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Space-filling Latin hypercube designs for computer experiments*

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Abstract

In the area of computer simulation, Latin hypercube designs play an important role. In this paper the classes of maximin and Audze-Eglais Latin hypercube designs are considered. Up to now only several two-dimensional designs and a few higher dimensional designs for these classes have been published. Using periodic designs and the Enhanced Stochastic Evolutionary algorithm of Jin et al. (2005), we obtain new results which we compare to existing results. We thus construct a database of approximate maximin and Audze-Eglais Latin hypercube designs for up to ten dimensions and for up to 300 design points. All these designs can be downloaded from the website `http://www.spacefillingdesigns.nl`.

Keywords: Audze-Eglais, computer experiment, Enhanced Stochastic Evolutionary algorithm, Latin hypercube design, maximin, non-collapsing, packing problem, simulated annealing, space-filling.
JEL Classification: C90.

1 Introduction

A k-dimensional Latin hypercube design (LHD) of n points, is a set of n points $x_i = (x_{i1}, x_{i2}, \ldots, x_{ik}) \in \{0, \ldots, n-1\}^k$ such that for each dimension j all x_{ij} are distinct. An LHD is called maximin when the separation distance $\min_{i \neq j} d(x_i, x_j)$ is maximal among all LHDs of given size n, where d is a certain distance measure. In this paper, we concentrate on the Euclidean (or ℓ^2) distance measure, i.e.

$$d(x_i, x_j) = \sqrt{\sum_{l=1}^{k} (x_{il} - x_{jl})^2}, \tag{1}$$

since this measure is often the first choice in practice.

Besides maximin LHDs, we also treat Audze-Eglais LHDs. These LHDs minimize the following objective:

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{d(x_i, x_j)^2}, \tag{2}$$

where $d(x_i, x_j)$ is again the Euclidean distance between points x_i and x_j. By minimizing this objective, we can also obtain LHDs with uniformly distributed points (Bates et al. (2004)).

For both classes of LHDs, we aim to construct a database of the best designs known in literature. We do this by generating new designs and comparing them with existing results. These designs are often approximate maximin or Audze-Eglais designs in the sense that optimality of the objective is not guaranteed. The reason for this is that optimization over the total set of LHDs can be very time-consuming for larger values of k and n. Therefore, in order to find good designs, optimization is often done over a certain class of LHDs or heuristics are used which do not guarantee optimality. The periodic LHDs

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described in this paper are a good example of the first case. Examples of the second case are simulated annealing used by Morris and Mitchell (1995), the permutation genetic algorithm of Bates et al. (2004) and the Enhanced Stochastic Evolutionary (ESE) algorithm of Jin et al. (2005).

The designs which are best according to the comparison in this paper are added to the website `http://www.spacefillingdesigns.nl` where they can be downloaded for free. As far as we know this is the first extensive online catalogue of maximin and Audze-Eglais LHDs, although there are several catalogues for classical design of experiments, see e.g. the WebDOE$^{\text{TM}}$ website of Crary (2008). Crary et al. (2000) developed I-OPT$^{\text{TM}}$ to generate designs with minimal integrated mean squared error (IMSE). They found that IMSE-optimal designs can have proximate design points, which they call "twin points"; see also Crary (2002).

Our main motivation for investigating this subject is that maximin and Audze-Eglais Latin hypercube designs are extremely useful in the area of computer simulation. One important area where computer simulation is used a lot is engineering. Engineers are confronted with the task of designing products and processes. Since physical experimentation is often expensive and difficult, computer models are frequently used for simulating physical characteristics. The engineer often needs to optimize the product or process design, i.e. to find the best settings for a number of design parameters that influence the critical quality characteristics of the product or process. A computer simulation run is usually time-consuming and there is a great variety of possible input combinations. For these reasons, meta-models that model the quality characteristics as explicit functions of the design parameters are constructed. Such a meta-model, also called a (global) approximation model or surrogate model, is obtained by simulating a number of design points. Well-known meta-model types are polynomials and Kriging models. Since a meta-model evaluation is much faster than a simulation run, in practice such a meta-model is used, instead of the simulation model, to gain insight into the characteristics of the product or process and to optimize it. A review of meta-modeling applications in structural optimization can be found in Barthelemy and Haftka (1993), and in multidisciplinary design optimization in Sobieszczanski-Sobieski and Haftka (1997).

As observed by many researchers, there is an important distinction between designs for computer experiments and designs for the more traditional response surface methods. Physical experiments exhibit random errors and computer experiments are often deterministic (cf. Simpson et al. (2004)). This distinction is crucial and much research is therefore aimed at obtaining efficient designs for computer experiments.

As is recognized by several authors, such a design for computer experiments should at least satisfy the following two criteria (see Johnson et al. (1990) and Morris and Mitchell (1995)). First of all, the design should be *space-filling* in some sense. When no details on the functional behavior of the response parameters are available, it is important to be able to obtain information from the entire design space. Therefore, design points should be "evenly spread" over the entire region. One of the measures often used to obtain space-filling designs is the maximin measure. The Audze-Eglais measure is another measure used for this purpose. Secondly, the design should be *non-collapsing*. When one of the design parameters has (almost) no influence on the function value, two design points that differ only in this parameter will "collapse", i.e. they can be considered as the same point that is evaluated twice. For deterministic simulation models this is not a desirable situation. Therefore, two design points should not share any coordinate values when it is not known a priori which dimensions are important. Note that in other fields of research such designs are referred to as *low discrepancy* designs. To obtain non-collapsing designs the Latin hypercube structure is often enforced. It can be shown that if the function of interest is independent of one or more of the k parameters then, after removal of the irrelevant parameters, the projection of the LHD onto the reduced design space retains good spatial properties; see Koehler and Owen (1996). Maximin LHDs are frequently used in practical applications, see e.g. the examples given in Driessen et al. (2002), Den Hertog and Stehouwer (2002), Alam et al. (2004), and Rikards and Auzins (2004).

Only a few authors consider the construction of maximin LHDs. For example, Morris and Mitchell (1995) use simulated annealing to find approximate maximin LHDs for up to five dimensions and up to 12 design points, and a few larger values, with respect to the ℓ^1- and ℓ^2-distance measure. Van Dam et al. (2007) derive general formulas for two-dimensional maximin LHDs, when the distance measure is ℓ^∞ or ℓ^1, while for the ℓ^2-distance measure (approximate) maximin LHDs up to 1000 design points are obtained by using a branch-and-bound algorithm and constructing (adapted) periodic designs. Ye et al. (2000) propose an exchange algorithm for finding approximate maximin symmetric LHDs. The symmetry property is used as a compromise between computing effort and design optimality. Jin et al. (2005) describe an enhanced stochastic evolutionary (ESE) algorithm for finding approximate maximin LHDs. They also apply their method to other space-filling criteria. The Statistics Toolbox of Matlab also contains a function `lhsdesign` to generate approximate maximin LHDs. This function randomly generates a number of LHDs and picks the one with the largest separation distance. Although this method is very fast, other methods generally result in much better space-filling LHDs. To asses the quality of approximate maximin LHDs, Van Dam et al. (2007) generate upper bounds on the separation distance for certain classes of maximin LHDs. By comparing the separation distances of LHDs to these bounds, we can get an indication of their quality.

There is much more literature related to maximin designs that are not restricted to LHDs. Note that a maximin design is certainly space-filling, but not necessarily non-collapsing.

First of all, the problem of finding the maximal common radius of n circles which can be packed into a square is equivalent to the maximin design problem in two dimensions. Melissen (1997) gives a comprehensive overview of the historical developments and state-of-the-art research in this field. For the ℓ^2-distance measure in the two-dimensional case, optimal solutions are known for $n \leq 30$ and $n = 36$, see e.g. Kirchner and Wengerodt (1987), Peikert et al. (1991), Nurmela and Östergård (1999), and Markót and Csendes (2005). Furthermore, many good approximating solutions have been found for $n \geq 31$; see the Packomania website of Specht (2008). Baer (1992) solved the maximum ℓ^∞-circle packing problem in a k-dimensional unit cube. The ℓ^1-circle packing problem in a square has been solved for many values of n; see Fejes Tóth (1971) and Florian (1989).

Secondly, the maximin design problem has been studied in location theory. In this area of research, the problem is usually referred to as the *max-min facility dispersion problem* (see Erkut (1990)). Facilities are placed such that the minimal distance to any other facility is maximal. Again, the resulting solution is certainly space-filling, but not necessarily non-collapsing. A few papers consider maximin designs in higher dimensions, e.g. Trosset (1999), Locatelli and Raber (2002), Stinstra et al. (2003), and Dimnaku et al. (2005). These papers describe nonlinear programming heuristics to find approximate maximin designs.

Audze-Eglais LHDs are also constructed by only a few authors. The criterion was first introduced by Audze and Eglais (1977) and is based on the analogy of minimizing forces between charged particles. In Bates et al. (2004), the problem of finding Audze-Eglais LHDs is formulated and a permutation genetic algorithm is used to generate them. Liefvendahl and Stocki (2006) compare maximin and Audze-Eglais LHDs and recommend the Audze-Eglais criterion over the maximin criterion. Examples of practical applications of Audze-Eglais LHDs can be found in Rikards et al. (2001), Bulik et al. (2004), Stocki (2005), and Hino et al. (2006).

There are several other measures proposed in the literature besides maximin and Audze-Eglais, e.g. maximum entropy, minimax, IMSE, and discrepancy. For a good overview, we refer to Koehler and Owen (1996). In statistical environments, Latin hypercube sampling (LHS) is often used. In such an approach, points on the grid are sampled without replacement, thereby deriving a random permutation for each dimension; see McKay et al. (1979). Giunta et al. (2003) give an overview of pseudo- and quasi-Monte Carlo sampling, LHS, orthogonal array sampling, and Hammersley sequence sampling. They notice that the basic LHS technique can lead to designs with poor space-filling properties. Extensions to the basic LHS technique are therefore necessary to obtain better designs but these are unfortunately not standard yet in all software packages. Bates et al. (1996) obtain designs for computer experiments by exploring so-called lattice points and using results from number theory.

Several papers combine space-filling criteria with the Latin hypercube structure. Jin et al. (2005) describe an enhanced stochastic evolutionary algorithm for finding maximum entropy and uniform designs. Van Dam (2005) derives interesting results for two-dimensional minimax LHDs.

In literature different designs for computer experiments have been compared and the overall conclusion tends to be that the maximum entropy and distance-based criteria, such as maximin and Audze-Eglais, often perform best; see e.g. Simpson et al. (2001), Santner et al. (2003), and Bursztyn and Steinberg (2006).

This paper is organized as follows. Section 2 describes how periodic designs can be used to obtain good approximate maximin and Audze-Eglais LHDs. In Section 3, we shortly describe some heuristics found in literature used for this same purpose. The ESE-algorithm of Jin et al. (2005) described in this section and periodic designs are used to generate new approximate maximin and Audze-Eglais LHDs. Computational results for up to ten dimensions and for up to 300 design points, as well as a comparison of the new and existing results, are provided in Section 4. Finally, Section 5 contains conclusions.

2 Periodic designs

Van Dam et al. (2007) show that two-dimensional maximin Latin hypercube designs often have a nice, periodic structure. By constructing (adapted) periodic designs, many maximin LHDs and, otherwise, good LHDs, are found for up to 1000 points. Therefore, extending this idea to higher dimensions seems natural.

Let a k-dimensional Latin hypercube design of n points be represented by the sequences $y_1, \ldots, y_k$, with every y_i a permutation of the set $\{0, \ldots, n-1\}$. As in the two-dimensional case, a design is constructed by fixing the first dimension, without loss of generality, to the sequence $y_1 = (0, \ldots, n-1)$ and assigning (adapted) periodic sequences to all other dimensions. Two types of periodic sequences are considered. The first one is the sequence $(v_0, \ldots, v_{n-1})$, where

$$v_i = (i+1)p \bmod (n+1) - 1, \text{ for } i = 0, \ldots, n-1. \tag{3}$$

Here, p is the period of the sequence, which is chosen such that $n+1$ and p have no common divisor, i.e. $\gcd(n+1, p) = 1$, resulting in a permutation of the set $\{0, \ldots, n-1\}$.

Note that the periodic designs obtained in this way resemble *lattices*; see e.g. Bates et al. (1996). The main difference is that lattices are infinite sets of points, which may collapse, and, hence, to construct a (finite) Latin hypercube design a proper subset of non-collapsing lattice points should be chosen. For given n, the structure of the lattice will, however, not always lead to a Latin hypercube design with a sufficient number of points. This is in contrast to periodic designs, for which the modulo-operator insures that for every combination of periods p_j, with $\gcd(n+1, p_j) = 1$, $j = 2, \ldots, k$, a feasible Latin hypercube design is obtained.

The second type of sequence that is considered is the more general sequence $(w_0, \ldots, w_{n-1})$, where $w_i = (s + ip) \bmod n$ (note that we changed the modulus), for $i = 0, \ldots, n-1$. In this case, all starting points $s = 0, \ldots, p$ and all periods $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ will be considered. Note, however, that the resulting sequence w may no longer be one-to-one, i.e. some values may occur more than once, and, hence, the resulting design may no longer be an LHD. Now, let $r > 0$ be the smallest value for which $w_r = w_0$; it then follows that $r = \frac{n}{\gcd(n,p)}$. When $r < n$ a way to construct a one-to-one sequence of length n is by shifting parts of the sequence by, say, q, and repeating this when necessary. To formulate this more explicitly, for the updated sequence w it now holds that

$$w_i = (s + ip + jq) \bmod n, \text{ for } i = jr, \ldots, (j+1)r - 1, \text{ and } j = 0, \ldots, \gcd(n,p) - 1. \tag{4}$$

Let m represent the modulus and, hence, the type of sequence used, i.e. $m = n+1$ corresponds to the first type and $m = n$ to the second. For given n, we now have to set the parameters (p, q, s, m) for every sequence $y_2, \ldots, y_k$.

Dimension	Class A	Class B	Class C
3	$2 \le n \le 70$	$71 \le n \le 100$	–
4	$2 \le n \le 25$	$26 \le n \le 100$	–
5	–	$2 \le n \le 80$	$81 \le n \le 100$
6	–	$2 \le n \le 35$	$36 \le n \le 100$
7	–	–	$2 \le n \le 100$

Table 1: Different classes of periodic sequences are checked to generate maximin designs for each dimension.

To find the best settings for the parameters it would be best to test all values. However, when the dimension and the number of points increase the number of possibilities increases rapidly. Hence, computing all possibilities gets very time-consuming or even impossible. Therefore, three classes of parameter settings (named A, B, and C) are distinguished. The largest one, class A, consists of checking the following parameter values: $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, $q = 1-p, \ldots, p-1$, $s = 0, \ldots, p$, and $m \in \{n, n+1\}$. Testing in three and four dimensions indicated that almost all adapted periodic maximin designs are based on a shift of

$1-p$, -1, or 1 (as was the case for two dimensions; see Van Dam et al. (2007)). Furthermore, most maximin designs are found to have a starting point equal to either $p-1$ or p. Class B is therefore set up to be a subset of class A with the aforementioned restrictions on the parameters q and s. Finally, for the dimensions 5 to 7 the number of possibilities has to be reduced even further, leading to parameter class C, which (based on some more test results) restricts class B to the values $q=1$ and $s=p$, leaving the other parameters unchanged. Table 1 shows the different classes used in the computations of the approximate maximin LHDs for each dimension. For the approximate Audze-Eglais LHDs only class C is used.

As an example, consider a three-dimensional adapted periodic LHD of 22 points. For the maximin criterion, a best parameter setting (class A) is found to be $(p_2, q_2, s_2, m_2) = (8, -7, 7, 22)$ and $(p_3, q_3, s_3, m_3) = (3, 0, 2, 23)$ and, hence, the corresponding maximin LHD, with separation distance 69, is defined by the sequences

$$\begin{array}{lcl} y_1 & = & (\,0\,,\,1\,,\,2\,,\,3\,,\,4\,,\,5\,,\,6\,,\,7\,,\,8\,,\,9\,,\,10\,,\,11\,,\,12\,,\,13\,,\,14\,,\,15\,,\,16\,,\,17\,,\,18\,,\,19\,,\,20\,,\,21\,), \\ y_2 & = & (\,7\,,\,15\,,\,1\,,\,9\,,\,17\,,\,3\,,\,11\,,\,19\,,\,5\,,\,13\,,\,21\,,\,0\,,\,8\,,\,16\,,\,2\,,\,10\,,\,18\,,\,4\,,\,12\,,\,20\,,\,6\,,\,14\,), \\ y_3 & = & (\,2\,,\,5\,,\,8\,,\,11\,,\,14\,,\,17\,,\,20\,,\,0\,,\,3\,,\,6\,,\,9\,,\,12\,,\,15\,,\,18\,,\,21\,,\,1\,,\,4\,,\,7\,,\,10\,,\,13\,,\,16\,,\,19\,). \end{array} \tag{5}$$

Thus, y_3 is a periodic sequence, with $m = n+1$, and y_2 is an adapted periodic sequence, with $m = n$ and $q_2 = -7$. Note that to obtain a one-to-one sequence, the second part of y_2, i.e. $(0, 8, \ldots, 14)$, is formed by shifting the first part of y_2, i.e. $(7, 15, \ldots, 21)$, by -7. The periods and shift are clearly visible in the two-dimensional projection of the LHD in Figure 1. In this figure the y_3-values are depicted at the design points.

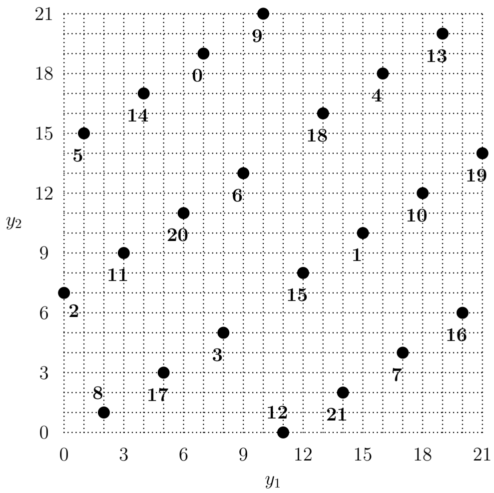

Figure 1: Two-dimensional projection of the three-dimensional LHD (y_1, y_2, y_3) of 22 points.

Like in the two-dimensional case, it may happen that for a given n the corresponding maximin LHD has a separation distance that is smaller than the distance of a design of $n-1$ points. For these n, however, better designs can usually be derived by adding an extra “corner point” to the LHD of $n-1$ points. In this way, a monotone nondecreasing sequence of separation distances was found for all dimensions; see Table 5.

3 Other methods

3.1 Enhanced stochastic evolutionary algorithm

Besides restricting ourselves to a certain class of LHDs, we can also generate good maximin or Audze-Eglais LHDs using heuristics. The ESE-algorithm of Jin et al. (2005) is one of the methods developed for this purpose and is used in this paper to generate new approximate maximin and Audze-Eglais LHDs.

This method starts with an initial design and tries to find better designs by iteratively changing the current design. To determine if a new design is accepted, a threshold-based acceptance criterion is used. This criterion is controlled in the outer loop of the algorithm. In the inner loop of the algorithm new designs are explored.

The inner loop explores the design space as follows. At each iteration, the algorithm creates a fixed number of new designs by exchanging two randomly chosen elements. The new design with the largest separation distance or with the smallest Audze-Eglais objective value is then compared to the current design with a threshold criterion. The criterion is such that it ensures that better designs are always accepted and that worse designs can also be accepted with a certain probability. If the new design is accepted, it replaces the current design. This proces is repeated for a user defined number of iterations.

The outer loop controls the threshold value. After the inner loop is completed, the outer loop determines how much improvement is made in the inner loop. If the amount of improvement is above a certain level, the algorithm starts an improving process in which it tries to rapidly find a local optimum. It does this by lowering the threshold value and thus accepting less deteriorations in the inner loop. If too little improvement is made, an exploration process is started which is intended to escape from a local optimum. The threshold value is first rapidly increased to move away from a local optimum and later slowly decreased to find better designs after moving away. The final design of the algorithm is the best design found during all iterations of the inner loop.

For a more detailed description of the algorithm, we refer to the original paper of Jin et al. (2005). To find maximin and Audze-Eglais LHDs, we implemented the ESE-algorithm in Matlab. The parameters of the algorithm were set to the values suggested in Jin et al. (2005). The only adjustment we made to the original algorithm is in the choice of stopping criterion. Instead of stopping after a fixed number of runs of the outer loop, our criterion is to stop when in the last 1000 runs of the outer loop no improvement is made.

3.2 Simulated Annealing

Another heuristic used to find maximin LHDs is simulated annealing. Morris and Mitchell (1995) were the first to apply simulated annealing for this purpose. The simulated annealing method tries to find good designs by iteratively changing a random starting design. A key characteristic of simulated annealing is that not only improvements are accepted but that also changes which result in worse designs can be accepted. This enables simulated annealing to escape from local optima.

Besides Morris and Mitchell (1995), also Husslage (2006) uses simulated annealing for finding maximin LHDs. One of the main differences between the two methods is the used objective function. Husslage (2006) directly uses the separation distance of a design, whereas Morris and Mitchell (1995) use a surrogate measure ϕ_p. This measure also takes into account the number of pairs of points with a certain distance between them. By including this information, it is easier to decide which design is best if they have the same separation distance. This surrogate measure is also used by other authors like Jin et al. (2005) and Palmer and Tsui (2001).

3.3 Permutation Genetic Algorithm

To obtain Audze-Eglais LHDs, Bates et al. (2004) use a permutation genetic algorithm. The genetic algorithm uses a population of 10 designs and creates new generations of designs by applying different crossover methods. Results of the algorithm are reported for eight different combinations of n and k. In Section 4, we make a comparison between these results and our designs obtained with periodic designs and ESE.

4 Computational results

Using (adapted) periodic designs and the ESE-algorithm, approximate maximin and Audze-Eglais LHDs have been obtained for the cases described in Table 2. All computations have been performed on PCs with a 2.8-GHz Pentium D processor. For the cases with $n > 100$, a limit of 6 hours was imposed on the calculation time.
Table 5 shows the squared ℓ^2-separation distance of the (approximate) maximin LHDs that were obtained by applying periodic designs and the ESE-algorithm. From this table it can be seen that (adapted) periodic designs work particularly well for larger values of n. For dimension 2 to 4 a break-even point, i.e. a point (or, better, an interval) where the preference shifts from the designs found by ESE to (adapted) periodic designs, is clearly visible in the table. Furthermore, these break-even points seem to increase with the dimension of the design and it is to be expected that break-even points for k-dimensional designs, with

dimension	3	4	5	6	7	8	9	10
Maximin PD	300	300	100	100	100			
Maximin ESE	300	300	100	100	100	100	100	100
Audze-Eglais PD	100	100	100					
Audze-Eglais ESE	100	100	100	100	100	100	100	100

Table 2: Largest values of n for which LHDs were generated using periodic designs (PD) and the ESE-algorithm.

$k \geq 5$, will occur for larger values of n, i.e. $n > 250$. This behavior could be explained by the "border effect", i.e. the irregularity of designs that is caused by the borders of the design space. Clearly, the number of "borders" of the k-dimensional box region increases exponentially, with respect to k. However, due to the Latin hypercube structure the number of design points that are located on or near these borders is limited. This, in turn, leads to very irregular optimal Latin hypercube designs when the number of design points is small with respect to the number of borders (which again depends on k). Hence, the nice, periodic structure that is sought for by our periodic heuristic only works well when the number of design points is relatively large, when compared to the dimension. Van Dam et al. (2007) already show the presence of this particular behavior in two-dimensional maximin Latin hypercube designs, i.e. the optimal designs found can all be represented by periodic designs. The results of Table 5 suggest that this behavior also occurs in higher dimensions. ESE, however, does not depend on an underlying structure and can therefore often find better designs. Since all six- and seven-dimensional (adapted) periodic designs, of 3 to 100 points, are dominated by the designs found by ESE, the former are not computed for larger dimensions.

n	3 dim		4 dim		5 dim		6 dim		7 dim		8 dim		9 dim	
	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE
3	**6**	**6**	**7**	**7**	**8**	**8**								
4	**6**	**6**	**12**	**12**	**14**	**14**								
5	**11**	**11**	**15**	**15**	**24**	**24**								
6	**14**	**14**	**22**	**22**	**32**	**32**	**40**	**40**						
7	**17**	**17**	**28**	**28**	**40**	**40**			**61**	**61**				
8	**21**	**21**	**42**	**42**	**50**	**50**					**91**	89		
9	**22**	**22**	**42**	**42**	**61**	**61**							**126**	**126**
10	**27**	**27**	**50**	47	**82**	**82**								
11	29	**30**	**55**	**55**	**80**	**80**								
12	**36**	**36**	**63**	**63**	**91**	**91**	**139**	136						
13														
14									**219**	215				

Table 3: Squared ℓ^2-separation distance of designs found by Morris and Mitchell (1995) and the ESE-algorithm.

With the ESE-algorithm, we are able to match the results of Morris and Mitchell (1995) for most combination of k and n. Only for the cases $k = 4, n = 10$, $k = 6, n = 12$, $k = 7, n = 14$, and $k = 8, n = 8$ slightly worse designs are obtained. Morris and Mitchell (1995), however, already observe that designs that satisfy $n = k$ or $n = 2k$ exhibit special symmetric properties; they refer to them as *foldover designs*. For the case $k = 3, n = 11$, we obtained an improved (and optimal) design. Furthermore, using a branch-and-bound algorithm, the three-dimensional designs of up to 14 points have been verified to be optimal (Van Dam et al. (2007)).

When comparing the ESE results with the SA results in Husslage (2006), we again see that ESE gives better or equally good results for most combination of k and n. For only nine combinations, the results of the SA algorithm are better. However, especially for larger values of n, the ESE algorithm finds designs with significantly higher separation distances.

The obtained results for the Audze-Eglais measure are given in Table 6. We can easily see that the results of the ESE-algorithm are better for almost all cases. It is likely that by running ESE for some more starting solutions, better or equally good designs can be found for all cases. The ESE algorithm thus outperforms the periodic designs for the Audze-Eglais measure.

Table 4: Audze-Eglais values of designs found by Bates et al. (2004) and the ESE-algorithm.

$n \times k$	2×5	2×10	2×120	3×5	3×10	3×120	5×50	5×120
PermGA	**1.2982**	**2.0662**	5.5174	**0.7267**	1.0242	1.9613	0.7270	0.7930
ESE	**1.2982**	**2.0662**	**5.4941**	**0.7267**	**1.0199**	**1.9328**	**0.7195**	**0.7840**

When we compare the results with those found by Bates in Table 4, we see that the ESE-algorithm gives better or equally good results. This shows that the ESE-algorithm is quite successful in finding LHDs with a good Audze-Eglais value.

5 Conclusions

This paper discusses existing and new results in the field of maximin and Audze-Eglais Latin hypercube designs. Such designs play an important role in the area of computer simulation. The new results are obtained using two heuristics. The first heuristic is based on the observation that many optimal LHDs, and two-dimensional LHDs in particular, exhibit a periodic structure. By considering periodic and adapted periodic designs, approximate maximin LHDs for up to seven dimensions and for up to 300 design points are constructed. The second heuristic uses the ESE-algorithm of Jin et al. (2005) to find approximate maximin LHDs for up to ten dimensions. These new results are compared to existing results obtained with simulated annealing and permutation genetic algorithms. In most cases, the ESE-algorithm resulted in the best maximin and Audze-Eglais LHDs. However when the number of design points is large with respect to the dimension of the design, the periodic designs tend to work better. In Appendix A, all the obtained squared ℓ^2-separation distances and Audze-Eglais values can be found. All corresponding designs can be downloaded from the website `http://www.spacefillingdesigns.nl`.

Appendix: Tables of numerical results

n	3 dim		4 dim		5 dim		6 dim		7 dim		8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE
2	**3**	**3**	**4**	**4**	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**9**	**10**
3	**6**	3	**7**	4	**8**	5	**12**	6	**13**	7	**14**	**18**	**19**
4	**6**	**6**	**12**	**12**	**14**	11	**20**	15	**21**	16	**26**	**28**	**33**
5	**11**	6	**15**	12	**24**	11	**27**	15	**32**	16	**40**	**43**	**50**
6	**14**	**14**	**22**	16	**32**	23	**40**	28	**47**	29	**53**	**61**	**68**
7	**17**	14	**28**	16	**40**	23	**52**	28	**61**	31	**70**	**80**	**89**
8	**21**	**21**	**42**	25	**50**	32	**63**	42	**79**	46	**90**	**101**	**114**
9	**22**	21	**42**	25	**61**	39	**75**	45	**92**	47	**112**	**126**	**142**
10	**27**	21	**47**	36	**82**	55	**91**	62	**109**	68	**131**	**154**	**171**
11	**30**	24	**55**	39	**80**	55	**108**	62	**129**	69	**152**	**178**	**206**
12	**36**	30	**63**	46	**91**	62	**136**	91	**152**	95	**177**	**204**	**235**
13	**41**	35	**70**	51	**103**	64	**138**	91	**178**	95	**205**	**235**	**268**
14	**42**	35	**77**	70	**114**	86	**154**	104	**215**	119	**236**	**268**	**305**
15	**45**	42	**87**	71	**129**	88	**171**	111	**220**	129	**273**	**309**	**347**
16	**50**	42	**93**	85	**151**	101	**190**	130	**241**	155	**317**	**352**	**393**
17	**53**	42	**99**	85	**158**	113	**208**	131	**266**	161	**332**	**396**	**442**
18	**56**	50	**108**	94	**170**	123	**231**	155	**291**	186	**361**	**451**	**496**
19	**59**	57	**119**	94	**184**	136	**256**	169	**323**	195	**390**	**469**	**554**
20	**65**	57	**130**	106	**206**	139	**279**	210	**349**	226	**425**	**506**	**625**
21	**68**	65	**145**	116	**223**	165	**302**	210	**380**	236	**463**	**548**	**650**
22	**72**	69	**150**	117	**235**	174	**325**	223	**418**	270	**501**	**595**	**691**
23	**75**	72	**159**	130	**250**	178	**348**	236	**448**	273	**542**	**640**	**747**
24	**81**	76	**170**	138	**266**	201	**374**	258	**481**	308	**585**	**690**	**800**
25	86	**91**	**178**	156	**285**	205	**400**	286	**520**	350	**626**	**739**	**857**
26	86	**91**	**188**	156	**302**	226	**426**	296	**548**	365	**664**	**791**	**910**
27	90	**91**	**198**	157	**310**	238	**447**	310	**585**	382	**712**	**840**	**976**
28	**94**	**94**	**210**	174	**331**	258	**479**	339	**620**	406	**766**	**898**	**1041**
29	**101**	94	**221**	174	**349**	269	**507**	346	**654**	417	**817**	**956**	**1100**
30	**105**	**105**	**233**	194	**367**	310	**531**	390	**691**	458	**849**	**1019**	**1173**
31	**110**	107	**244**	212	**405**	310	**563**	390	**728**	482	**900**	**1104**	**1241**
32	110	**114**	**253**	212	**413**	341	**587**	419	**778**	518	**966**	**1139**	**1318**
33	**117**	114	**264**	215	**426**	341	**622**	430	**814**	537	**1010**	**1201**	**1396**
34	125	**133**	**273**	230	**445**	358	**648**	470	**851**	561	**1072**	**1270**	**1478**
35	126	**133**	**286**	234	**467**	366	**683**	495	**914**	586	**1113**	**1326**	**1555**
36	131	**133**	**297**	250	**486**	400	**719**	518	**939**	636	**1181**	**1405**	**1647**
37	138	**152**	**309**	266	**520**	408	**744**	528	**976**	668	**1236**	**1477**	**1721**
38	142	**152**	**321**	283	**541**	415	**788**	561	**1028**	709	**1286**	**1534**	**1790**
39	146	**152**	**330**	283	**566**	439	**816**	561	**1084**	726	**1344**	**1609**	**1870**
40	152	**155**	**342**	291	**575**	492	**876**	632	**1122**	786	**1416**	**1675**	**1946**
41	158	**162**	**355**	293	**596**	492	**882**	632	**1156**	802	**1496**	**1765**	**2058**
42	161	**168**	**367**	319	**626**	496	**907**	670	**1209**	903	**1526**	**1843**	**2149**
43	**171**	168	**383**	323	**666**	520	**947**	670	**1256**	903	**1597**	**1905**	**2224**
44	179	**186**	**396**	331	**680**	548	**992**	696	**1336**	903	**1653**	**1994**	**2319**
45	182	**186**	**407**	347	**698**	565	**996**	737	**1366**	926	**1723**	**2079**	**2415**
46	186	**189**	**421**	366	**723**	592	**1064**	797	**1408**	985	**1794**	**2155**	**2507**
47	**189**	**189**	**438**	378	**754**	611	**1088**	797	**1459**	985	**1847**	**2244**	**2600**
48	**201**	189	**450**	413	**763**	632	**1119**	857	**1531**	1054	**1924**	**2336**	**2732**
49	**203**	196	**464**	415	**803**	634	**1167**	893	**1592**	1074	**1989**	**2397**	**2828**
50	206	**213**	**478**	415	**830**	663	**1203**	893	**1639**	1113	**2041**	**2492**	**2893**
51	206	**213**	**490**	421	**850**	692	**1230**	917	**1662**	1161	**2132**	**2566**	**3006**
52	**217**	213	**504**	455	**883**	709	**1274**	1003	**1734**	1231	**2203**	**2686**	**3134**
53	**219**	216	**515**	455	**894**	716	**1340**	1003	**1808**	1241	**2234**	**2713**	**3261**
54	209	**233**	**534**	477	**932**	760	**1359**	1019	**1856**	1288	**2356**	**2805**	**3339**
55	230	**243**	**546**	483	**956**	760	**1421**	1082	**1896**	1325	**2429**	**2935**	**3452**
56	230	**243**	**558**	515	**982**	784	**1431**	1104	**2003**	1358	**2444**	**3021**	**3551**
57	249	**261**	**574**	515	**1007**	846	**1488**	1136	**2024**	1479	**2554**	**3119**	**3651**
58	245	**261**	**594**	539	**1035**	846	**1554**	1166	**2043**	1479	**2650**	**3187**	**3795**
59	254	**266**	**609**	544	**1063**	849	**1564**	1223	**2136**	1509	**2733**	**3297**	**3889**
60	261	**273**	**618**	568	**1094**	904	**1631**	1242	**2232**	1577	**2796**	**3420**	**4090**
61	266	**274**	**630**	620	**1128**	904	**1667**	1258	**2266**	1615	**2868**	**3525**	**4158**
62	269	**283**	**657**	620	**1150**	934	**1715**	1306	**2345**	1680	**2977**	**3636**	**4313**
63	281	**297**	**670**	620	**1178**	967	**1781**	1380	**2376**	1680	**3056**	**3690**	**4355**
64	278	**297**	**684**	625	**1206**	985	**1804**	1430	**2452**	1769	**3097**	**3820**	**4514**
65	290	**314**	**694**	630	**1216**	997	**1868**	1430	**2492**	1786	**3219**	**3932**	**4581**
66	299	**314**	**718**	666	**1261**	1050	**1874**	1476	**2543**	1857	**3279**	**4004**	**4769**
67	294	**314**	**735**	666	**1299**	1072	**1954**	1482	**2638**	1868	**3399**	**4081**	**4942**
68	306	**314**	**746**	685	**1330**	1087	**1983**	1538	**2693**	1940	**3453**	**4212**	**4995**
69	306	**324**	**765**	698	**1351**	1112	**2028**	1588	**2746**	1965	**3520**	**4317**	**5127**
70	314	**325**	**779**	716	**1378**	1150	**2094**	1633	**2838**	2130	**3588**	**4464**	**5276**

Table 5: Squared ℓ^2-separation distance found using periodic designs (PD) and the ESE-algorithm (ESE).

n	3 dim		4 dim		5 dim		6 dim		7 dim		8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE
71	314	**325**	**793**	716	**1413**	1150	**2141**	1644	**2871**	2130	**3749**	**4548**	**5437**
72	314	**341**	**810**	750	**1430**	1203	**2136**	1768	**2960**	2177	**3810**	**4666**	**5556**
73	329	**350**	**834**	759	**1462**	1229	**2197**	1768	**3042**	2206	**3932**	**4776**	**5661**
74	341	**350**	**842**	767	**1512**	1229	**2291**	1774	**3120**	2244	**3941**	**4915**	**5817**
75	341	**350**	**867**	771	**1530**	1274	**2303**	1862	**3157**	2295	**4073**	**5006**	**5937**
76	341	**363**	**882**	813	**1569**	1300	**2387**	1935	**3218**	2375	**4178**	**5179**	**6111**
77	341	**363**	**894**	823	**1591**	1308	**2433**	1947	**3323**	2403	**4266**	**5222**	**6272**
78	371	**387**	**910**	844	**1621**	1382	**2479**	2014	**3387**	2505	**4390**	**5385**	**6384**
79	374	**387**	**927**	848	**1639**	1382	**2498**	2037	**3474**	2525	**4465**	**5535**	**6466**
80	374	**403**	**949**	873	**1691**	1395	**2554**	2037	**3550**	2590	**4565**	**5577**	**6653**
81	381	**406**	**963**	916	**1730**	1406	**2648**	2064	**3619**	2642	**4679**	**5748**	**6780**
82	374	**406**	**989**	938	**1742**	1475	**2680**	2141	**3669**	2753	**4719**	**5859**	**6935**
83	374	**417**	**1002**	940	**1762**	1501	**2696**	2141	**3723**	2767	**4848**	**5976**	**7094**
84	406	**426**	**1021**	967	**1818**	1534	**2790**	2229	**3870**	2838	**4920**	**6119**	**7256**
85	413	**426**	**1043**	967	**1866**	1552	**2819**	2232	**3919**	2874	**5032**	**6212**	**7357**
86	413	**428**	**1053**	967	**1882**	1573	**2875**	2375	**3958**	3103	**5164**	**6346**	**7532**
87	413	**428**	**1073**	976	**1934**	1598	**2913**	2375	**4095**	3103	**5225**	**6469**	**7639**
88	434	**437**	**1086**	1050	**1954**	1685	**2975**	2398	**4166**	3183	**5340**	**6660**	**7877**
89	426	**443**	**1102**	1050	**1990**	1690	**3067**	2400	**4176**	3183	**5450**	**6750**	**7950**
90	446	**481**	**1134**	1060	**2027**	1710	**3104**	2516	**4308**	3190	**5576**	**6901**	**8128**
91	434	**481**	**1134**	1089	**2031**	1748	**3143**	2516	**4379**	3234	**5626**	**6950**	**8330**
92	446	**481**	**1149**	1089	**2100**	1805	**3216**	2599	**4428**	3277	**5758**	**7067**	**8442**
93	446	**481**	**1171**	1098	**2130**	1813	**3283**	2604	**4512**	3361	**5832**	**7342**	**8601**
94	470	**481**	**1199**	1124	**2169**	1881	**3348**	2747	**4581**	3474	**6007**	**7436**	**8774**
95	**482**	481	**1219**	1135	**2206**	1901	**3335**	2747	**4703**	3531	**6064**	**7469**	**8877**
96	486	**509**	1250	**1261**	**2227**	1965	**3451**	2769	**4808**	3639	**6222**	**7645**	**9146**
97	474	**515**	1258	**1261**	**2299**	1965	**3514**	2817	**4848**	3639	**6304**	**7781**	**9379**
98	485	**531**	**1283**	1261	**2299**	1965	**3560**	2850	**4936**	3690	**6376**	**7896**	**9381**
99	489	**531**	**1298**	1261	**2338**	2009	**3628**	2878	**4999**	3731	**6448**	**8023**	**9617**
100	494	**554**	**1305**	1261	**2401**	2053	**3648**	3000	**5040**	3903	**6617**	**8228**	**9835**
105	521	**563**	**1395**	1329									
110	566	**626**	**1510**	1414									
115	594	**650**	**1591**	1499									
120	629	**702**	**1708**	1603									
125	629	**713**	**1798**	1750									
130	693	**766**	**1906**	1872									
135	729	**780**	**1995**	1909									
140	758	**845**	**2103**	2089									
145	779	**894**	2185	**2225**									
150	825	**934**	**2310**	2278									
155	842	**986**	2365	**2367**									
160	854	**1002**	2486	**2548**									
165	904	**1041**	2582	**2648**									
170	914	**1121**	2659	**2869**									
175	965	**1132**	2771	**2902**									
180	1011	**1208**	2897	**3077**									
185	1026	**1224**	2970	**3267**									
190	1061	**1298**	3094	**3325**									
195	1086	**1350**	3210	**3492**									
200	1106	**1371**	3257	**3596**									
205	1166	**1425**	3273	**3708**									
210	1196	**1473**	3377	**3767**									
215	1229	**1538**	3476	**3983**									
220	1259	**1544**	3543	**4159**									
225	1293	**1611**	3661	**4292**									
230	1329	**1646**	3703	**4326**									
235	1305	**1706**	3815	**4532**									
240	1350	**1806**	3893	**5061**									
245	1397	**1891**	3986	**5061**									
250	1412	**1901**	3990	**5075**									
255	1417	**1923**	4100	**5122**									
260	1445	**1971**	4164	**5236**									
265	1449	**2021**	4182	**5519**									
270	1464	**2144**	4361	**5656**									
275	1478	**2150**	4487	**5746**									
280	1493	**2184**	4388	**6023**									
285	1501	**2209**	4607	**6094**									
290	1476	**2269**	4722	**6380**									
295	1526	**2354**	4726	**6590**									
300	1542	**2409**	4898	**6604**									

Table 5: Squared ℓ^2-separation distance found using periodic designs (PD) and the ESE-algorithm (ESE).

n	2dim		3dim		4 dim		5 dim		6 dim	7 dim	8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE	ESE
2	0.500	0.500	0.333	0.333	0.250	0.250	0.200	0.200	0.167	0.143	0.125	0.111	0.100
3	0.900	0.900	0.611	0.611	0.386	0.450	0.321	0.362	0.250	0.230	0.193	0.200	0.151
4	1.000	1.000	0.642	0.642	0.454	0.489	0.367	0.382	0.300	0.260	0.225	0.201	0.180
5	1.298	1.390	0.727	0.891	0.509	0.658	0.401	0.527	0.336	0.287	0.250	0.222	0.200
6	1.521	1.521	0.794	0.800	0.561	0.594	0.431	0.476	0.358	0.307	0.268	0.238	0.215
7	1.598	1.598	0.867	0.975	0.599	0.694	0.464	0.532	0.376	0.322	0.282	0.250	0.225
8	1.804	1.879	0.921	0.960	0.619	0.696	0.488	0.538	0.398	0.334	0.292	0.260	0.234
9	1.935	1.935	0.971	1.052	0.660	0.742	0.504	0.567	0.414	0.349	0.301	0.267	0.240
10	2.066	2.066	1.020	1.085	0.686	0.744	0.515	0.556	0.425	0.360	0.311	0.273	0.246
11	2.196	2.279	1.069	1.137	0.709	0.785	0.536	0.612	0.434	0.369	0.319	0.281	0.250
12	2.273	2.273	1.095	1.163	0.724	0.785	0.551	0.589	0.441	0.375	0.326	0.287	0.256
13	2.401	2.487	1.128	1.191	0.746	0.825	0.563	0.632	0.453	0.381	0.331	0.292	0.261
14	2.476	2.476	1.167	1.252	0.762	0.829	0.575	0.635	0.462	0.385	0.335	0.296	0.265
15	2.578	2.643	1.194	1.255	0.775	0.818	0.583	0.636	0.470	0.393	0.339	0.299	0.268
16	2.666	2.683	1.221	1.290	0.791	0.848	0.589	0.642	0.477	0.398	0.341	0.302	0.271
17	2.721	2.721	1.246	1.340	0.805	0.866	0.600	0.656	0.483	0.404	0.347	0.305	0.273
18	2.819	2.848	1.271	1.337	0.816	0.875	0.609	0.655	0.488	0.408	0.350	0.307	0.275
19	2.890	2.984	1.292	1.374	0.827	0.895	0.615	0.667	0.492	0.413	0.354	0.310	0.277
20	2.959	2.962	1.318	1.394	0.835	0.907	0.620	0.681	0.496	0.416	0.358	0.313	0.278
21	3.025	3.033	1.339	1.408	0.847	0.914	0.625	0.671	0.501	0.419	0.361	0.316	0.281
22	3.070	3.070	1.357	1.426	0.856	0.922	0.632	0.687	0.505	0.422	0.363	0.318	0.283
23	3.138	3.159	1.377	1.454	0.868	0.925	0.638	0.693	0.510	0.425	0.366	0.321	0.285
24	3.197	3.201	1.396	1.458	0.875	0.931	0.644	0.677	0.513	0.427	0.368	0.323	0.287
25	3.254	3.293	1.412	1.485	0.884	0.940	0.648	0.701	0.516	0.430	0.370	0.324	0.289
26	3.309	3.332	1.428	1.480	0.891	0.947	0.653	0.707	0.518	0.432	0.372	0.326	0.290
27	3.360	3.383	1.442	1.499	0.898	0.957	0.657	0.708	0.521	0.435	0.373	0.328	0.292
28	3.405	3.420	1.454	1.503	0.906	0.961	0.660	0.712	0.524	0.437	0.375	0.329	0.293
29	3.458	3.539	1.468	1.543	0.912	0.978	0.664	0.716	0.527	0.439	0.376	0.330	0.294
30	3.505	3.515	1.481	1.528	0.919	0.974	0.667	0.716	0.530	0.441	0.378	0.331	0.295
31	3.543	3.550	1.493	1.563	0.925	0.976	0.671	0.719	0.533	0.443	0.380	0.333	0.296
32	3.589	3.623	1.505	1.562	0.931	0.996	0.674	0.729	0.535	0.444	0.381	0.334	0.297
33	3.636	3.642	1.517	1.588	0.935	0.990	0.678	0.732	0.537	0.446	0.383	0.335	0.298
34	3.676	3.713	1.528	1.565	0.941	1.005	0.682	0.735	0.540	0.447	0.384	0.336	0.299
35	3.716	3.786	1.539	1.601	0.946	1.003	0.685	0.731	0.542	0.449	0.385	0.337	0.300
36	3.758	3.774	1.549	1.600	0.950	1.005	0.688	0.734	0.543	0.450	0.386	0.338	0.301
37	3.794	3.819	1.558	1.599	0.956	1.019	0.691	0.736	0.545	0.452	0.387	0.339	0.301
38	3.828	3.828	1.568	1.623	0.959	1.023	0.694	0.746	0.547	0.453	0.388	0.340	0.302
39	3.868	3.879	1.578	1.646	0.965	1.025	0.696	0.742	0.548	0.455	0.389	0.341	0.303
40	3.906	3.918	1.587	1.636	0.968	1.019	0.699	0.742	0.550	0.456	0.390	0.341	0.303
41	3.939	4.009	1.596	1.639	0.971	1.033	0.701	0.751	0.551	0.457	0.391	0.342	0.304
42	3.974	3.974	1.604	1.658	0.975	1.031	0.703	0.742	0.552	0.458	0.392	0.343	0.305
43	4.007	4.045	1.612	1.675	0.979	1.042	0.705	0.752	0.554	0.460	0.393	0.344	0.306
44	4.029	4.029	1.621	1.670	0.983	1.040	0.708	0.754	0.555	0.461	0.394	0.344	0.306
45	4.063	4.074	1.628	1.678	0.986	1.044	0.710	0.752	0.557	0.462	0.394	0.345	0.307
46	4.096	4.115	1.636	1.693	0.990	1.044	0.712	0.753	0.559	0.463	0.395	0.346	0.307
47	4.130	4.179	1.643	1.695	0.993	1.055	0.714	0.761	0.560	0.464	0.396	0.346	0.308
48	4.160	4.206	1.650	1.699	0.997	1.052	0.716	0.759	0.561	0.464	0.397	0.347	0.308
49	4.187	4.187	1.657	1.711	1.001	1.059	0.718	0.762	0.563	0.465	0.398	0.347	0.309
50	4.216	4.254	1.665	1.713	1.004	1.058	0.720	0.765	0.564	0.466	0.398	0.348	0.309
51	4.246	4.280	1.671	1.729	1.007	1.063	0.721	0.768	0.566	0.467	0.399	0.348	0.310
52	4.273	4.277	1.678	1.730	1.010	1.062	0.723	0.765	0.567	0.468	0.400	0.349	0.310
53	4.302	4.343	1.685	1.734	1.013	1.072	0.725	0.771	0.568	0.468	0.400	0.349	0.310
54	4.331	4.341	1.690	1.739	1.016	1.070	0.726	0.769	0.569	0.469	0.401	0.350	0.311
55	4.355	4.413	1.697	1.755	1.018	1.073	0.728	0.773	0.570	0.470	0.401	0.350	0.311
56	4.382	4.404	1.703	1.756	1.022	1.071	0.729	0.772	0.571	0.470	0.402	0.351	0.312
57	4.404	4.427	1.708	1.760	1.024	1.079	0.731	0.776	0.572	0.471	0.403	0.351	0.312
58	4.431	4.437	1.714	1.763	1.027	1.076	0.732	0.776	0.573	0.472	0.403	0.352	0.312
59	4.458	4.498	1.719	1.777	1.030	1.087	0.734	0.780	0.574	0.473	0.404	0.352	0.313
60	4.482	4.490	1.725	1.772	1.032	1.079	0.735	0.777	0.575	0.473	0.404	0.353	0.313
61	4.499	4.530	1.731	1.778	1.034	1.087	0.736	0.776	0.576	0.474	0.405	0.353	0.314
62	4.526	4.576	1.736	1.786	1.036	1.087	0.738	0.781	0.576	0.475	0.405	0.354	0.314
63	4.556	4.576	1.742	1.789	1.039	1.094	0.739	0.783	0.577	0.475	0.406	0.354	0.314
64	4.573	4.590	1.746	1.794	1.041	1.087	0.740	0.784	0.578	0.476	0.406	0.354	0.315
65	4.595	4.599	1.751	1.802	1.043	1.095	0.742	0.786	0.579	0.477	0.407	0.355	0.315
66	4.619	4.635	1.757	1.804	1.045	1.093	0.742	0.785	0.580	0.477	0.407	0.355	0.315
67	4.636	4.642	1.761	1.812	1.047	1.100	0.744	0.787	0.581	0.478	0.407	0.355	0.315
68	4.661	4.681	1.766	1.819	1.049	1.096	0.745	0.790	0.581	0.478	0.407	0.356	0.316
69	4.683	4.713	1.771	1.818	1.052	1.104	0.746	0.791	0.582	0.479	0.408	0.356	0.316
70	4.703	4.710	1.775	1.818	1.053	1.100	0.747	0.790	0.583	0.480	0.408	0.356	0.316

Table 6: Audze-Eglais values found using periodic designs (PD) and the ESE-algorithm (ESE).

n	2dim		3dim		4 dim		5 dim		6 dim	7 dim	8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE	ESE
71	**4.727**	4.742	**1.780**	1.831	**1.055**	1.108	**0.747**	0.795	**0.584**	**0.480**	**0.409**	**0.357**	**0.317**
72	**4.743**	4.746	**1.784**	1.829	**1.057**	1.103	**0.749**	0.791	**0.584**	**0.481**	**0.409**	**0.357**	**0.317**
73	**4.763**	4.781	**1.789**	1.836	**1.059**	1.106	**0.749**	0.794	**0.585**	**0.481**	**0.409**	**0.357**	**0.317**
74	**4.781**	4.820	**1.793**	1.842	**1.061**	1.112	**0.751**	0.795	**0.586**	**0.482**	**0.410**	**0.358**	**0.317**
75	**4.803**	4.817	**1.796**	1.847	**1.063**	1.112	**0.752**	0.797	**0.586**	**0.482**	**0.410**	**0.358**	**0.318**
76	**4.823**	4.828	**1.801**	1.850	**1.064**	1.110	**0.753**	0.796	**0.587**	**0.483**	**0.410**	**0.358**	**0.318**
77	**4.838**	4.853	**1.805**	1.851	**1.066**	1.112	**0.755**	0.799	**0.587**	**0.483**	**0.411**	**0.359**	**0.318**
78	**4.863**	4.883	**1.809**	1.847	**1.068**	1.114	**0.755**	0.795	**0.588**	**0.484**	**0.411**	**0.359**	**0.318**
79	**4.882**	4.934	**1.812**	1.863	**1.070**	1.119	**0.756**	0.801	**0.589**	**0.484**	**0.411**	**0.359**	**0.318**
80	**4.895**	4.922	**1.816**	1.864	**1.071**	1.117	**0.757**	0.799	**0.589**	**0.484**	**0.412**	**0.359**	**0.319**
81	**4.920**	4.942	**1.820**	1.869	**1.072**	1.120	**0.758**	0.802	**0.590**	**0.485**	**0.412**	**0.360**	**0.319**
82	**4.936**	4.944	**1.824**	1.862	**1.074**	1.120	**0.759**	0.801	**0.590**	**0.485**	**0.413**	**0.360**	**0.319**
83	**4.949**	4.949	**1.827**	1.879	**1.076**	1.126	**0.760**	0.805	**0.591**	**0.486**	**0.413**	**0.360**	**0.319**
84	**4.968**	4.992	**1.831**	1.876	**1.077**	1.122	**0.761**	0.802	**0.591**	**0.486**	**0.413**	**0.360**	**0.320**
85	**4.985**	5.014	**1.834**	1.879	**1.079**	1.124	**0.761**	0.804	**0.592**	**0.486**	**0.414**	**0.360**	**0.320**
86	**5.003**	5.014	**1.838**	1.882	**1.081**	1.125	**0.762**	0.804	**0.592**	**0.487**	**0.414**	**0.361**	**0.320**
87	**5.019**	5.060	**1.842**	1.891	**1.082**	1.130	**0.763**	0.808	**0.593**	**0.487**	**0.414**	**0.361**	**0.320**
88	**5.034**	5.047	**1.845**	1.885	**1.083**	1.130	**0.764**	0.805	**0.594**	**0.487**	**0.414**	**0.361**	**0.320**
89	**5.056**	5.096	**1.848**	1.895	**1.085**	1.133	**0.765**	0.810	**0.594**	**0.488**	**0.415**	**0.361**	**0.321**
90	5.070	**5.063**	**1.852**	1.885	**1.086**	1.131	**0.766**	0.807	**0.594**	**0.488**	**0.415**	**0.361**	**0.321**
91	**5.086**	5.113	**1.854**	1.890	**1.088**	1.134	**0.766**	0.809	**0.595**	**0.489**	**0.415**	**0.362**	**0.321**
92	**5.104**	5.114	**1.858**	1.902	**1.089**	1.135	**0.767**	0.809	**0.595**	**0.489**	**0.416**	**0.362**	**0.321**
93	**5.119**	5.122	**1.861**	1.903	**1.090**	1.136	**0.768**	0.810	**0.596**	**0.489**	**0.416**	**0.362**	**0.321**
94	**5.130**	5.143	**1.864**	1.900	**1.092**	1.138	**0.769**	0.810	**0.596**	**0.490**	**0.416**	**0.362**	**0.321**
95	**5.151**	5.177	**1.867**	1.909	**1.093**	1.138	**0.769**	0.813	**0.597**	**0.490**	**0.416**	**0.362**	**0.322**
96	**5.163**	5.183	**1.870**	1.910	**1.094**	1.139	**0.770**	0.811	**0.597**	**0.490**	**0.417**	**0.363**	**0.322**
97	**5.177**	5.179	**1.872**	1.915	**1.096**	1.138	**0.771**	0.814	**0.598**	**0.490**	**0.417**	**0.363**	**0.322**
98	**5.198**	5.223	**1.876**	1.915	**1.097**	1.142	**0.771**	0.812	**0.598**	**0.491**	**0.417**	**0.363**	**0.322**
99	**5.211**	5.244	**1.879**	1.923	**1.098**	1.143	**0.772**	0.815	**0.599**	**0.491**	**0.417**	**0.363**	**0.322**
100	5.223	**5.221**	**1.882**	1.921	**1.099**	1.143	**0.773**	0.812	**0.599**	**0.491**	**0.418**	**0.363**	**0.322**

Table 6: Audze-Eglais values found using periodic designs (PD) and the ESE-algorithm (ESE).

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