak topology on Ω_K:

$$\lim_{k\to\infty} \sup_{F\in\mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| = 0.$$

The sequence $(\nu^k)_{k\in\mathbb{N}}$ is easy to construct. If μ has finite support in T, i.e., if the set $\{t \in T \mid q_\mu(t) > 0\}$ is finite, then simply set $\nu^k = \mu$ for all $k \in \mathbb{N}$. Otherwise, we construct $(\nu^k)_{k\in\mathbb{N}}$ as follows. For each $k \in \mathbb{N}$, define

$$\mathcal{G}^{(k)} := \{g \in \mathcal{G} \mid \forall i \in V(g), D_i(g) \leq k\}$$

to be the set of networks in which the maximum connectivity is k. Note that the sequence $(\mathcal{G}^{(k)})_{k\in\mathbb{N}}$ is increasing. For each $g \in \mathcal{G}$, let

$$\nu^k(g)\begin{cases} \frac{\mu(g)}{\mu(\mathcal{G}^{(k)})}, & \text{if } g \in \mathcal{G}^{(k)} \text{ and } \mu(\mathcal{G}^{(k)}) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that (a) is satisfied. To see that (b) is also satisfied, first recall that $\mathscr{C}^{(k)}$ is the collection of isomorphism classes in $\mathcal{G}^{(k)}$. For each $k \in \mathbb{N}$ such that $\mu(\mathcal{G}^{(k)}) > 0$, we have

$$\begin{aligned}
\sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| &= \sup_{F \in \mathscr{F}_K} \left| \frac{\sum_{C \in \mathscr{C}} \mu(C) n_C(F)}{\sum_{C \in \mathscr{C}} \mu(C) n_C(\Omega_K)} - \frac{\sum_{C \in \mathscr{C}} \nu^k(C) n_C(F)}{\sum_{C \in \mathscr{C}} \nu^k(C) n_C(\Omega_K)} \right|, \\
&= \sup_{F \in \mathscr{F}_K} \left| \frac{\sum_{C \in \mathscr{C}} \mu(C) n_C(F)}{\sum_{C \in \mathscr{C}} \mu(C) n_C(\Omega_K)} - \frac{\sum_{C \in \mathscr{C}^{(k)}} \nu^k(C) n_C(F)}{\sum_{C \in \mathscr{C}^{(k)}} \nu^k(C) n_C(\Omega_K)} \right|, \\
&\leq \frac{1}{\bar{n}} \sup_{F \in \mathscr{F}_K} |\sum_{C \in \mathscr{C}} \mu(C) n_C(F) - \sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(F)| + \\
&\qquad \left(\frac{1}{\bar{n}} - \frac{1}{\sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(\Omega_K)} \right) \sup_{F \in \mathscr{F}_K} \sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(F), \\
&\leq \frac{1}{\bar{n}} \sup_{F \in \mathscr{F}_K} \left(\sum_{C \in \mathscr{C} \setminus \mathscr{C}^{(k)}} \mu(C) n_C(F) \right) + 1 - \frac{\bar{n}}{\sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(\Omega_K)}.
\end{aligned}$$

As for all $F \in \mathscr{F}_K$,

$$\lim_{k \to \infty} \sum_{C \in \mathscr{C}^{(k)}} \mu(C) n_C(F) = \sum_{C \in \mathscr{C}} \mu(C) n_C(F),$$

it follows that (b) holds.

Since μ is insensitive to small probability events, we also have that $d_1(\mu, \nu^k) \to 0$. Hence, for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$\sup_{F \in \mathscr{F}_K} |q_\mu(F) - q_{\nu^k}(F)| \leq \frac{\varepsilon}{3} \tag{B.11}$$

and

$$\inf\big\{\delta \in [0,1] \mid q_\mu\big(\Theta(C_\mu^{1-\delta}(T^\delta_{\mu,\nu^k}))\big) \geq 1 - \delta\big\} \leq \frac{\varepsilon}{3}. \tag{B.12}$$

Let $k > K$, and define

$$\hat{T}^\varepsilon_{\mu,\nu^k} := \{t \in T^\varepsilon_{\mu,\nu^k} \mid q_{\nu^k}(t) > 0\}$$

to be the set of types in $T^\varepsilon_{\mu,\nu^k}$ that have positive probability under ν^k. By (B.12) and using that the local p-belief operator is monotonic and continuous in p,

$$q_\mu\big(\Theta(C_\mu^{1-\varepsilon}(T^\varepsilon_{\mu,\nu^k}))\big) \geq q_\mu\big(\Theta(C_\mu^{1-\varepsilon/3}(T^\varepsilon_{\mu,\nu^k}))\big) \geq q_\mu\big(\Theta(C_\mu^{1-\varepsilon/3}(T^{\varepsilon/3}_{\mu,\nu^k}))\big) \geq 1 - \frac{\varepsilon}{3},$$

so that by (B.11),

$$q_{\nu^k}\big(\Theta(C_\mu^{1-\varepsilon}(\hat{T}^\varepsilon_{\mu,\nu^k}))\big) = q_{\nu^k}\big(\Theta(C_\mu^{1-\varepsilon}(T^\varepsilon_{\mu,\nu^k}))\big) \geq 1 - \frac{2\varepsilon}{3},$$

and hence (using (B.11) again),

$$q_\mu\big(\Theta(C_\mu^{1-\varepsilon}(\hat{T}^\varepsilon_{\mu,\nu^k}))\big) \geq 1 - \varepsilon.$$

By definition, $\hat{T}^{\varepsilon}_{\mu,\nu^k}$, and hence $C^{1-\varepsilon}_{\mu}\big(\hat{T}^{\varepsilon}_{\mu,\nu^k}\big)$, is finite. Moreover, by Lemma 4.1, $C^{1-\varepsilon}_{\mu}\big(\hat{T}^{\varepsilon}_{\mu,\nu^k}\big)$ is $(1-\varepsilon)$-closed. Hence, by setting

$$S_{\varepsilon} = C^{1-\varepsilon}_{\mu}\big(\hat{T}^{\varepsilon}_{\mu,\nu^k}\big)$$

we obtain the desired result. $\square$

References

Bollobás, B., S. Janson, and O. Riordan (2007). The phase transition in inhomogeneous random graphs. *Random Structures and Algorithms*. Forthcoming.

Coleman, J., E. Katz, and H. Menzel (1966). *Medical Innovation: A Diffusion Study* (2nd ed.). New York, NY: Bobbs-Merrill.

Conley, T. and C. Udry (2005). Learning about a new technology: Pineapple in Ghana. Working Paper, Yale University.

Ellison, G. (1993). Learning, local interaction, and coordination. *Econometrica 61*, 1047–1071.

Engl, G. (1995). Lower hemicontinuity of the Nash equilibrium correspondence. *Games and Economic Behavior 9*, 151 – 160.

Erdős, P. and A. Rényi (1959). On random graphs I. *Publicationes Matematicae 6*, 290–297.

Fafchamps, M. and S. Lund (2003). Risk sharing networks in rural Philippines. *Journal of Development Economics 71*, 261 – 287.

Friedkin, N. (1983). Horizons of observability and limits of informal control in organizations. *Social Forces 62*, 54–77.

Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2006). Network games. Working Paper.

Granovetter, M. (1974). *Getting a Job: A Study of Contacts and Careers*. Cambridge, MA: Harvard University Press.

Grimmett, G. R. and D. R. Stirzaker (1992). *Probability and Random Processes* (Second ed.). Oxford: Oxford University Press.

Hagedoorn, J. (2002). Inter-firm R&D partnerships: An overview of major trends and patterns since 1960. *Research Policy 31*, 477 – 492.

Ioannides, Y. (2004). Random graphs and social networks: An economics perspective. *Working Paper, Tufts University*.

Jackson, M. O. (2006). The economics of social networks. In R. Blundell, W. Newey, and T. Persson (Eds.), *Proceedings of the 9th World Congress of the Econometric Society*. Cambridge: Cambridge University Press.

Jackson, M. O. and L. Yariv (2007). Diffusion of behavior and equilibrium properties in network games. *American Economic Review (Papers and Proceedings) 97*(2), 92 – 98.

Janicik, G. A. and R. P. Larrick (2005). Social network schemas and the learning of incomplete networks. *Journal of Personality and Social Psychology 88*, 348 364.

Johnson, J. and M. Orbach (2002). Perceiving the political landscape: Ego biases in cognitive political networks. *Social Networks 24*, 291 – 310.

Kajii, A. and S. Morris (1998). Payoff continuity in incomplete information games. *Journal of Economic Theory 82*, 267 – 276.

Kalai, E. (2004). Large robust games. *Econometrica 72*, 1631 – 1666.

Kets, W. (2007a). Ph. D. thesis, CentER, Tilburg University. In preparation.

Kets, W. (2007b). Convergence of beliefs in Bayesian network games. CentER Discussion Paper, Tilburg University.

Krackhardt, D. and J. Hanson (1993). Informal networks: The company behind the chart. *Harvard Business Review 71*, 104 – 111.

Kumbasar, E., A. Rommey, and W. Batchelder (1994). Systematic biases in social perception. *American Journal of Sociology 100*, 477 – 505.

Littler, D., F. Leverick, and M. Bruce (1995). Factors affecting the process of collaborative product development: A study of UK manufacturers of information and communications technology products. *Journal of Product Innovation Management 12*, 16 – 32.

Mailath, G., L. Samuelson, and A. Shaked (1997). Correlated equilibria and local interactions. *Economic Theory 9*, 551 – 556.

McAfee, R. and J. McMillan (1987). Auctions with a stochastic number of bidders. *Journal of Economic Theory 43*, 1 – 19.

Milchtaich, I. (2004). Random-player games. *Games and Economic Behavior 47*, 353 – 388.

Milgrom, P. R. and R. J. Weber (1985). Distributional strategies for games with incomplete information. *Mathematics of Operations Research 10*, 619 – 632.

Monderer, D. and D. Samet (1989). Approximating common knowledge with common beliefs. *Games and Economic Behavior 1*, 170 – 190.

Monderer, D. and D. Samet (1996). Proximity of information in games with incomplete information. *Mathematics of Operations Research 3*, 707 – 725.

Morris, S. (1997). Interaction games: A unified analysis of incomplete information, local interaction and random matching games. *Santa Fe Institute Working Papers 97-08-072E.*

Morris, S. (2000). Contagion. *Review of Economic Studies 67*, 57–78.

Myerson, R. B. (1998). Population uncertainty and Poisson games. *International Journal of Game Theory 27*, 375 – 392.

Ok, E. A. (2007). *Real Analysis with Economic Applications.* Princeton, NJ: Princeton University Press.

Powell, W. W., K. W. Koput, and L. Smith-Doerr (1996). Interorganizational collaboration and the locus of innovation: Networks of learning in biotechnology. *Administrative Science Quarterly 41*, 116 – 145.

Powell, W. W., D. R. White, K. W. Koput, and J. Owen-Smith (2005). Network dynamics and field evolution: The growth of inter-organizational collaboration in the life sciences. *American Journal of Sociology 110*, 1132 – 1205.

Rothschild, C. G. (2005). Payoff continuity in incomplete information games: A comment. *Journal of Economic Theory 120*, 270 – 274.

Sundararajan, A. (2005). Local network effects and network structure. Working Paper.

Tucker, C. (2005). Formal and informal influence in the adoption of technologies with network effects. Working Paper.

Vega-Redondo, F. (2007). *Diffusion, Search, and Play in Complex Social Networks.* Econometric Society Monographs. Cambridge: Cambridge University Press.